REGULARITY OF FINITE-DIMENSIONAL REALIZATIONS FOR EVOLUTION EQUATIONS

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Abstract. We show that a continuous local semiflow of C^k-maps on a finite-dimensional C^k-manifold M with boundary is in fact a local C^k-semiflow on M and can be embedded into a local C^k-flow around interior points of M under some weak assumption. This result is applied to an open regularity problem for finite-dimensional realizations of stochastic interest rate models.

1. Introduction

Let $k \geq 1$ be given. We consider a Banach space X and a *continuous local semiflow Fl of C^k-maps* on an open subset $V \subset X$, i.e.

i) There is $\varepsilon > 0$ and $V \subset X$ open with $Fl : [0, \varepsilon[\times V \to X$ a continuous map.
ii) $Fl(0, x) = x$ and $Fl(s, Fl(t, x)) = Fl(s + t, x)$ for $s, t, s + t \in [0, \varepsilon[$ and $x, Fl(t, x) \in V$.
iii) The map $Fl_t : V \to X$ is C^k for $t \in [0, \varepsilon[$.

To shorten terminology we say that "Fl" is a continuous local semiflow of C^k-maps on X if for any $x \in X$ there is an open neighborhood $V \subset X$ of x and a continuous local semiflow $Fl = Fl^{(V)}$ of C^k-maps on V, such that $Fl^{(V_1)} = Fl^{(V_2)}$ on $V_1 \cap V_2$. Continuous local semiflows of C^k-maps appear naturally as mild solutions of nonlinear evolution equations (see Appendix A). The continuous local semiflow Fl is called C^k or *local C^k-semiflow* if $Fl : [0, \varepsilon[\times V \to X$ is C^k.

We assume that we are given a finite-dimensional C^k-submanifold M with boundary of X such that M is *locally invariant* for Fl, i.e. for every $x \in M \cap V$ there is $\delta_x \in]0, \varepsilon[$ such that $Fl(t, x) \in M$ for $0 \leq t \leq \delta_x$. In this case Fl restricts in a small open neighborhood of any point $x \in M \cap V$ to a continuous local semiflow of C^k-maps on M, see Lemma 1.3 below, where we make the restriction precise. A continuous local semiflow of C^k-maps on M is defined as above, the same for local C^k-semiflows: notice however that the manifold might have a boundary (for the notions of analysis in this case see for example [6]). By T_xM we denote the (full) tangent space at $x \in M$, even at the boundary. By $(T_xM)_{\geq 0}$ we denote the halfspace of inward pointing tangent vectors for $x \in \partial M$. The boundary subspace of this halfspace is the tangent space of the C^k-submanifold (without boundary) ∂M, these are the tangent vectors parallel to the boundary.

We shall prove that the restriction of a continuous local semiflow of C^k-maps Fl to a C^k-submanifold with boundary M is jointly C^k and can in particular be embedded in a local C^k-flow around any interior point of M. We shall apply classical

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methods from [9] developed to solve the fifth Hilbert problem. Nevertheless we have to face the difficulty that Fl is only a continuous *local semi*flow. We can prove the result under a weak assumption, which will always be satisfied with respect to our applications. This problem arises in several contexts, for example recently in interest rate theory, see [2, 3].

We first cite the classical results from Dean Montgomery and Leo Zippin [9] and draw a simple conclusion, which illustrates, what are going to do, namely proving a non-linear version of Example 1.2.

Theorem 1.1. *Let M be a finite-dimensional C^k-manifold and $Fl : \mathbb{R} \times M \to M$ a continuous flow of C^k-maps on M, then Fl is a C^k-flow on M.*

Example 1.2. *Let S be a strongly continuous group on a Banach space X and assume that M is a locally S-invariant finite-dimensional C^k-submanifold of X. Then $M \subset D(A^k)$, where A denotes the infinitesimal generator of S, and the restriction of A to M is a C^{k-1}-vector field on M.*

The paper is organized as follows. In Section 2 we prove the extension of Theorem 1.1 for continuous local semiflows of C^k-maps. In Section 3 we apply this result to a problem that arises in connection with stochastic interest rate models as it has been announced in [2]: finite-dimensional realizations are highly regular objects, namely given by submanifolds with boundary of $D(A^\infty)$, where A is the generator of a strongly continuous semigroup. The appendix contains a regularity result for the dependence of solutions to evolution equations on the initial point.

We end this section by the announced lemma. Let M be a finite-dimensional C^k-submanifold with boundary and let M be locally invariant for Fl, as defined above. We denote by $\mathbb{R}^n_{\geq 0}$ the halfspace $\{x \in \mathbb{R}^n ; x_n \geq 0\}$, consequently $\mathbb{R}_{\geq 0}$ is the positive halfline including 0.

Lemma 1.3. *For every $x \in M \cap V$ there exists an open neighborhood $V' \subset X$ of x and $\varepsilon' > 0$, such that $Fl(t, y) \in M$ for all $(t, y) \in [0, \varepsilon'[\times (V' \cap M)$.*

Proof. Take $x \in M$ and a C^k-submanifold chart $u : U \subset X \to X$ with $u(U \cap M) = \{0\} \times W \subset \{0\} \times \mathbb{R}^n_{\geq 0}$, where $U \subset \overline{U} \subset V$ is open and $x \in U$, $W \subset \mathbb{R}^n_{\geq 0}$ is open, convex. Here n denotes the dimension of M. We may assume that u has a continuous extension on $\overline{U}$ with $u(\overline{U \cap M}) = u(\overline{U} \cap M) = \{0\} \times \overline{W}$ by restriction of U. The closure of $U \cap M$ is taken in M.

For $y \in U$ define the lifetime in $U \cap M$

$$T(y) := \sup\{0 < t < \varepsilon \mid \ \forall 0 \leq s < t : \ Fl(s, y) \in U \cap M, \}.$$

By continuity of Fl we have $Fl(T(y), y) \in \overline{U \cap M} \setminus (U \cap M)$ if $T(y) < \varepsilon$. We claim that there exists an open neighborhood $V' \subset U$ of x in X and $\varepsilon' > 0$ such that $T(y) \geq \varepsilon'$ for all $y \in V'$. Indeed, otherwise we could find a sequence (x_n) in $U \cap M$ with $x_n \to x$ and $\varepsilon > T(x_n) \to 0$. But this means that $u(Fl(T(x_n), x_n)) \in \{0\} \times (\overline{W} \setminus W)$ converges to $u(Fl(0, x)) = u(x) \in \{0\} \times W$, a contradiction. Whence the claim, and the lemma follows. ∎

2. The classical proof revisited

Since we are treating local questions as differentiability, we can – without any restriction – assume that $f : [0, \varepsilon[\times V \to \mathbb{R}^n_{\geq 0}$ is a given continuous local semiflow of C^k-maps, where V is open, convex in $\mathbb{R}^n_{\geq 0}$. For the notion of differentiability on manifolds with boundary see for example [6]. We do not make a difference in notation between right derivatives and derivatives, even though on the boundary points in space or time, respectively, we only calculate right derivatives. We shall always assume in this section that f is continuous and $f(t,.)$ is C^k for all $t \in [0,\varepsilon[$, for some $k \geq 1$. We shall write $f(t,x) = (f_1(t,x),...,f_n(t,x))$ for $(t,x) \in [0,\varepsilon[\times V$.

Assumption (crucial). *We assume that for any $x \in V$ there is $\varepsilon_x > 0$ such that $D_x f(t,x)$ is invertible for $0 \leq t \leq \varepsilon_x$ ($D_x f$ denotes the derivative with respect to x).*

Lemma 2.1. *The mapping $(t,x) \mapsto D_x f(t,x)$ is continuous.*

Proof. For the proof we proceed from the Baire category Theorem and Lemma 2 of [9] on p. 198. We then have the following result:

Let Z be any compact interval, V the open set in $\mathbb{R}^n_{\geq 0}$ and let $F : Z \times V \to \mathbb{R}$ be a continuous real valued function, such that $F(g,.)$ is C^1 for any $g \in Z$. Given $a \in V$ and $1 \leq i \leq n$, the set of points $g_0 \in Z$ such that $\frac{\partial}{\partial x_i} F$ is continuous at (g_0, a) is dense in Z, even more, the set where it is not continuous is of first category in Z.

Let now $a \in V$ be fixed, then the set of points $t_0 \in [0,\varepsilon[$ such that $f_{ij} := \frac{\partial}{\partial x_i} f_j$ is continuous at (t_0, a), for all $1 \leq i,j \leq n$, is everywhere dense in $[0,\varepsilon[$. We shall denote this set by I_a. In addition the determinant $\det(f_{ij})$ is continuous at these points, too. We want to show now that for fixed $a \in V$ the mappings f_{ij} are continuous at $(0,a)$. Notice that the determinant at any point of continuity (t_0,a), with $t_0 \in I_a$ small enough, is bounded away from zero in a neighborhood.

We fix $a \in V$, then for $t_0 \in [0,\varepsilon[$

$$f(t_0 + h, a + y) = f(t_0, f_1(h, a+y), ..., f_n(h, a+y))$$

for $h \geq 0$ and $y \in \mathbb{R}^n_{\geq 0}$, both sufficiently small, hence

$$D_x f(t_0 + h, a + y) = D_x f(t_0, f(h, a+y)) \cdot D_x f(h, a+y).$$

There is $t_0 \in I_a$ such that $D_x f(t_0, z)$ is invertible in a neighborhood of a, hence

$$D_x f(t_0, f(h, a+y))^{-1} \cdot D_x f(t_0 + h, a+y) = D_x f(h, a+y)$$

and therefore

$$id = \lim_{h \downarrow 0, y \to 0} D_x f(t_0, f(h, a+y))^{-1} \cdot D_x f(t_0 + h, a+y) = \lim_{h \downarrow 0, y \to 0} D_x f(h, a+y)$$

by continuity of $D_x f$ at (t_0, a), continuity of f in both variables and the continuity of the inversion of matrices. So $0 \in I_a$ for all $a \in V$.

Now we can conclude for arbitrary $t \in]0,\varepsilon[$ in the following way:

$$D_x f(t+h, a+y) = D_x f(t, f(h, a+y)) \cdot D_x f(h, a+y)$$

for $h \geq 0$ and $y \in \mathbb{R}^n_{\geq 0}$ sufficiently small, hence by continuity at $(0,a)$

$$\lim_{h \downarrow 0, y \to 0} D_x f(t+h, a+y) = \lim_{h \downarrow 0, y \to 0} D_x f(t, f(h, a+y)) \cdot D_x f(h, a+y) = D_x f(t,a).$$

For left continuity we apply

$$D_x f(t, a+y) = D_x f(h, f(t-h, a+y)) \cdot D_x f(t-h, a+y)$$

for $h \geq 0$ and $y \in \mathbb{R}^n_{\geq 0}$ sufficiently small, hence by continuity of $D_x f$ at $(0,a)$ and $(0, f(t,a))$, the continuity of $D_x f$ in the second variable and the existence of the inverse for small h

$$\lim_{h\downarrow 0, y\to 0} D_x f(t-h, a+y) = \lim_{h\downarrow 0, y\to 0} D_x f(h, f(t-h, a+y))^{-1} \cdot D_x f(t, a+y) = D_x f(t,a).$$

Consequently the desired assertion holds. □

In the next step we shall show that there is a derivative at 0.

Lemma 2.2. *The right-hand derivative* $\frac{d}{dt} f(t,x)|_{t=0}$ *exists for* $x \in V$, *and for small* $h \geq 0$ *we have the formula*

$$f(h,x) - x = \int_0^h D_x f(t,x) dt \cdot (\frac{d}{dt} f(0,x)).$$

Moreover, $\frac{d}{dt} f(t,.)|_{t=0} : V \to \mathbb{R}^n$ *is continuous.*

Proof. We may differentiate with respect to x under the integral sign by Lemma 2.1 und uniform convergence, so

$$T(h,x) := \int_0^h f(t,x) dt$$
$$D_x T(h,x) := \int_0^h D_x f(t,x) dt.$$

By the mean value theorem we obtain

$$T(h,y) - T(h,x) = D_x T(h, \widetilde{x})(y - x),$$

where $\widetilde{x} \in [x, y]$. Now we take $y = f(p,x)$, then

$$\begin{aligned} T(h,y) - T(h,x) &= \int_p^{h+p} f(t,x) dt - \int_0^h f(t,x) dt \\ &= \int_h^{h+p} f(t,x) dt - \int_0^p f(t,x) dt, \end{aligned}$$

which finally yields

$$\frac{1}{p}(\int_0^p f(t+h,x) dt - \int_0^p f(t,x) dt) = D_x T(h, \widetilde{x})[\frac{1}{p}(f(p,x) - x)].$$

This equation can be solved by joint continuity of $(h,z) \mapsto \frac{1}{h}\int_0^h D_x f(t,z) dt$: we obtain for small h and a compact set in x that the expression is in a small neighborhood of the identity matrix. So inversion leads to the desired result and then to the given formula.

The formula asserts again by inversion, that the derivative is continuous with respect to x. □

By the semigroup-property and the chain rule, the result of Lemma 2.2 can be extended for $0 < t < \varepsilon$, and the derivative of $f(\cdot, x)$ exists for all $t \in [0, \varepsilon[$. Indeed,

$$\begin{aligned}
\lim_{p\downarrow 0} \frac{f(t+p,x) - f(t,x)}{p} &= \lim_{p\downarrow 0} \frac{f(p, f(t,x)) - f(t,x)}{p} = \frac{d}{dt} f(0, f(t,x)) \\
\lim_{p\downarrow 0} \frac{f(t,x) - f(t-p,x)}{p} &= \lim_{p\downarrow 0} \frac{f(p, f(t-p,x)) - f(t-p,x)}{p} \\
&= \lim_{p\downarrow 0} \frac{1}{p} (\int_0^p D_x f(t, f(t-p,x)) dt) \frac{d}{dt} f(0, f(t-p,x)) \\
&= \frac{d}{dt} f(0, f(t,x))
\end{aligned}$$

by Lemmas 2.1 and 2.2. Consequently for small $h \geq 0$

$$\frac{d}{dt} f(t,x) = \frac{d}{dt} f(0, f(t,x)) = (\int_0^h D_x f(s, f(t,x)) ds)^{-1} \cdot (f(h,x) - x). \tag{2.1}$$

In particular $(t,x) \mapsto \frac{d}{dt} f(t,x)$ is continuous in both variables on the whole domain of definition.

Lemma 2.3. *The semiflow f is C^k in both variables.*

Proof. If $f(t,.)$ is C^k for $t \in [0, \varepsilon[$, then the r-jet

$$(f(t,x_0), D_x f(t,x_0) \cdot x_1, ..., D_x^r f(t,x_0) \cdot x_1 \cdot ... \cdot x_r)$$

for $(t, x_0, x_1, ..., x_r) \in [0, \varepsilon[\times V \times \mathbb{R}^n \times ... \times \mathbb{R}^n$ is a local semiflow of C^{k-r}-maps, for $0 \leq r \leq k-1$. For $r = 1$, the 1-jet is a continuous, local semiflow of C^{k-1}-maps, by Lemma 2.1. Assume that for $r < k$ the r-jet is a continuous, local semiflow then, by Lemma 2.1 again, the $(r+1)$-jet is continuous. By induction

$$(t,x) \mapsto D_x^r f(t,x)$$

is continuous in both variables for $0 \leq r \leq k$.

If we apply the above results to the r-jet for $r < k$, we conclude by equation (2.1) that $D_x^r f(t,x)$ can be $(k-r)$ times differentiated with respect to the t-variable, and these derivatives are continuous. Hence f is C^k in both variables. $\square$

Theorem 2.4. *Let $k \geq 1$ be given and let $Fl : [0, \varepsilon[\times U \to M$ be a local semiflow on a finite-dimensional C^k-manifold M with boundary, which satisfies the following conditions:*

i) *The semiflow $Fl : [0, \varepsilon[\times U \to M$ is continuous with $U \subset M$ open.*
ii) *The mapping $Fl(t,.)$ is C^k.*
iii) *For fixed $x \in U$ there exists $\varepsilon_x > 0$ such that $T_x Fl(t,.)$ is invertible for $0 \leq t \leq \varepsilon_x$.*

Then Fl is C^k and for any $x_0 \in U \setminus \partial M$ there is a local C^k-flow $\tilde{Fl} :]-\delta, \delta[\times V \to M$ with $V \subset U \setminus \partial M$ open around x_0 and $\delta \leq \varepsilon$ such that $Fl(y,t) = \tilde{Fl}(y,t)$ for $y \in V$ and $0 \leq t \leq \delta$. This also holds for the smooth case ($k = \infty$).

Proof. By the previous lemmas the map $Fl : [0, \varepsilon[\times U \to M$ is a C^k-semiflow on M. We fix $x_0 \in U \setminus \partial M$, then there is $0 < \delta < \varepsilon$ and $W \subset U$ open in $M \setminus \partial M$, such that $(t,x) \mapsto (t, Fl(t,x))$ is C^k-invertible on $[0, \delta[\times W$ by the C^k-inverse function theorem on manifolds with boundary (see [6]). We then choose an open neighborhood $V \subset \cap_{0 \leq t < \delta} Fl(t, W)$ of x_0 in $M \setminus \partial M$. Therefore we can define $\tilde{Fl}(-t, y) := \{Fl(.,.)\}^{-1}(t,y)$ for $t \in [0, \delta[$ and $y \in V$. Since this is the unique solution in z of the C^k-equation $Fl(t,z) = y$, we obtain a C^k-map $\tilde{Fl}$. The flow property holds by uniqueness, too. Notice that V can be chosen independent of k. $\square$

Remark 2.5. *Remark that for evolutions (which correspond in the differentiable case to time-dependent vector fields) we can pass to the extended phase space and apply the results thereon.*

3. Applications

The following application has been announced in [2] in connection with finite-dimensional realizations for stochastic models of the interest rates. Let X be a Banach space, S a strongly continuous semigroup on X with infinitesimal generator $A : D(A) \to X$, and let $P : X \to X$ be a locally Lipschitz map. For $x \in D(A)$ we write

$$\mu(x) := Ax + P(x).$$

Proposition A.2 yields the existence of a continuous, local semiflow Fl^μ of mild solutions to the evolution equation

$$\frac{d}{dt}x(t) = \mu(x(t)). \tag{3.1}$$

That is, for every $x_0 \in X$ there exists a neighborhood U of x_0 in X and $T > 0$ such that $Fl^\mu \in C([0,T] \times U, X)$ and

$$Fl^\mu(t,x) = S_t x + \int_0^t S_{t-s} P(Fl^\mu(s,x))ds, \quad \forall (t,x) \in [0,T] \times U. \tag{3.2}$$

Now let $k \geq 1$, and M be a finite-dimensional C^k-submanifold with boundary in X, which is locally invariant for Fl^μ. Hence, by Lemma 1.3, $x_0 \in M$ implies $Fl^\mu(t,x) \in M$ for all $(t,x) \in [0,T] \times (U \cap M)$, for some open neighborhood U of x_0 in X and $T > 0$. By the methods of [1] (see also Remark 3.3 below) we obtain that necessarily $M \subset D(A)$ and

$$\forall x \in M : \mu(x) \in T_x M \text{ and } \forall x \in \partial M : \mu(x) \in (T_x M)_{\geq 0}, \tag{3.3}$$

since μ has to be additionally inward pointing. We now can strengthen this result.

Theorem 3.1. *Suppose*

$$P \in \bigcap_{r=0}^{k} C^{k-r}(X, D(A^r)) \tag{3.4}$$

and $D_x^k P$ is locally Lipschitz continuous. Then $M \subset D(A^k)$ and $\mu|_M$ is a C^{k-1}-vector field on M.

Proof. Let $x_0 \in M$, and U, T as above. Hence $Fl^\mu(t,x) \in M$ for all $(t,x) \in [0,T] \times (U \cap M)$. By the assumptions we made, Theorem A.3 applies and we may assume that $Fl^\mu(t,\cdot)$ is C^k on U for all $t \in [0,T]$. Now let $x \in U \cap M$. We claim that there exists $\varepsilon_x > 0$ such that

$$D_x Fl^\mu(t,x) : T_x M \to T_{Fl^\mu(t,x)} M \text{ is invertible for } 0 \leq t \leq \varepsilon_x. \tag{3.5}$$

Indeed, let $y \in X$. The directional derivative $D_x Fl^\mu(t,x)y$ is continuous in t on $[0,T]$, see (A.5). Hence (3.2) and dominated convergence imply that

$$D_x Fl^\mu(t,x)y = S_t y + \int_0^t S_{t-s} DP(Fl^\mu(s,x)) D_x Fl^\mu(s,x)y \, ds.$$

By the bound (A.6) we conclude that

$$\sup_{y\in T_xM,\,\|y\|\le 1} \|D_x Fl^\mu(t,x)y - y\| \le \sup_{y\in T_xM,\,\|y\|\le 1} \|S_t y - y\| + O(t),$$

where $O(t)\to 0$ for $t\to 0$. Since T_xM is finite-dimensional, it is easy to see that

$$\sup_{y\in T_xM,\,\|y\|\le 1} \|S_t y - y\| \to 0 \quad \text{for } t\to 0.$$

Hence there exists $\varepsilon_x > 0$ such that $D_x Fl^\mu(t,x)$ restricted to T_xM is injective, and hence invertible, for all $t\in[0,\varepsilon_x]$. This yields the claim (3.5).

Therefore Theorem 2.4 applies and $Fl^\mu : [0,T]\times(U\cap M)\to M$ is C^k. In particular, since $\mu(x) = \partial_t Fl^\mu(0,x)$, $\mu|_M$ is a C^{k-1}-vector field on M, and $Fl^\mu(\cdot,x_0)$ is C^k on $[0,T]$.

¿From (3.2) we have

$$S_t x_0 = Fl^\mu(t,x_0) - \int_0^t S_{t-s} P(Fl^\mu(s,x_0))\,ds, \quad t\in[0,T].$$

By (3.4), the integral on the right is C^k in $t\in[0,T]$. Indeed, we obtain inductively by dominated convergence

$$\partial_t^r \int_0^t S_{t-s}P(Fl^\mu(s,x_0))\,ds = \partial_t^{r-1}P(Fl^\mu(t,x_0)) + \partial_t^{r-2}AP(Fl^\mu(t,x_0)) + \cdots + A^{r-1}P(Fl^\mu(t,x_0)) + \int_0^t S_{t-s}A^rP(Fl^\mu(s,x_0))\,ds,$$

for $r\le k$. We conclude that $S_t x_0$ is C^k in $t\in[0,T]$. But this means that $x_0\in D(A^k)$ and the theorem is proved. □

We now consider a setup that is given in [2]. Let W be a connected open set in X, $d\ge 1$ and $\sigma = (\sigma_1,\dots,\sigma_d)$ such that

(A1): P and σ_i are Banach maps from X into $D(A^\infty)$, for $1\le i\le d$.

(A2): $\mu,\sigma_1,\dots,\sigma_d$ are pointwise linearly independent on $W\cap D(A^\infty)$.

For the definition of a Banach map see [2, 4]. The Banach map principle ([4, Theorem 5.6.3]) yields that each σ_i generates a local flow Fl^{σ_i} on X with the following property: for every $x_0\in X$ there exists an open neighborhood V of x_0 in X and $T>0$ such that

$$Fl^{\sigma_i}\in C^\infty(]-T,T[\times V, X) \quad\text{and}\quad Fl^{\sigma_i}\in C^\infty(]-T,T[\times V', D(A^\infty)),$$

where $V' := V\cap D(A^\infty)$ is considered as an open set in $D(A^\infty)$, and $Fl^{\sigma_i}(\cdot,x)$ is the unique solution of

$$\frac{d}{dt}x(t) = \sigma_i(x(t)),\quad x(0)=x, \qquad (t,x)\in]-T,T[\times V.$$

Local invariance for Fl^{σ_i} is defined as for Fl^μ above.

Theorem 3.2. *Let $M\subset W$ be a $(d+1)$-dimensional C^∞-submanifold with boundary of X. If M is locally invariant for Fl^μ, $Fl^{\sigma_1},\dots,Fl^{\sigma_d}$, then M is a C^∞-submanifold with boundary of $D(A^\infty)$.*

Proof. Theorem 3.1 implies that $M\subset D(A^\infty)$ and $\mu|_M$ is a C^∞-vector field on M with respect to the given differentiable structure as submanifold with boundary of X. Furthermore $\sigma_1,\dots,\sigma_d$ restrict to smooth vector fields on M and $\sigma_i(x)\in T_x\partial M$ for $x\in\partial M$, since $\sigma_i(x)$ and $-\sigma_i(x)$ have to be inward pointing by local invariance.

We do also have integral curves for μ and $\sigma_1, \dots, \sigma_d$ on $D(A^\infty)$ which coincide with the respective integral curves on X on the intersection of the domains of definition if they start from the same point.

We have to construct submanifold charts for M, such that M is also a submanifold with boundary of $D(A^\infty)$. We shall do this by constructing smooth parametrizations $\alpha : U \to D(A^\infty)$ for any point $x_0 \in M$, then we apply [2, Lemma 3.1].

Let $x_0 \in M \backslash \partial M$. From [2, Section 2] we know that the vector field μ generates a smooth local semiflow on $D(A^\infty)$, which coincides locally by uniqueness of integral curves with the local C^∞-flow Fl^{μ_M} of $\mu|_M$ on a neighborhood of x_0. This means in particular that $t \mapsto Fl^{\mu_M}(t, x_0)$ is smooth with respect to the topology of $D(A^\infty)$. As in the proof of [2, Theorem 3.9] it follows, by **(A2)**, that

$$\alpha(u, x_0) := Fl^{\sigma_1}_{u_1} \circ \cdots \circ Fl^{\sigma_d}_{u_d} \circ Fl^{\mu_M}_{u_{d+1}}(x_0) : U \to D(A^\infty),$$

where U is an open, convex (sufficiently small) neighborhood of 0 in $\mathbb{R}^{d+1}$, is a diffeomorphism (it has maximal rank) to an open submanifold $N \subset M$ with respect to the differentiable structure as submanifold with boundary of X. But α is additionally a smooth parametrization of a submanifold $N \subset M \subset D(A^\infty)$, therefore we constructed for the open subset N of M an appropriate chart as submanifold of $D(A^\infty)$ by [2, Lemma 3.1].

For the boundary points $x_0 \in \partial M$ the argument is simpler: first we observe that $\sigma_i(x)$ are parallel to the boundary for $x \in \partial M$. In this case it is sufficient to define α on an open, convex subset $U \subset \mathbb{R}^d \times \mathbb{R}_{\geq 0}$. Again α is a smooth diffeomorphism to an open submanifold with boundary $N \subset M$ with respect to the original differentiable structure, but by [2, Lemma 3.1] this is also a smooth parametrization of a N as a submanifold of $D(A^\infty)$.

Whence we have constructed submanifold charts with respect to $D(A^\infty)$, so $M \subset D(A^\infty)$ is also a submanifold with boundary of $D(A^\infty)$. □

Remark 3.3. *The Nagumo type consistency results in* [1] *have been derived for submanifolds without boundary. These results can be extended to submanifolds with boundary. There are two key points. First, any submanifold with boundary M can be smoothly embedded in a submanifold without boundary, say $\tilde{M}$, of the same dimension. Then the main arguments in* [1] *carry over: to derive the Nagumo type consistency conditions at a point $x \in M$ it is enough to have local viability of the process with initial point $x \in M$ in M (and hence in $\tilde{M}$). Consequently we obtain the Nagumo type conditions for the whole of M (including the boundary!). Second, the coefficients of a diffusion process (i.e. the coordinate process) viable in a half space have to satisfy the appropriate inward pointing conditions at the boundary. We refer to* [7] *for the rigorous analysis.*

Appendix A. Regular Dependence on the Initial Point

Let X be a Banach space, S a strongly continuous semigroup on X with infinitesimal generator A, and $P : \mathbb{R}_{\geq 0} \times X \to X$ a continuous map. In this section we shall provide the basic existence, uniqueness and regularity results for the evolution equation

$$\frac{d}{dt}x(t) = Ax(t) + P(t, x(t)). \tag{A.1}$$

We first recall a classical existence and uniqueness result (see [8, Theorem 1.2, Chapter 6]).

Theorem A.1. *Let $T > 0$. Suppose $P : [0,T] \times X \to X$ is uniformly Lipschitz continuous (with constant C) on X. Then for every $x \in X$ there exists a unique mild solution $x(t)$, $t \in [0,T]$, to (A.1) with $x(0) = x$. If $x(t)$ and $y(t)$ are two mild solutions of (A.1) with $x(0) = x$ and $y(0) = y$ then*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\| \le M e^{MCT} \|x - y\|, \tag{A.2}$$

where

$$M := \sup_{t\in[0,T]} \|S_t\|. \tag{A.3}$$

There is an immediate local version of Theorem A.1. We say that $P : \mathbb{R}_{\ge 0} \times X \to X$ is *locally Lipschitz continuous on* X if for every $T \ge 0$ and $K \ge 0$ there exists $C = C(T,K)$ such that

$$\|P(t,x) - P(t,y)\| \le C\|x - y\|$$

for all $t \in [0,T]$, and $x, y \in X$ with $\|x\| \le k$ and $\|y\| \le k$.

Proposition A.2. *Suppose $P : \mathbb{R}_{\ge 0} \times X \to X$ is locally Lipschitz continuous on X. Let $x_0 \in X$. Then there exist a neighborhood U of x_0 and $T > 0$ such that, for every $x \in U$, equation (A.1) has a unique mild solution $x(t)$, $t \in [0,T]$, with $x(0) = x$. If $x(t)$ and $y(t)$ are two mild solutions of* (A.1) *with $x(0) = x \in U$ and $y(0) = y \in U$ then* (A.2) *holds, for M as in* (A.3) *and some $C = C(T,U)$.*

Proof. Set $K := 2\|x_0\|$ and fix $T' > 0$. Define

$$\tilde{P}(t,x) := \begin{cases} P(t,x), & \text{if } \|x\| \le K, \\ P(t, Kx/\|x\|), & \text{if } \|x\| > K. \end{cases}$$

Then $\tilde{P} : [0,T'] \times X \to X$ is uniformly Lipschitz continuous on X with constant $C = C(T',K)$. Hence Theorem A.1 yields existence and uniqueness of mild solutions for equation (A.1) where P is replace by $\tilde{P}$. By (A.2) there exists $0 < T \le T'$ and a neighborhood U of x_0 such that $\sup_{t\in[0,T]} \|x(t)\| \le K$ for every mild solution $x(t)$ with $x(0) \in U$. It is now easy to see that T and U satisfy the assertions of the proposition. □

Here is the announced regularity result.

Theorem A.3. *Let $k \ge 1$. Suppose $P : \mathbb{R}_{\ge 0} \times X \to X$ is C^k in x, and $D^k_x P$ is continuous on $\mathbb{R}_{\ge 0} \times X$ and locally Lipschitz continuous on X. Let $x_0 \in X$. Then there exists an open neighborhood U of x_0 and $T > 0$, and a map $F \in C([0,T] \times U, H)$ such that, for every $x \in U$, $F(\cdot, x)$ is the unique mild solution of (A.1) with $F(0,x) = x$. Moreover $F(t,\cdot) \in C^k(U,X)$ for all $t \in [0,T]$.*

Proof. By assumption, $D^r_x P$ is continuous on $\mathbb{R}_{\ge 0} \times X$ and locally Lipschitz continuous on X, for all $r \le k$. Hence Proposition A.2 yields the existence of U, T and $F \in C([0,T] \times U, H)$ such that $F(\cdot, x) \in C([0,T], H)$ is the unique mild solution of (A.1) with $F(0,x) = x$, for all $x \in U$. It remains to show regularity of $F(t,\cdot)$.

Let $x \in U$ and $y \in X$. The candidate, say $\psi(t,x,y)$, for the Gateaux directional derivative $D_x F(t,x)y$ is given by the linear evolution equation

$$\begin{aligned} \frac{d}{dt}\psi(t,x,y) &= A\psi(t,x,y) + D_x P(t,F(t,x))\psi(t,x,y) \\ \psi(0,x,y) &= y. \end{aligned} \tag{A.4}$$

Since $C_1 = C_1(x) := \sup_{t\in[0,T]} \|D_x P(t,F(t,x))\| < \infty$, Theorem A.1 yields the existence of a unique mild solution

$$\psi(\cdot,x,y) \in C([0,T],X) \tag{A.5}$$

to (A.4), and by (A.2)

$$\sup_{t\in[0,T]} \|\psi(t,x,y)\| \le M e^{MC_1 T}\|y\|. \tag{A.6}$$

Now let $t \in [0,T]$ and (x_n) be a sequence in U converging to x. We claim that

$$\sup_{y\in X,\, \|y\|\le 1} \|\psi(t,x_n,y) - \psi(t,x,y)\| \to 0, \quad \text{for } n \to \infty. \tag{A.7}$$

Indeed, $\Delta_n(t) := \psi(t,x_n,y) - \psi(t,x,y)$ satisfies

$$\Delta_n(t) = \int_0^t S_{t-s}\left(D_x P(s,F(s,x_n))\psi(s,x_n,y) - D_x P(s,F(s,x))\psi(s,x,y)\right) ds.$$

Hence

$$\|\Delta_n(t)\| \le MC_2 \int_0^t \|\Delta_n(s)\|\, ds + M^2 C_3 e^{M(C_0+C_1)T}\|y\|\|x_n - x\|,$$

where C_0 and C_3 are local Lipschitz constants of P and $D_x P$, respectively, and $C_2 := \sup_n \sup_{s\in[0,T]} \|D_x P(s,F(s,x_n))\|$. By Gronwall's inequality

$$\|\Delta_n(t)\| \le M^2 C_3 e^{M(C_0+C_1+C_2)T}\|y\|\|x_n - x\|,$$

whence (A.7).

Next, we claim that

$$D_x F(t,x)y = \psi(t,x,y). \tag{A.8}$$

Let $\varepsilon_0 > 0$ be such that $x + \varepsilon y \in U$ for all $\varepsilon \in [0,\varepsilon_0]$. For such ε we write $\delta(t,\varepsilon) := F(t,x+\varepsilon y) - F(t,x) - \varepsilon\psi(t,x,y)$, and obtain

$$\begin{aligned} \delta(t,\varepsilon) &= \int_0^t S_{t-s}(P(s,F(s,x+\varepsilon y)) - P(s,F(s,x)))\, ds \\ &\quad - \varepsilon \int_0^t S_{t-s} D_x P(s,F(s,x))\psi(s,x,y)\, ds \\ &= \int_0^t S_{t-s}\left(D_x P(s,F(s,x))\delta(s,\varepsilon) + \Delta(s,\varepsilon)\right) ds, \end{aligned}$$

where

$$\Delta(s,\varepsilon) := P(s,F(s,x+\varepsilon y)) - P(s,F(s,x)) - D_x P(s,F(s,x))(F(s,x+\varepsilon y) - F(s,x)).$$

By regularity of P and in view of (A.2) there exists $C_4 = C_4(T,U)$ such that

$$\sup_{t\in[0,T]} \|\Delta(t,\varepsilon)\| \le C_4 \varepsilon.$$

Hence, writing $C_5 := \sup_{t\in[0,T]} \|D_x P(t, F(t,x))\|$,

$$\|\delta(t,\varepsilon)\| \le C_5 MT \int_0^t \|\delta(s,\varepsilon)\|\, ds + C_4 MT\varepsilon,$$

and by Gronwall's inequality $\lim_{\varepsilon\to 0} \|\delta(t,\varepsilon)\| = 0$, whence (A.8).

By (A.8) it follows that $D_x F(t,x)y$ is well defined for all $x \in U$ and $y \in X$, and by (A.7) the mapping $D_x F(t,\cdot) : U \mapsto L(X)$ is continuous, hence $F(t,\cdot) \in C^1(U,X)$ for all $t \in [0,T]$.

Higher order regularity is shown by induction of the above argument. We only sketch the case C^2. Let $x \in U$ and $y_1, y_2 \in X$, and write $\psi_2(x, y_1, y_2)$ for the candidate of $D_x^2 F(t,x)(y_1,y_2)$, which solves the inhomogeneous linear evolution equation

$$\begin{aligned} \frac{d}{dt}\psi_2(t,x,y_1,y_2) &= A\psi_2(t,x,y_1,y_2) + D_x P(t,F(t,x))\psi_2(t,x,y_1,y_2) \\ &\quad + D_x^2 P(t,F(t,x))(D_x F(t,x)y_1, D_x F(t,x)y_2) \\ \psi_2(0,x,y_1,y_2) &= 0. \end{aligned} \tag{A.9}$$

Notice that the inhomogeneous part, $D_x^2 P(t,F(t,x))(D_x F(t,x)y_1, D_x F(t,x)y_2)$, is continuous in $t \in [0,T]$ by induction. Hence $\psi_2(\cdot,x,y_1,y_2) \in C([0,T],X)$ is the unique mild solution of (A.9) by Theorem A.1. Now let $t \in [0,T]$. One shows first that $\psi_2(t,\cdot,y_1,y_2) : U \to X$ is continuous, uniformly in $y_1, y_2 \in X$ with $\|y_1\| \le 1$, $\|y_2\| \le 1$ (see (A.7)). Then the identity $D_x^2 F(t,x)(y_1,y_2) = \psi_2(t,x,y_1,y_2)$ is proved (see (A.8)), whence $F(t,\cdot) \in C^2(U,X)$. □

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