TOTALLY GEODESIC SUBGROUPS OF DIFFEOMORPHISMS

STEFAN HALLER, JOSEF TEICHMANN AND CORNELIA VIZMAN

Abstract. We determine the Riemannian manifolds for which the group of exact volume preserving diffeomorphisms is a totally geodesic subgroup of the group of volume preserving diffeomorphisms, considering right invariant L^2-metrics. The same is done for the subgroup of Hamiltonian diffeomorphisms as a subgroup of the group of symplectic diffeomorphisms in the Kähler case. These are special cases of totally geodesic subgroups of diffeomorphisms with Lie algebras big enough to detect the vanishing of a symmetric 2-tensor field.

1. Introduction

Euler equation for an ideal fluid flow is just the geodesic equation on the group of volume preserving diffeomorphisms with right invariant L^2-metric, see [A]. In their book Arnold and Khesin [AK] state, that the subgroup of exact volume preserving diffeomorphisms is totally geodesic in the group of volume preserving diffeomorphisms on compact, oriented surfaces. This means that the group of Hamiltonian diffeomorphisms is totally geodesic in the group of symplectic diffeomorphisms, which has a physical interpretation, namely the existence of a single valued stream function for the velocity field at the initial moment implies the existence of a single valued stream function at any other moment of time. We will prove that a closed, oriented surface having this property either has the first Betti number zero (the Lie algebras of symplectic and Hamiltonian vector fields coincide) or it is a flat 2-torus.

A much more general classification result is actually true: twisted products of a torus by a Riemannian (resp. Kähler) manifold with vanishing first Betti number are the only Riemannian (resp. Kähler) manifolds, where the exact volume preserving diffeomorphisms lie totally geodesic in the Lie group of volume preserving diffeomorphisms (resp. the Hamiltonian diffeomorphisms in the symplectic diffeomorphisms), see Definition 2, Theorem 1 and Theorem 2 in section 2.

The problem can be formulated in the setting of regular Fréchet-Lie groups: given a regular Fréchet-Lie group in the sense of Kriegl-Michor, see [KM], and a (bounded, positive definite) scalar product $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$, we can define a right invariant metric on G by

$$G_x(\xi, \eta) := g((T_x\rho^x)^{-1}\xi, (T_x\rho^x)^{-1}\eta) \quad \text{for } \xi, \eta \in T_xG,$$

1991 *Mathematics Subject Classification.* 58D05, 58B20.

Key words and phrases. groups of diffeomorphisms as manifolds, geodesic equations on infinite dimensional Lie groups.

This paper was mainly written while the first author stayed at the Ohio State University. He would also like to thank D. Burghelea for raising the question of Proposition 1 below.

The second author would like to thank Peter W. Michor and Walter Schachermayer for the perfect working atmosphere, he acknowledges the support by the research grant FWF Z-36 "Wittgenstein prize" awarded to Walter Schachermayer.

where ρ^x denotes the right translation on G. The energy functional of a smooth curve $c : \mathbb{R} \to G$ is defined by

$$E(c) = \int_a^b G_{c(t)}(c'(t), c'(t))dt = \int_a^b g(\delta^r c(t), \delta^r c(t))dt,$$

where δ^r denotes the right logarithmic derivative on the Lie group G.

Assuming $c : [a, b] \to G$ to be a geodesic with respect to the right invariant (weak) Riemannian metric, variational calculus yields

$$\frac{d}{dt} X_t = -\operatorname{ad}(X_t)^\top X_t$$
$$X_t = \delta^r c(t),$$

where $\operatorname{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint with respect to the Hilbert scalar product of $\operatorname{ad}(X)$, see [KM], which we assume to exist as bounded linear map $\operatorname{ad}(\cdot)^\top : \mathfrak{g} \to L(\mathfrak{g})$. A Lie subgroup $H \subseteq G$ is totally geodesic if any geodesic c with $c(a) = e$ and $c'(a) \in \mathfrak{h}$ stays in H. This is the case if $\operatorname{ad}(X)^\top X \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. If there is a geodesic in G in any direction of $\mathfrak{h}$, then the condition is necessary and sufficient.

The setting for the whole article is the following: Given a regular Fréchet-Lie group G with Lie algebra $\mathfrak{g}$ and a bounded, positive definite scalar product on $\mathfrak{g}$. We assume, that $\operatorname{ad}(\cdot)^\top : \mathfrak{g} \to L(\mathfrak{g})$ exists and is bounded. Furthermore we are given a splitting subalgebra $\mathfrak{h}$, i.e. $\mathfrak{h}$ has an orthogonal complement $\mathfrak{h}^\perp$ in $\mathfrak{g}$ with respect to the scalar product. We only assume $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as orthogonal direct sum, in the algebraic sense. It follows, that $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are closed, and that the orthogonal projections onto $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are bounded with respect to the Fréchet space topology. $\mathfrak{h}$ is called totally geodesic in $\mathfrak{g}$ if $\operatorname{ad}(X)^\top X \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. The following reformulation of the condition provides the main condition for our work:

Lemma 1. *In the situation above, $\mathfrak{h}$ is totally geodesic in $\mathfrak{g}$ iff $\langle [X, Y], X\rangle = 0$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{h}^\perp$.*

In this article we consider the following important examples of the outlined situation: Let M be a closed, connected and oriented manifold. The regular Fréchet-Lie group $\operatorname{Diff}(M)$ is modeled on the vector fields $\mathfrak{X}(M)$, a Fréchet space. The Lie algebra is $\mathfrak{X}(M)$ with the negative of the usual Lie bracket. The symbol $[\cdot, \cdot]$ will denote the usual Lie bracket and $\operatorname{ad}(X)Y = [X, Y]$. With these conventions, the geodesic equation on diffeomorphism groups has no minus sign. The following subgroups are regular Fréchet-Lie subgroups of $\operatorname{Diff}(M)$, see [KM]:

1. The group $\operatorname{Diff}(M, \mu)$ of volume preserving diffeomorphisms of (M, μ), where μ is a volume form on M; its Lie algebra is $\mathfrak{X}(M, \mu)$ the Lie algebra of divergence free vector fields.
2. The group $\operatorname{Diff}_{\mathrm{ex}}(M, \mu)$ of exact volume preserving diffeomorphisms of (M, μ) with Lie algebra
$$\mathfrak{X}_{\mathrm{ex}}(M, \mu) = \{X \in \mathfrak{X}(M) : i_X\mu \text{ is an exact differential form}\}.$$
3. The group $\operatorname{Diff}(M, \omega)$ of symplectic diffeomorphisms of the symplectic manifold (M, ω) with Lie algebra $\mathfrak{X}(M, \omega)$ of symplectic vector fields.
4. The group $\operatorname{Diff}_{\mathrm{ex}}(M, \omega)$ of Hamiltonian diffeomorphisms of (M, ω) with Lie algebra $\mathfrak{X}_{\mathrm{ex}}(M, \omega)$ of Hamiltonian vector fields.

Let (M, g) denote a closed connected and orientable Riemannian manifold with Riemannian metric g, ∇ the Levi-Civita covariant derivative and μ the canonical volume form on M induced by the metric g and a choice of orientation. By $\sharp_g : T^*M \to TM$ we denote the geometric lift $g(\sharp_g\alpha, \cdot) = \alpha$ and by $\flat_g$ its inverse. We will omit the index g when no confusion is possible. The Hodge-$*$-operator is given with respect to the volume form μ such that $g(\beta, \eta)\mu = \beta \wedge *\eta$ for β, η k-forms, where g denotes the respective scalar product on the forms. The exterior derivative is denoted by d, the codifferential by $\delta = (-1)^{n(k+1)+1} * d*$ on k-forms. With this convention d and δ are adjoint with respect to scalar product on forms. Furthermore $\Delta = d\delta + \delta d$.

In the case of $G = \operatorname{Diff}(M)$ the adjoint of $\operatorname{ad}(X)$, with respect to the induced right invariant L^2-structure is given by the expression

$$\operatorname{ad}(X)^\top X = -\nabla_X X - (\operatorname{div} X)X - \frac{1}{2}\operatorname{grad}(g(X, X)).$$

We apply here the notions of gradient of a function $\operatorname{grad} f = \flat(df)$ and divergence of a vector field $\operatorname{div} X = -\delta(\flat X)$, i.e. $L_X\mu = \operatorname{div}(X)\mu$. If $H \subseteq G$ is a subgroup with splitting subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, then for $X \in \mathfrak{h}$

$$\operatorname{ad}(X)|_{\mathfrak{h}}^\top X = \pi(\operatorname{ad}(X)|_{\mathfrak{g}}^\top X),$$

where $\pi : \mathfrak{g} \to \mathfrak{h}$ denotes the orthogonal projection.

In particular we obtain for the volume preserving diffeomorphisms $\operatorname{Diff}(M, \mu)$

$$\operatorname{ad}(X)|_{\mathfrak{X}(M,\mu)}^\top X = -\nabla_X X - (\operatorname{grad} p)(X)$$
$$\Delta p = \operatorname{div}(\nabla_X X).$$

Such a function p exists, is unique up to a constant and smooth by application of the smooth inverse of the Laplacian on its range. Hence $(\operatorname{grad} p)(X)$ is a well defined smooth vector field. Remark, that the orthogonal complement to divergence free vector fields are gradients of some functions, with respect to the L^2-metric, which is easily seen due to the orthogonal Hodge decomposition $\mathfrak{X}(M) = \sharp_g d\Omega^0(M) \oplus \sharp_g \ker \delta$, where $\sharp_g$ denotes the geometric lift. Consequently the geodesic equation on $\operatorname{Diff}(M, \mu)$ with right invariant L^2-metric is

$$\frac{d}{dt}X_t = -\nabla_{X_t} X_t - (\operatorname{grad} p)(X_t),$$

the Euler equation for an ideal fluid flow, see [A].

Let M be an closed, connected almost Kähler manifold (M, g, ω, J), i.e. the symplectic form ω, the almost complex structure J and the Riemannian metric g satisfy the relation $g(X, Y) = \omega(X, JY)$. Note, that a Kähler manifold has a natural orientation given by J. Moreover we have $\flat_\omega(X) = -(\flat_g X) \circ J = \flat_g(JX)$ and $\sharp_\omega \varphi = -J\sharp_g\varphi = \sharp_g(\varphi \circ J)$, especially $\sharp_\omega : T_x^*M \to T_xM$ is an isometry, for J and $\sharp_g$ are.

For the symplectic diffeomorphisms $\operatorname{Diff}(M, \omega)$ we obtain the following adjoint

$$\operatorname{ad}(X)|_{\mathfrak{X}(M,\omega)}^\top X = -\nabla_X X - \frac{1}{2}\operatorname{grad}(g(X, X)) - \sharp_\omega((\delta\alpha)(X))$$
$$d((\delta\alpha)(X)) = -di_{\nabla_X X + \frac{1}{2}\operatorname{grad}(g(X,X))}\omega$$

and the geodesic equation

$$\frac{d}{dt}X_t = -\nabla_{X_t} X_t - \frac{1}{2}\operatorname{grad}(g(X_t, X_t)) - \sharp_\omega((\delta\alpha)(X_t)).$$

Remark that via the symplectic lift the orthogonal Hodge decomposition of

$$\Omega^1(M) = d\Omega^0(M) \oplus \mathcal{H}^1(M) \oplus \delta\Omega^2(M)$$

can be carried to the vector fields. Symplectic vector fields are those with $L_X\omega = d(\flat_\omega X) = 0$, i.e. $\flat_\omega X \in \ker d = d\Omega^0(M) \oplus \mathcal{H}^1(M)$. So there is some (symplectic) harmonic part and some Hamiltonian part, see [KM]. In the above formula $(\delta\alpha)(X)$ is uniquely determined and smoothly dependent for $X \in \mathfrak{X}(M,\omega)$ by the Hodge decomposition. The divergence part is zero, since symplectic diffeomorphisms are volume preserving.

2. Statement and proof of the results

In this section we develop the necessary notions and prove the asserted main results of the article. We shall provide several equivalent conditions, geometric and analytic ones, in the Riemannian and Kähler case, such that the presented subgroups are totally geodesic.

For a 1-form φ we set

$$(\nabla\varphi)^{\mathrm{sym}}(X,Y) := (\nabla_X\varphi)(Y) + (\nabla_Y\varphi)(X),$$

the symmetric part of $\nabla\varphi$. Note that $\sharp\varphi$ is a Killing vector field, i.e. generates a flow of isometries, if and only if $(\nabla\varphi)^{\mathrm{sym}} = 0$. Note also, that

$$d\varphi(X,Y) = (\nabla_X\varphi)(Y) - (\nabla_Y\varphi)(X),$$

the skew symmetric part of $\nabla\varphi$, and $\mathrm{tr}(\nabla\varphi)^{\mathrm{sym}} = 2\,\mathrm{div}\,\sharp\varphi = -2\delta\varphi$.

Lemma 2. *Let (M,g) be a closed oriented n-dimensional Riemannian manifold and $\mathfrak{g} \subseteq \mathfrak{X}(M)$ a closed subalgebra, such that $\mathrm{ad}(X)^\top : \mathfrak{g} \to \mathfrak{g}$ exists for all $X \in \mathfrak{g}$. Then one has*

$$2\int_M g(\mathrm{ad}(X)^\top(X),Y)\mu = \int_M \big((\nabla\flat Y)^{\mathrm{sym}} + (\mathrm{div}\,Y)g\big)(X,X)\mu.$$

Moreover

$$\mathrm{tr}\,\big((\nabla\flat Y)^{\mathrm{sym}} + (\mathrm{div}\,Y)g\big) = (n+2)\,\mathrm{div}\,Y.$$

Especially $(\nabla\flat Y)^{\mathrm{sym}} + (\mathrm{div}\,Y)g = 0$ iff $(\nabla\flat Y)^{\mathrm{sym}} = 0$, i.e. Y is Killing.

Proof. We have:

$$\begin{aligned}
\int g(\mathrm{ad}(X)^\top(X),Y)\mu &= \int g(X,[X,Y])\mu \\
&= \int \big(g(X,\nabla_X Y) - g(X,\nabla_Y X)\big)\mu \\
&= \int \big(i_X\nabla_X\flat Y - \tfrac{1}{2}L_Y g(X,X)\big)\mu \\
&= \int \tfrac{1}{2}(\nabla\flat Y)^{\mathrm{sym}}(X,X)\mu + \tfrac{1}{2}g(X,X)L_Y\mu \\
&= \frac{1}{2}\int \big((\nabla\flat Y)^{\mathrm{sym}} + (\mathrm{div}\,Y)g\big)(X,X)\mu
\end{aligned}$$

The second statement follows from $\mathrm{tr}\,\big(\nabla(\flat Y)^{\mathrm{sym}}\big) = 2\,\mathrm{div}\,Y$ and $\mathrm{tr}(g) = n$. □

Definition 1. We say $\mathfrak{g} \subseteq \mathfrak{X}(M)$ is big enough to detect the vanishing of a symmetric 2-tensor field, if a symmetric 2-tensor field $T \in \Gamma(S^2T^*M)$ vanishes if

$$\int_M T(X,Y)\mu = 0 \quad \text{for all } X, Y \in \mathfrak{g}.$$

Remark 1. Let $\mathfrak{g} \subseteq \mathfrak{X}(M)$ be the Lie algebra of a Lie group of diffeomorphisms G, such that $\mathrm{ad}(\cdot)^\top : \mathfrak{g} \to L(\mathfrak{g})$ is bounded. Suppose $H \subseteq G$ is a Lie subgroup with splitting Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and assume that $\mathfrak{h}$ is big enough to detect the vanishing of a symmetric 2-tensor. It will then be easy to decide if $\mathfrak{h}$ is totally geodesic in $\mathfrak{g}$, for the condition of being totally geodesic, $\mathrm{ad}(X)^\top X \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, translates in this case to $(\nabla \flat Y)^{\mathrm{sym}} = 0$ for all $Y \in \mathfrak{h}^\perp$, i.e. Y is Killing for all $Y \in \mathfrak{h}^\perp$.

Note also, that if $\mathfrak{h}$ has an orthogonal complement in $\mathfrak{X}(M)$, i.e. $\mathfrak{h} \oplus \mathfrak{h}' = \mathfrak{X}(M)$, then $\mathfrak{h}$ is a splitting subalgebra of $\mathfrak{g}$, since $\mathfrak{h}^\perp = \mathfrak{h}' \cap \mathfrak{g}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.

Lemma 3. *Let (M, ω) be a symplectic manifold. Then the Lie algebra of compactly supported Hamiltonian vector fields is big enough to detect the vanishing of a symmetric 2-tensor field.*

Proof. Suppose T is a symmetric 2-tensor field, which does not vanish at a point in M. Using Darboux's theorem, and rescaling ω and T by constants, we may choose a chart $M \supseteq U \to (-1,1)^{2n} \subseteq \mathbb{R}^{2n}$, such that

$$\omega = dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n}$$

and $T_{22}(x) > 0$ for all $x \in (-1,1)^{2n}$, where $T = \sum T_{ij} dx^i \otimes dx^j$. Now choose a bump function $b : \mathbb{R} \to [0,1]$, such that $b(t) = 0$ for $|t| \geq \frac{1}{2}$ and $b(0) = 1$. For $0 < \varepsilon \leq 1$ we define

$$\lambda_\varepsilon(x^1, \ldots, x^{2n}) := b(\tfrac{x^1}{\varepsilon}) b(x^2) \cdots b(x^{2n})$$

and $Z_\varepsilon := \sharp_\omega d\lambda_\varepsilon$. Since the support of λ_ε is contained in $(-\varepsilon, \varepsilon) \times (-1,1)^{2n-1}$, Z_ε extends by zero to a compactly supported Hamiltonian vector field on M. An easy calculation shows

$$\lim_{\varepsilon \to 0} \varepsilon \int_M T(Z_\varepsilon, Z_\varepsilon)\omega^n = \int_{(-1,1)^{2n}} \big(b'(x^1) b(x^2) \cdots b(x^{2n})\big)^2 T_{22}(0, x^2, \ldots, x^{2n})\omega^n > 0$$

and hence $\int_M T(Z_\varepsilon, Z_\varepsilon)\omega^n \neq 0$ for ε small enough. □

Lemma 4. *Let (M, μ) be a manifold with volume form, $\dim(M) > 1$. Then the Lie algebra of compactly supported exact divergence free vector fields is big enough to detect the vanishing of a symmetric 2-tensor field.*

Proof. As in the proof of Lemma 3, we choose a chart $M \supseteq U \to (-1,1)^n \subseteq \mathbb{R}^n$, such that

$$\mu = dx^1 \wedge \cdots \wedge dx^n$$

and such that $T_{22} > 0$ on $x \in (-1,1)^n$. Take a bump function b as above and set

$$\lambda_\varepsilon := b(\tfrac{x^1}{\varepsilon}) b(x^2) \cdots b(x^n).$$

Now define $i_{Z_\varepsilon}\mu := d(\lambda_\varepsilon dx^3 \wedge \cdots \wedge dx^n)$. Then Z_ε is a compactly supported exact divergence free vector field on M and

$$Z_\varepsilon = b(\tfrac{x^1}{\varepsilon}) b'(x^2) b(x^3) \cdots b(x^n) \tfrac{\partial}{\partial x^1} - \tfrac{1}{\varepsilon} b'(\tfrac{x^1}{\varepsilon}) b(x^2) \cdots b(x^n) \tfrac{\partial}{\partial x^2}.$$

Again we get

$$\lim_{\varepsilon\to 0}\varepsilon\int_M T(Z_\varepsilon,Z_\varepsilon)\mu=\int_{(-1,1)^n}\big(b'(x^1)b(x^2)\cdots b(x^n)\big)^2 T_{22}(\varepsilon x^1,x^2,\dots,x^n)\mu>0,$$

and hence $\int_M T(Z_\varepsilon,Z_\varepsilon)\mu\neq 0$ for ε small enough. □

Definition 2 (Twisted products)**.** Let $T^k=\mathbb{R}^k/\Lambda$ be a flat torus, equipped with the metric induced from the Euclidean metric on $\mathbb{R}^k$. Suppose F is an oriented Riemannian manifold and that Λ acts on F by orientation preserving isometries. The total space of the associated fiber bundle $\mathbb{R}^k\times_\Lambda F\to T^k$ is an oriented Riemannian manifold in a natural way. Locally over T^k the metric is the product metric. We call every manifold obtained in this way a twisted product of a flat torus and the oriented Riemannian manifold F.

If k is even, F Kähler and Λ acts by isometries preserving the Kähler structure then $\mathbb{R}^k\times_\Lambda F\to T^k$ is a Kähler manifold in a natural way and we call it a twisted product of a flat torus with the Kähler manifold F.

Theorem 1. *Let (M,g) be a closed, connected and oriented Riemannian manifold with volume form μ. Then the following are equivalent:*

1. *The group of exact volume preserving diffeomorphisms is a totally geodesic subgroup in the group of all volume preserving diffeomorphisms.*
2. *Every harmonic 1-form is parallel.*
3. $\operatorname{ric}(\beta_1,\beta_2)=0$ *for all harmonic 1-forms β_1,β_2.*
4. (M,g) *is a twisted product of a flat torus and a closed, connected, oriented Riemannian manifold with vanishing first Betti number.*
5. *For all 2-forms α and all harmonic 1-forms β one has*

$$\int_M g(d\delta\alpha,\delta\alpha\wedge\beta)\mu=0.$$

Proof of Theorem 1. Recall that the orthogonal complement of $\mathfrak{X}_{\mathrm{ex}}(M,\mu)$ in $\mathfrak{X}(M,\mu)$ is $\{\sharp\beta:\beta \text{ harmonic 1-form}\}$. The equivalence (1) ⇔ (2) now follows immediately from Remark 1, Lemma 4 and the fact that for closed 1-forms $(\nabla\beta)^{\mathrm{sym}}=0$ is equivalent to $\nabla\beta=0$.

(2) ⇒ (3) is obvious from the definition of the curvature $R_{X,Y}Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$ and $\operatorname{ric}=-\operatorname{tr}_{13}R$.

The integrated Bochner equation on 1-forms, see for example [LM], takes the form

$$\langle\Delta\alpha,\alpha\rangle=\|\nabla\alpha\|^2+\int_M\operatorname{ric}(\alpha,\alpha)\mu,$$

and (3) ⇒ (2) follows.

(4) ⇒ (2): Suppose $M=\mathbb{R}^k\times_\Lambda F$. Since $H^1(F;\mathbb{R})=0$ it follows from the Leray-Serre spectral sequence that the projection $M\to T^k$ induces an isomorphism $H^1(M;\mathbb{R})\cong H^1(T^k;\mathbb{R})$. So every harmonic 1-form comes from T^k and hence is parallel.

(2) ⇒ (4), cf. Theorem 8.5 in [LM] and [CG]: Suppose (M,g) is a closed, connected and oriented Riemannian manifold, such that every harmonic 1-form is parallel. Choose an orthonormal base $\{\beta_1,\dots,\beta_k\}$ of harmonic 1-forms. Since they are parallel they are orthonormal at every point in M. Choose a base point $x_0\in M$, let $U\subseteq\pi_1(M,x_0)$ be the kernel of the Huréwicz-homomorphism

$$\pi_1(M,x_0)\to H_1(M;\mathbb{Z})\to H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))\cong\mathbb{Z}^k,$$

and let $\pi : \tilde{M} \to M$ be the covering of M, which has U as characteristic subgroup. This is a normal covering, the group of deck transformations is $\mathbb{Z}^k$ and $\pi^*\beta_i = df_i$ for smooth functions $f_i : \tilde{M} \to \mathbb{R}$. Let z_0 be a base point in $\tilde{M}$ sitting above x_0 and assume $f_i(z_0) = 0$. Consider the mapping

$$f : \tilde{M} \to \mathbb{R}^k, \quad f(z) = \big(f_1(z), \dots, f_k(z)\big).$$

Obviously this is a proper, surjective submersion and $F := f^{-1}(0)$ is a compact submanifold. Let $X_i := \sharp_g \pi^*\beta_i$. Then the X_i are orthonormal at every point and they are all parallel, especially they commute. Consider

$$\kappa : F \times \mathbb{R}^k \to \tilde{M}, \quad \kappa(z,t) := \big(\operatorname{Fl}^{X_1}_{t_1} \circ \cdots \circ \operatorname{Fl}^{X_k}_{t_k}\big)(z).$$

Of course we have $f \circ \kappa = \operatorname{pr}_2$, and it follows easily, that κ is a diffeomorphism. So F is closed, connected, oriented and $H^1(F;\mathbb{R}) = H^1(\tilde{M};\mathbb{R}) = 0$. Moreover $\kappa^* g$ is the product metric of the induced metric on F and the standard metric on $\mathbb{R}^k$. Every deck transformation of $\tilde{M}$ is of the form

$$F \times \mathbb{R}^k \to F \times \mathbb{R}^k, \quad (z,t) \mapsto (\varphi_\lambda(z), t + \lambda),$$

where $\lambda \in \Lambda \subseteq \mathbb{R}^k$ and φ_λ is an orientation preserving isometry of F. So M is a twisted product, as claimed.

To see (1) $\Leftrightarrow$ (5) let $\sharp\delta\alpha$ be an exact volume preserving vector field, $\alpha \in \Omega^2(M)$, and let β be a harmonic 1-form. Then

$$\begin{aligned}\int_M g(\operatorname{ad}(\sharp\delta\alpha)^\top \sharp\delta\alpha, \sharp\beta)\mu &= \int_M g(\sharp\delta\alpha, [\sharp\delta\alpha, \sharp\beta])\mu \\ &= -\int_M g(\sharp\delta\alpha, \sharp\delta(\delta\alpha \wedge \beta))\mu \\ &= -\int_M g(d\delta\alpha, \delta\alpha \wedge \beta)\mu,\end{aligned}$$

where we used

$$\delta(\varphi_1 \wedge \varphi_2) - (\delta\varphi_1) \wedge \varphi_2 + \varphi_1 \wedge \delta\varphi_2 = -\flat[\sharp\varphi_1, \sharp\varphi_2] \quad \text{for } \varphi_1, \varphi_2 \in \Omega^1(M)$$

to obtain $[\sharp\delta\alpha, \sharp\beta] = -\sharp\delta(\delta\alpha \wedge \beta)$ for the second equality. ☐

Theorem 2. *Let (M, g, J, ω) be a closed, connected Kähler manifold. Then the following are equivalent:*

1. *The group of Hamiltonian diffeomorphisms is a totally geodesic subgroup in the group of all symplectic diffeomorphisms.*
2. *Every harmonic 1-form is parallel.*
3. $\operatorname{ric}(\beta_1, \beta_2) = 0$ *for all harmonic 1-forms β_1, β_2.*
4. *(M, g, J, ω) is a twisted product of a flat torus and a closed connected Kähler manifold with vanishing first Betti number.*
5. *For all functions f and all harmonic 1-forms β one has*

$$\int_M (\Delta f) df \wedge \beta \wedge \omega^{n-1} = 0.$$

Proof of Theorem 2. Recall that the orthogonal complement of $\mathfrak{X}_{\text{ex}}(M,\omega)$ in $\mathfrak{X}(M,\omega)$ is $\{\sharp_\omega\beta : \beta \text{ harmonic 1-form}\}$. By Remark 1 and Lemma 3, (1) is equivalent to $\nabla(\beta \circ J)^{\text{sym}} = \nabla(\flat_g \sharp_\omega \beta)^{\text{sym}} = 0$ for all harmonic 1-forms β. On a Kähler manifold one has $\Delta(\varphi \circ J) = (\Delta\varphi) \circ J$ for 1-forms φ. Particularly the space of harmonic

1-forms is J-invariant, and so (1) is equivalent to $(\nabla\beta)^{\mathrm{sym}} = 0$ and since harmonic 1-forms are closed this is equivalent to (2).

(2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (2) are as in the proof of Theorem 1. For (4) $\Rightarrow$ (2) one needs some extra arguments: One observes, that the span of the X_i constructed in the proof of Theorem 1, is J-invariant and so is its orthogonal complement. Hence F is a complex submanifold and therefore a Kähler submanifold. Moreover the complex structure is, locally over T^k, the product structure and so is the symplectic structure as well.

(1) $\Leftrightarrow$ (5) follows from the following computation for a function f and a closed 1-form β:

$$\begin{aligned}\int_M g(\mathrm{ad}(\sharp_\omega df)^\top \sharp_\omega df, \sharp_\omega\beta)\omega^n &= \int_M g(\sharp_\omega df, [\sharp_\omega df, \sharp_\omega\beta])\omega^n \\ &= -\int_M g(\sharp_\omega df, \sharp_\omega d(L_{\sharp_\omega\beta} f))\omega^n \\ &= -\int_M g(df, d(L_{\sharp_\omega\beta} f))\omega^n \\ &= -\int_M (\Delta f)(L_{\sharp_\omega\beta} f)\omega^n \\ &= -n\int_M (\Delta f) df\wedge\beta\wedge\omega^{n-1}\end{aligned}$$

For the second equality we used $[\sharp_\omega\varphi_1, \sharp_\omega\varphi_2] = -\sharp_\omega(L_{\sharp_\omega\varphi_2}\varphi_1)$ for closed 1-forms φ_1, φ_2, a relation derived from $i_{[X,Y]} = L_X i_Y - i_Y L_X$. □

Remark 2. The fact that M is Kähler was only used to show, that the space of harmonic 1-forms is invariant under J. In the almost Kähler case the arguments in the proof of Theorem 2 show, that following are equivalent:

1. The group of Hamiltonian diffeomorphisms is a totally geodesic subgroup in the group of all symplectic diffeomorphisms.
2. $\sharp_\omega\beta = \sharp_g(\beta\circ J) = -J\sharp_g\beta$ is Killing for every harmonic 1-form β.
3. For all functions f and all harmonic 1-forms β one has
$$\int_M (\Delta f) df\wedge\beta\wedge\omega^{n-1} = 0.$$

Remark 3. The computation in the proof of Theorem 2 shows, that for a function f, $\mathrm{ad}(\sharp_\omega df)^\top\sharp_\omega df = 0$ if and only if

$$\int_M (\Delta f) df\wedge\beta\wedge\omega^{n-1} = 0 \quad \text{for all closed 1-forms } \beta,$$

even on almost Kähler manifolds. If f is a 'generalized eigenvector' of the Laplacian, i.e. $\Delta f = h\circ f$ for some smooth function $h \in C^\infty(\mathbb{R},\mathbb{R})$, then

$$\int_M (\Delta f) df\wedge\beta\wedge\omega^{n-1} = \int_M (h\circ f) df\wedge\beta\wedge\omega^{n-1} = \int_M d\big((H\circ f)\beta\wedge\omega^{n-1}\big) = 0$$

with H an integral of h, consequently the condition is satisfied. So the geodesic is given by an exponential. These are examples of a more general method how to solve the geodesic equation: In the general setting any finite dimensional submanifold $S \subset \mathfrak{g}$ such that $\mathrm{ad}(X)^\top X \in T_X S$ for $X \in S$ admits the calculation of flowlines in the manifold S. In the above case the submanifold S is given by a single point.

3. Final Remarks

Since Killing vector fields are divergence free, Remark 1 immediately implies

Corollary 1. *Let M be a closed, connected and oriented Riemannian manifold. Then there does not exist a closed Lie subalgebra $\mathfrak{X}(M,\mu) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that* $\operatorname{ad}(\cdot)^\top : \mathfrak{g} \to L(\mathfrak{g})$ *is bounded and such that $\mathfrak{X}(M,\mu)$ is totally geodesic in $\mathfrak{g}$.*

On a closed Riemannian manifold the Lie algebra of Killing vector fields is finite dimensional. So Remark 1 also implies

Corollary 2. *Let (M, g, J, ω) be a closed, connected almost Kähler manifold. Then the symplectic diffeomorphisms are not totally geodesic in the group of volume preserving diffeomorphisms, provided* $\dim(M) > 2$.

Remark 4. Let K be a compact Lie group acting by isometries on the closed connected orientable manifold (M, g). In [V] it is shown that the group of K-equivariant diffeomorphisms is a totally geodesic subgroup of $\operatorname{Diff}(M)$. Its Lie algebra, K-invariant vector fields on M, is split, a complement is $\{X \in \mathfrak{X}(M) : \int_K k^* X dk = 0\}$, infinite dimensional. This does not contradict the arguments above, for the Lie algebra of K-invariant vector fields on M does not detect the vanishing of a symmetric 2-tensor field.

Let $\mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega)$ denote the compactly supported Hamiltonian vector fields.

Lemma 5. *Let (M, ω) be a connected symplectic manifold and let $\alpha \in \Omega^2(M)$. If*

$$L_X \alpha = 0, \quad \textit{for all } X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega)$$

then there exists $\lambda \in \mathbb{R}$ such that $\alpha = \lambda\omega$.

Proof. Choose a Darboux chart centered at $z \in M$, such that

$$\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$$

and write

$$\alpha = \sum_{i<j} a_{ij} dx^i \wedge dx^j + \sum_{i<j} b_{ij} dy^i \wedge dy^j + \sum_{i,j} c_{ij} dx^i \wedge dy^j.$$

Let h be a compactly supported function on M, such that $h = x^i$ resp. $h = y^i$ locally around z. Then the condition $L_{\sharp_\omega dh} \alpha = 0$ shows that a_{ij}, b_{ij} and c_{ij} are all constant locally around z. Using $h = (x^i)^2$ one sees, that $b_{ij} = 0$ and $c_{ij} = 0$ for $i \neq j$. Using $h = (y^i)^2$ yields $a_{ij} = 0$. Finally, using $h = x^i x^j$ shows $c_{ii} = c_{jj}$. So $\alpha = \lambda\omega$ locally around z, for some constant $\lambda \in \mathbb{R}$. Since M is connected this is true globally. ∎

We denote by

$$\mathfrak{X}(M, [\omega]) := \{X \in \mathfrak{X}(M) : \exists \lambda \in \mathbb{R} : L_X \omega = \lambda\omega\}.$$

Notice that for closed M we have $\mathfrak{X}(M, [\omega]) = \mathfrak{X}(M, \omega)$. Moreover if $L_X \omega = f\omega$ for some function f and if $\dim(M) > 2$ then f is constant, due to the non-degeneracy of ω.

Lemma 6. *Let (M, ω) be a symplectic manifold and let $Z \in \mathfrak{X}(M)$. If*

$$[Z, X] \in \mathfrak{X}(M, [\omega]), \quad \textit{for all } X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M, \omega)$$

then $Z \in \mathfrak{X}(M, [\omega])$.

Proof. Set $\alpha := L_Z\omega \in \Omega^2(M)$. Then for every $X \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(M,\omega)$ we have

$$L_X\alpha = L_X L_Z\omega = L_{[X,Z]}\omega = \lambda\omega = 0.$$

Here λ has to vanish, since $[X,Z]$ has compact support. So by the previous lemma there exists $\tilde{\lambda} \in \mathbb{R}$, such that $L_Z\omega = \alpha = \tilde{\lambda}\omega$, i.e. $Z \in \mathfrak{X}(M,[\omega])$. □

Proposition 1. *Let (M,ω) be a symplectic manifold. Then there does not exists a Lie subalgebra $\mathfrak{X}(M,[\omega]) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathfrak{X}(M,[\omega])$ has finite codimension in $\mathfrak{g}$.*

Proof. Suppose $\mathfrak{g}$ is bigger than $\mathfrak{X}(M,[\omega])$. Then there exists $Z \in \mathfrak{g}$ and an open subset $U \subseteq M$, such that $Z|_V \notin \mathfrak{X}(V,[\omega])$, for every all open $V \subseteq U$. For any $k \in \mathbb{N}$ we choose disjoint subsets $V_1,\dots,V_k \subseteq U$. Since $Z|_{V_i} \notin \mathfrak{X}(V_i,[\omega])$ Lemma 6 yields $X_i \in \mathfrak{X}^{\mathrm{c}}_{\mathrm{ex}}(V_i,\omega)$, such that $Y_i := [Z,X_i] \notin \mathfrak{X}(V_i,[\omega])$. But $Y_i \in \mathfrak{g}$ and obviously $\{Y_1,\dots,Y_k\}$ are linearly independent in $\mathfrak{g}/\mathfrak{X}(M,[\omega])$. Hence the codimension of $\mathfrak{X}(M,[\omega])$ in $\mathfrak{g}$ is at least k. Since k was arbitrary we are done. □

Corollary 3. *Let (M,g,J,ω) be a closed, connected almost Kähler manifold. Then there does not exist a closed Lie subalgebra $\mathfrak{X}(M,\omega) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathrm{ad}(\cdot)^{\mathsf{T}} : \mathfrak{g} \to L(\mathfrak{g})$ is bounded and such that $\mathfrak{X}(M,\omega)$ is totally geodesic in $\mathfrak{g}$.*

Proof. Suppose conversely such a $\mathfrak{g}$ exists. By Remark 1 and Lemma 3, $\mathfrak{X}(M,\omega)$ has an orthogonal complement in $\mathfrak{g}$, consisting of Killing vector fields. So this complement has to be finite dimensional, but this contradicts Proposition 1. □

For a manifold with volume form (M,μ) we let

$$\mathfrak{X}(M,[\mu]) := \{X \in \mathfrak{X}(M) : \exists\lambda \in : L_X\mu = \lambda\mu\}.$$

Notice, that for closed M one has $\mathfrak{X}(M,\mu) = \mathfrak{X}(M,[\mu])$. Similarly, although it does not yield anything new for our totally geodesic subgroups, one shows

Proposition 2. *Let (M,μ) be a manifold with volume form and $\dim(M) > 1$. Then there does not exist a Lie subalgebra $\mathfrak{X}(M,[\mu]) \subset \mathfrak{g} \subseteq \mathfrak{X}(M)$, such that $\mathfrak{X}(M,[\mu])$ has finite codimension in $\mathfrak{g}$.*

References

[A] V.I. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier, 16(1966), 319–361.

[AK] V.I. Arnold and B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, 1998.

[CG] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom., 6 (1971), 119–128.

[KM] A. Kriegl and P.W. Michor, *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs 53, AMS (1997).

[LM] H.B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, New Jersey, 1989.

[MR] P.W. Michor and T. Ratiu, *On the geometry of the Virasoro-Bott group*, Journal of Lie Theory, 8(1998), 293–309.

[V] C. Vizman, *Geodesics and curvature of diffeomorphism groups*, Proceedings of the Fourth International Workshop on Differential Geometry, Brasov, Romania, 1999, 298–305.

Stefan Haller, Institute of Mathematics, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria.

E-mail address: stefan@mat.univie.ac.at

Josef Teichmann, Institute of financial and actuarial mathematics, Technical University of Vienna, Wiedner Hauptstrasse 8–10, A-1040 Vienna, Austria.
E-mail address: josef.teichmann@fam.tuwien.ac.at

Cornelia Vizman, West University of Timisoara, Department of Mathematics, Bd. V.Parvan 4, 1900 Timisoara, Romania.
E-mail address: vizman@math.uvt.ro