Convenient Analysis and Frobenius Theorems

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1 Infinite dimensional Problems

We shall present general methods for infinite dimensional analysis. Infinite dimensional Spaces appear to be important as

- Configuration spaces of many well-known problems from quantum mechanis (Schrödinger's equation), hydromechanics (Euler and Navier-Stokes equation), financial mathematics (Interest Rate Theory),...

- independent objects of research in differential geometry (Klein's Erlangen program performed for symplectic geometry), analysis (variational problems),...

2 Three particular Problems

1. The structure of the group of diffeomorphisms *Diff*(M) on a compact manifold M is described by a smooth Lie group structure on the manifold *Diff*(M), which is modeled on a particular Fréchet space. Is it possible to describe these basic facts in a reasonably simple way?

2. Given a Lie algebra $\mathfrak{g}$ and a Fréchet space F such that there is homomorphism $\phi :\rightarrow L(F)$ and a basis $(a_i)_{i=1,\ldots,n}$ of $\mathfrak{g}$ such that $A_i = \phi(a_i)$ generates nice one-parameter groups. Does there exist $\Phi : G \rightarrow GL(F)$ integrating the homorphism, where G is the simply connected Lie group associated to $\mathfrak{g}$?

3. Does there exist a Frobenius Theorem on Fréchet manifolds? In financial mathematics possible Frobenius Theorems on Fréchet spaces lead to a classification of "relevant" interest rate models (Björk, Svensson, Filipovic, Teichmann).

3 Classical Analysis

- Fréchet calculus of approximations by a linear map, the derivative. A map $f : U \subset X \to Y$ is called differentiable at $x_0 \in U$ if
$$||f(x_0 + h) - f(x_0) - Ah|| = o(||h||)$$
as $h \to 0$. This works well up to Banach spaces but causes ambiguities on more general spaces, since there is more than one seminorm.

- Gateaux calculus as calculus of directional derivatives. A map $f : U \subset X \to Y$ is called Gateaux-differentiable at $x_0 \in U$ if
$$d_v f(x_0) := \frac{d}{dt}|_{t=0} f(x_0 + th)$$
exists for $h \in X$ and is bounded linear in h.

For smooth mappings of lot of essentially different calculi have been developed on locally convex vector spaces, since there is a gap between smoothness and continuity...

4 Smooth Curves

Convenient Analysis provides a satisfactory, natural and unique solution of the question how to do analysis on a large class of locally convex vector spaces.

- E, F, G denote locally convex vector spaces.

- The notion of smooth curves poses no problems, we shall denote them by $C^\infty(\mathbb{R}, E)$.

- The continuous (bounded) linear functionals on E are denoted by E'_c (E', respectively).

We have the following two fundamental assertions:

- $B \subset E$ is bounded if and only if $l(B)$ is bounded for $l \in E'_c$.

- If there exists a bounded neighborhood $U \subset E$, then E is normable.

Theorem 4.1 (Mean Value Theorem) *Let $c : [a, b] \longrightarrow E$ be continuous and differentiable at all points and let $h : [a, b] \longrightarrow \mathbb{R}$ be continuous, monotone and differentiable. Let $A \subset E$ be convex and closed and assume $c'(t) \in h'(t)A$ for $t \in [a, b]$, then*

$$c(b) - c(a) \in (h(b) - h(a))A.$$

From this we conclude immediately for $c \in C^\infty(\mathbb{R}, E)$

$$\frac{1}{t}\left(\frac{c(t) - c(0)}{t} - c'(0)\right) \in B$$

for some bounded, convex, closed set $B \subset E$ and t in some compact set of real numbers.

5 Convenient Vector Spaces

For smooth curves the convergence of the difference quotient to the derivative is "better" than usual convergence in locally convex vector spaces.

A sequence $\{x_n\}_{n\geq 0}$ is called *Mackey-convergent* to $x \in E$ (a *Mackey-Cauchy* sequence) if there is a bounded, absolutely convex set $B \subset E$, a real sequence $a_n \downarrow 0$ (a real double sequence $a_{nm} \downarrow 0$) such that

$$x_n - x \in a_n B$$

for $n \geq 0$ or

$$x_n - x_m \in a_{nm} B$$

as $n, m \to \infty$, respectively.

A locally convex vector space is called *convenient* if every Mackey-Cauchy sequence converges. Difference quotients of smooth curves converge Mackey to their respective limits.

6 Fundamental Characterization

Theorem 6.1 *Let E be a locally convex vector space, then the following assertions are equivalent:*

1. *E is convenient.*

2. *If $c : \mathbb{R} \to E$ is a curve such that $l \circ c$ is smooth for all $l \in E'_c$, then $c \in C^\infty(\mathbb{R}, E)$.*

3. *Any smooth curve is locally Riemann-integrable.*

The second assertion states that the concept of weakly smooth and smooth curves are equivalent, even more is true.

For any $c : \mathbb{R} \to E$ we have that $l \circ c$ is smooth for all $l \in E'$ if and only if $c \in C^\infty(\mathbb{R}, E)$.

7 c^∞-Topology

We introduce a topology naturally associated to questions of analysis: let E be a convenient locally convex vector space, $U \subset E$ is called c^∞-open if $c^{-1}(U)$ is open for all $c \in C^\infty(\mathbb{R}, E)$, i.e the final topology with respect to all smooth curves.

Every Mackey sequence is c^∞-converging and the c^∞-topology is finer than any locally convex topology on E with the same bounded sets,

$$c^\infty E \hookrightarrow E_{born} \hookrightarrow E.$$

We expect good news for Fréchet spaces, namely

$$c^\infty E = E_{born} = E,$$

since on Fréchet spaces every converging sequence converges Mackey to its limit.

8 Smooth Maps

Now we are able to introduce smooth maps by the following "geometric" approach, which will be very useful for a lot of applications:

Definition 8.1 *Let E, F be convenient vector spaces and $U \subset E$ a c^∞-open subset. A map $f : U \subset E \to F$ is called smooth if*

$$f \circ c : \mathbb{R} \to F$$

is a smooth curve for any smooth curve $c \in C^\infty(\mathbb{R}, U)$.

We shall denote smooth maps by $C^\infty(U, F)$ and introduce the following initial locally convex topology

$$C^\infty(U, F) \stackrel{c^* \circ l_*}{\to} C^\infty(\mathbb{R}, \mathbb{R})$$
$$f \mapsto l \circ f \circ c$$

for $c \in C^\infty(\mathbb{R}, U)$ and $l \in F'$. We obtain again a convenient vector space. Inheritance as it should be!

9 Main Structure Theorem

Theorem 9.1 *Let E, F, G be convenient vector spaces and $U \subset E, V \subset F$ be c^∞-open subsets.*

1. *Smooth maps are continuous with respect to the c^∞-topology.*

2. *Multilinear maps are smooth if and only if they are bounded.*

3. *Let $f : U \subset E \to F$ be a smooth map, then*

$$Df : U \times E \to F$$

is smooth, where

$$Df(x) \cdot h := \frac{d}{dt}|_{t=0} f(x + th)$$

is the derivative along the affine line.

4. *The chain rule holds.*

5. *Taylor's formula holds, i.e.*

$$f(x+h) = \sum_{i=0}^{n} \frac{1}{i!} D^i f(x) \cdot h^{(i)} + \\ + \int_0^1 \frac{(1-t)^n}{n!} D^{n+1} f(x+th) \cdot h^{(n+1)} dt.$$

6. *The exponential law holds (cartesian closedness), i.e.*

$$C^\infty(U \times V, G) \simeq C^\infty(U, C^\infty(V, G)).$$

7. *Evaluation and composition are smooth.*

8. *The map* $f : U \subset E \to L(F, G)$ *is smooth if and only if* $\mathrm{ev}_v(f) : U \to G$ *is smooth for all* $v \in F$.

10 Problems

Analysis works well as far as basic notions are concerned. Existence theorems in turn need a very careful investigation, in general we have

- no existence theorem for differential equations,
- no inverse function theorems,
- no exponential series.

At hand one has "hard" inverse function theorems (which one should consult only for rare occasions) or particular classes of vector fields where flows exist. In addition to "algebraic" conditions one has to solve an analytic problem, too, to obtain results.

11 Applications

- Foundations of Global Analysis (Kriegl, Michor).
- Regularity of Lie groups (existence of exponential maps).
- Trotter-type existence formulas for semigroups (an existence theorem for abstract Cauchy problems).
- Kato-Rellich smoothness results (Alekseevsky, Kriegl, Losik, Michor) for eigenvalues of curves of unbounded operators.

12 Vector fields I

A vector field X on a convenient manifold M is a section of the tangent bundle $TM \to M$. We can associate a differential equation to X namely

$$\begin{aligned} c'(t) &= X(c(t)), \\ c(0) &= x. \end{aligned}$$

If the problem is uniquely solvable on $U \subset M$ up to time ϵ and the dependence on initial values x is smooth, we can define a local flow and vice versa.

The Lie bracket can be defined chartwise by the following local formula for smooth vector fields $X, Y : U \subset E \to E$

$$[X, Y](x) = DX(x) \cdot Y(x) - DY(x) \cdot X(x)$$

for $x \in U$. If a vector field $X \in C^\infty(M \leftarrow TM)$ admits a local flow Fl^X around any point, then

$$[X, Y] = \frac{d}{dt}|_{t=0}(Fl^X_{-t})^* Y$$

for all $Y \in C^\infty(M \leftarrow TM)$.

13 Vector fields II

Let M, N be convenient manifolds and $f : M \to N$ a smooth map. We call two vector fields $X \in C^\infty(M \leftarrow TM)$ and $Y \in C^\infty(N \leftarrow TN)$ f-related if

$$T_x f \cdot X_x = Y_{f(x)}$$

for $x \in M$. Given X_1, X_2 vector fields on M and Y_1, Y_2 vector fields on N. If X_i f-related to Y_i for $i = 1, 2$, then

$$[X_1, X_2] \text{ is } f\text{-related to } [Y_1, Y_2].$$

Frobenius Problem: Given vector fields $X_1, \dots, X_m$ on a manifold M. Does there exist a finite dimensional submanifold $N \subset M$ such that $X_i(x) \in T_x N$ for $x \in N$ and how can the minimal dimension of the manifold be calculated? Is it possible to find such N for any "initial" value?

14 Submanifolds

As the concept of convenient manifolds, the concept of submanifolds (with boundary) can be carried over with any changes. Surprinsingly even the parametrization results hold for finite-dimensional submanifolds.

Theorem 14.1 *Let $\phi : U \subset \mathbb{R}^n_{\geq 0} \to M$ be an immersion, i.e. the tangent map $T_x\phi$ is injective for $x \in U$. Then for any $x \in U$ there is an open neighborhood V of x such that $\phi|_V : V \to M$ is an embedding, in particular $\phi(V)$ is a submanifold (with boundary).*

Given $N \subset M$ a submanifold and X, Y vector fields on M such that $X_x, Y_x \in T_xN$ for $x \in N$, then $[X, Y](x) \in T_xN$. This leads immediately to the Frobenius condition for distributions to be tangent to a submanifold.

15 Distributions

We shall consider the *Frobenius problem* in full generality on convenient manifolds, which can be defined straight forward with respect to convenient calculus:

- A distribution is a collection of subspaces $\mathcal{D}_x \subset T_xM$ for $x \in M$. If all $\mathcal{D}_x$ are finite dimensional, we call the distribution finite dimensional.

- A finite dimensional distribution is called smooth if for any point $x_0 \in M$ there is an open neighborhood V of x_0 and pointwise linearly independent vector fields $X^1, ..., X^n$ such that
$$\left\langle X^1(x), ..., X^n(x) \right\rangle_{\mathbb{R}} = D_x$$
for $x \in V$. This is called a local frame at x_0.

Given a finite dimensional distribution $\mathcal{D}$ on M: is it possible to find for any $x_0 \in M$ a finite dimensional submanifold (with boundary) $N \subset M$ such that $x_0 \in N$ and $\mathcal{D}_y \subset T_yN$ for $y \in N$?

16 Fundamental Lemma

Let $\mathcal{D}$ be a finite-dimensional smooth distribution of rank n on a convenient manifold M and assume that $\mathcal{D}$ is *involutive*, i.e. that for vector fields $X, Y \in C^\infty(M \leftarrow TM)$ with values in $\mathcal{D}$ the Lie bracket takes values in $\mathcal{D}$. Given $X \in C^\infty(M \leftarrow \mathcal{D})$, which admits a local flow Fl^X on $U \subset M$, and take $Y \in C^\infty(M \leftarrow \mathcal{D})$ arbitrary, then

$$(Fl_t^X)^*(Y)(x) \in \mathcal{D}_x$$

for $x \in U$ for small t.

The proof applies the generality of convenient calculus: one has to evaluate

$$\frac{d}{dt}(Fl_t^X)^* X^i = \sum_{i=1}^{n} (f_j^i \circ Fl_t^X)(Fl_t^X)^* X^j$$

for small t pointwise at $x \in U$.

17 Frobenius Theorem I

Let $\mathcal{D}$ be a finite-dimensional smooth distribution of rank n on a convenient manifold M and assume that $\mathcal{D}$ is involutive. Assume additionally that around each point $x_0 \in M$ there is a local frame $X_1, \ldots, X_n$ of vector fields admitting local flows. Then we obtain a foliation on M in the classical sense.

In the proof the power of convenient calculus gets obvious again, since we have to navigate around an inverse function theorem. This works nicely by pulling back the problem to finitely many dimensions.

18 Frobenius Theorem II

Let $\mathcal{D}$ be a finite-dimensional smooth distribution of rank n on a convenient manifold M and assume that $\mathcal{D}$ is involutive. Assume additionally that around each point $x_0 \in M$ there is a local frame $X_1, \ldots, X_n$ (defined on U_{x_0}) of vector fields such that $X_1, \ldots, X_{n-1}$ admit local flows and X_n admits a local semi-flow. Then for any $y \in U_{x_0}$ there is a submanifold N with boundary such that

1. $y \in N$,

2. For all $z \in N$ the equality $\mathcal{D}_z = T_z N$ holds.

Here we loose the classical foliation structure due to generically infinite dimensional gap phenomena.

19 Remarks

Convenient Calculus was developed by Andreas Kriegl and Peter Michor, all the concepts can be found in

Andreas Kriegl, Peter Michor, *The convenient Setting of Global Analysis*, Mathematical Surveys and Monographs 53 (1997).

The geometric approach to interest rate theory was invented by Tomas Björk, Bent Jesper Christensen and Lars Svenson! Frobenius theory on Hilbert spaces was applied to the problem by Tomas Björk and Lars Svenson in a series of articles.

The main contents of this talk can be found in articles:

Damir Filipovic, Josef Teichmann, *Existence of invariant manifolds for stochastic equations in infinite dimensions*, Journal of Functional Analysis, to appear (2002).

Damir Filipovic, Josef Teichmann, *Regularity of finite dimensional realizations for evolution equations*, Journal of Functional Analysis, to appear (2002).

Damir Filipovic, Josef Teichmann, *On the Geometry of the Term structure of Interest Rates*, Proceedings of the London Mathematical Society, invited contribution (2002).

Josef Teichmann, *A Frobenius Theorem on convenient manifolds*, Monatshefte für Mathematik 134 (2001).

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