Affine Processes & non-linear PDEs

(joint work with Christa Cuchiero, Georg Grafendorfer)

(1) Non-linear PDEs in Finance:

Credit Valuation Adjustment, American Option Problems, Utility optimization problems, Superhedging Problems, ...

... lead to non-linear PDEs.

e.g. pricing of an American option with payoff g and maturity T:

$$\partial_t u_t(x) + \mathcal{L} u_t(x) = 1_{\{g(x) \geq u_t(x)\}} \cdot \mathcal{L} g(x)$$

generator of Markov process

maturity

$$u_T(x) = g(x)$$

$u_t(x)$ price of AO at time $t < T$

(2) Numerical Methods for (non-) linear PDEs

(Q) MC techniques

$$\partial_t u_t(x) + \mathcal{L} u_t(x) = 0$$

$$u_T(x) = g(x)$$

$\mathcal{L}$ ... generator of Markov process X

$$u_t(x) = E_{t,x}\left[g(X_T) \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} g\Big(\underbrace{X_T^{(i),t,x}}_{}\Big)$$

independent samples

of values of X

at time T when

starting at t with x.

... a very robust & universal &

rather dimension-free method of solving

PDEs numerically.

Alternatives :

finite difference or finite element

methods.

... deterministic bounds and quick convergence

but works only in low dimension.

Numerical methods for non-linear PDEs:

$$\partial_t u_t(x) + \mathcal{L} u_t(x) + F(u_t(x)) = 0$$

$$u_t(x) = E_{t,x}\left[g(X_T) \right] + \int_t^T E_{t,x}\left[F(u_s(X_s)) \right] ds$$

... which leads to a Backwards algorithm, i.e. nested MC.

A Branching diffusion approach :

(Mc Kean , Dynkin , ...)

$$0 = \partial_t u_t(x) + \mathcal{L} u_t(x) + \sum_{k=0}^{M} p_k\, u_t^k(x) - u_t(x)$$

$$\boxed{\begin{array}{c} p_k \geq 0 \\ \sum p_k = 1 \end{array}}$$

$$u_T(x) = g(x)$$

$$\mathbb{E}[u^Y] = \sum_{k=0}^{M} p_k\, u^k$$

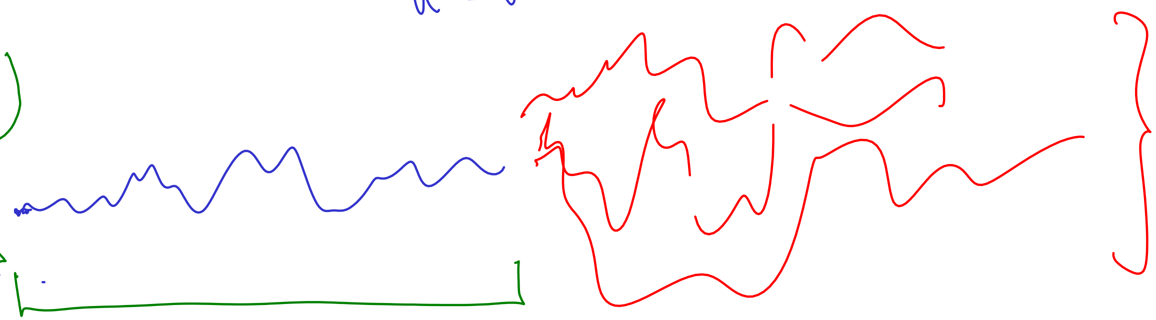
Branching mechanism !

$$\mu_t(x) = E_{t,x}\left[e^{-(T-t)} g(X_T)\right] + $$

$$+ E_{t,x}\left[\int_t^T e^{-(T-s)} \sum_{k=0}^{M} p_k \, \mu_s^k(X_s)\, ds\right]$$

$$= e^{-(T-t)} \cdot E_{t,x}\left[g(X_T)\right] + (1-e^{-(T-t)}) \cdot \sum_{k=0}^{M} p_k \, E_{t,x}\left[\prod_{i=1}^{k} g(X_T^{(i)})\right] + O((T-t)^2)$$

exp(1)



$\mathbb{P}[Y = k] = p_k$

Which equations can be treated by this method ?

$$0 = \partial_t \mu_t(x) + \mathcal{L}\mu_t(x) + F(\mu_t(x)) - \mu_t(x)$$

- $F(u) = \sum_{k=0}^{\infty} p_k u^k \qquad \sum_{k=0}^{\infty} p_k = 1, \quad p_k \geq 0.$
- $F(u) = \sum_{k=0}^{\infty} a_k u^k \qquad \sum_{k=0}^{\infty} a_k < \infty \qquad a_k \geq 0$
- $F(u) = \sum_{k=0}^{\infty} a_k u^k \qquad \sum_{k=0}^{\infty} |a_k| < \infty$

$$\mu_t(x) = E_{t,x}\left[\prod_{i=1}^{N_T} g\left(X_T^{(i)}\right) * \prod_{k=0}^{M}\left(\frac{a_k}{p_k}\right)^{\#\{\text{branching of type } k\}}\right]$$

auxiliary branching mechanism.

... very involved statistical properties but a universal technique to solve semi-linear PDEs numerically. Looks ad hoc ...

③ Relationship to affine processes:

The above branching mechanism can be seen from an affine process point of view.

$$\beta = 0 \qquad (\text{constant Markov process } X)$$

turn time around!

$$\partial_t \mu_t(x) = \sum_{k=0}^{\infty} \mu_t(x)^k - \mu_t(x)$$

$$\mu_t(x) := \exp(\psi_t(x))$$

Cole-Hopf-transform

$$\left(\partial_t \varphi_t(x)\right) e^{\varphi_t(x)} = \sum_{k=0}^{\infty} p_k e^{k \varphi_t(x)} - e^{\varphi_t(x)}$$

$$\partial_t \varphi_t(x) = \sum_{k=0}^{\infty} p_k e^{(k-1)\varphi_t(x)} - 1$$

$$= \int \left(e^{y \varphi_t(x)} - 1\right) \nu(dy)$$

$$\nu = \mathrm{Law}(Y - 1)$$

$$\partial_t \varphi_t(x) = \Psi\left(\varphi_t(x)\right)$$

... which is a generalized RICCATI EQ.

for an affine process jumping with law $Y-1$ and intensity $(1 \times N_t)$ at time t.

This is precisely the process N_T describing the number of offsprings at time T.

What are affine processes?

$(X_t)_{t \geq 0}$ Markov process with state space

D (usually a cone)

- stochastically continuous, time homogeneous

- $$\mathbb{E}_x \left[\exp(\langle f, X_t \rangle) \right] = \exp\left(\langle \psi_t(f), x \rangle + \phi_t(f) \right)$$

ϕ & ψ satisfy in FD generalized RICCATI ODEs

$$\partial_t \psi_t = R(\psi_t), \quad \psi_0(p) = p$$

$$\partial_t \phi_t = F(\psi_t), \quad \phi_0(p) = 0$$

R & F are of Lévy–Khintchine form.

Classification understood on

$$D = \mathbb{R}_{\geq 0}^{m} \times \mathbb{R}^{n}$$

$$= \mathrm{Sym}_{n}(\mathbb{R})$$

$$= \dots$$

describing admissible forms of R & F such that N exists.

Affine point of view:

calculate ϕ, ψ $\Rightarrow$ understand the marginal distribution of N

... turning it around:

simulate N $\Rightarrow$ represent the solutions ϕ, ψ stochastically.

Henry-Labordère / Touzi / Wang

$\Leftrightarrow$ (branching) simulation of AP

on infinite dimensional state spaces

RICCATI ODEs $\rightsquigarrow$ RICCATI PDEs

$$\partial_t \psi_t(f) = \mathcal{L}(f) + R(\psi_t(f))$$

R is of Lévy-Khintchine form

$$\psi_0(f) = f \in C_b(\mathbb{R}^d).$$

④ Beyond generalized Riccati equations

The LK form of F & R is binding!

Henry-Labordère & Touzi & Wang go in their special case beyond. How is this possible?

Take again the FD point of view

$$R^i(f) = \int_{\mathbb{R}^d} \left(e^{\langle f, x\rangle} - 1\right) \nu^i(dx)$$

state space $D \subseteq \mathbb{R}^d$,

$$\tilde{R}(f, z^1, \dots, z^d) = \int_{\mathbb{R}^d \times (\mathbb{Z}\sqrt{-1}\,\pi)^d} \left(e^{\langle f, x\rangle + \sum_i z^i v^i} - 1\right) \tilde{\nu}(dx, dv)$$

$$\mathbb{R}^d = D^i_+ \cup D^i_- \; ; \; i = 1, \dots, d.$$

$$\tilde{F}^i(A,B) = \nu(A)\, 1_{A\cap D_-^i}\, \delta_{\sqrt{-1}\pi}(B) + \nu(A)\, 1_{A\cap D_+^i}\, \delta_0(B)$$

$i = 1, \dots, d$; otherwise $\tilde{F}^j = 0$; $j = d+1, \dots, 2d$

This describes a self-acting linear process, which jumps with jump measure ν on D and which jumps by $\sqrt{-1}\pi$ in $\sqrt{-1}\pi\mathbb{Z}$ whenever a jump in D_- occurs.

$$\partial_t \tilde{\psi}^i_t(f, 1) = \tilde{R}^i\left(\tilde{\psi}_t(f, 1)\right)$$

$$= \begin{pmatrix} \vdots \\ \int_{D_+^i} e^{\langle x, \psi_t(f)\rangle} r^i(dx) - \int_{D_-^i} e^{\langle x, \psi_t(f)\rangle} r^i(dx) - 1 \\ \vdots \\ 0 \end{pmatrix}$$

... which generalizes generalized RICCATI EQ substantially.

Theorem (Cudiero, Grafendorfer, JT)

N Poisson process on D with

measure

$$\partial \psi_t^i(f) = \int_{\mathbb{R}^d} \left(e^{\langle \psi_t(f),\, x \rangle} - 1 \right) \nu^i(dx)$$

$$\mathbb{R}^d = D_+^i \cup D_-^i \qquad i = 1, \dots, d$$

$\Rightarrow$ $\tilde N$ Poisson process on $D \times (\sqrt{-1}\,\mathbb{Z})^d$

$\tilde x = (x, 0)$

and that

$$\langle \tilde\psi_t(f, 1), \tilde x \rangle = \lim_{N \to \infty} \log \left(\frac{1}{N} \sum_{i=1}^{N} \exp\left(\langle f, \tilde N_t \rangle \right) \right)$$

where $\psi_t(f, 1)$ satisfies a generic ODE:

$$\partial_t \psi_t^i(f,1) = \int_{D_+^i} e^{\langle u, \psi_t(f,1)\rangle} \nu^i(du) - \int_{D_-^i} e^{\langle u, \psi_t(f,1)\rangle} \nu^i(du) - 1$$

$$\tilde{\psi}_0(f,1) = f$$

$$\tilde{\psi}_t(f,1) = \begin{pmatrix} \psi_t(f,1) \\ 1 \end{pmatrix}.$$

A simple application :

$$V^i(\mu) = \sum_{|k| \leq M} a^i_k \mu^k \qquad i = 1, \dots, d$$

$\mu \in \mathbb{R}^d$

polynomial vector field on $\mathbb{R}^d$.

$$\partial_t \mu_t(g) = V(\mu_t(g))$$

$$\mu_0(g) = g \in \mathbb{R}^d$$

... Construct a AP representing $\mu_t(g)$.

$$v^i = \sum_{|k| \le M} |a_k^i| \, \delta_k$$

define a self-exciting AP $(N_t)_{t \ge 0}$ with jump characteristic

$$\sum_{i=1}^{d} N_t^i \, v^i \, .$$

Define also

$i = 1, \dots, d$ (zero otherwise)

$$\tilde{r}^i = \sum_{\substack{|k| \le M \\ a_k^i < 0}} |a_k^i| \, \delta_{(k, \sqrt{-1}\,\pi\, e_i)} + \sum_{\substack{|k| \le M \\ a_k^i \ge 0}} |a_k^i| \, \delta_{(k, 0)}$$

on $\mathbb{R}^d \times (\sqrt{-1}\,\mathbb{Z})^d$,

which defines a self-acting AT $(\tilde{N}_t)_{t \geq 0}$

$$f^i := \log g^i$$

Then $\exp\left(\tilde{\varphi}^i_t(f, 1)\right) =: \mu^i_t(g)$ solves

the polynomial VF equation

$$\partial_t \mu_t(g) = V(\mu_t(g)).$$

In other words $(\tilde{N}_t)$ started at c_i represents $\mu_t^i(g)$ so

$$\mathbb{E}_{c_i}\left[\exp\left(\langle(g,1),\tilde{N}_t\rangle\right)\right]=\mu_t^i(g),$$

hence the solution of EVERY POLYNOMIAL ODE can be represented stochastically.

High-dim. ODE ≈ non-linear PDE