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Trotter's formula on infinite dimensional Lie groups

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Abstract. A theory for the solution of non-autonomous linear differential equations on convenient vector spaces is presented. The theory generalizes the Hille-Yosida theorem and several non-autonomous versions of it. It is special feature of the theory, that the conditions can be applied in the case of locally convex spaces, which are not well understood from a functional analytic point of view. The main application is the investigation of a sufficient condition for the existence of exponential and evolution mappings on infinite dimensional Lie groups.

Trotter's formula in the classical functional analytic sense is a synonym for building complicated semigroups from simple ones (see [3], for some recent progress see [5]). We want to point out the existence aspect, i.e. provided some boundedness condition Trotter-type-approximations converge to a smooth semigroup on a huge class of locally convex spaces. The results remind at a first sight the results in [2], even though there is no direct connection. The main point of the presented results is the fact, that we obtain convergence in all derivatives to a smooth semigroup, which allows several non-trivial conclusions.

We apply the result in particular to infinite dimensional Lie groups in the sense of Kriegl, Michor, Milnor, Omori et al. (see [4], [6], [7]), even though the theory is more general. Beyond Banach spaces there is a lack of methods, how to solve ordinary differential equations (however Fréchet spaces are the only reasonable model spaces for diffeomorphism groups on compact manifolds). We tried to get reasonable results on solutions of non-autonomous equations by investigating the problem on the vector space of smooth real valued functions on a given Lie group G. This can be viewed as a linearization of the problem.

Given a Lie group G, which is assumed to be smoothly regular, the space $C^\infty(G,\mathbb{R})$ is a convenient vector space. The right regular representation

$$\begin{aligned}\rho : G &\to L(C^\infty(G,\mathbb{R}))\\ g &\mapsto (f \mapsto f(.g))\end{aligned}$$

is smooth and initial (for initiality of the map ρ, see [4]), i.e. a curve $d:\mathbb{R}\to G$ is smooth if the curve of linear mappings $\rho\circ d$ is smooth. If we want to solve a differential equation of the type

$$\delta^r c(t) = X(t)$$

where X is a smooth curve in the Lie algebra $\mathfrak{g}$ and δ^r denotes the right logarithmic derivative on G, then we can equally analyse this non-linear problem on the convenient vector space $C^\infty(G,\mathbb{R})$. This programme is performed in the last section for the abstract case of smooth semigroups. In [7] strong analytic conditions in the charts have to be assumed for a given Lie group, such that non-autonomous equations of the above type admit flows. The presented theory allows to formulate "inner" conditions on the Lie group, such that flows exist. These ideas have been worked out in [8].

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1. Convenient Calculus

Convenient calculus (see [4], ch.I) provides the setting for infinite dimensional analysis and geometry. The concept of a smooth curve with values in a locally convex space is straight forward, namely all derivative of arbitrary order exist as limits of difference quotients. The concept of Mackey-convergence will be of high importance: Given a locally convex vector space E, a sequence $\{x_n\}_{n\geq 0}$ is called Mackey-convergent to x, if there is a bounded set $B \subset E$ and a sequence of positive real numbers $\mu_n \downarrow 0$ such that $x_n - x \in \mu_n B$ for $n \geq 0$. A sequence $\{x_n\}_{n\geq 0}$ is called Mackey-Cauchy, if there is a bounded set $B \subset E$ and a double-sequence of positive real numbers $\mu_{nm} \downarrow 0$ as $n,m \to \infty$, such that $x_n - x_m \in \mu_{nm} B$ for $n,m \geq 0$. A locally convex space is called Mackey-complete if every Mackey-Cauchy-sequence converges.

It is a nice observation that the class of locally convex spaces, where curves are smooth if and only if they are weakly smooth (the composition with a continuous functional yields a smooth curve with values in $\mathbb{R}$), is exactly given by the class of Mackey-complete locally convex spaces (see [4], ch.I, 2.14). In the sequel we refer to these spaces as convenient spaces, in particular all sequentially complete locally convex spaces are convenient. A linear mapping on a locally convex vector space is called bounded if the image of bounded sets is bounded. The vector space of bounded linear maps between convenient vector spaces E and F is denoted by $L(E,F)$, the dual of bounded linear functionals by E'. On $L(E,F)$ we shall always consider the locally convex topology of uniform convergence on bounded sets of E. We obtain a convenient vector space (see [4] for details). Remark that we have the following uniform boundedness principle for convenient vector spaces: A set of linear maps in $L(E,F)$ is uniformly bounded on bounded sets if and only if it is pointwise bounded.

The final topology with respect to all smooth curves into E is denoted by $c^\infty E$. Remark that up to Fréchet-spaces the c^∞-topology coincides with the given locally convex topology, however, there are examples, where the $c^\infty E$ is not even a topological vector space (see [4], ch.I, 4.11). Even though a convenient vector space is given through its system of bounded sets, we consider several times locally convex topologies on the convenient vector space compatible with this system of bounded sets. Mackey-sequences converge in each of these compatible topologies.

Definition 1.1. Let $U \subset E$ be a c^∞-open subset of the convenient vector space E, $f : U \to F$ a mapping. f is called smooth if smooth curves are mapped to smooth curves. Let $f : U \to F$ be a smooth mapping, then the differential $df : U \times E \to F$ is defined as directional derivative

$$df(x,v) := \frac{d}{dt}|_{t=0} f(x+tv)$$

This rather categorical definition of smoothness coincides with all reasonable concepts of smoothness up to Fréchet-spaces, however already on $\mathbb{R}^2$ it is not obvious how to prove the equivalence (see [4], ch.I, 3.2). Originally the idea stems from [1]. Smooth functions are continuous in the c^∞-topology, but not necessarily in one of the compatible topologies. The following theorem is a condensation of results from convenient analysis, which will be applied in this article, for the proof see [4], ch.I.

Theorem 1.2. *Let E, F, G be convenient vector spaces, $U \subset E$, $V \subset F$ c^∞-open, then we obtain:*

1. *Multilinear mappings are smooth if and only if they are bounded.*
2. *If $f : U \to F$ is smooth, then $df : U \times E \to F$ and $df : U \to L(E,F)$ are smooth.*
3. *The chain rule holds.*

4. *The vector space* $C^\infty(U,F)$ *of smooth mappings* $f: U \to F$ *is again a convenient vector space (inheritance property) with the following initial topology:*

$$C^\infty(U,F) \to \prod_{c \in C^\infty(\mathbb{R},U)} C^\infty(\mathbb{R},F) \to \prod_{c \in C^\infty(\mathbb{R},U),\, \lambda \in F'} C^\infty(\mathbb{R},\mathbb{R}).$$

5. *The exponential law holds, i.e.*

$$i : C^\infty(U, C^\infty(V,G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces. Usually we write $i(f) = \hat{f}$ *and* $i^{-1}(f) = \check{f}$.

6. *The smooth uniform boundedness principle is valid: A linear mapping* $f : E \to C^\infty(V,G)$ *is smooth (bounded) if and only if* $ev_v \circ f : E \to G$ *is smooth for* $v \in V$, *where* $ev_v : C^\infty(V,G) \to G$ *denotes the evaluation at the point* $v \in V$.

7. *The smooth detection principle is valid:* $f : U \to L(F,G)$ *is smooth if and only if* $ev_x \circ f : U \to G$ *is smooth for* $x \in F$ *(This is a reformulation of the smooth uniform boundedness principle by cartesian closedness).*

8. *Taylor's formula is true, if one defines by applying cartesian closedness and obvious isomorphisms the multilinear-mapping-valued higher derivatives* $d^n f : U \to L^n(E,F)$ *of a smooth function* $f \in C^\infty(U,F)$, *more precisely for* $x \in U$, $y \in E$ *so that* $[x, x+y] = \{x + sy | 0 \leq s \leq 1\} \subset U$ *we have the formula*

$$f(y) = \sum_{i=0}^{n} \frac{1}{i!} d^i f(x) y^{(i)} + \int_0^1 \frac{(1-t)^n}{n!} d^{n+1} f(x+ty)\,(y^{(n+1)}) dt$$

for all $n \in \mathbb{N}$.

In the sequel we apply the Landau-like symbols to shorten the proof : Given a mapping c from a set M to a convenient vector space E we write $c = O(d)$ if there is a mapping $d : M \to \mathbb{R}_{\geq 0}$ and a bounded absolutely convex set $B \subset E$ such that $c(m) \in d(m)B$ for all $m \in M$. Remember that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Mackey-convergent if there is a sequence of positive real numbers $\{\mu_n\}_{n\in\mathbb{N}}$ with $\mu_n \downarrow 0$ such that $x_n = O(\mu_n)$.

A convenient algebra is a convenient vector space A with a bounded multiplication being associative and unital. In particular $L(E,E) := L(E)$ is a convenient algebra, since it is a convenient vector space and the multiplication is bounded by the uniform boundedness principle.

Given a smooth curve $X : \mathbb{R} \to A$ we try to solve the following ordinary differential equation

$$\frac{d}{dt} x(t) = X(t) x(t) \tag{R}$$

with initial value $x(s) = x_s$ at the point s for $t \geq s$. If there is a smooth family of solutions c_s for initial value e at any point s, then there is a smooth family of solutions for all initial values x at any point s given through the curves $t \mapsto c_s(t)x$ for $t \geq s$. If there is a smooth family of smooth solutions c_s for initial value e at any point s with the propagation condition

$$c_s(t) c_r(s) = c_r(t) \text{ for } t \geq s \geq r$$

then the solutions of the equation are unique for all initial values at any point in time. From the defining property of the solution family $c_s(t)$ we obtain

$$0 = \frac{d}{ds} c_s(t) c_r(s) = (\frac{d}{ds} c_s(t)) c_r(s) + c_s(t) X(s) c_r(s)$$

which yields evaluated at $s = r$

$$\frac{d}{ds}c_s(t) = -c_s(t)X(s)$$

for $s \leq t$. So $s \mapsto c_s(t)$ is a smooth family of solutions for initial value e at any point in time of a ordinary differential equation of the type

$$\frac{d}{ds}y(s) = y(s)Y(s) \tag{L}$$

but in the negative time direction. Consequently we obtain by looking at another smooth solution $\widetilde{c}_r(t)$ from r to t

$$\frac{d}{ds}c_s(t)\widetilde{c}_r(s) = (\frac{d}{ds}c_s(t))\widetilde{c}_r(s) + c_s(t)X(s)\widetilde{c}_r(s) = 0$$

which allows the desired conclusion of uniqueness. If a smooth solution family satisfying the propagation condition exists for (R) we call it the right evolution of the curve X. If a smooth solution family satisfying the propagation condition exists for (L), we call it the left evolution of the non-autonomous curve Y.

2. The Approximation Theorem

With the concept of product integrals (see [6], [7] on Lie groups) we try to approximate right evolutions of a given curve X to obtain an existence theorem.

Definition 2.1. Let A be a convenient algebra. Given a smooth curve $X\colon \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R}^2 \to A$ with $h(s,0) = e$ and $\frac{\partial}{\partial t}|_{t=0}h(s,t) = X(s)$, then we define the following finite products of smooth curves

$$p_n(s,t,h) := \prod_{i=0}^{n-1} h\left(s + \frac{(n-i)(t-s)}{n}, \frac{t-s}{n}\right)$$

for $a, s, t \in \mathbb{R}$. If p_n converges in all derivatives to a smooth curve $c : \mathbb{R} \to A$, then c is called the product integral of X or h and we write $c(s,t) = \prod_s^t \exp(X(s)ds)$ or $c(s,t) =: \prod_s^t h(s,ds)$. The case $h(s,t) = c(t)$ with $p_n(s,t,h) = c(\frac{t-s}{n})^n$ is referred to as simple product integral.

Theorem 2.2. *Let A be convenient algebra. Given $X : \mathbb{R} \to A$ and a smooth mapping $h : \mathbb{R} \times \mathbb{R}_{\geq 0} \to A$ with $h(r,0) = e$ and $\frac{\partial}{\partial t}|_{t=0}h(r,t) = X(r)$. Suppose that for every fixed $s_0 \in \mathbb{R}$, there is $t_0 > s_0$ such that $p_n(s,t,h) = O(1)$ on $\mathbb{N} \times \{(s,t) \in [s_0,t_0]^2 \,|\, s \leq t\}$. Then the product integral $\prod_s^t h(r,dr)$ exists and the convergence is Mackey in all derivatives on compact (s,t)-sets for $s \leq t$. Furthermore the product integral is the right evolution of X.*

Remark 2.3. The hypothesis on the product integrals will be referred to as *boundedness condition.*

Proof. Literally the condition on the approximations is the following: There is an absolutely convex bounded and closed set B such that for $s_0 \leq s \leq t \leq t_0$ and $n \in \mathbb{N}$

$$p_n(s,t,h) \in B$$

We have to derive some more subtle boundedness conditions: Therefore we apply Taylor expansion successively to necessary estimates. First we show that the first derivative of the product integral is bounded:

$$\frac{d}{d\delta}\prod_{i=0}^{n-1} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) \in$$

$$\sum_{j=0}^{n-1}\prod_{i=0}^{j-1} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) \cdot C\frac{1}{n} \cdot \prod_{i=j+1}^{n-1} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)$$

for $\delta \in [0, t_0 - s_0]$ with a closed, absolutely convex and bounded set C, such that

$$\begin{aligned}\frac{d}{d\delta} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) &= \partial_1 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{(n-i)}{n} + \partial_2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n} \\ &= \frac{(n-i)\delta}{n^2}\int_0^1 \partial_2\partial_1 h\left(s+\frac{(n-i)\delta}{n}, r\frac{\delta}{n}\right) dr + \\ &\quad + \partial_2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n} \in C\frac{1}{n}\end{aligned}$$

for $\delta \in [0, t_0 - s_0]$ and $n \in \mathbb{N}$.

Remark that $\partial_1^k h(s, 0) = 0$ for $s \in \mathbb{R}$, $k \geq 1$. The other factors in the above sum are of type p_m with adjusted lower and upper bound and step width,

$$\prod_{i=0}^{j-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) = p_j\left(s+\frac{(n-j)\delta}{n}, s+\delta, h\right)$$

$$\prod_{i=j+1}^{n-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) = p_{n-j-1}\left(s, s+\frac{(n-j-1)\delta}{n}, h\right),$$

so bounded by assumption on the respective intervals in $[s_0, t_0]$. This allows to conclude boundedness of the first derivative with respect to t.

Repeating this procedure we obtain by induction for $k = 1, 2$ that

$$\frac{d^k}{d\delta^k}\prod_{i=0}^{n-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) = O(1)$$

for $\delta \in [0, t_0 - s_0]$ and $n \in \mathbb{N}$, since

$$\begin{aligned}\frac{d^2}{d\delta^2} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) &= \partial_1^2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{(n-i)^2}{n^2} + \\ &+ 2\partial_1\partial_2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n}\frac{(n-i)}{n} + \partial_2^2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n^2} \\ = \frac{(n-i)^2\delta}{n^3}\int_0^1 \partial_2\partial_1^2 h\left(s+\frac{(n-i)\delta}{n}, r\frac{\delta}{n}\right) dr &+ 2\partial_1\partial_2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n}\frac{(n-i)}{n} + \\ &+ \partial_2^2 h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\frac{1}{n^2} \in D\frac{1}{n}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{d\delta^2}\prod_{i=0}^{n-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) &\in \\ \sum_{0\leq j<k\leq n-1}\prod_{i=0}^{j-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\cdot C\frac{1}{n}\cdot &\prod_{i=j+1}^{k-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\cdot \\ C\frac{1}{n}\cdot \prod_{i=k+1}^{n-1} h&\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) + \\ + \sum_{j=0}^{n-1}\prod_{i=0}^{j-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\cdot D\frac{1}{n}\cdot &\prod_{i=j+1}^{n-1} h\left(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}\right)\end{aligned}$$

where D is a bounded closed absolutely convex subset of A. This means in particular that

$$h(s+\delta, \delta) - p_m(s, s+\delta, h) = O(\delta^2) \tag{E}$$

by Taylor's formula up to second order uniformly in m, since $h(s, 0) = e$ and $\frac{d}{d\delta}|_{\delta=0} h(s+\delta, \delta) = X(s)$ and $\frac{d}{d\delta}|_{\delta=0}\prod_{i=0}^{n-1} h(s+\frac{(n-i)\delta}{n}, \frac{\delta}{n}) = X(s)$.

Now we calculate

$$p_n(s,t,h) - p_{nm}(s,t,h) = \sum_{j=0}^{n-1} \prod_{i=0}^{j-1} h\left(s + \frac{(n-i)(t-s)}{n}, \frac{t-s}{n}\right) \cdot$$

$$\left(h\left(s + \frac{(n-j)(t-s)}{n}, \frac{t-s}{n}\right) - \prod_{i=jm}^{(j+1)m-1} h\left(s + \frac{(nm-i)(t-s)}{nm}, \frac{t-s}{nm}\right) \right) \cdot$$

$$\prod_{i=(j+1)m}^{nm-1} h\left(s + \frac{(nm-i)(t-s)}{nm}, \frac{t-s}{nm}\right)$$

in the spirit of the following formula

$$a_1 \cdot \ldots \cdot a_n - b_1 \cdot \ldots \cdot b_n = \sum_{j=1}^{n} a_1 \cdot \ldots \cdot a_{j-1}(a_j - b_j) b_{j+1} \cdot \ldots \cdot b_n \tag{S}$$

which is true in any associative algebra. For the middle factor of the above series we observe that

$$h\left(s + \frac{(n-j)(t-s)}{n}, \frac{t-s}{n}\right) = h\left(s + \frac{(n-j-1)(t-s)}{n} + \frac{t-s}{n}, \frac{t-s}{n}\right)$$

The third term is the m-th approximation for a product integral with lower border $s + \frac{(n-j-1)(t-s)}{n}$ and upper border $s + \frac{(n-j)(t-s)}{n}$.

Via the estimate (E) with $\delta = \frac{t-s}{n}$ we arrive at

$$p_n(s,t,h) - p_{nm}(s,t,h) = O\left(\frac{(t-s)^2}{n}\right)$$

on $\{m | m \in \mathbb{N}\} \times \{(s,t) \in [s_0,t_0]^2 \mid s \leq t\}$, which provides the Mackey-Cauchy property for the above sequence.

Convergence in all derivatives follows by redoing the above program: Calculating the derivative of order k needs the binomial formula

$$\begin{aligned}
\frac{d^k}{d\delta^k} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) &= \sum_{j=0}^{k} \binom{k}{j} \partial_1^j \partial_2^{k-j} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) \left(\frac{(n-i)}{n}\right)^j \left(\frac{1}{n}\right)^{k-j} \\
&= X^{(k)}\left(s + \frac{(n-i)\delta}{n}\right) \frac{\delta}{n} \left(\frac{(n-i)}{n}\right)^k + \\
&+ kX^{(k-1)}\left(s + \frac{(n-i)\delta}{n}\right) \left(\frac{(n-i)}{n}\right)^{k-1} \frac{1}{n} + O\left(\frac{1}{n^2}\right) \\
&= X^{(k)}\left(s + \frac{(n-i-1)\delta}{n}\right) \frac{\delta}{n} \left(\frac{(n-i)}{n}\right)^k + \\
&+ kX^{(k-1)}\left(s + \frac{(n-i-1)\delta}{n}\right) \left(\frac{(n-i)}{n}\right)^{k-1} \frac{1}{n} + O\left(\frac{1}{n^2}\right)
\end{aligned}$$

for $k \geq 1$, where we applied Taylor expansions. The terms of order $\frac{1}{n}$ will cancel out in our summation procedure (S), since this formula holds for all smooth h with $h(s,0) = e$ and $\partial_2 h(s,0) = X(s)$, hence also for $p_m(s-t,s,h)$ (remark that $p_n(s,s+t,p_m(s-t,s,h)) = p_{nm}(s,s+t,h)$). We shall work with the following estimate

$$\frac{d^k}{d\delta^k} h\left(s + \frac{(n-i)\delta}{n}, \frac{\delta}{n}\right) - \frac{d^k}{d\delta^k} p_m\left(s + \frac{(n-i-1)\delta}{n}, s + \frac{(n-i)\delta}{n}, h\right) = O\left(\frac{1}{n^2}\right),$$

which is valid uniformly in m by the boundedness condition. Deriving with respect to s we obtain the same estimate uniformly in m. Hence this is sufficient for convergence by applying (S) to

$$\frac{\partial^k}{\partial t^k}(p_n(s,t,h) - p_{nm}(s,t,h)) = O\left(\frac{1}{n}\right)$$

for $k \geq 1$. The limit will be denoted by $c(s,t)$ on the interval $[s_0, t_0]$. "Sufficient for convergence" will be explained literally: Differentiating k-times yields with the above summation procedure (S) n^{k+1} terms to sum up (see the formula for the second derivative above). There are n terms where order of differentiation k appears, $O(n^2)$ terms where two orders smaller than k appear, but with sum k, $O(n^3)$ terms where three orders smaller than k with sum k appear,... Applying our summation procedure (S) to the n terms where order k of differentiation appears we get n^2 terms: n terms involve k-th derivative, so the difference is of order $\frac{1}{n^2}$, the other $n^2 - n$ terms involve ordinary factors, so the difference is of order $\frac{1}{n^2}$, but there is some outer factor $\frac{1}{n}$. So we get inductively our order estimate:

$$O\left((n^2-n)\frac{1}{n^2}\frac{1}{n} + n\frac{1}{n^2} + (n^3-2n^2)\frac{1}{n^2}\frac{1}{n^2} + 2n^2\frac{1}{n}\frac{1}{n^2} + \right.$$
$$\left. +(n^4-3n^3)\frac{1}{n^2}\frac{1}{n^3} + 3n^3\frac{1}{n^2}\frac{1}{n^2} + \ldots + (n^{k+1}-kn^k)\frac{1}{n^2}\frac{1}{n^k} + kn^k(\frac{1}{n^2})^k\right) = O\left(\frac{1}{n}\right)$$

Consequently $c(s,t)$ is smooth for $s \leq t$ in $[s_0, t_0]$ and the convergence takes place as Mackey-convergence in $C^\infty(\{(s,t) \in [s_0,t_0]^2 \,|\, s \leq t\}, A)$ with quality $\frac{1}{n}$ in each derivative.The propagation condition follows with standard arguments on continuity with respect to the smooth topology:

$$c(r,t) = c(s,t)c(r,s)$$

for $(t-r)q + r = s$ with $q \in \mathbb{Q}$, $0 < q < 1$ by construction and everywhere by continuity.

We calculate $\partial_2 c(t,t) = X(t)$ via uniform convergence of the derivative, then derivation of the propagation condition yields the result

$$\frac{\partial}{\partial t}c(r,t) = X(t)c(r,t)$$

for $t = s$ and $r \leq s \leq t \in [s_0, t_0]$, so $c(r,t)$ is smooth in t. Looking at the situation of an arbitrary interval we can multiply existing product integrals to get an arbitrary one: Given $s < t$ we can cover this compact interval by intervals of length $\frac{t-s}{k}$ for k large enough, such that on the cover-intervals our estimates are valid.

$$p_{mk}(s,t,h) = p_m\left(s, s+\frac{t-s}{k}, h\right) \cdot \ldots \cdot p_m\left(t-\frac{t-s}{k}, t, h\right)$$

and $p_{mk}(s,t,h) - p_{mk+r}(s,t,h) = O(\frac{t-s}{mk})$ for $0 \leq r < k$, so we get the desired boundedness condition on the interval $[s,t]$. ∎

The next corollary asserts smooth dependence on the smooth curve X, which will be useful in the sequel, here we apply the main features of convenient calculus, namely cartesian closedness (Theorem 1.2.5) and the very definition of smoothness: A mapping is smooth if its composition with smooth curves is smooth.

Corollary 2.4. *Let A be convenient algebra. Given a smooth curve $X : \mathbb{R}^2 \to A$ and a smooth mapping $h : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to A$ with $h(r_1, r_2, 0) = e$ and $\frac{\partial}{\partial t}h(r_1,r_2,0) = X(r_1,r_2)$. Suppose that for every fixed $s_0 \in \mathbb{R}$ and a compact r_1-interval, there is $t_0 > s_0$ such that $p_n(s,t,h)(r_1) = O(1)$ on $\mathbb{N} \times \{(s,t) \in [s_0,t_0]^2 \,|\, s \leq t\}$ and the compact r_1-intervals. Then the product integral $\prod_s^t h(r_1,r_2,dr_2)$ exists as smooth mapping on $\mathbb{R} \times \{(s,t) \,|\, s \leq t\}$ and the convergence is Mackey in all derivatives on compact (r_1,s,t)-sets for $s \leq t$.*

Proof. By inheritance (see Theorem 1.2.4) we obtain that $C^\infty(\mathbb{R}, A)$ is a convenient algebra and the above condition means, that the product integrals lie in a bounded set in this algebra on compact (s,t)-sets for $s \leq t$ and $n \in \mathbb{N}$. Consequently we arrive at the desired result. The boundedness in $C^\infty(\mathbb{R}, A)$ follows from direct calculation since $\partial_1^k h(r_1, r_2, t) = O(t)$ on compact (r_1, r_2)-sets. ∎

3. Refinements and Applications

In this section we collect several (simple) refinements and comments to the approximation theorem. For a convenient Hille-Yosida-Theory see [9]. We want to point out the fact that the famous reproduction formula in Hille-Yosida-Theory, namely

$$s - \lim_{n\to\infty} \left(id - A\frac{t}{n}\right)^{-n} = T_t$$

for a strongly continuous semigroup T with infinitesimal generator A is a corollary of the above result on the space $D(A^\infty)$ with the smooth curve $c(t) := (id - At)^{-1}$. Considering semigroup problems it seems to be useful sometimes to have the possibility to pass to a locally convex vector space (see the last example on infinite dimensional heat semigroups in [9]).

Remark 3.1. We provide some (simple) examples of convenient algebras:

1. The boundedness condition is always satisfied up to the level of unital locally m-convex convenient algebras A (the only completeness assumption is Mackey-completeness). Let $c : \mathbb{R}_+ \to A$ be a smooth curve passing through the identity at zero. Let $p : A \to \mathbb{R}$ be a continuous seminorm satisfying $p(ab) \le p(a)p(b)$ and $p(e) = 1$. A set of seminorms of this type can be chosen on any unital locally m-convex convenient algebra A to generate the topology. Then we obtain for a given $s \in \mathbb{R}_+$
$$p\left(c\left(\frac{t}{n}\right)^n\right) \le p\left(c\left(\frac{t}{n}\right)\right)^n \le \left(1 + \frac{Kt}{n}\right)^n \le \exp(Kt)$$
for $t \in [0, s]$. The constant K depends on c and s, in fact $K = \sup_{t\in[0,s]} p(c'(t))$. In this case we obtain a smooth one-parameter group in each direction.
2. It is easy to construct examples, where the boundedness condition is not satisfied: Take $A = L(s)$ the unital convenient algebra of bounded (which is equivalent to continuous on Fréchet spaces) operators on the space of rapidly decreasing sequences s. We take for $a : s \to s$ the following bounded operator $a(x_1, x_2, x_3, ...) = (0, 1^2x_1, 2^2x_2, 3^2x_3, ...)$, then the Abstract Cauchy Problem associated to a has no nontrivial solutions. Consequently no semigroup with generator a exists. Anyway a can be decomposed into two nilpotent operators of order 2:
$$a_1(x_1, x_2, x_3, ...) = (0, 1^2x_1, 0, 3^2x_3, 0, ...)$$
$$a_2(x_1, x_2, x_3, ...) = (0, 0, 2^2x_2, 0, 4^2x_4 ...)$$
$a = a_1 + a_2$, ${a_1}^2 = 0$ and ${a_2}^2 = 0$. We define $c(t) = \exp(a_1t)\exp(a_2t)$ for $t \in \mathbb{R}$. For this smooth curve the boundedness condition cannot be satisfied, otherwise a smooth semigroup with generator a would exist, which is a contradiction. So the set of operators which admit a smooth semigroup is not linear space on s.

Corollary 3.2. *Let E be a convenient vector space and $c : \mathbb{R}_{\ge 0} \to L(E)$ a smooth curve with $c(0) = id_E$. If there is $s > 0$ so that for every $x \in E$ the set*

$$\{c\left(\frac{t}{n}\right)^n x | 0 \le t \le s\}$$

is bounded in E, then the boundedness condition is satisfied for c in $L(E)$.

The main theorem of the previous part is in fact an existence theorem. The question, what is implied by the existence of a semigroup, was not treated, more precisely: Let E be a convenient vector space, T a semigroup of bounded linear operators on E with infinitesimal generator $a \in L(E)$. If $c : \mathbb{R}_{\ge 0} \to E$ is a smooth curve, so that $c(0) = id$ and $c'(0) = a$ are satisfied, does the sequence $\{c(\frac{t}{n})^n\}_{n\in\mathbb{N}}$ converge in some sense to the semigroup T? The question seems to be difficult.

Proposition 3.3. *Let E be a convenient vector space, T a semigroup of bounded linear operators on E with infinitesimal generator $a \in L(E)$ and $c : \mathbb{R}_{\geq 0} \to E$ a smooth curve, so that $c(0) = id$, $c'(0) = a$. Given furthermore $s_0 > 0$ such that for every $x \in E$ there exists $k \in \mathbb{N}$ and $s \geq s_0$, so that the set*

$$\{\frac{1}{n^k} c\left(\frac{t}{n}\right)^n x | n \in \mathbb{N},\ 0 \leq t \leq s\}$$

is bounded in E. Then the boundedness condition is satisfied for the curve c, consequently the sequence $\{c(\frac{t}{n})^n\}_{n \in \mathbb{N}}$ converges to the smooth semigroup T_t uniformly on compact subsets of $[0, \infty[$ in all derivatives.

Proof. We apply the same methods as in the proof of the main theorem. We use the formulas pointwise. Let $x \in E$ be given, then we obtain

$$T_t x - c\left(\frac{t}{n}\right)^n x = \sum_{i=1}^{n} T_{\frac{t(i-1)}{n}} \left(T_{\frac{t}{n}} - c\left(\frac{t}{n}\right)\right) c\left(\frac{t}{n}\right)^{n-i} x$$

for $n \in \mathbb{N}$ and $t \geq 0$. The middle term is estimated in the usual way

$$T_{\frac{t}{n}} - c\left(\frac{t}{n}\right) \in \frac{t^2}{n^2} C \text{ for all } k \in \mathbb{N}, t \in [0, s]$$

for a given $s > 0$ by Taylor expansion. By hypothesis there is a bounded set B and positive number $k := k(x) \in \mathbb{N}$ so that on $[0, s] := [0, s(x)]$ with $s(x) \geq s_0$

$$c\left(\frac{t}{n}\right)^n x \in n^k B.$$

Inserting all estimates we obtain

$$T_t x - c\left(\frac{t}{n}\right)^n x \in \frac{t^2}{n^2} \sum_{i=1}^{n} T_{\frac{t(i-1)}{n}} C (n-i)^k B,$$

which means that the assumed estimate can be improved on $[0, s]$ by the uniform boundedness principle ($T_r x$ is bounded on any fixed compact r-set by smoothness, so $T_r CB$ is bounded on any fixed compact r-set). We arrive finally at

$$c\left(\frac{t}{n}\right)^n x \in n^{k-1} B'$$

on the interval $[0, s]$. Repeating this procedure k times we arrive at the result that for any $x \in E$ there is $s \geq s_0$ so that

$$\{c\left(\frac{t}{n}\right)^n x | n \in \mathbb{N},\ 0 \leq t \leq s\}$$

is bounded in E. ∎

By the same methods we can prove a version of this proposition on convenient algebras:

Proposition 3.4. *Let A be a convenient algebra, T a smooth semigroup with infinitesimal generator a and $c : \mathbb{R}_{\geq 0} \to A$ a smooth curve, so that $c(0) = e$ and $c'(0) = a$. If there exists $s > 0$ and $k \in \mathbb{N}$ so that*

$$\{\frac{1}{n^k} c\left(\frac{t}{n}\right)^n | n \in \mathbb{N},\ 0 \leq t \leq s\}$$

is bounded in A, then the boundedness condition is satisfied for the given curve c.

4. Infinite dimensional smooth semigroups

Definition 4.1. A non-empty set X, a set of curves $C_X \subset Map(\mathbb{R}, X)$ and a set of mappings $F_X \subset Map(X, \mathbb{R})$ are called a Frölicher space if the following conditions are satisfied (see [4], section 23):

1. A map $f : X \to \mathbb{R}$ belongs to F_X if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for $c \in C_X$.
2. A curve $c : X \to \mathbb{R}$ belongs to C_X if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for $f \in F_X$

Let X be a Frölicher space, then C_X is called the set of smooth curves, F_X the set of smooth real valued functions. Mappings between Frölicher spaces are called smooth if the compositions with smooth curves is smooth. Let X, Y be Frölicher spaces then $C^\infty(X, Y)$ has a natural structure of a Frölicher space by the following definition

$$C^\infty(X, Y) \overset{C(f,c)}{\to} C^\infty(\mathbb{R}, \mathbb{R}) \overset{\lambda}{\to} \mathbb{R}$$

is a smooth map for $c \in C_X$, $f \in F_Y$ and $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})'$, where $C(f, c)(\phi) := f \circ \phi \circ c$. The Frölicher space structure generated by these smooth maps is the canonical structure on $C^\infty(X, Y)$. Given a convenient vector space E and a Frölicher space X, then

$$C^\infty(X, E) \overset{C(f,c)}{\to} C^\infty(\mathbb{R}, \mathbb{R}) \overset{\lambda}{\to} \mathbb{R}$$

with $c \in C_X$, $f \in E'$, the dual space of bounded linear functionals, and $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})'$ induces a convenient structure on the space $C^\infty(X, E)$. An algebraic semigroup is an associative monoid with identity, a smooth semigroup is a smooth space with algebraic semigroup-structure, such that the algebraic structures are smooth. We shall assume that smooth semigroups are *smoothly Hausdorff*, i.e. the smooth functions separate points. The natural topology on a smooth semigroup is given by the final topology with respect to all smooth curves, however, in this topology the smooth semigroup is not necessarily a topological semigroup (take a convenient vector space E as abelian group, such that $c^\infty E$ is not a topological vector space) . We shall denote this topology on a smooth semigroup G by $c^\infty G$.

The definition of product integrals on semigroups is done in the same way as in algebras, see section 2. The right regular representation of G

$$\begin{aligned} \rho : G &\to L(C^\infty(G, \mathbb{R})) \\ g &\mapsto (f \mapsto f(.g)) \end{aligned}$$

in the bounded operators on $C^\infty(G, \mathbb{R})$ is smooth and initial, i.e. a curve $d : \mathbb{R} \to G$ is smooth if $\rho \circ d$ is smooth. This is clear by definition, since smoothness of $\rho \circ d$ means that for all smooth functions $f \in C^\infty(G, \mathbb{R})$ and all $g \in G$ the mapping $t \mapsto f(gd(t))$ is smooth, so d is smooth by the definition of a Frölicher space.

Theorem 4.2. *Let G be a smooth (smoothly Hausdorff!) semigroup. Given a smooth mapping $c : \mathbb{R}^3_{\geq 0} \to G$ with $c(r_1, r_2, 0) = e$ such that the approximations $p_n(s, t, c)(r_1)$ lie in a sequentially c^∞-compact set on compact (r_1, s, t)-sets with $s \leq t$, then there is a smooth curve $d : \mathbb{R}_{\geq 0} \times \{(s, t) \mid s \leq t\} \to G$ with $d(r_1, s, t)d(r_1, r, s) = d(r_1, r, t)$ (propagation condition), $p_n(s, t, c)(r_1) \overset{n\to\infty}{\to} d(r_1, s, t)$ in $c^\infty G$ and $\rho[p_n(s, t, c)(r_1)] \overset{n\to\infty}{\to} \rho[d(r_1, s, t)]$ uniformly on compact (r_1, s, t)-sets with $s \leq t$ in all derivatives. Given another curve $\widetilde{c} : \mathbb{R}^3_{\geq 0} \to G$ such that the approximations $p_n(s, t, \widetilde{c})(r_1)$ lie in a sequentially c^∞-compact set on compact (r_1, s, t)-sets with $s \leq t$ and $\frac{d}{dt}|_{t=0}\rho(c(r_1, r_2, t)) = \frac{d}{dt}|_{t=0}\rho(\widetilde{c}(r_1, r_2, t))$ for $r_1, r_2 \geq 0$, then $p_n(s, t, \widetilde{c})(r_1) \overset{n\to\infty}{\to} d(r_1, s, t)$.*

Proof. Sequentially c^∞-compact sets are mapped to bounded sets under the smooth representation ρ. Consequently we can apply the approximation theorem to conclude that $\rho[p_n(s, t, c)(r_1)]$ converges uniformly on compact (r_1, s, t)-sets with $s \leq t$ in all derivatives. But the sequence $p_n(s, t, c)(r_1)$ has at least one adherence point in G for fixed parameters by sequential compactness, which has to be unique by the Hausdorff-property. Consequently there is a mapping $d : \mathbb{R}_{\geq 0} \times \{(s,t) \mid s \leq t\} \to G$ given by the limits of $p_n(s,t,c)(r_1)$ for the respective parameter values, but $\rho[p_n(s,t,c)(r_1)] \overset{n\to\infty}{\to} \rho[d(r_1,s,t)]$ pointwise implies that $\rho \circ d$ is the limit of $p_n(s,t,c)(r_1)$, so d is smooth by initiality and the convergence is uniform in all derivatives by the approximation theorem 2.2. Since propagations are unique on $L(C^\infty(G,\mathbb{R}))$ the second assertion is proved either. ■

Corollary 4.3. *Let G be a smooth semigroup and $c : \mathbb{R}_{\geq 0} \to G$ a smooth curve with $c(0) = e$ such that $c(\frac{t}{n})^n$ lie in a sequentially c^∞-compact set on compact t-sets, then there is a smooth semigroup $d : \mathbb{R}_{\geq 0} \to G$ ($d(s)d(t) = d(s+t)$ for $s,t \geq 0$) with $\lim_{n\to\infty} c(\frac{t}{n})^n = d(t)$ in G and $\rho(c(\frac{t}{n})^n) \to \rho(d(t))$ uniformly on compact t-sets in all derivatives.*

Remark 4.4. If there is a sort of "Lie cone" on the smooth semigroup, then the compactness condition on the approximations implies the existence of an "exponential map". This was applied to strong ILH-groups, which are Frölicher spaces in particular (since they are smoothly regular, see [4]), to formulate "inner" conditions on the existence of evolution mappings (see [8]). Remark that "all" up to now known Fréchet-Lie groups are strong ILH-groups. Even if a group does not admit charts, these procedures can be applied. For some general ideas in this setting, how to obtain, that the approximations remain in a sequentially compact set see the preprint [10].

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