Well-balanced schemes for equilibrium flows

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Joint work with L. Grosheintz & S. Mishra

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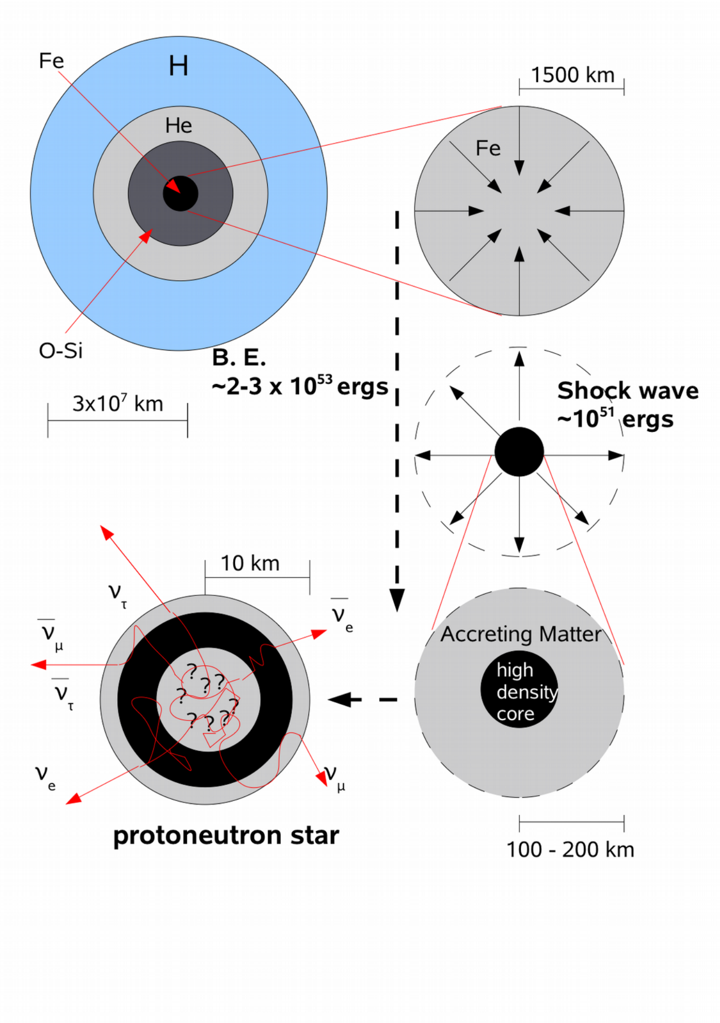
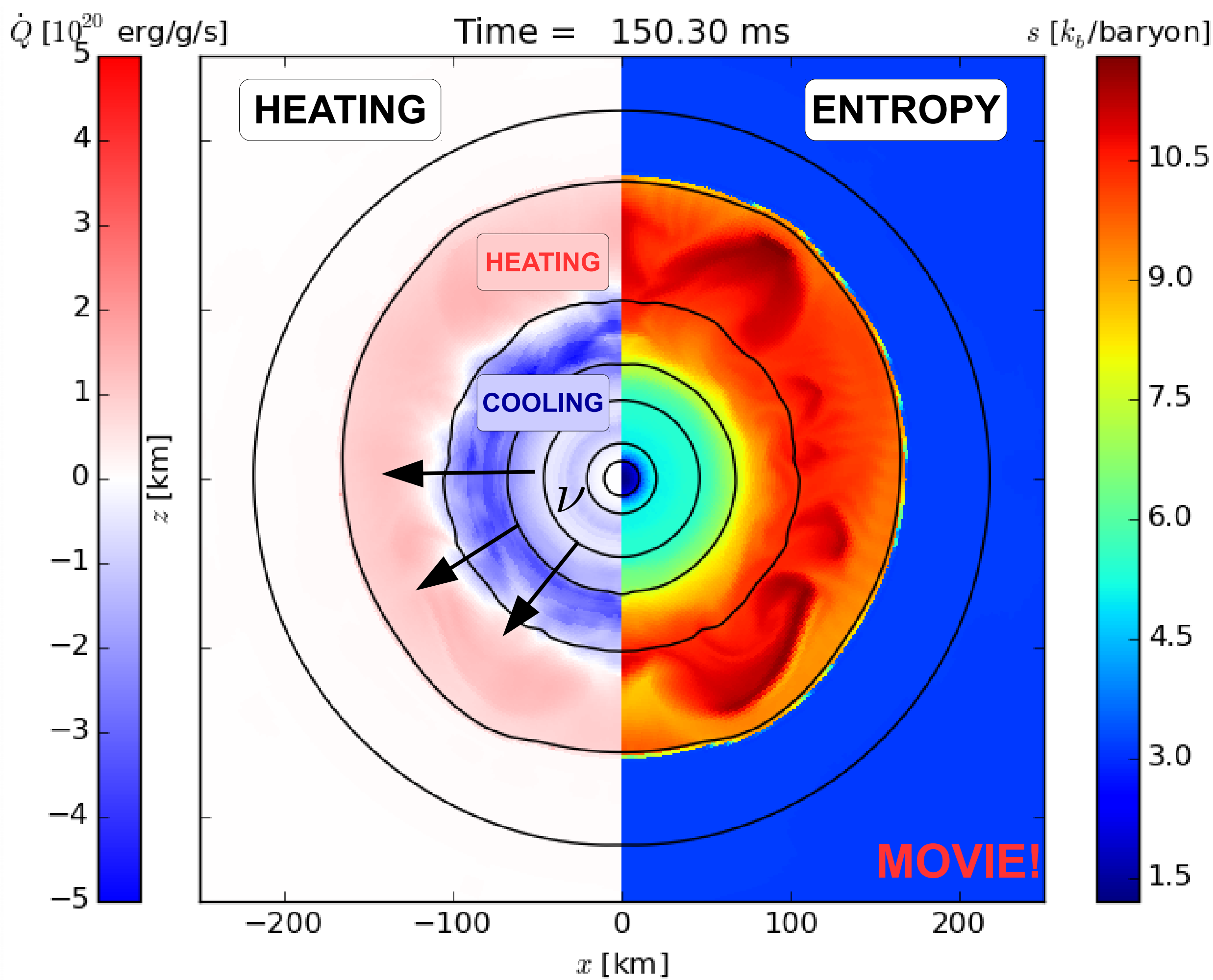
Outline

- **Introduction & Motivation**
- **Well-balanced schemes**
  - Arbitrary stratification
- **Astrophysical applications**
- **Higher-order & Moving steady states**
- **Conclusions**

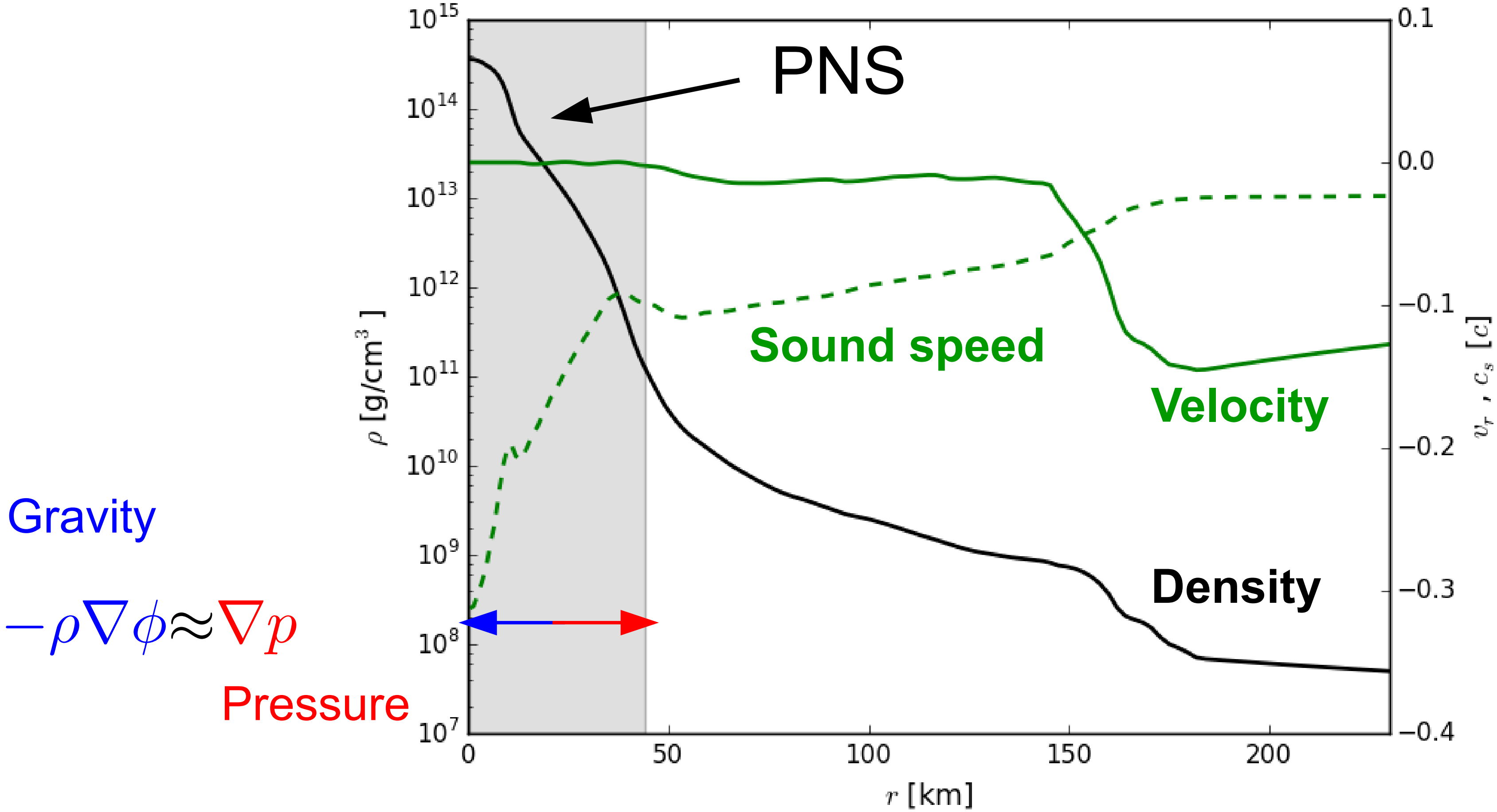
Core-collapse Supernova





Core-collapse Supernova

- The problem:



Ability to maintain near hydrostatic equilibrium for a long time!

$$\tau_{\mathrm{dyn}} = (G\bar{\rho})^{-1/2} \approx 1\mathrm{ms} \quad \Longleftrightarrow \quad \tau_{\mathrm{expl}} \gtrsim 100\mathrm{ms}$$

Hydrostatic equilibrium

- Consider 1D hydrodynamics eqs with gravity

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S}$$

$$\boldsymbol{u} = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (E + p)v \end{bmatrix} \quad \boldsymbol{S} = -\begin{bmatrix} 0 \\ \rho \\ \rho v \end{bmatrix} \frac{\partial \phi}{\partial x}$$

- Classical solution algorithm:
  - Solve homogeneous eqs with Godunov type method (i.e. solve Riemann problem)
  - Account for source term in second step (split/unsplit)

Hydrostatic equilibrium (2)

- Classical solution algorithm:

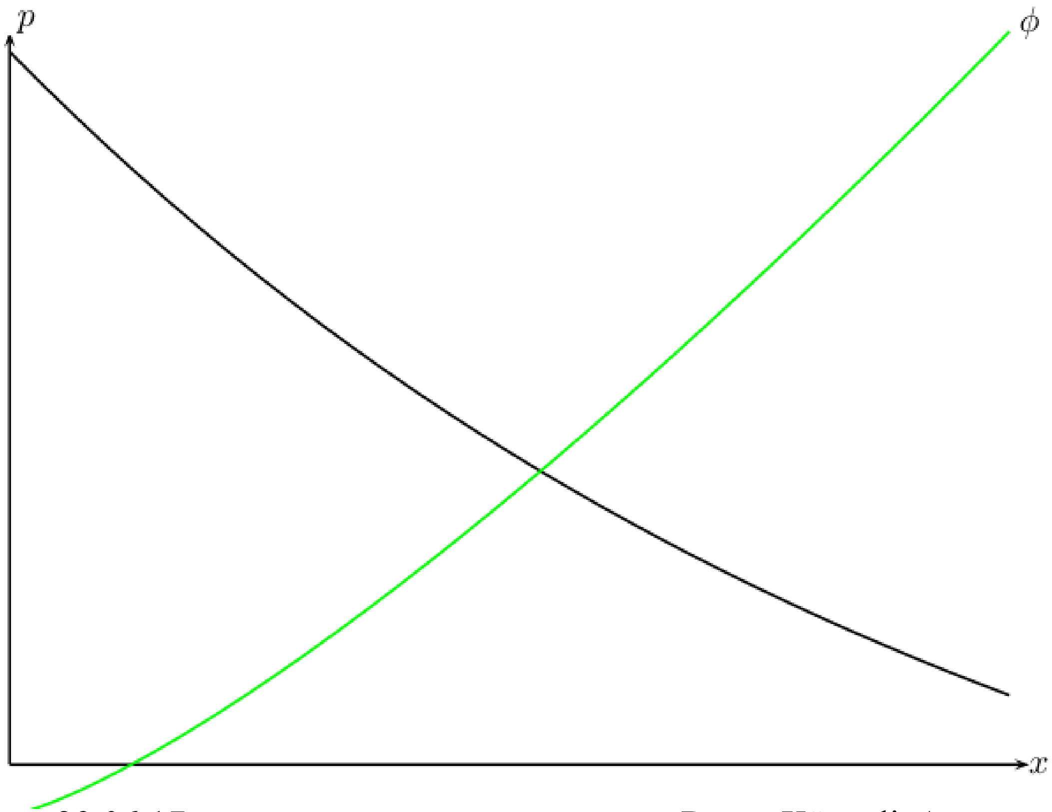
$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^n - \boldsymbol{F}_{i-1/2}^n\right) + \Delta t \boldsymbol{S}_i^n$$

- Numerical flux $\boldsymbol{F}_{i\pm1/2}^n = \mathcal{F}(\boldsymbol{u}_{i\pm1/2}^{n,L}, \boldsymbol{u}_{i\pm1/2}^{n,R})$ from (approximate) Riemann solver, e.g.
  - (Local) Lax-Friedrichs Lax (1954), Rusanov (1961)
  - HLL (C) Harten, Lax and van Leer (1983), Toro et al. (1994)
  - Roe Roe (1981)
  - ...

Hydrostatic equilibrium (3)

Interested in hydrostatic equilibrium: $$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



Hydrostatic equilibrium (4)

Interested in hydrostatic equilibrium: $$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \Longrightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

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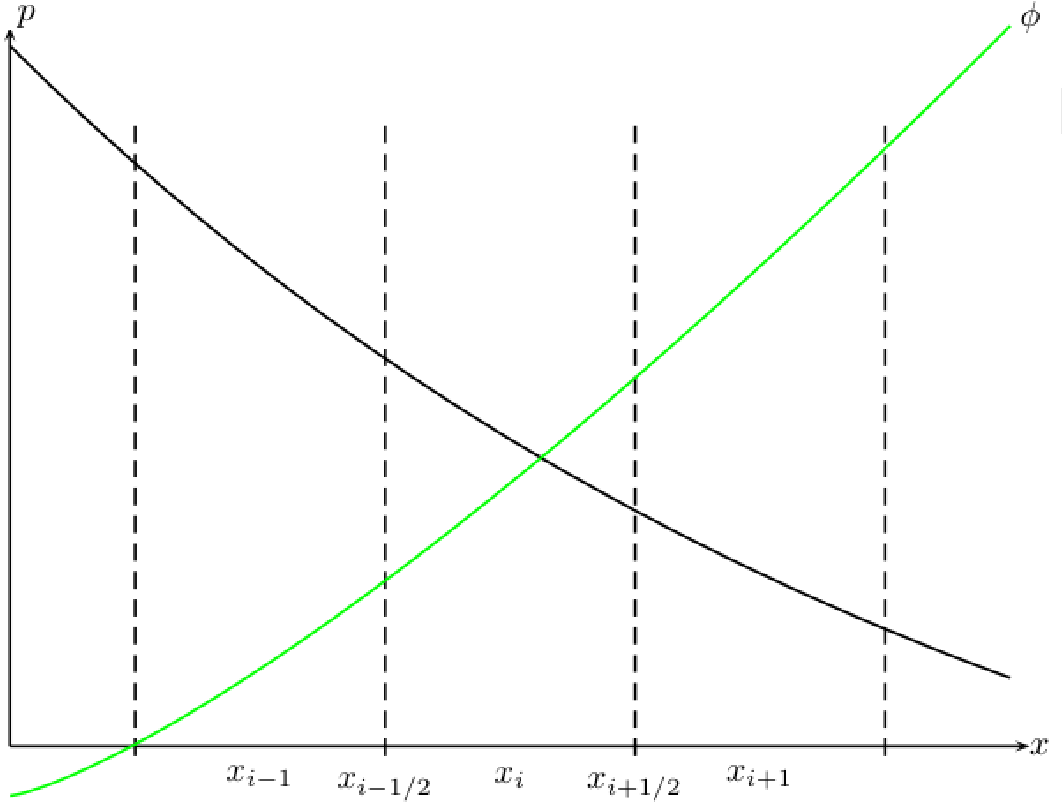


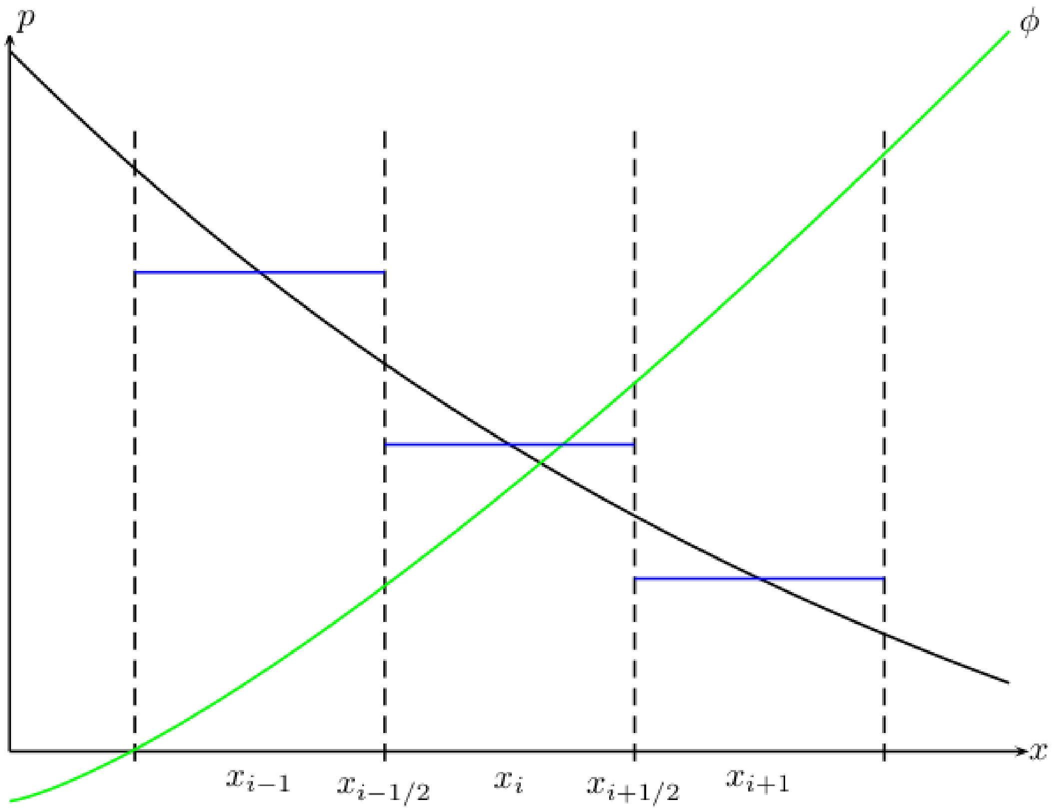
Discretise in cells $[x_{i-1/2}, x_{i+1/2}]$

Hydrostatic equilibrium (5)

Interested in hydrostatic equilibrium:

$$\frac{\partial \boldsymbol{F}}{\partial x} = \boldsymbol{S} \Rightarrow \quad \frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

EoS: $p = p(\rho, e)$



Discretise in cells $\left[x_{i-1/2}, x_{i+1/2}\right]$

Define cell averages

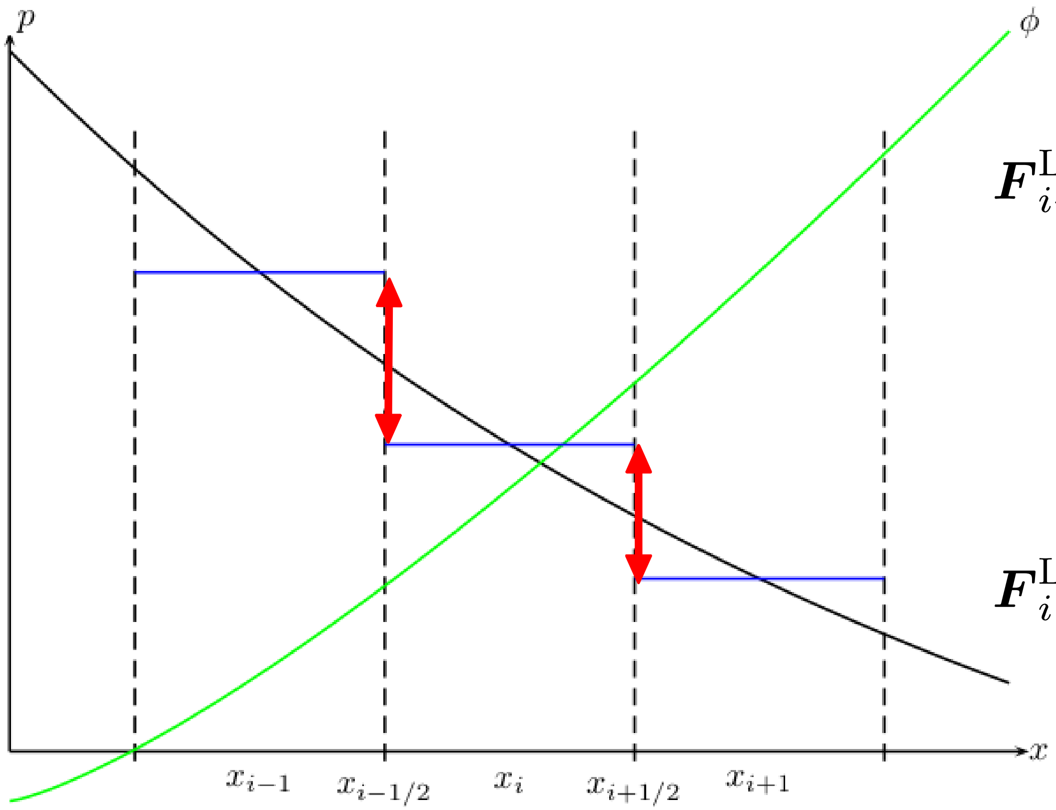
$$\boldsymbol{u}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{u}(x, t^n) x$$

$$\boldsymbol{S}_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \boldsymbol{S}\left(\boldsymbol{u}(x, t)\right) x$$

Hydrostatic equilibrium (6)

Interested in hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) \stackrel{?}{=} \boldsymbol{S}^n_i$$



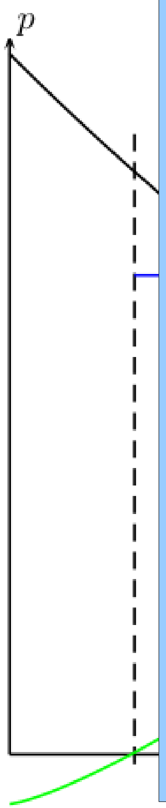
$$\boldsymbol{F}^{\mathrm{LxF}}_{i+1/2} = \frac{1}{2}\left(\boldsymbol{F}_i + \boldsymbol{F}_{i+1}\right) - \frac{S_{\max}}{2}\underbrace{\left(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i\right)}$$

Contains also gravity induced gradient!

$$\boldsymbol{F}^{\mathrm{LxF}}_{i-1/2} = \frac{1}{2}\left(\boldsymbol{F}_{i-1} + \boldsymbol{F}_i\right) - \frac{S_{\max}}{2}\overbrace{\left(\boldsymbol{u}_i - \boldsymbol{u}_{i-1}\right)}$$

Hydrostatic equilibrium (6)

Intere
equil



Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \qquad \rho(x) = \left[\rho_0^{\gamma-1} - \frac{g}{K}\frac{\gamma-1}{\gamma}x\right]^{\frac{1}{\gamma-1}} \qquad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0, 2]$$

Error in pressure:

(after 2 sound crossing times!!!)

N	1st	2ndTVD
128	2.1E-02	6.5E-05
256	1.1E-02	1.6E-05
512	5.3E-03	4.1E-06
1024	2.6E-03	1.0E-06
2048	1.3E-03	2.6E-07

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

HLLC numerical flux

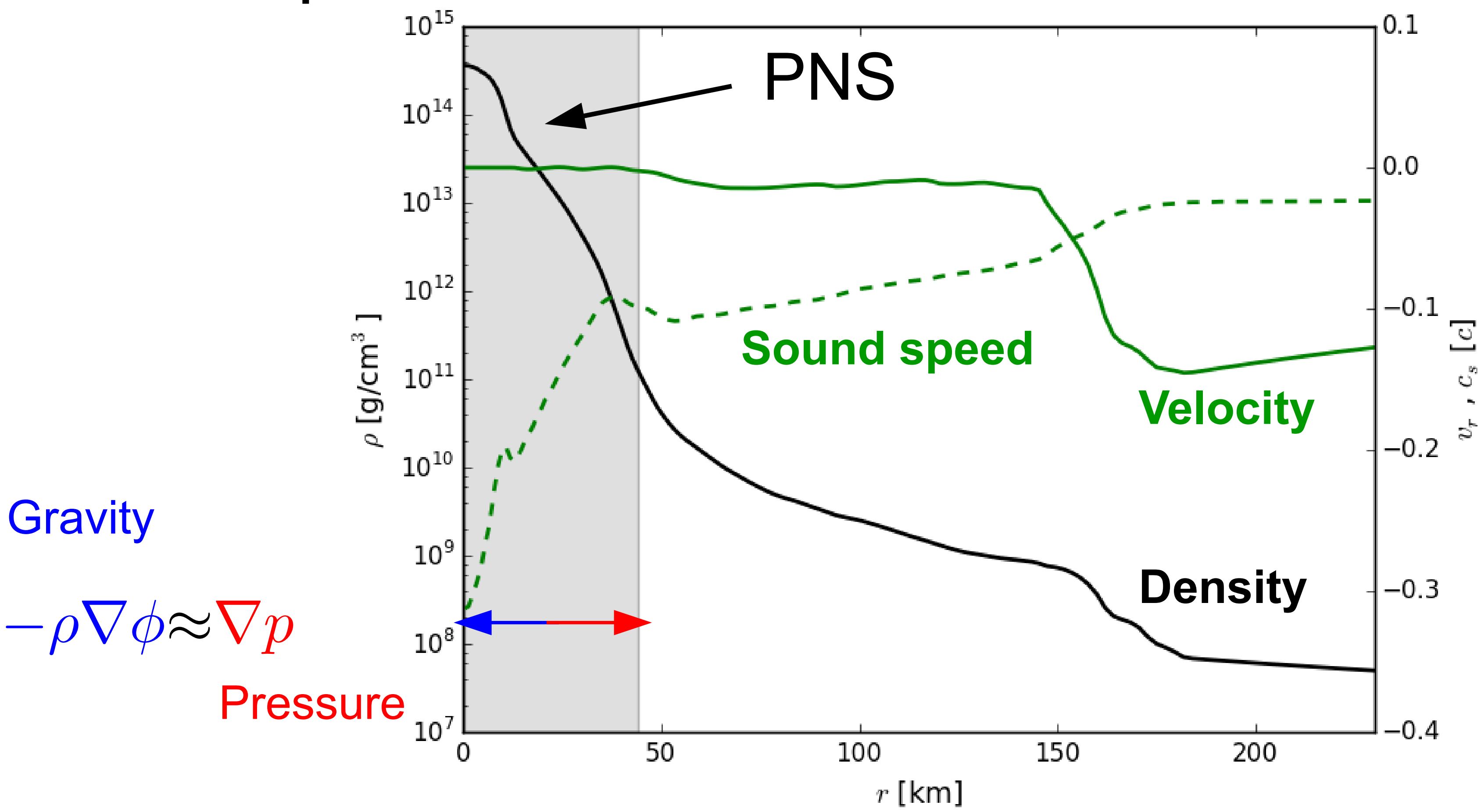
$_1 - \boldsymbol{u}_i)$

avity
!

$- \boldsymbol{u}_{i-1})$

Core-collapse Supernova

- The problem:



Ability to maintain near hydrostatic equilibrium for a long time!

$$\tau_{\rm dyn} = (G\bar{\rho})^{-1/2} \approx 1{\rm ms} \qquad \Leftrightarrow \qquad \tau_{\rm expl} \gtrsim 100{\rm ms}$$

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Well-balanced schemes

- Solutions:
  - Define a **global** stationary state $u_0(\boldsymbol{x})$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(\boldsymbol{x})$
  - Steady state preserving reconstructions, well-balanced schemes

e.g. Cargo & LeRoux (1994), LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010), Xing & Shu (2013), Vides et al. (2013), Käppeli & Mishra (2014), Desveaux et al. (2014), Chandrashekar & Klingenberg (2015), Desveaux et al. (2016), Li & Xing (2015/2016), ...

See also Mellema et al. (1991), Zingale et al. (2002), Kastaun (2006), Castro et al. (2007), Käppeli et al. (2011), Freytag et al. (2012), Gosse (2015)

Note: there are many, many more, e.g. especially for shallow-water with bottom topography, nozzle flow with variable cross-section, ...

See also L. Gosse, “Computing Qualitatively Correct Approximations of Balance Laws”, 2013

Well-balanced schemes

- Solutions:
  - Define a **global** stationary state $u_0(\boldsymbol{x})$ **at each time step** and evolve $u(\boldsymbol{x}) - u_0(\boldsymbol{x})$
  - Steady state preserving reconstructions, well-balanced schemes e.g. Cargo & LeRoux (1994), LeVeque (1998), LeVeque & Bale (1998), Botta et al. (2004), Fuchs et al. (2010), Xing & Shu (2013), Vides et al. (2013), Käppeli & Mishra (2014), ... g (2015),

... aun (2006), ... 012),

Not ... raphy, noz ...

See a ... 2013

Requirements

- Equilibrium (usually) not known in advance (self-gravity)
- Extensible for general EoS
- (At least) second order accuracy
- Preserve robustness of shock capturing base scheme

Well-balanced scheme

- Hydrostatic equilibrium

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial \phi}{\partial x}$$

  Describes only a mechanical equilibrium...

- Can we directly start from the above without any assumption on entropy/temperature?

Well-balanced scheme (2)

Interested in **numerical** hydrostatic equilibrium:

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) = \boldsymbol{S}^n_i$$

Standard centered differences

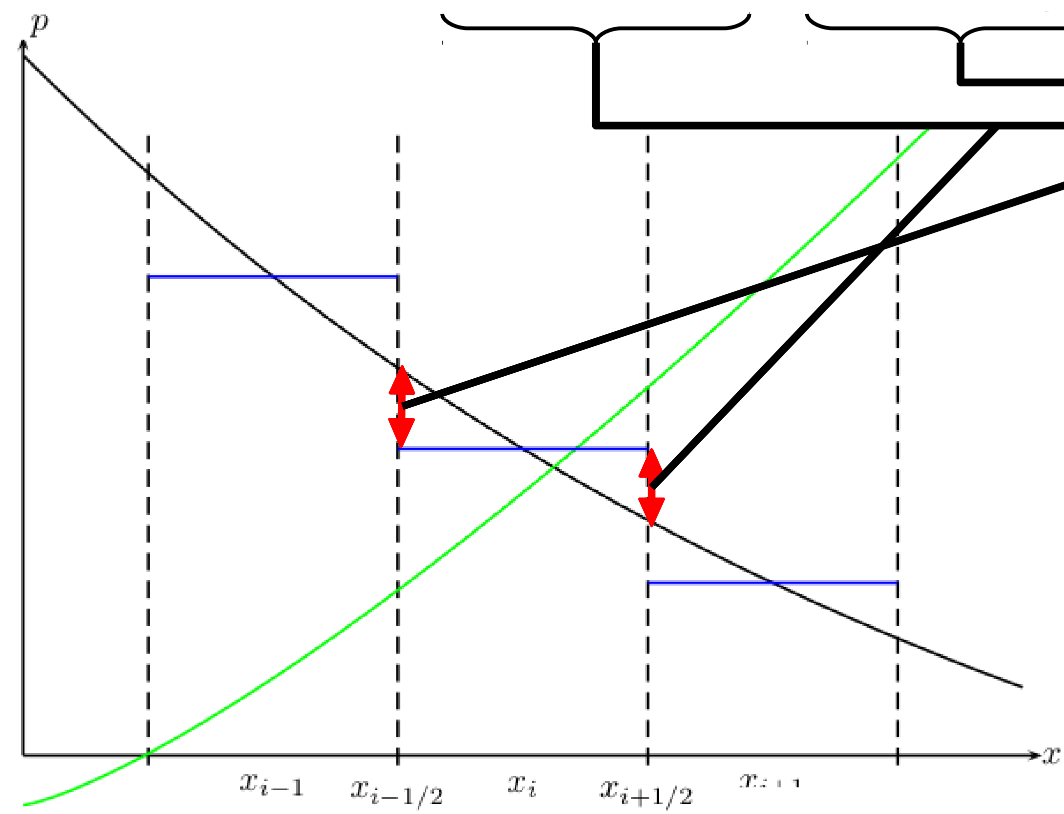
$$\frac{\partial p}{\partial x} + O(\Delta x^2) = \frac{p_{i+1/2} - p_{i-1/2}}{\Delta x} = -\rho_i \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = -\rho \frac{\partial \phi}{\partial x} + O(\Delta x^2)$$

$$\frac{(p_{i+1/2} - p_i) - (p_{i-1/2} - p_i)}{\Delta x} = -\frac{\rho_i}{2} \frac{(\phi_{i+1} - \phi_i) - (\phi_{i-1} - \phi_i)}{\Delta x}$$

Well-balanced scheme (3)

Interested in **numerical** hydrostatic equilibrium:

$$\frac{p_{i+1/2} - p_i}{\Delta x} - \frac{p_{i-1/2} - p_i}{\Delta x} = -\frac{\rho_i}{2}\left(\frac{\phi_{i+1} - \phi_i}{\Delta x} - \frac{\phi_{i-1} - \phi_i}{\Delta x}\right)$$



Equilibrium pressure reconstruction:

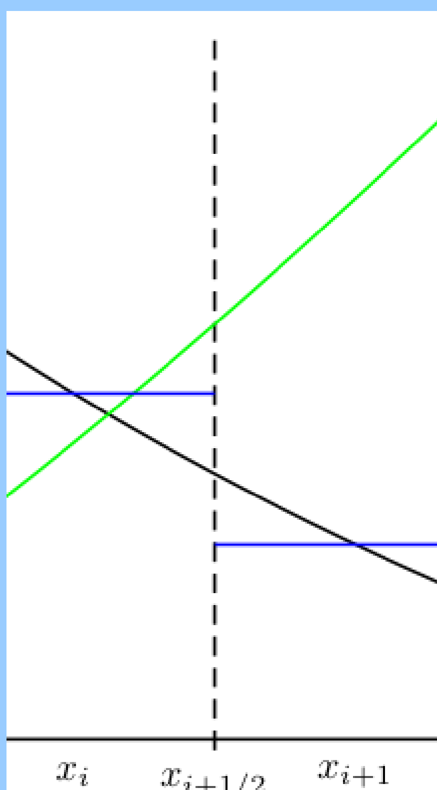
$$p_{i+1/2-} = p_i - \frac{\Delta x}{2}\rho_i \frac{\phi_{i+1} - \phi_i}{\Delta x}$$

$$p_{i-1/2+} = p_i + \frac{\Delta x}{2}\rho_i \frac{\phi_i - \phi_{i-1}}{\Delta x}$$

Well-balanced scheme (3)

Interested in **numerical** hydrostatic equilibrium:

Equilibrium?



$$p_{i+1/2-} \overset{!}{=} p_{i+1/2+}$$

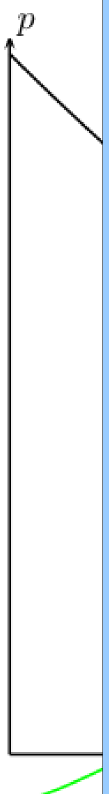
$$\Downarrow$$

$$\frac{p_{i+1}-p_i}{\Delta x} = -\frac{\rho_i+\rho_{i+1}}{2}\frac{\phi_{i+1}-\phi_i}{\Delta x}$$

Discrete HydroStatic Equilibrium

Well-balanced scheme (3)

Interested in **numerical** hydrostatic equilibrium:

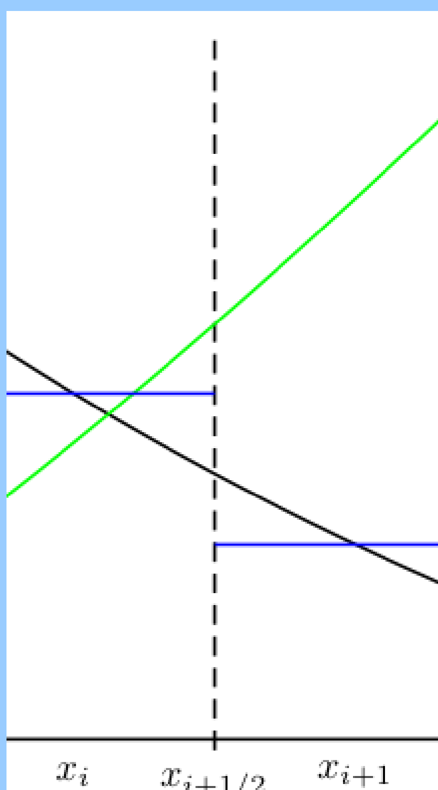


n:

ϕ_i

-1

Equilibrium?



$$p_{i+1/2-} \overset{!}{=} p_{i+1/2+}$$

Requirement on Riemann solver:

$$\boldsymbol{F}^n_{i\pm 1/2} = \boldsymbol{\mathcal{F}}\left(\begin{bmatrix} \rho_{i+1/2-} \\ 0 \\ p_{i+1/2} \end{bmatrix}, \begin{bmatrix} \rho_{i+1/2+} \\ 0 \\ p_{i+1/2} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ p_{i+1/2} \\ 0 \end{bmatrix}$$

e.g. HLLC, Roe

Discrete HydroStatic Equilibrium

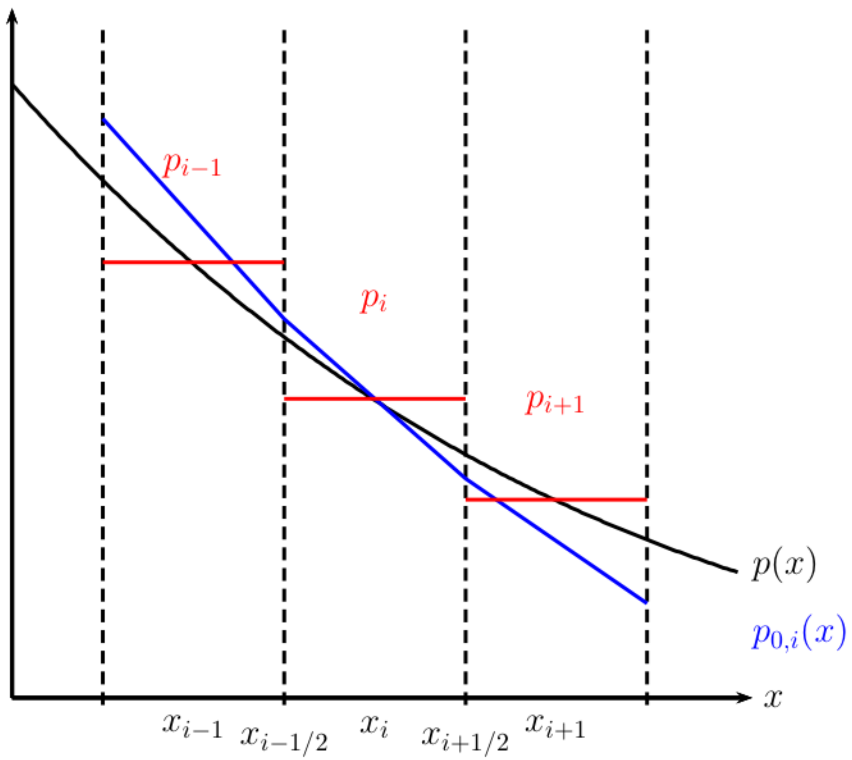
Higher-order extension

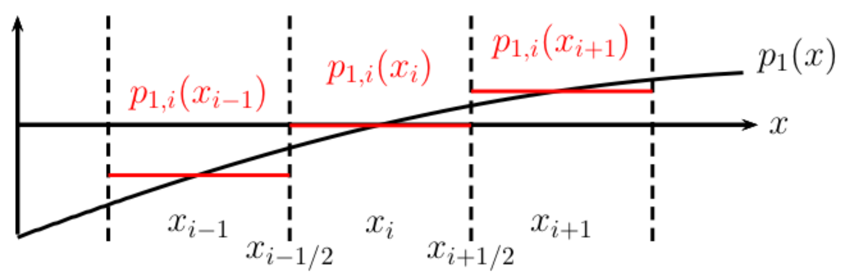
- Second order extension: $r_{1,i}(x_j) = r_j - r_{0,i}(x_j)$

$r =$ pressure, density

Eq. perturbation — Data — Equilibrium

Stencil: $j = ..., i-1, i, i+1, ...$





Apply a high-order reconstruction on perturbation!
E.g. piecewise-linear, ...

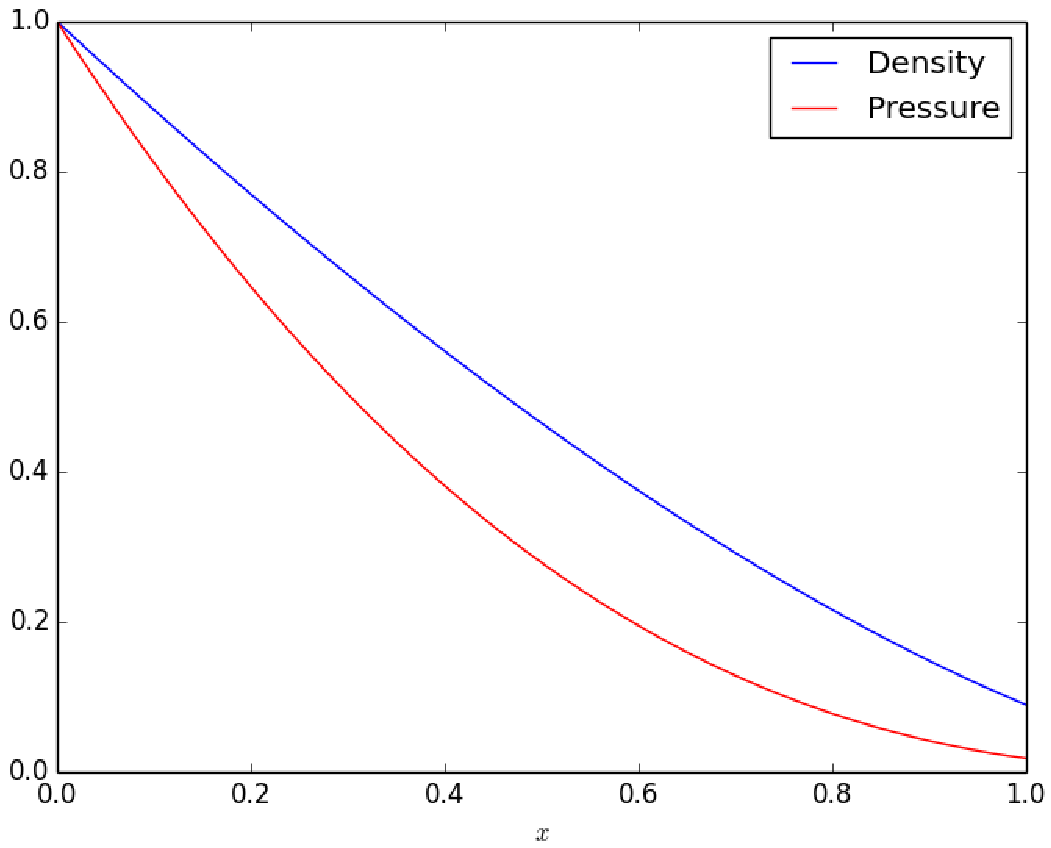
Example 1

Hydrostatic atmosphere in a constant gravitational field

$$\phi(x) = gx \qquad \frac{p_{i+1} - p_i}{\Delta x} = -\frac{\rho_i + \rho_{i+1}}{2}\frac{\phi_{i+1} - \phi_i}{\Delta x} \qquad p = \frac{p_0}{\rho_0^\gamma}\rho^\gamma = K\rho^\gamma$$

$$x \in [0,1] \qquad g = 2 \qquad \gamma = 5/3$$

$K = const$ ~ entropy



Error in pressure:

MC limiter

N	1st	2ndTVD
128	2.4E-02 / **2.7E-16**	2.6E-07 / **5.9E-16**
256	1.2E-02 / **6.2E-15**	3.3E-08 / **3.1E-16**
512	5.9E-03 / **2.7E-14**	4.2E-09 / **3.4E-15**
1024	3.0E-03 / **2.0E-14**	5.2E-10 / **1.5E-15**
2048	1.5E-03 / **5.5E-14**	6.5E-11 / **1.3E-14**
rate	1.00 / -	3.00 / -

$$Err = \frac{1}{N}\sum_i |p_i - p_i^0|$$

Example 2

Hydrostatic atmosphere in a constant gravitational field

+ small perturbation $$v(t, x = 0) = 10^{-8} \sin\left(6 \frac{2\pi t}{t_f}\right)$$

Title:test_small.eps
Creator:matplotlib version 1.3.1, http:/
CreationDate:Thu Jul 31 14:24:10 2014

Example 2 (2)

Hydrostatic atmosphere in a constant gravitational field

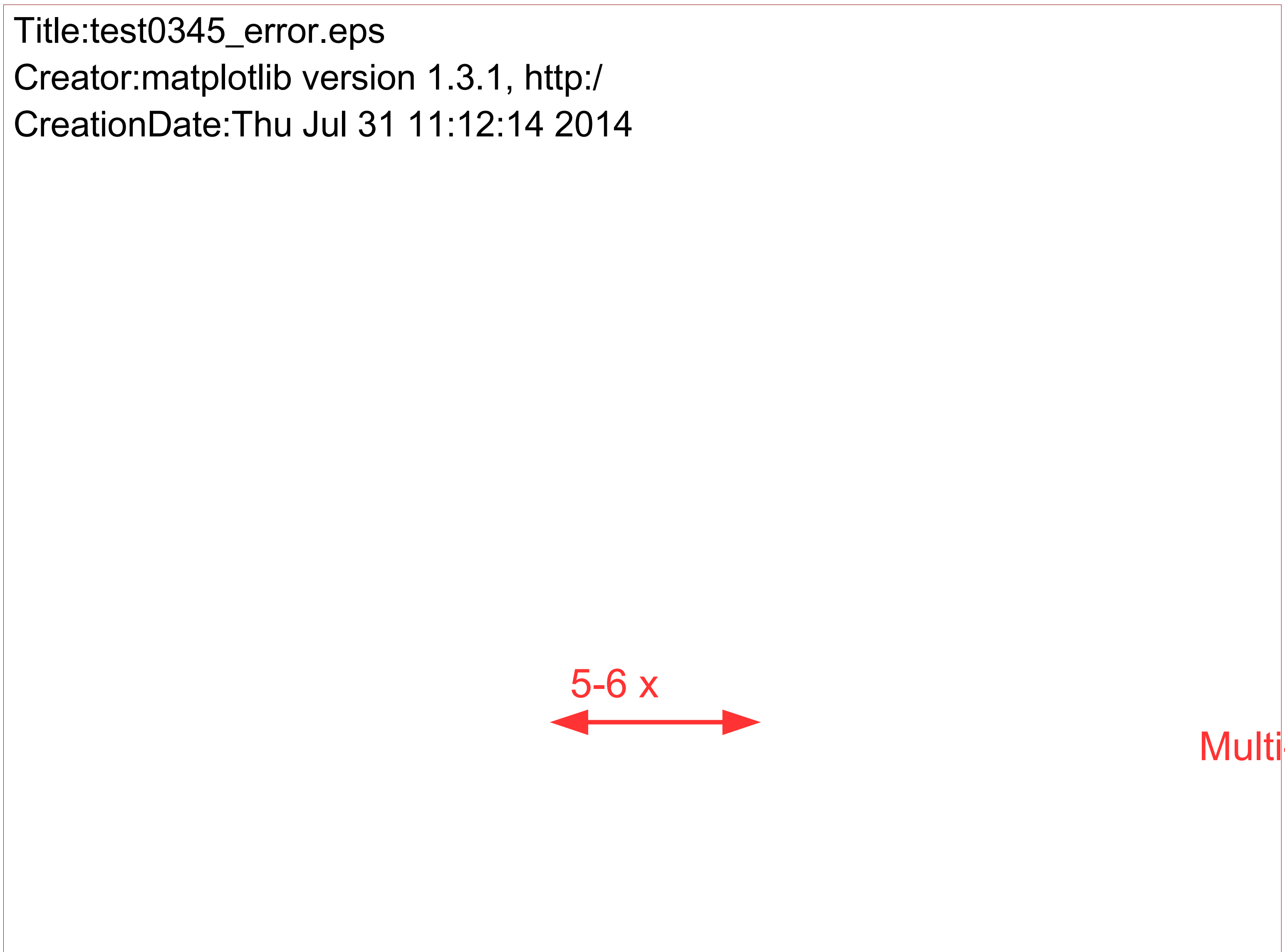
+ large perturbation $$v(t, x = 0) = 10^{-1} \sin\left(6\frac{2\pi t}{t_f}\right)$$

Title:test_large.eps
Creator:matplotlib version 1.3.1, http:/
CreationDate:Thu Jul 31 14:24:11 2014

NO loss of robustness!

Example 2 (3)

Hydrostatic atmosphere in a constant gravitational field



Multi-D...

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CCSN simulation

In collaboration with R. Cabezon (Basel) & A. Perego (Darmstadt)

2D axisymmetric

Resolution:

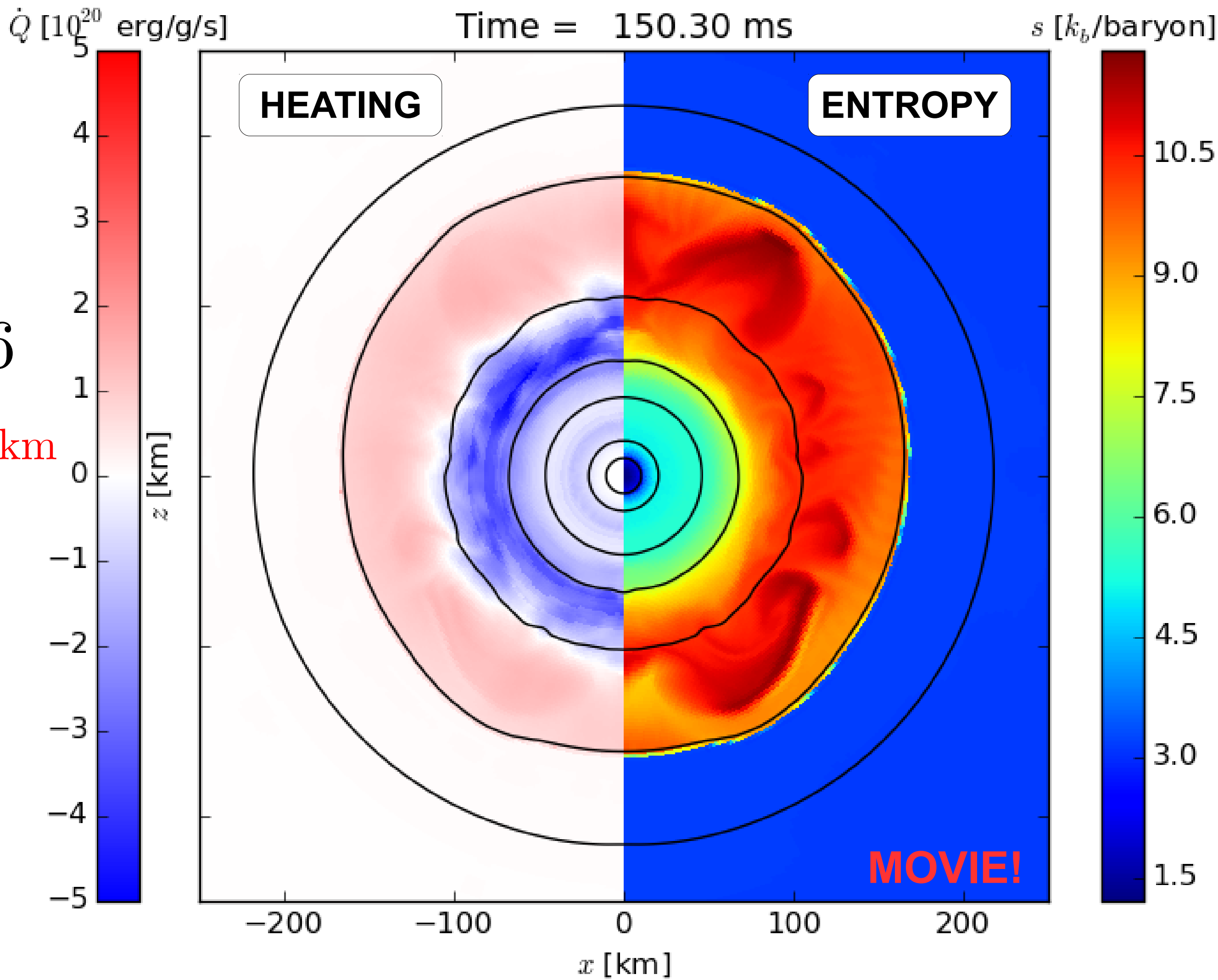
$$N_r \times N_\theta = 512 \times 256$$

Radial: logarithmic spacing $\Delta r_1 = 1\mathrm{km}$

Polar: regular spacing

Neutrino transport:

Spectral leakage scheme with heating



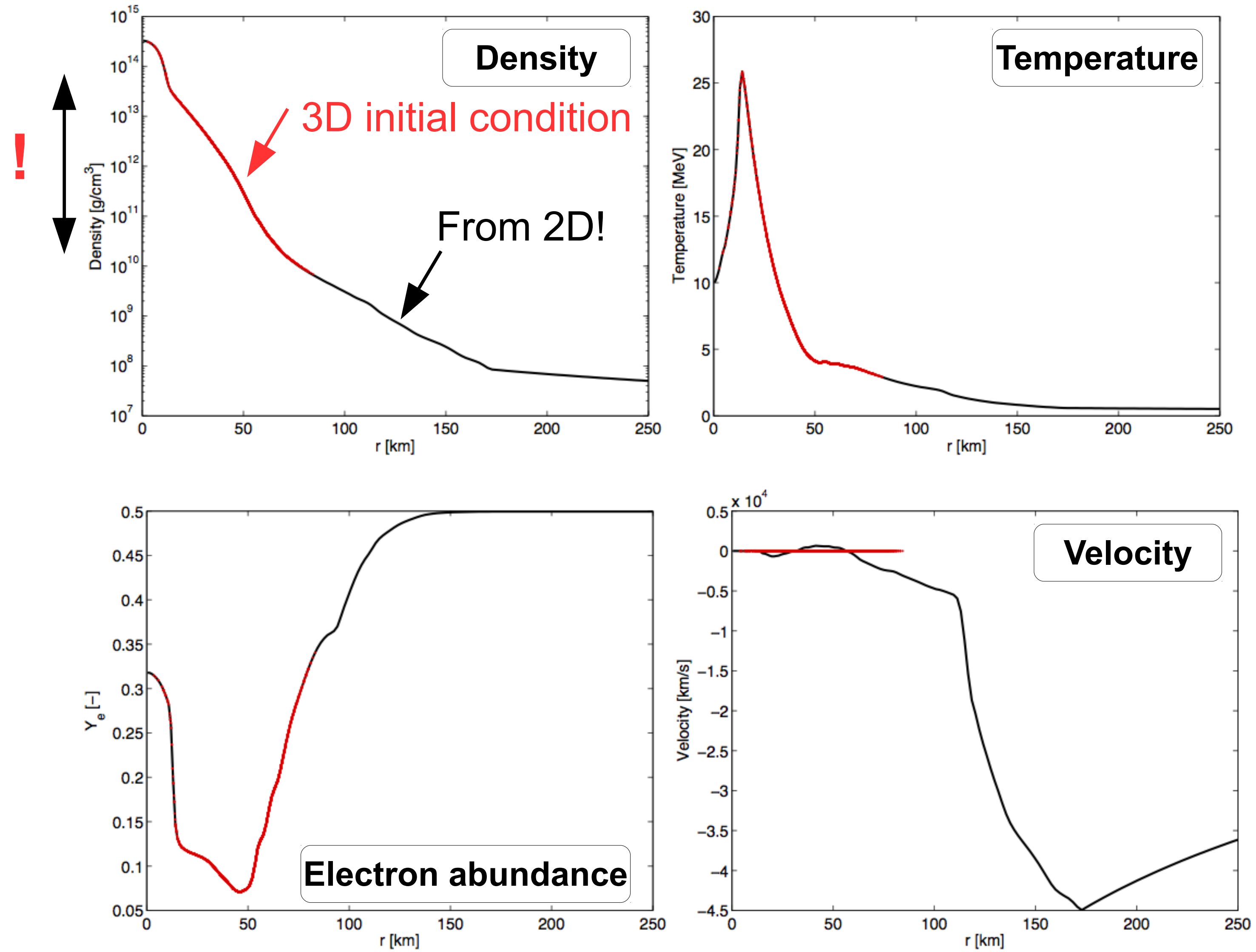
Perego, Cabezon & RK (2016)

"CCSN" simulation

Actually, just the simulation of a stationary PNS!

3D (Cartesian)

"No" physics, but "realistic" EoS

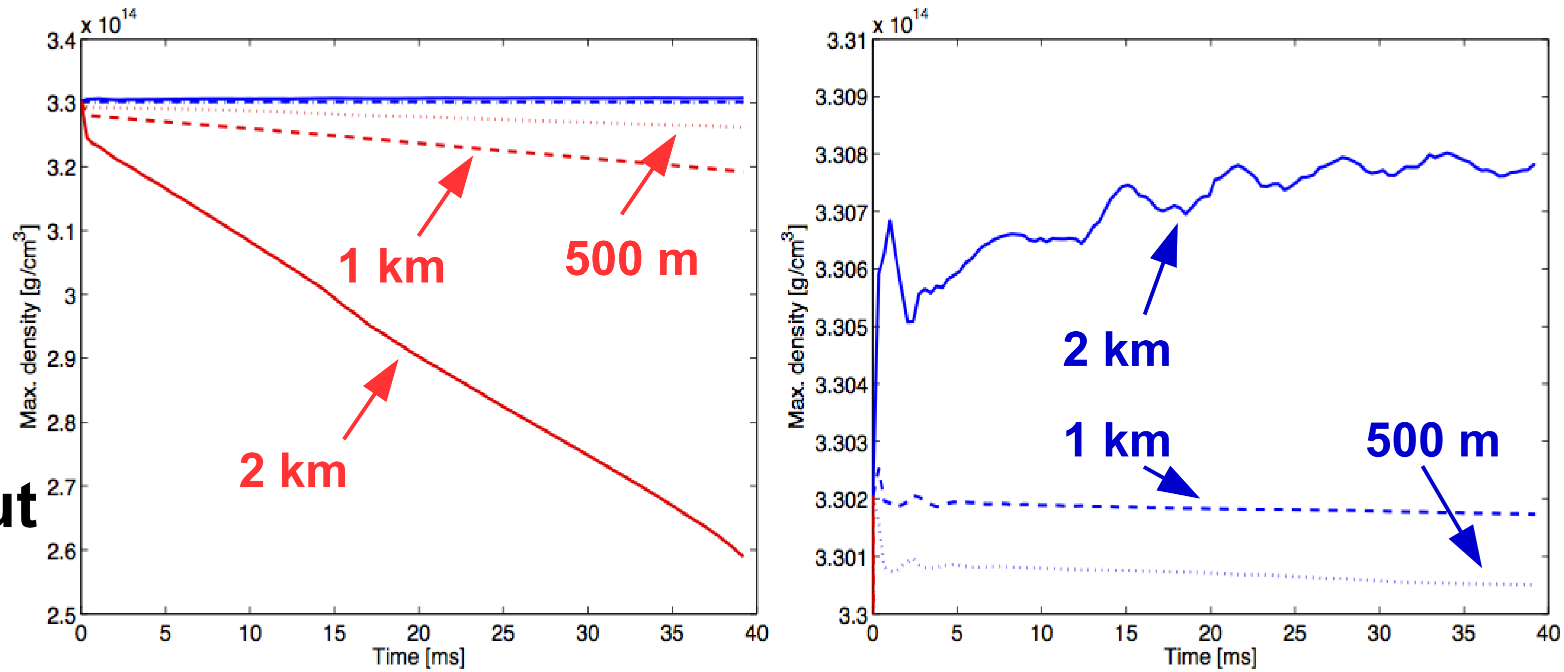


“CCSN” simulation

Actually, just the simulation of a stationary PNS!

3D (Cartesian)

Maximal density evolution



“No” physics, but “realistic” EoS

“Real” CCSN simulation evolve for 10-20 times more!

NO HSE

WITH HSE

For explicit schemes: keep CFL condition in mind!

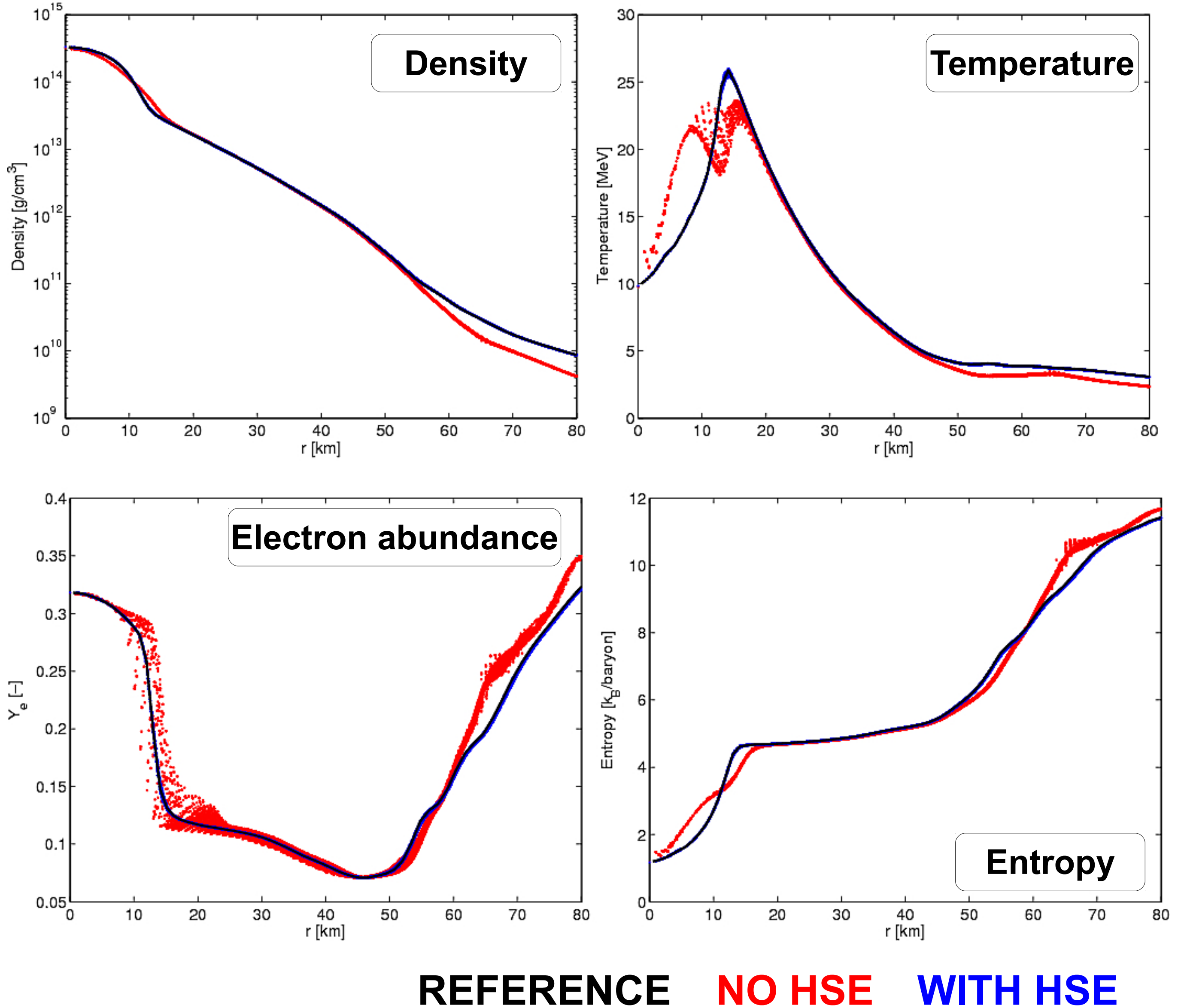
"CCSN" simulation

Actually, just the simulation of a stationary PNS!

3D (Cartesian)

$$\Delta x = \Delta y = \Delta z = 1 \text{ km}$$

"No" physics, but "realistic" EoS



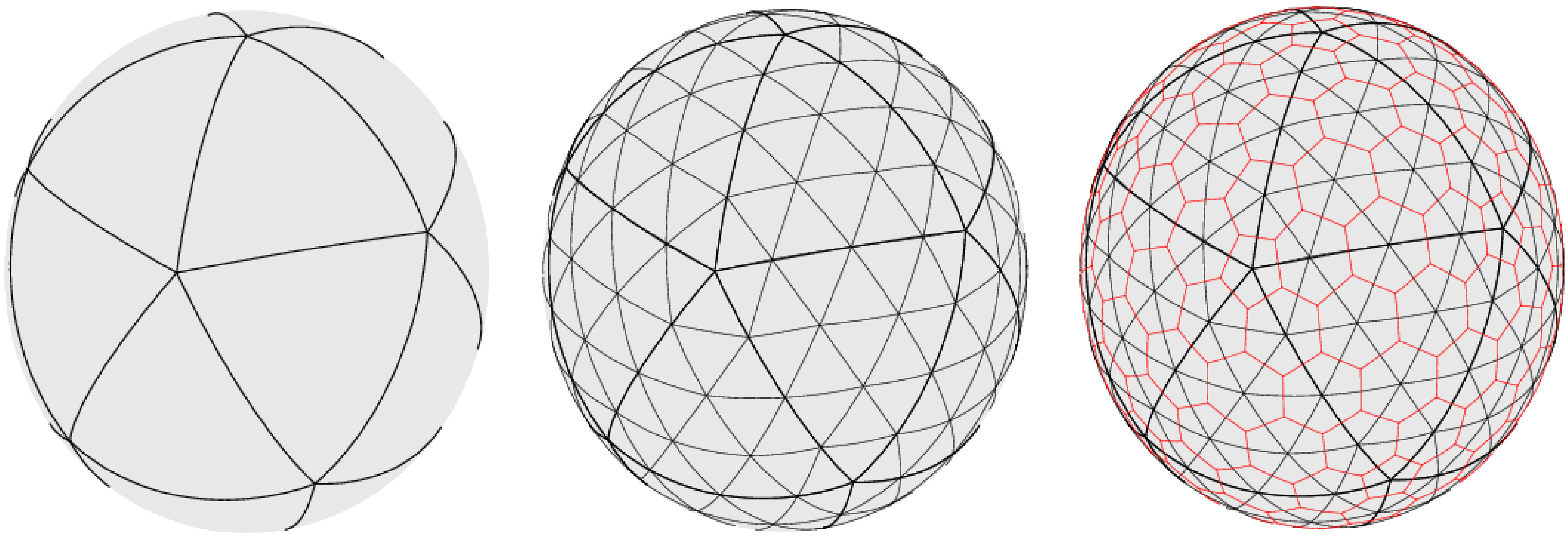
Climate on exoplanets

In collaboration with K. Heng's group (Bern)

Luc Grosheintz

- Well-balancing on icosahedral grid

Avoids axis issues...



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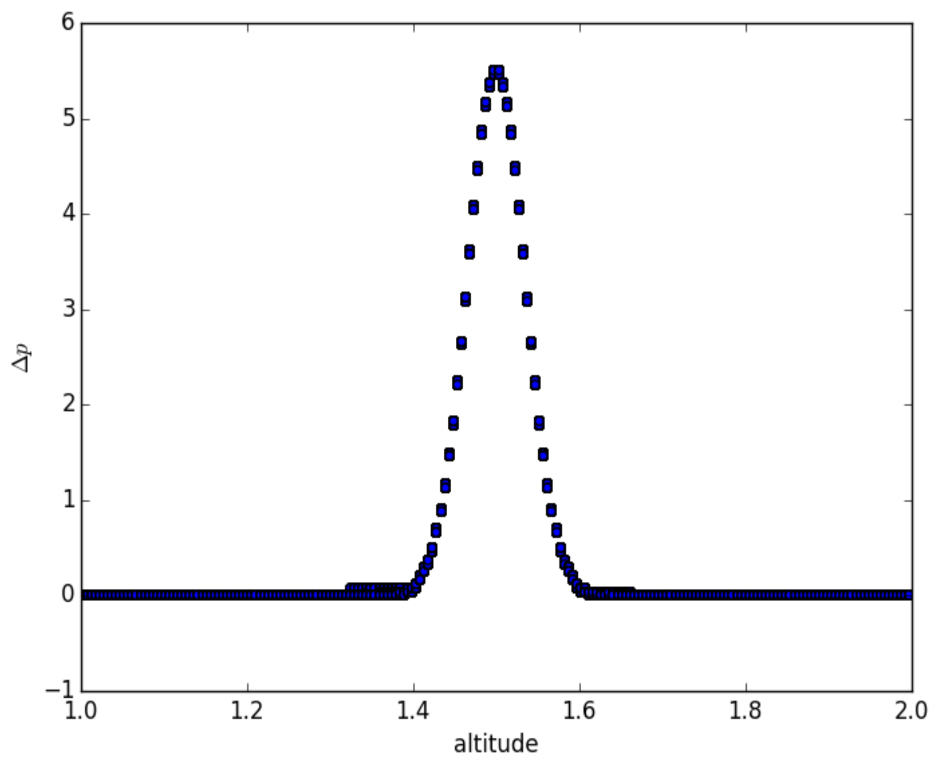

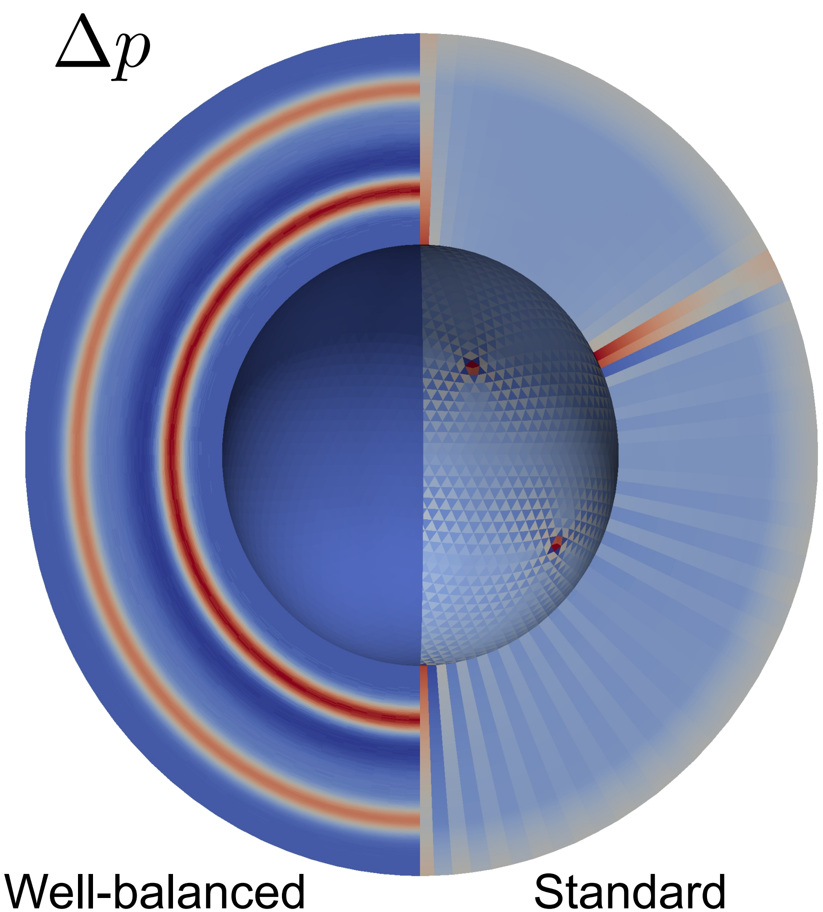

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Isentropic well-balanced scheme

- Consider constant entropy profile

- Using the thermodynamic relation

$$\mathrm{d}h = T\mathrm{d}s + \frac{\mathrm{d}p}{\rho} \qquad h = e + \frac{p}{\rho}$$ Enthalpy

- Hydrostatic eq.

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial h}{\partial x} = -\frac{\partial \phi}{\partial x}$$

- Or simply

$$h + \phi = const$$

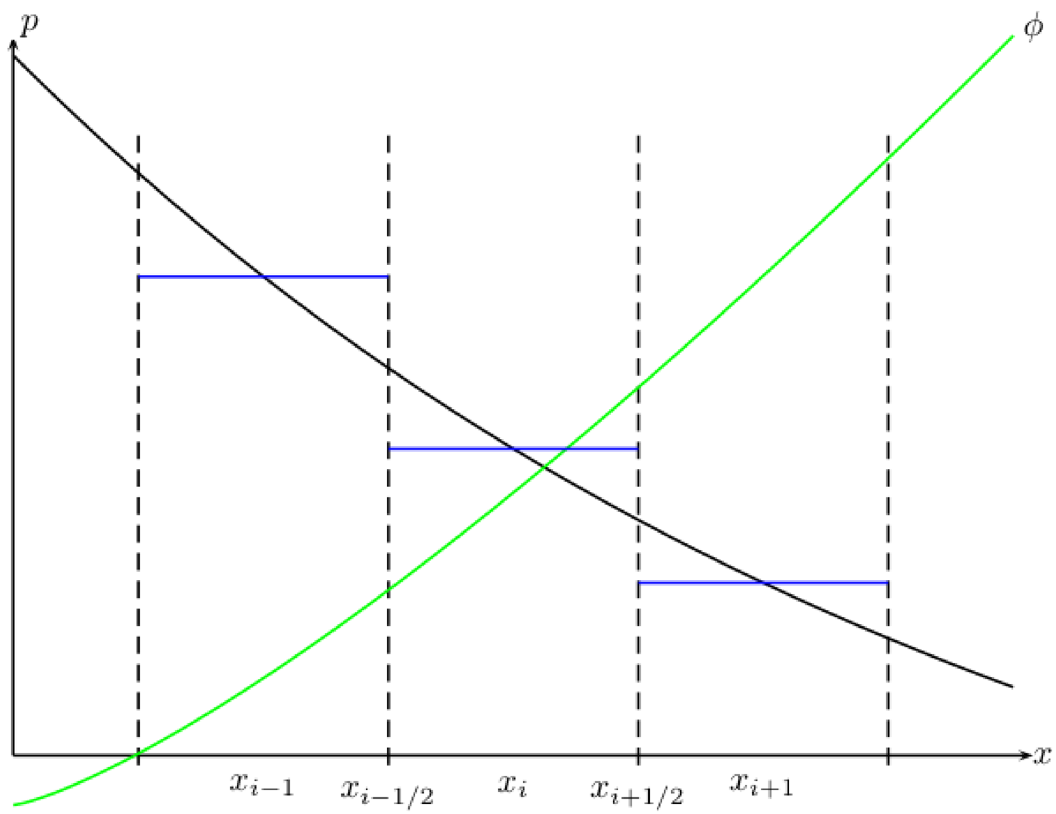
Isentropic well-balanced scheme (2)

Cell avg.

Perform local equilibrium reconstruction: $h_{0,i}(x) = \bar{h}_i + \bar{\phi}_i - \phi(x)$

Equilibrium enthalpy

$$h + \phi = const$$



EoS $h_{0,i}(x) = h\left(\bar{s}_i, p_{0,i}(x)\right)$

May need a non-linear solve...

$p_{0,i}(x)$ & $\rho_{0,i}(x)$

$$\boldsymbol{w}^n_{i\pm 1/2\mp} = \begin{bmatrix} \rho^n_{0,i}(x_{i\pm 1/2}) \\ v^n_{x,i} \\ p^n_{0,i}(x_{i\pm 1/2}) \end{bmatrix}$$

Equilibrium reconstructed primitive variables

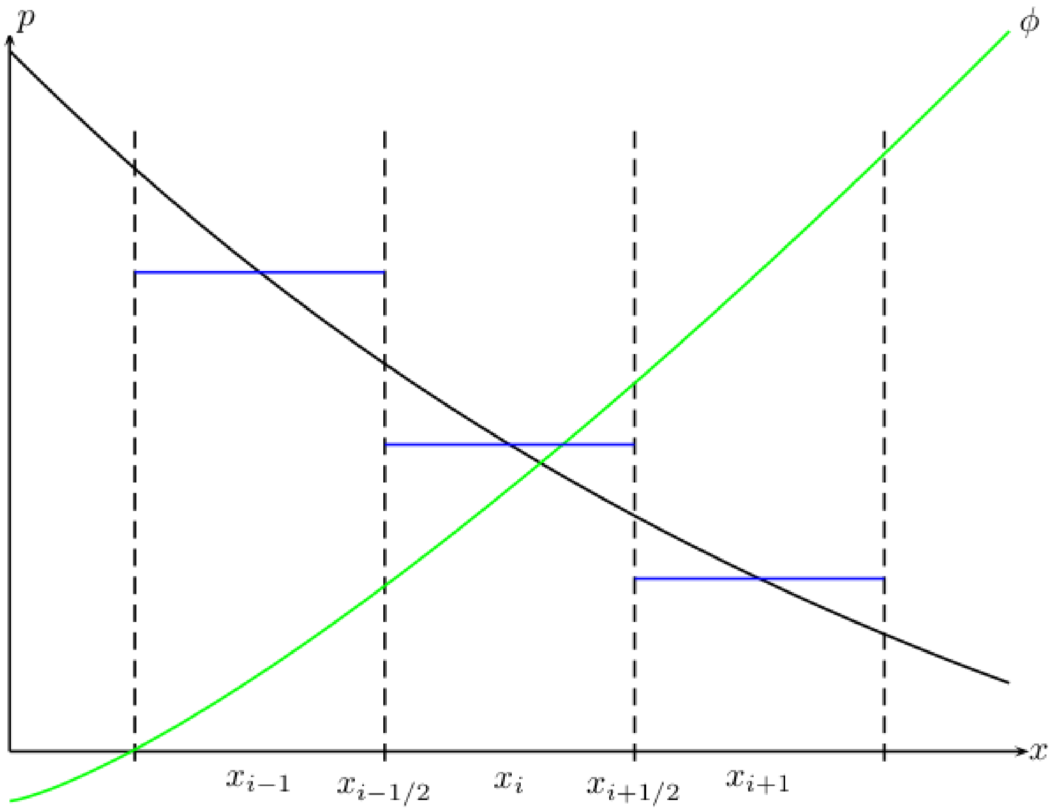
Isentropic well-balanced scheme (2)

Eq. point values

Perform local equilibrium reconstruction: $h_{0,i}(x) = h_i + \phi_i - \phi(x)$

Equilibrium enthalpy

$$h + \phi = const$$



EoS

$$h_{0,i}(x) = h\left(\bar{s}_i, p_{0,i}(x)\right)$$

May need a non-linear solve...

$$p_{0,i}(x) \quad \& \quad \rho_{0,i}(x)$$

$$\boldsymbol{w}^n_{i\pm 1/2\mp} = \begin{bmatrix} \rho^n_{0,i}(x_{i\pm 1/2}) \\ v^n_{x,i} \\ p^n_{0,i}(x_{i\pm 1/2}) \end{bmatrix}$$

Equilibrium reconstructed primitive variables

Isentropic well-balanced scheme (3)

- Well-balanced discretization of momentum source term

$$S^n_{\rho v,i} = \frac{p^n_{0,i}(x_{i+1/2}) - p^n_{0,i}(x_{i-1/2})}{\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + O(\Delta x^2)$$

- Then for data satisfying $h + \phi = const$, $v_x = 0$ and any consistent numerical flux

$$\frac{1}{\Delta x}\left(\boldsymbol{F}^n_{i+1/2} - \boldsymbol{F}^n_{i-1/2}\right) = \boldsymbol{S}^n_i$$

Well-balanced wrt isentropic hydrostatic equilibrium!

Isentropic well-balanced scheme (3)

- Well-balanced discretization of momentum source term

$$S_{\rho v,i}^{n} = \frac{p_{0,i}^{n}(x_{i+1/2}) - p_{0,i}^{n}(x_{i-1/2})}{\Delta x} = -\int_{x_{i-1/2}}^{x_{i+1/2}} \rho \frac{\partial \phi}{\partial x} \mathrm{d}x + C_1 \Delta x^2 + C_2 \Delta x^4 + \ldots$$

Richardson extrapolation...

Like Noelle et al. (2006)

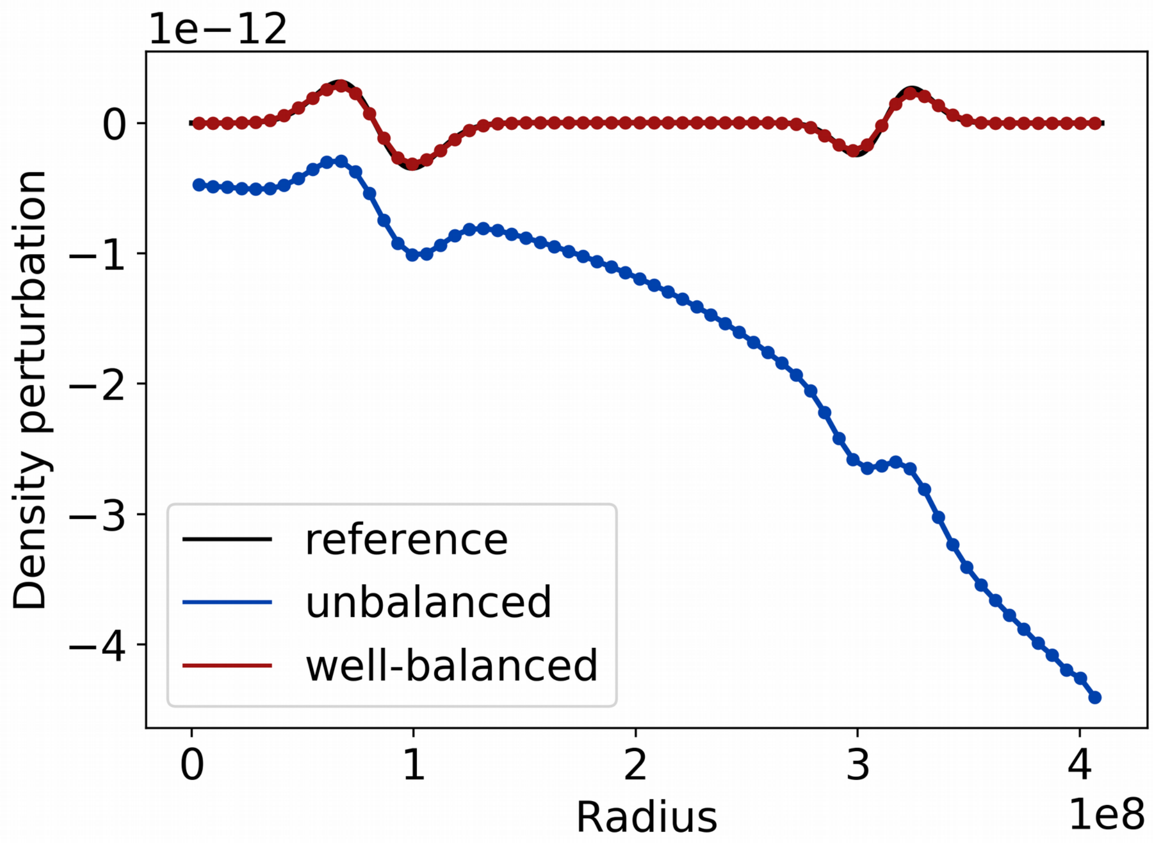
- Then for data satisfying $h + \phi = const$, $v_x = 0$
  and any consistent numerical flux

$$\frac{1}{\Delta x}\left(\boldsymbol{F}_{i+1/2}^{n} - \boldsymbol{F}_{i-1/2}^{n}\right) = \boldsymbol{S}_{i}^{n}$$

Well-balanced wrt isentropic hydrostatic equilibrium!

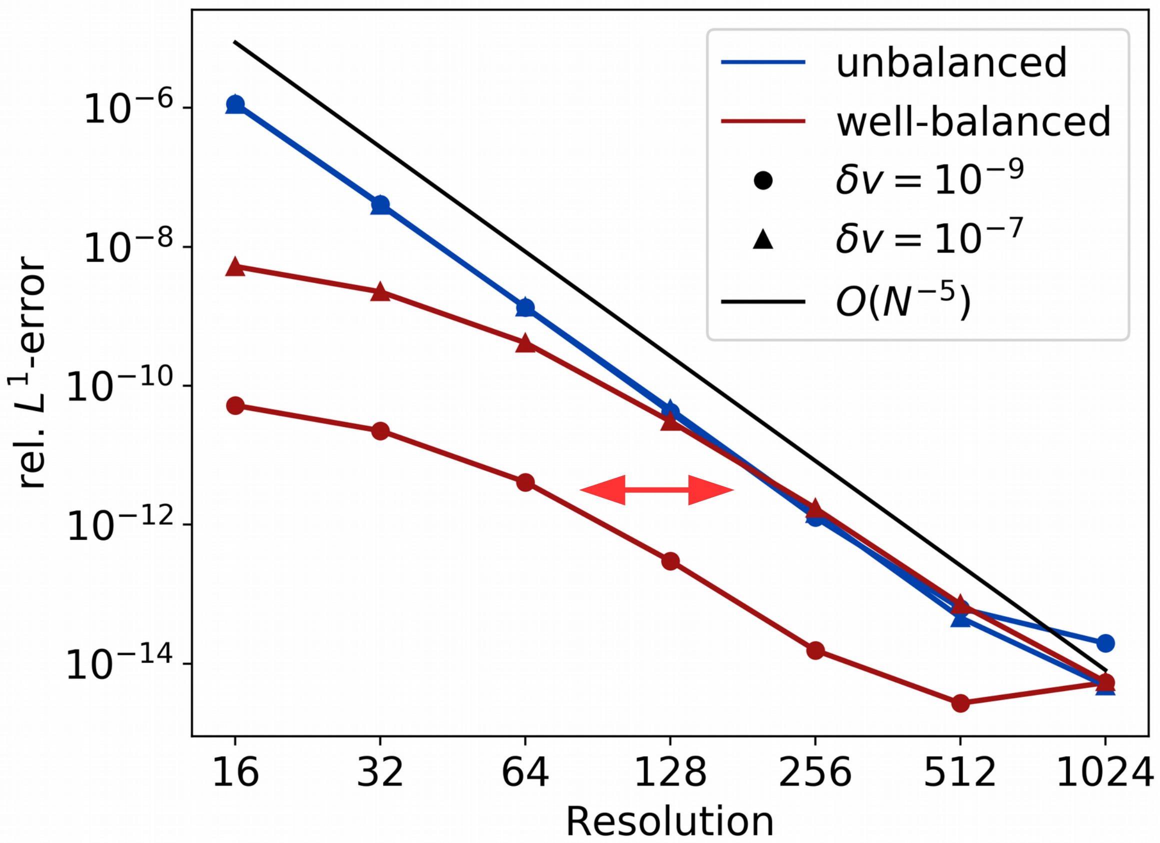
Isentropic well-balanced scheme (4)

- Hot Jupiter atmosphere + perturbation



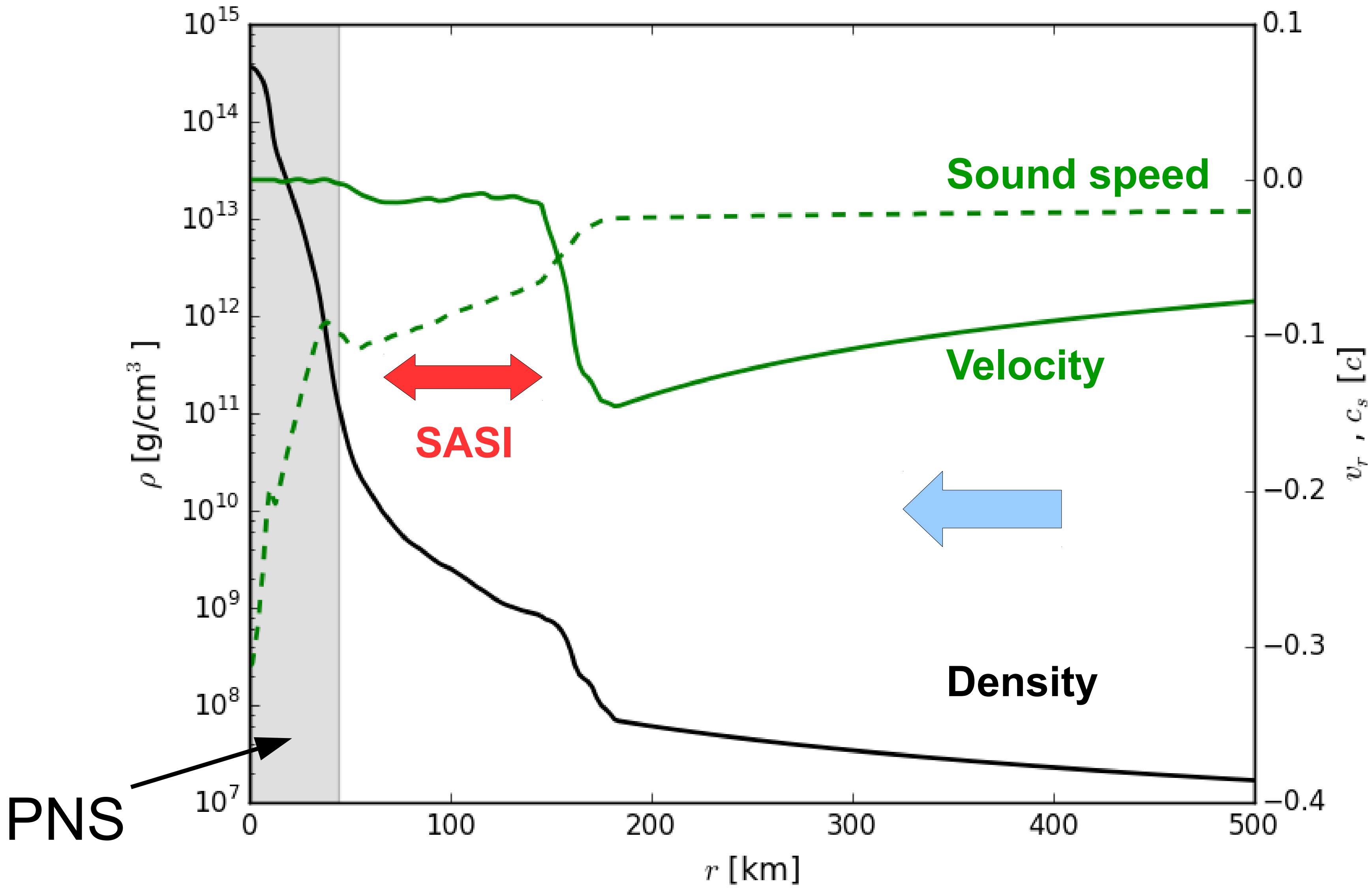
Isentropic well-balanced scheme (4)

- Hot Jupiter atmosphere + perturbation



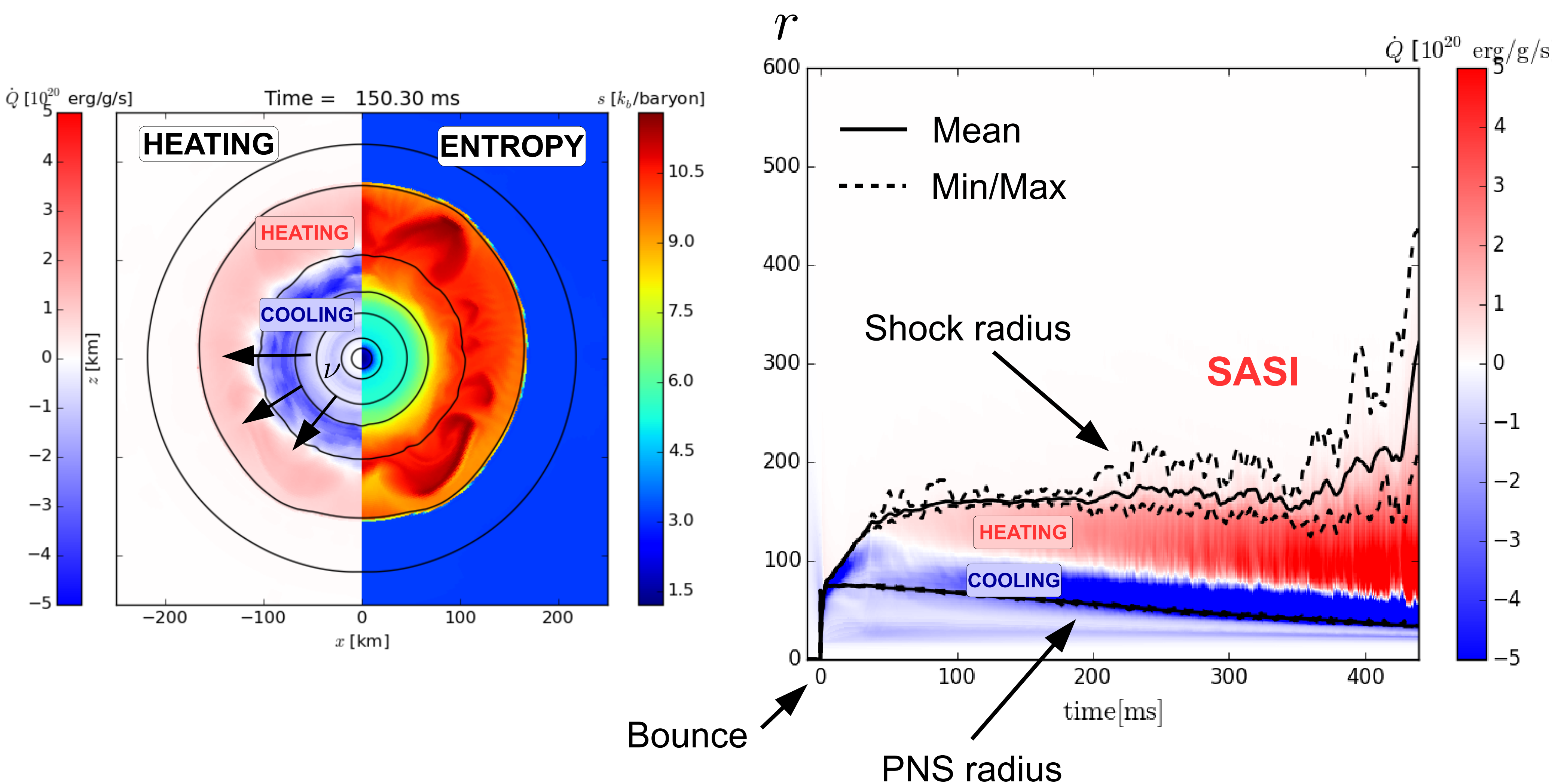
Core-collapse Supernova

- Steady accretion:



Core-collapse Supernova

- Steady accretion:



Standing Accretion Shock Instability (SASI)... See e.g. Foglizzo et al. (2015) and refs therein

Moving steady states

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \rho \boldsymbol{v}) + \nabla p = -\rho \nabla \phi$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)\, \boldsymbol{v}] = -\rho \boldsymbol{v} \cdot \nabla \phi$$

STEADY

$$\rho v = m = const$$

$$\frac{v^2}{2} + h + \phi = const$$

Specific enthalpy

L. Euler

D. Bernoulli

Images from wikipedia...

Well-balanced scheme for moving equi.

Perform local equilibrium reconstruction:

Actually: $p(\rho_{0,i}(x), \bar{s}_i)$

$$\left.\begin{aligned} \rho v = m = \bar{m}_i \\ \frac{v^2}{2} + h + \phi = \overline{Ber}_i \end{aligned}\right\} \frac{1}{2}\left(\frac{\bar{m}_i}{\rho_{0,i}(x)}\right)^2 + h(\rho_{0,i}(x), \bar{s}_i) + \phi(x) = \overline{Ber}_i$$

Solve (nonlinear!)

$\rho_{0,i}(x)$ & $v_{0,i}(x)$ & $p_{0,i}(x)$

$$\boldsymbol{w}_{i\pm 1/2\mp} = \begin{bmatrix} \rho_{0,i}(x_{i\pm 1/2}) \\ v_{0,i}(x_{i\pm 1/2}) \\ p_{0,i}(x_{i\pm 1/2}) \end{bmatrix}$$

Equilibrium reconstructed primitive variables

Analogous to shallow water & Euler-Poisson case:

See e.g. Gosse (2000), Jin (2001), Russo (2001), Wen (2006), Noelle et al. (2007), Bouchut & Morales (2010), Gosse (2013), Castro et al. (2013), ...

Outline

- **Introduction & Motivation**
- **Well-balanced schemes**
  - Arbitrary stratification
- **Astrophysical applications**
- **Higher-order & Moving steady states**
- **Conclusions**

Conclusions

- (Close to) Equilibrium flows are relevant in many astrophysical applications
- Well-balanced scheme for hydrostatic equilibrium with arbitrary thermal stratifications Käppeli & Mishra, A&A, 587, 2016
- (Arbitrary) High-order well-balanced scheme for isentropic stratifications Grosheintz et al., in preparation
- Moving equilibria... Käppeli et al., in preparation

Thank you for your attention!!!