Structure Preserving Schemes

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Introduction

The Euler equations of hydrodynamics in the presence of a gravitational field are a system of balance laws expressing the conservation of mass, momentum and total energy:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) &= 0\\ \frac{\partial \rho v}{\partial t} + \nabla\cdot(v\rho v) + \nabla p &= -\rho\nabla\phi\\ \frac{\partial E}{\partial t} + \nabla\cdot[(E+p)v] &= -\rho v\cdot\nabla\phi\end{aligned}$$

The source terms on the right hand side of the momentum and energy equations detail the effect of the gravitational forces on the fluid. They are proportional to the gradient of the gravitational potential, which can either be a given function or it can also be determined by the Poisson equation

$$\nabla^2\phi = 4\pi G\rho$$

The Euler equations (in 2D for simplicity) can be solved numerically by standard finite volume/difference schemes of the form

$$\begin{aligned}\frac{\mathrm{d}\mathbf{u}_{i,j}}{\mathrm{d}t} = &- \frac{1}{\Delta x}\left(\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}\right)\\ &- \frac{1}{\Delta y}\left(\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}\right)\\ &+ \mathbf{S}_{i,j}\end{aligned}$$

for the conserved variables. The numerical fluxes at cell interfaces are obtained by solving Riemann problems

$$\begin{aligned}\mathbf{F}_{i+1/2,j} &= \mathcal{F}(\mathbf{u}_{i+1/2-,j}, \mathbf{u}_{i+1/2+,j})\\ \mathbf{G}_{i,j+1/2} &= \mathcal{G}(\mathbf{u}_{i,j+1/2-}, \mathbf{u}_{i,j+1/2+})\end{aligned}$$

The cell interface conserved variables are obtained through a polynomial non-oscillatory reconstruction from the cell averaged/point values.

However, the Euler equations are a non-linear system of balance laws, which feature certain structures in the form of companion laws that are fulfilled at the analytical level. When the system is discretized in a straightforward manner as above, these companion laws are generally violated.

The purpose of structure preserving numerical schemes is then to fulfill a discrete form of these companion laws. In the following we will consider two such structures, which are of particular interest for numerical simulations of a wealth of applications ranging from (exo-) climate modeling to the explosive death of massive stars: the preservation of stationary states and the conservation of angular momentum.

Well-balanced schemes

The first structure concerns the steady states of the Euler equation with gravity. Of particular interest is the hydrostatic equilibrium case

$$\nabla p = -\rho\nabla\phi$$

Maintaining this subtle balance between pressure and gravitational force is not assured in standard numerical schemes. The reason for this is straightforward. The hydrostatic pressure and density distributions are in general not simple polynomial functions and therefore cannot be accurately represented by standard high-order non-oscillatory reconstructions.

However, the above hydrostatic equilibrium is not uniquely prescribed, because it implies only a mechanical equilibrium. An assumption on a thermodynamic quantity such as entropy or temperature has to be supplemented. We assume a constant entropy distribution, which is a practical assumption in many applications. Then the hydrostatic equilibrium reduces to a scalar equation relating the specific enthalpy with the gravitational potential

$$h + \phi = \text{const}$$

where the specific enthalpy is a function of density and specific entropy

$$h = h(\rho, s)$$

By performing the reconstruction in respect of the above equilibrium a scheme results, which is capable of exactly resolving a hydrostatic equilibrium, i.e. a well-balanced scheme. For details see **[1]**.

An example is shown in figure **A** below. A periodic velocity perturbation running up an atmosphere. Only the well-balanced scheme is able to accurately follow the wave pattern.

Angular momentum conservation

The second structure concerns the conservation of angular momentum. The conservation law for the angular momentum density follows by simply taking the cross product of the position vector and the equation of conservation of linear momentum:

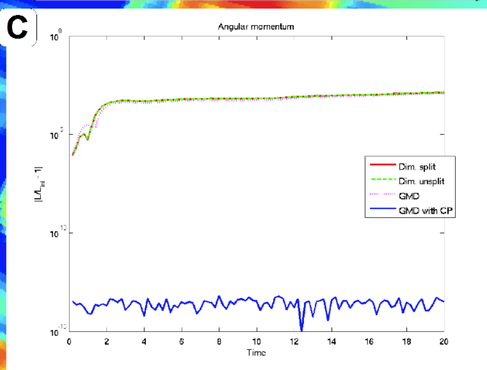
$$\frac{\partial \mathbf{j}}{\partial t} + \nabla\cdot[\mathbf{x}\times(\rho\mathbf{v}\otimes\mathbf{v} + p\mathbf{I})] = -\rho\mathbf{x}\times\nabla\phi$$

The crucial property used in deriving the above equation is the symmetry of the stress tensor. Unfortunately, this symmetry property is generally not retained by the discretized equations and the conservation of angular momentum is lost. For smooth solution the conservation violation is of the order of the spatial accuracy of the scheme, but at flow discontinuities (as shocks or contacts) it is of order O(1).

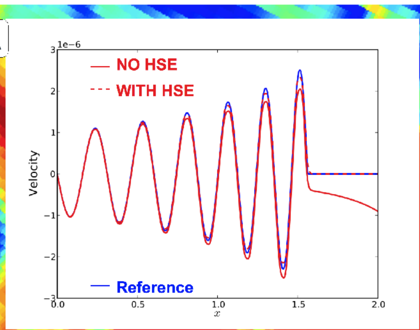
To preserve angular momentum conservation at the discrete level, we use the genuinely multidimensional (GMD) schemes [3]. In these schemes, the numerical fluxes are computed by averaging so-called numerical potentials, which are centered at the grid's vertices. The numerical potentials can then be chosen so as to fulfill the discrete symmetry properties to guarantee angular momentum conservation. For details we refer to [2].

In figure **B** below the interaction of a shock wave with three isentropic vortices is shown. The angular momentum preserving properties of several standard schemes are shown in figure **C**. It results that the constraint preserving GMD scheme can conserve angular momentum up to the machine precision. All other schemes suffer spurious deviations.

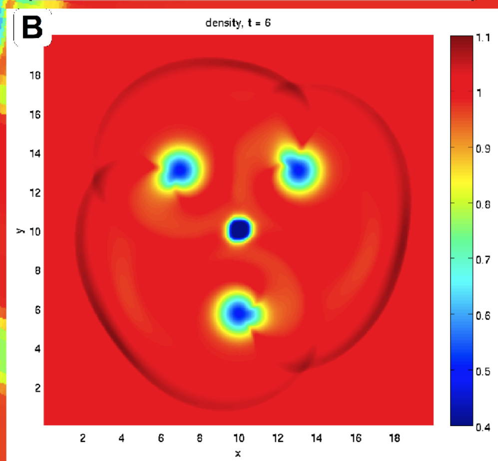
C



A



B



References:

[1] Well-balanced schemes for the Euler equations, R. Käppeli, S. Mishra, JCP, 2014

[2] Structure preserving schemes, R. Käppeli, S. Mishra, SAM Report, 2014-02, 2014

[3] Constraint Preserving Schemes Using Potential-Based Fluxes. II. Genuinely Multidimensional Systems of Conservation Laws, S. Mishra, E. Tadmor, SIAM Journal on Numerical Analysis, 2011

Background: simulation of the Rayleigh-Taylor instability on an axisymmetric spherical grid with the well-balanced schemes