A FUNNY IDENTITY

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The identity (3), and its corollaries (1) and (2) below, are undoubtedly well-known, and as indicated by T. Rivoal, they are special cases of more general formulas. The first one is useful, for instance, to check directly that the probabilistic interpretation of the Haar measure on $SU(N)$ due to Bourgade, Hughes, Nikeghbali and Yor[1] implies the Keating-Snaith formula for integral moments of values of the characteristic polynomials of uniformly-distributed random unitary matrices (though it is probably better, as in their paper, to use more general hypergeometric identities to derive in one go the formulas for complex moments). The pleasant aspect of the expression to evaluate (the "rolling denominators") is nice enough to deserve to spend some space to a complete argument.

The first goal is to prove that

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{k}{k+\ell}\frac{k-1}{k+\ell-1}\cdots\frac{k-j+1}{k+\ell-j+1}=0 \tag{1}$$

for integers $k\geqslant 1$ and $0\leqslant \ell\leqslant k-1$. (As usual, the empty products for $j=0$ are interpreted as equal to 1). The left-hand sum is undefined for $\ell=-1$ (for $j=k$), and the range of ℓ is best possible because the second goal will be to check that for $\ell=k$ we have

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{k}{2k}\frac{k-1}{2k-1}\cdots\frac{k-j+1}{2k-j+1}=\frac{1}{\binom{2k}{k}}=\frac{(k!)^2}{(2k)!}. \tag{2}$$

In fact, we can perform the computation for arbitrary ℓ, and we will prove that

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{k}{k+\ell}\frac{k-1}{k+\ell-1}\cdots\frac{k-j+1}{k+\ell-j+1}&=\frac{k!}{(k+\ell)!}\ell(\ell-1)\cdots(\ell-k+1)\\&=\frac{\ell!k!}{(\ell-k)!(\ell+k)!},\\&=\frac{\Gamma(\ell+1)\Gamma(k+1)}{\Gamma(\ell-k+1)\Gamma(\ell+k+1)}\end{aligned} \tag{3}$$

where the first expression is valid for all integers ℓ, the second for $\ell\geqslant k$, and writing it in terms of the gamma function leads to an expression valid for all $\ell\in\mathbf{C}$ as identity among meromorphic (in fact, rational) functions of ℓ.

Note that, when $\ell=0$, (1) becomes a trivial application of the binomial theorem. Define more generally the polynomial

$$T(k,\ell,x)=\sum_{j=0}^{k}\binom{k}{j}\frac{k(k-1)\cdots(k-j+1)}{(k+\ell)\cdots(k+\ell-j+1)}x^j$$

so that to prove (1), we need to show that $T(k,\ell,-1)=0$ for $0\leqslant\ell\leqslant k-1$.

[1] See *The characteristic polynomial of a random unitary matrix: a probabilistic approach*, `arXiv:0706.0333`.

Obviously, we have

$$(4)\quad (k+\ell)!T(k,\ell,x) = \sum_{j=0}^{k}\binom{k}{j}x^j \times \{k(k-1)\cdots(k-j+1)(k+\ell-j)(k+\ell-j-1)\cdots 2\cdot 1\}$$

$$= (-1)^\ell k!\sum_{j=0}^{k}\binom{k}{j}x^j A(k,\ell,j)$$

where

$$A(k,\ell,t) = (t-k-1)\cdots(t-k-\ell)$$

is a monic polynomial of degree ℓ with respect to the variable $t=j$. Hence there exist coefficients $\alpha_{k,\ell,i}$ such that

$$A(k,\ell,t) = \sum_{i=0}^{\ell}\alpha_{k,\ell,i}t(t-1)\cdots(t-i+1)$$

and therefore we have

$$T(k,\ell,x) = \frac{(-1)^\ell k!}{(k+\ell)!}\sum_{i=0}^{\ell}\alpha_{k,\ell,i}\Big\{\sum_{j=0}^{k}\binom{k}{j}j(j-1)\cdots(j-i+1)x^j\Big\}$$

$$= \frac{(-1)^\ell k!}{(k+\ell)!}\sum_{i=0}^{\ell}\alpha_{k,\ell,i}x^i\frac{d^i}{dx^i}(1+x)^k.$$

It is now obvious that

$$T(k,\ell,-1) = 0$$

if $0\leqslant \ell < k$, since $x\mapsto (1+x)^k$ has a zero of order k at $x=-1$, and we have proved (1).

To derive (3) and (2), we look again at (4). From the latter, it follows that

$$S(\ell) = (k+\ell)!T(k,\ell,-1)$$

is, for fixed k, a polynomial in ℓ, of degree $\leqslant k$, the monomial of highest degree of which is obtained from the term $j=0$ in the sum (the j-term has degree $k-j$ with respect to ℓ).

But we know from the previous argument that $S(\ell)=0$ for $0\leqslant \ell\leqslant k-1$, hence there exists a constant c_k such that

$$(5)\qquad (k+\ell)!T(k,\ell,-1) = c_k\ell(\ell-1)\cdots(\ell-k+1)$$

for *all* ℓ.

To find c_k, we look at the asymptotic behavior as $\ell\to+\infty$. Since the monomial of highest degree, as stated, comes from $j=0$ in (4), we have

$$(k+\ell)!T(k,\ell,-1) \sim \binom{k}{0}\cdot(-1)^0\cdot k!\cdot \ell^k, \qquad \text{as } \ell\to+\infty,$$

and comparing with (5) shows that $c_k = k!$. Hence from (5), we have

$$T(k,\ell,-1) = \frac{k!}{(k+\ell)!}\ell(\ell-1)\cdots(\ell-k+1)$$

$$= \frac{\ell!k!}{(\ell-k)!(\ell+k)!}, \qquad \text{for } \ell\geqslant k,$$

which leads to (3) and (2) in particular.

Remark. Spelling out, for instance, what happens for $k = 5$, $1 \leqslant j \leqslant 5$, we have:

$$1 - 5\frac{5}{6} + 10\frac{5\cdot 4}{6\cdot 5} - 10\frac{5\cdot 4\cdot 3}{6\cdot 5\cdot 4} + 5\frac{5\cdot 4\cdot 3\cdot 2}{6\cdot 5\cdot 4\cdot 3} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2} = 0,$$
$$1 - 5\frac{5}{7} + 10\frac{5\cdot 4}{7\cdot 6} - 10\frac{5\cdot 4\cdot 3}{7\cdot 6\cdot 5} + 5\frac{5\cdot 4\cdot 3\cdot 2}{7\cdot 6\cdot 5\cdot 4} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3} = 0,$$
$$1 - 5\frac{5}{8} + 10\frac{5\cdot 4}{8\cdot 7} - 10\frac{5\cdot 4\cdot 3}{8\cdot 7\cdot 6} + 5\frac{5\cdot 4\cdot 3\cdot 2}{8\cdot 7\cdot 6\cdot 5} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4} = 0,$$
$$1 - 5\frac{5}{9} + 10\frac{5\cdot 4}{9\cdot 8} - 10\frac{5\cdot 4\cdot 3}{9\cdot 8\cdot 7} + 5\frac{5\cdot 4\cdot 3\cdot 2}{9\cdot 8\cdot 7\cdot 6} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{9\cdot 8\cdot 7\cdot 6\cdot 5} = 0,$$
$$1 - 5\frac{5}{10} + 10\frac{5\cdot 4}{10\cdot 9} - 10\frac{5\cdot 4\cdot 3}{10\cdot 9\cdot 8} + 5\frac{5\cdot 4\cdot 3\cdot 2}{10\cdot 9\cdot 8\cdot 7} - \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{10\cdot 9\cdot 8\cdot 7\cdot 6} = \frac{1}{252}.$$