Gross–Stark units for totally real number fields

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Contents

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Chapter 1

Introduction

The German mathematician Leopold Kronecker (1823–1891) made the famous statement that "God created the integers, all else is the work of man," by which he referred to the general process of constructing numbers. Namely, if one starts with the integers $0, \pm 1, \pm 2, \ldots$, one can add and subtract numbers but cannot always divide. However, one can fix this by introducing the set of rational numbers, denoted by $\mathbb{Q}$, which enlarges the integers and consists of all possible ratios of integers with nonzero denominator. Now division by a nonzero number is always possible. The construction of the rationals is thus purely algebraic, starting from the integers.

Still, the set $\mathbb{Q}$ does not contain all the numbers in which one is interested. For example, one would like to work with a number α such that $\alpha^2 = 2$, but there is no such α in the set $\mathbb{Q}$. The fix is simply to *introduce* an additional number, denoted $\alpha = \sqrt{2}$, and to enlarge $\mathbb{Q}$ to a bigger set $F = \mathbb{Q}(\sqrt{2})$ which consists of all formal expressions $a + b\sqrt{2}$, where a and b are in $\mathbb{Q}$. When $a = 0, b = 1$, we have indeed $\sqrt{2} \in F$. Elements in F can be added and multiplied by the rule $(\sqrt{2})^2 = 2$. The key to this construction is the fact that $\sqrt{2}$ is a root of the polynomial $x^2 - 2$. Similarly, one can construct square-roots $\sqrt{3}, \sqrt{5}$ by the purely *algebraic* process of introducing a root of the corresponding polynomial $x^2 - 3, x^2 - 5$.

Another important class of numbers that one is interested in constructing consists of the *roots of unity*. By definition, a root of unity is a number ζ such that $\zeta^m = 1$ for some integer m. Besides ± 1, such numbers do not exist in $\mathbb{Q}$, but one can introduce them by a purely algebraic process similar to the one for $\sqrt{2}$ described above. In more detail, one can construct a number ζ (called a primitive m-th root of unity) whose m-th power is 1, but whose smaller powers are not 1. For example, one can construct a number ζ such that $\zeta^8 = 1$ but $\zeta^4 = -1$; thus, ζ is a solution of the polynomial equation $x^4 + 1 = 0$.

A particular case of a theorem by Kronecker–Weber states essentially that the construction of roots of unity is more fundamental than the one of square-roots. Namely, it turns out that by constructing the various roots of unity, we have already accounted also for all of the square-roots; i.e., any square-root of a rational number can be expressed in terms of roots of unity. For example, using the root of unity ζ from above, recalling that $\zeta^4 = -1$, we can write

$$\left(\zeta + \frac{1}{\zeta}\right)^2 = 2 + \zeta^2 + \frac{1}{\zeta^2} = 2 + \frac{\zeta^4 + 1}{\zeta^2} = 2.$$

Therefore,

$$\sqrt{2} = \zeta + \frac{1}{\zeta} \tag{1.1}$$

can be written in terms of the root of unity ζ.

Of special interest to number theory are numbers which generate abelian extensions[1]; the simplest example of such numbers are the square-roots, so we concentrate our discussion here on those. The Kronecker–Weber theorem is a very general and powerful result, asserting that any number which generates an abelian extension over $\mathbb{Q}$ can be expressed in terms of roots of unity. Class field theory is the study of such abelian extensions over general number fields. The Kronecker–Weber theorem, which is the simplest case of class field theory, can be viewed as a statement that the roots of unity constitute explicit class field theory over the field $\mathbb{Q}$.

On the other hand, one can construct the roots of unity by a process rather different in nature — namely, by looking at special values of an *analytic* function. These are functions described as convergent power series and whose values are obtained by an analytic process of approximation: by computing more and more terms in a power series expansion, one obtains better and better estimates for the value; the value is equal to the limit of all approximations. Such functions are in sharp contrast with polynomials, such as $x^2 - 2$ and $x^4 + 1$, where the value is computed directly and exactly through a finite process, as opposed to an infinite limiting one. In particular, the function

$$f(z) = e^{2\pi i z} = 1 + 2\pi i z + \frac{(2\pi i z)^2}{2!} + \frac{(2\pi i z)^3}{3!} + \dots \tag{1.2}$$

is analytic — it is given by a convergent power series. Remarkably, its special value at $z = \frac{1}{m}$ is $\zeta = e^{\frac{2\pi i}{m}}$, which is a primitive m-th root of unity. In other words, the roots of unity arise not only from a purely algebraic construction similar to the one which produces $\sqrt{2}$, but also naturally as certain special values of an analytic function. It is surprising that an *analytic* object, such as the function $e^{2\pi i z}$ accounts for the purely *algebraic* property that a number generates an abelian extension over the rationals (such as any square-root).

Kronecker was aware of this fact and asked more generally whether one can obtain all of the abelian extensions of a number field by considering special values of an analytic function. Kronecker's Jugendtraum ("dream of youth") is, given a number field such as $F = \mathbb{Q}(\sqrt{2})$, to produce a single analytic function such that any algebraic number that generates an abelian extension over F (in particular, any square-root one can extract from F) can be written in terms of special values of this analytic function. This question is solved only when $F = \mathbb{Q}$, by the Kronecker–Weber theorem and the function $e^{2\pi i z}$, and when F is imaginary quadratic. However, the problem is still a great mystery for any other number field. If one takes $F = \mathbb{Q}(\sqrt{2})$, the roots of unity are not sufficient any more because for example the number $\sqrt[4]{2} = \sqrt{\sqrt{2}}$, obtained simply by extracting a square-root of an element in F, cannot be written in terms of roots of unity and elements in F; i.e., there is no formula which generalizes (1.1) when $\sqrt{2}$ is replaced by $\sqrt[4]{2}$.

So, given F, one has to concoct an analytic function — finer than $e^{2\pi i z}$ — whose special values will play the role of the roots of unity over the rationals. The problem is that a special

[1] Galois extensions with abelian Galois group.

value of an analytic function is obtained by an infinite limiting process and is normally not one that on the other hand can be constructed by algebraic methods, as a root of a polynomial. Even if one has a candidate for an analytic function, one would expect difficulties in proving that the special value is algebraic (as in the case of Dasgupta's conjecture treated in this thesis). Finally, the constructed special values have to generate abelian extensions and to be substantial enough, so that any number which generates an abelian extension is already accounted for.

A conjecture by Stark (1970's) and especially its further refinements by Gross (1982, 1987) and Dasgupta (2007) give a hint of where this mysterious analytic function may come from ([13],[6],[13]). Consider an abelian extension K/F. Associated to the extension are a number of analytic functions, called partial zeta-functions. They generalize the Riemann zeta-function, which is an analytic function on the complex plane, holomorphic outside $s = 1$, and given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad Re(s) > 1.$$

The Riemann zeta-function would be associated to the trivial extension $K = F = \mathbb{Q}$, so it is a particular case of a partial zeta-function. It turns out that the special value of ζ at $s = 0$ is not just algebraic, but in fact a rational number:

$$\zeta(0) = -\frac{1}{2}.$$

The denominator 2 has arithmetic significance: it equals the number of roots of unity in $\mathbb{Q}$ (which are ± 1). The same phenomenon is observed with the partial zeta functions in general: their values at $s = 0$ are rational numbers with denominators dividing the number of roots of unity in the extension under consideration. Therefore, by multiplying by this number of roots of unity, we can produce a set of *integers* attached to the abelian extension we want to study, which come as special values (at $s = 0$) of analytic functions (the partial zeta functions).

Stark's conjecture states the existence of a special element, called a Stark unit, in the abelian extension K which plays the role of a root of unity. The Stark unit satisfies very strong arithmetic properties in terms of the data of these integers. Stark's conjecture is proven to be true in general only over the rational field and over imaginary quadratic fields: not by accident, these are precisely the instances when Kronecker's dream of youth is solved and when the abelian extension is known explicitly already. For example, over $\mathbb{Q}$, we can build the Stark unit out of roots of unity.

A further refinement by Gross predicts even more properties that this Stark unit has to satisfy. Finally, Dasgupta has proposed a yet stronger conjecture, which involves an explicit conjectural *formula* for the Stark unit. The previous conjectures only state that there exists an element with some properties, while Dasgupta's formula states that the element given by a certain explicit formula in fact satisfies all of the conjectural properties. The formula involves building up the Stark unit out of the data of the integers obtained as special values of partial zeta-functions and is analytic in nature, in the flavor of Kronecker's dream of youth. Naively, if one thinks of the Stark unit as playing the role of a root of unity in that it generates an abelian extension, Dasgupta's formula is an attempt to exhibit a function similar in nature to (1.2), which is analytic and yet produces an algebraic number. The essence of Dasgupta's conjecture is proving that the element the formula yields in fact lies in an abelian extension and comes from an algebraic process. We verify the conjectural formula computationally with a certain accuracy in two particular examples. This formula would be a deep link between special values of *analytic* functions and *algebraic* elements in abelian extensions, hence a step towards Kronecker's dream of youth.

Chapter 2

Outline of contents

The motivation for Stark's conjectures comes from Dirichlet's class number formula: let F be a number field and let

$$\zeta_F(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} N\mathfrak{a}^{-s} \quad \text{for} \quad Re(s) > 1$$

be the zeta-function attached to it; it extends to a meromorphic function on $\mathbb{C}$, and the Taylor expansion of ζ_F around the origin starts as follows:

$$\zeta_F(s) = -\frac{hR}{\omega} s^r + \ldots \text{ (higher–order terms)}, \tag{2.1}$$

where r is the rank of the unit group of $\mathcal{O}_F$, h is the class number of F, R is the regulator, and ω is the number of roots of unity in F. Stark formulated conjectures that would generalize this formula in the case of extensions K/F and the leading terms of L-functions attached to it. The abelian version of these conjectures is particularly striking.

Namely, the insight is that one can interpret (2.1) not as a formula for the value at $s = 0$ of an analytic function (as the formula suggests at first glance), and not as a formula for the class number h (as is often classically done), but as a formula for the regulator R in terms of a special value of an analytic function. As a motivating example, if F is real quadratic, equation (2.1) is a statement about the archimedean absolute value of the fundamental unit of $\mathcal{O}_F$. When one starts with an abelian extension K/F, Stark's conjecture is a statement about the existence of a special element $\epsilon \in K$ with prescribed absolute values, in terms of the values at 0 of the partial zeta functions associated to K/F.

Gross has formulated further refinements of the classical Stark's conjecture in the $\mathfrak{p}$-adic context, predicting more information for this special element ϵ. Finally, Dasgupta has an *explicit* analytical conjectural formula for the Gross–Stark unit. It can be viewed as an attempt for explicit class field theory over a totally real number field.

In Chapter 4, we discuss the formulation of the Gross–Stark conjecture. In Chapter 5, following [8], we present the proof over the rational field, which is the only totally real number field for which it is known. The proof relies on a result of Gross–Koblitz which relates the Gross–Stark unit to a special value of the p-adic Gamma function. This special value can then be understood due to the functional equation that the p-adic Gamma function satisfies and the conjecture can be verified. We identify a mistake in [8] and fix it with only small modifications of the argument.

Next, in Chapter 6, we present Dasgupta's approach of starting with the statement of Gross's conjecture and attempting to construct the Gross–Stark unit explicitly. Dasgupta interprets Gross's conjecture in fact as a *formula* for the Gross–Stark unit, but in a certain *quotient* of $F_{\mathfrak{p}}^*$. The main ingredient in this formula involves a summation over elements defined only up to the action of a certain group. The goal of Chapter 7 is to write down an exact formula for the unit in $F_{\mathfrak{p}}^*$. The key idea is the use of Shintani domains, which constitute a particular fundamental domain for the group action mentioned above, with a special geometric shape.

Finally, in Chapter 8, we provide computational evidence for Dasgupta's conjecture. The naive approach to calculate the unit from Dasgupta's formula is inefficient, and we construct a more general measure than the one explicitly involved in the formula and described in [4]. Thus, we had to slightly modify and generalize arguments from [12] (only for a certain analytic ingredient) and [4] (mainly) in order to obtain a formula for the more general measure and hence an algorithm for computing the conjectural Gross–Stark unit.

Chapter 3

Notation

3.1 Number fields

For a number field K, we denote by $\mathcal{O}_K$ the ring of integers of K, by $\mu(K)$ the group of roots of unity in K, and we set $W_K = \#\mu(K)$. For an integer $m \geq 1$, we let μ_m be the group of m-th roots of unity. For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we denote by $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$, and by $\mathcal{O}_\mathfrak{p}$ the valuation ring of $K_\mathfrak{p}$. For $m \geq 0$, set

$$U_{\mathfrak{p}^m} = \begin{cases} (1+\mathfrak{p}^m\mathcal{O}_\mathfrak{p})^*, & \text{if} \quad m \geq 1, \\ \mathcal{O}_\mathfrak{p}^*, & \text{if} \quad m = 0. \end{cases}$$

For a modulus $\mathfrak{m} = \mathfrak{m}_f\mathfrak{m}_\infty$, $I_K^\mathfrak{m}$ denotes the group of fractional ideals of K relatively prime to $\mathfrak{m}_f$. Also,

$$a \equiv b \pmod{\mathfrak{m}}$$

for $a, b \in K^*$ means that for each finite $\mathfrak{p}|\mathfrak{m}_f$, we have

$$\nu_\mathfrak{p}(\frac{a}{b} - 1) \geq \nu_\mathfrak{p}(\mathfrak{m}_f),$$

and for each real prime $v \mid \mathfrak{m}$, we have $(ab^{-1})_v > 0$. Similarly for $a \equiv b \pmod{\mathfrak{p}^k}$, where $a, b \in F_\mathfrak{p}^*$. Next,

$$K_{\mathfrak{m},1} = \{x \in K^* \mid x \equiv 1 \pmod{\mathfrak{m}}\},$$

and $i(K_{\mathfrak{m},1})$ is the image of $K_{m,1}$ under the map $i : K_{\mathfrak{m},1} \longrightarrow I_K^{\mathfrak{m}_f}$ given by $\alpha \mapsto (\alpha)$. When K is a totally real number field, we let $\infty = \prod_{v|\infty} v$, where the product is over all the infinite (real) primes of K.

For an abelian extension L/K and a fractional ideal $\mathfrak{b}$ of K relatively prime to the product of all ramified primes, $\sigma_\mathfrak{b} \in G(L/K)$ denotes the image of $\mathfrak{b}$ uder the Artin map. When $K = \mathbb{Q}$ and $b > 0$, we define $\sigma_b = \sigma_{(b)}$.

3.2 Measures

Let X be a compact Hausdorff, totally disconnected topological space and let μ be an additive $\mathbb{Z}$-valued measure on X, i.e., μ assigns an integer $\mu(U) \in \mathbb{Z}$ to each compact open subset

$U \subset X$, such that

$$\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2)$$

for any disjoint compact opens $U_1, U_2 \subset X$. Let $I = \lim_{\leftarrow} I_\alpha$ be a profinite group (each I_α is finite) and let $f : X \to I$ be a continuous map. Let $U_i \subset I$ denote the inverse image of an element $i \in I_\alpha$ under $I \to I_\alpha$. Define

$$\times\!\!\!\!\!\!\int_X f(x)d\mu(x) = \varprojlim \prod_{i \in I_\alpha} i^{\mu(f^{-1}(U_i))} \in I = \varprojlim I_\alpha;$$

it is the multiplicative integral of $f(x)$ over X.

Say X is a compact subset of some valuation ring $\mathcal{O}_\mathfrak{p}$, and $f : X \to F_\mathfrak{p}^*$ is a continuous map. Suppose that $a \equiv b \pmod{\mathfrak{p}^k}$ implies $f(a) \equiv f(b) \pmod{\mathfrak{p}^k}$ (this is the case we will be most interested in; one can define the multiplicative integral without this assumption). Let μ be a $\mathbb{Z}$-valued measure on X. Define

$$A = \times\!\!\!\!\!\!\int_X f(x)d\mu(x) \in F_\mathfrak{p}^*$$

as the unique element in $F_\mathfrak{p}^*$ such that whenever X is written as a disjoint union $X = \cup_{i=1}^d (x_i + \mathfrak{p}^N \mathcal{O}_\mathfrak{p})$, we have

$$A \equiv \prod_{i=1}^d f(x_i)^{\mu(x_i + \mathfrak{p}^N \mathcal{O}_\mathfrak{p})} \pmod{\mathfrak{p}^N}.$$

It is the limit of Riemann products over finer covers of X by compact intervals $x_i + \mathfrak{p}^N \mathcal{O}_\mathfrak{p}$.

We define the additive integral similarly: given a compact subset $X \subset \mathcal{O}_\mathfrak{p}$, a function $f : X \to F_\mathfrak{p}$ as above, and a $\mathbb{Z}$-valued measure μ on X, we can define $\int_X f(x)d\mu(x) \in F_\mathfrak{p}$ analogously as above, with product replaced by sum. One can prove the Riemann sums converge and hence the integral is well–defined.

Chapter 4

Statement of the Gross–Stark conjecture

Here we present the statement of Gross's conjecture as in [8].

4.1 An arithmetic preliminary

Proposition 1. *Let L/k be an abelian extension with $G = G(L/k)$, and S a finite set of primes of k containing the archimedean primes, the primes which ramify in L, and the ones dividing the order $e = \#\mu(L)$. Then the annihilator $Ann(L/k)$ of the $\mathbb{Z}[G]$-module $\mu(L)$ is generated as a $\mathbb{Z}$-module by the collection*

$$\{\sigma_{\mathfrak{p}} - N\mathfrak{p} \mid \mathfrak{p} \notin S\}.$$

Proof. First, let $\mathfrak{p} \notin S$ and let $\alpha \in \mu(L)$. If $\mathfrak{B}$ is a prime of L lying over $\mathfrak{p}$, we know that $b = \alpha^{\sigma_{\mathfrak{p}} - N\mathfrak{p}} \equiv 1 \pmod{\mathfrak{B}}$. To prove that in fact $b = 1$, we note that $b^e = 1$ in particular in $\mathcal{O}_{\mathfrak{B}}/\mathfrak{B}$, and invoke Hensel's lemma, which gives us uniqueness of the solution of the equation $x^e = 1$ with $x \equiv 1 \pmod{\mathfrak{B}}$, since $\mathfrak{B} \nmid e$.

If $A \in Ann(L/k)$, by Chebotariov's density theorem, we can write any $\sigma \in G$ as $\sigma = \sigma_{\mathfrak{p}}$ for some $\mathfrak{p} \notin S$, and hence

$$A = \sum_{\mathfrak{p} \notin S} a_{\mathfrak{p}}(\sigma_{\mathfrak{p}} - N\mathfrak{p}) + a,$$

for some integers $a_{\mathfrak{p}}$ and a, where the sum is of course finite. But then $a \in Ann(L/k)$ and so $e|a$. Thus, the statement will follow if we prove that

$$e = \gcd_{\mathfrak{p} \notin S,\ \sigma_{\mathfrak{p}} = 1} (1 - N\mathfrak{p}).$$

By above, it is clear that for $\mathfrak{p} \notin S$ with $\sigma_{\mathfrak{p}} = 1$, the integer $1 - N\mathfrak{p}$ is divisible by e, as it annihilates $\mu(L)$. If e' is a common divisor of all $(1 - N\mathfrak{p})$ with $\mathfrak{p} \notin S$ and $\sigma_{\mathfrak{p}} = 1$, consider the field $L' = L(\zeta)$, where ζ is a primitive e'-th root of unity. If $\sigma \in G(L'/L)$ is arbitrary, write $\sigma = \sigma_{\mathfrak{p}}$ for some $\mathfrak{p} \notin S$, unramified in L', where the Frobenius is taken in L'/k. Since $\sigma_{\mathfrak{p}}|_L$ is trivial, we must have $e'|1-N\mathfrak{p}$. But then by the first part of the Proposition, $\zeta^{\sigma_\mathfrak{p}} = \zeta^{N\mathfrak{p}} = \zeta$ and so $\sigma = \sigma_\mathfrak{p}$ is the identity. But, $\sigma \in G(L'/L)$ was arbitrary, hence $L' = L$ and $e'|e$, as desired. $\square$

4.2 The classical Stark's conjecture

Consider an abelian extension K/k with $G = G(K/k)$. Let S be a finite set of primes of k containing all archimedean primes, as well as the ones which ramify in K. For $\sigma \in G$, define the partial zeta function

$$\zeta_{K/k,S}(\sigma, s) = \sum_{\substack{I \subset \mathcal{O}_k \\ \sigma_I = \sigma}} NI^{-s},$$

where the series converges for $Re(s) > 1$ and admits a meromorphic continuation to the entire complex plane. Define

$$\theta_{K/k,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - \sigma_\mathfrak{p}^{-1} N\mathfrak{p}^{-s})^{-1} = \sum_{\sigma \in G} \zeta_{K/k,S}(\sigma, s)\sigma^{-1}.$$

Set $\theta_K = \theta_{K/k,S}(0)$. It is known (see [1]) that for any $A \in Ann(K/k)$, we have $A\theta_K \in \mathbb{Z}[G]$ (in particular, $\theta_K \in \mathbb{Q}[G]$). If $\mathfrak{p} \in S$ is unramified in K and $R = S - \{\mathfrak{p}\}$, then

$$\theta_{K/k,S} = (1 - \sigma_\mathfrak{p}^{-1})\theta_{K/k,R}.$$

In particular, if S contains a finite prime $\mathfrak{p}$ which splits completely in K, then $\zeta_{K/k,S}(\sigma, 0) = 0$ for all $\sigma \in G$.

Assume the set S contains a finite prime $\mathfrak{p}$ which splits completely in K, and set $R = S - \{\mathfrak{p}\}$. Fix a prime $\mathfrak{B}$ of K lying over $\mathfrak{p}$. Denote

$$U_\mathfrak{p} = \{x \in K^* \mid |x|_\mathfrak{B} = 1 \quad \text{for any (finite or infinite) prime} \quad \mathfrak{B} \nmid \mathfrak{p}\}.$$

The general abelian Stark's conjecture states as follows:

Conjecture 1. *(Stark) There exists an element $\epsilon = \epsilon(\mathfrak{B}, S) \in K^*$ such that*

$$(\epsilon) = \mathfrak{B}^{W_K \theta_{K/k,R}}$$

and $\epsilon \in U_\mathfrak{p}$ if $|S| \geq 3$. Moreover, $K(\epsilon^{\frac{1}{W_K}})$ is abelian over k.

The element $\epsilon \in K^*$ is called a Stark unit associated to the data $(K/k, S, \mathfrak{B})$ and is uniquely determined up to a root of unity in K. We now discuss Gross's refinement of Stark's conjecture, which involves first assuming that Conjecture 1 holds.

4.3 Gross's refinement

Consider an arbitrary abelian extension L/k with $G(L/k) = G$ and a finite set S of primes of k as before, which contains all archimedean primes, as well as all primes which ramify in L. Fix a finite prime $\mathfrak{p} \in S$ and a subfield K of L/k in which $\mathfrak{p}$ splits completely. Fix a prime $\mathfrak{B}$ of K lying over $\mathfrak{p}$, and let $A \in Ann(L/k)$. Let $\epsilon \in K$ be a Stark unit associated to $(K/k, S, \mathfrak{B})$, and let ζ be any root of unity in an algebraic closure of k containing L. If $\lambda^{W_K} = \zeta\epsilon$, the extension $K(\lambda)/k$ is abelian, and hence so is $L(\lambda)/k$. Thus, by Proposition 1, there exists $A_0 \in Ann(L(\lambda)/k)$ such that $A_0|_L = A$. Define

$$V_A = \{\lambda^{A_0} \mid A_0 \in Ann(L(\lambda)/k),\ A_0|_L = A\}.$$

By Galois theory, $V_A \subset K$. We now show that V_A is independent of the choices of ζ, ϵ, λ. The independence of λ is obvious because A_0 annihilates $\mu(K)$. Since ϵ is determined up to a root of unity in K, it suffices to show that V_A is independent of the choice of ζ. But, a different choice of ζ yields λ' with $\lambda' = \lambda\nu$, for some root of unity ν. Let $A_0' \in Ann(L(\lambda')/k)$ be such that $A_0'|_L = A$. We have to produce $A_0 \in Ann(L(\lambda)/k)$ such that $A_0|_L = A$ and $\lambda^{A_0} = (\lambda')^{A_0'}$. But, the extension $L(\lambda, \lambda')/k$ is abelian, hence there exists $\tilde{A} \in Ann(L(\lambda, \lambda')/k)$ such that $\tilde{A}|_{L(\lambda')} = A_0'$. Set $A_0 = \tilde{A}|_{L(\lambda)}$. Since $\tilde{A}$ annihilates the roots of unity in $L(\lambda, \lambda')$, we have

$$(\lambda')^{A_0'} = (\lambda')^{\tilde{A}} = \lambda^{\tilde{A}} = \lambda^{A_0},$$

as desired.

Write

$$A\theta_L = A\theta_{L/k,S} = \sum_{\sigma \in G} n(A, \sigma)\sigma,$$

with $n(A, \sigma) \in \mathbb{Z}$; we think of $n(A, \sigma)$ as a "shift" of zeta functions evaluated at $s = 0$.

Let $H = G(L/K)$, and let

$$r_{\mathfrak{B}} : K_{\mathfrak{B}}^* \longrightarrow H$$

be the reciprocity map of local class field theory.

Conjecture 2. *(Gross) Let $A \in Ann(L/k)$. Then*

$$r_{\mathfrak{B}}^{-1}(\epsilon_A) = \prod_{\sigma \in H} \sigma^{n(A,\sigma)}$$

for all $\epsilon_A \in V_A$.

An element $\epsilon_A \in V_A$ is called a Gross–Stark unit associated to the given data; we write $Gr_k(L/K, S, \mathfrak{p})$ if Conjecture 2 holds for any choice of $\mathfrak{B}$ over $\mathfrak{p}$.

Note that we can state Conjecture 2 equivalently as follows: for any $\epsilon_A \in V_A$ and any $\tau \in G$, we have

$$r_{\mathfrak{B}^\tau}^{-1}(\epsilon_A) = \prod_{\sigma \in H\tau} \sigma^{n(A,\sigma)}.$$

Indeed, for a given $\tau \in G$, consider $A' = \tau^{-1}A \in Ann(L/k)$ and notice that $V_{A'} = V_{\tau^{-1}A} = V_A^{\tau^{-1}}$. So, Conjecture 2 implies that

$$\begin{aligned} r_{\mathfrak{B}^\tau}^{-1}(\epsilon_A) = r_{\mathfrak{B}}^{-1}(\epsilon_A^{\tau^{-1}}) &= \prod_{\sigma \in H} \sigma^{n(\tau^{-1}A,\sigma)} \\ &= \prod_{\sigma \in H} \sigma^{n(A,\sigma\tau)} \\ &= \prod_{\sigma \in H\tau} \sigma^{n(A,\sigma)}, \end{aligned}$$

where we used that
$$\sum_{\sigma\in H} n(A,\tau\sigma) = 0 \quad \text{for any} \quad \tau \in G,$$
which follows from
$$\sum_{\sigma\in H} \zeta_{L/k,S}(\sigma\tau',0) = \zeta_{K/k,S}(H\tau',0) = 0,$$
since S contains a place $\mathfrak{p}$ which splits completely in K.

4.4 Functoriality properties

We now prove the first functoriality property of the Conjecture:

Lemma 1. *Suppose L_1, L_2 are abelian extensions of k, each containing the field K. Let $L = L_1L_2$. If $Gr_k(L_1/K,S,\mathfrak{p})$ and $Gr_k(L_2/K,S,\mathfrak{p})$ hold, then so does $Gr_k(L/K,S,\mathfrak{p})$.*

Proof. Let $A \in Ann(L_1L_2/k)$, and set $A_i = A|_{L_i}, i = 1,2$. If $\epsilon_A \in V_A$, then also $\epsilon_A \in V_{A_1}, \epsilon_A \in V_{A_2}$. Let $H = G(L_1L_2/K)$, $H_i = G(L_i/K)$, and $T_i = G(L_1L_2/L_i), i = 1,2$. Because of the inclusion $H \hookrightarrow H_1 \times H_2$, it suffices to check that for $i = 1,2$, we have
$$r_{\mathfrak{B}}^{-1}(\epsilon_A)|_{L_i} = \left(\prod_{\sigma\in H} \sigma^{n(A,\sigma)}\right)|_{L_i}.$$

By functoriality of the local Artin map, assuming $Gr_k(L_i/K,S,\mathfrak{p})$, the left–hand side above equals $\prod_{\sigma_1\in H/T_i} \sigma_1^{n(A_i,\sigma_1)}$ while if R_i is a system of coset representatives for H/T_i, the right–hand side above equals
$$\prod_{\sigma_1\in R_i} (\sigma_1|_{L_i})^{\sum_{\tau\in T_i} n(A,\sigma_1\tau)} = \prod_{\sigma_1\in R_i} (\sigma_1|_{L_i})^{n(A_i,\sigma_1 T_i)},$$
which proves the Lemma. □

It is now convenient to reformulate Conjecture 2 in the following way. Let $I \subset \mathbb{Z}[H]$ be the augmentation ideal,
$$I = \{a_1\sigma_1 + \cdots + a_k\sigma_k \in \mathbb{Z}[H] \mid a_1 + \cdots + a_k = 0\},$$
and consider the homomorphism
$$\text{Exp} : I \longrightarrow H \quad \text{given by}$$
$$a_1\sigma_1 + \cdots + a_k\sigma_k \longmapsto \sigma_1^{a_1}\ldots\sigma_k^{a_k}.$$

Let I_H be the kernel of the restriction homomorphism
$$\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H].$$

If R is a system of coset representatives for H in G, an element $\eta = \sum_{\tau\in R}\gamma_\tau\tau$ of $\mathbb{Z}[G]$ (with $\gamma_\tau \in \mathbb{Z}[H]$) belongs to I_H if and only if $\gamma_\tau \in I$ for all $\tau \in R$. So, an element $\eta \in I_H$

gives rise to a map $R \to I$ given by $\tau \to \gamma_\tau$. Since Exp is constant on H-orbits of I, we obtain a map $\mathrm{Exp} \circ \eta : G/H \to H$, which is independent of the choice of R. So, we obtain a homomorphism

$$\mathrm{Exp}_H : I_H \longrightarrow \mathrm{Fcts}(G/H, H) \quad \text{given by}$$
$$\eta \longmapsto [\mathrm{Exp} \circ \eta : G/H \to H],$$

where $\mathrm{Fcts}(G/H, H)$ is the group of functions $G/H \to H$ under pointwise multiplication.

Write

$$A\theta_L = \sum_{\tau \in R} \left(\sum_{h \in H} n(A, h\tau) h \right) \tau = \sum_{\tau \in R} \gamma_\tau \tau.$$

We had observed earlier that $\gamma_\tau \in I$ for all τ, so $A\theta_L \in I_H$. By definition,

$$\mathrm{Exp}_H(A\theta_L)(\tau) = \prod_{h \in H} h^{n(A,h\tau)} = \prod_{\sigma \in H\tau} \sigma^{n(A,\sigma)}.$$

This allows us to restate Conjecture 2 as follows:

Conjecture 3. *Notation as before, for any $A \in Ann(L/k)$ and any $\epsilon_A \in V_A$, the function $G/H \to H$ given by*

$$\tau \longmapsto r_{\mathfrak{B}^\tau}^{-1}(\epsilon_A)$$

is precisely $Exp_H(A\theta_L)$.

Moreover, by above, it suffices to verify this Conjecture only for the trivial coset $\tau = H$.

Lemma 2. *Suppose $L \subset L_*$ and $K \subset K_*$, the prime $\mathfrak{p} \in S$ splits completely in K_*, and L_*/k is unramified outside S. Then $Gr_k(L_*/K_*, S, \mathfrak{p})$ implies $Gr_k(L/K, S, \mathfrak{p})$.*

Proof. Assume $Gr_k(L_*/K_*, S, \mathfrak{p})$. Fix a prime $\mathfrak{B}$ of K lying over $\mathfrak{p}$ and take $A \in Ann(L/k)$. Choose a prime $\mathfrak{B}_*$ of K_* lying over $\mathfrak{B}$, and let ϵ_* be a Stark unit associated to $(K_*/k, S, \mathfrak{B}_*)$. Set $W = \#\mu(K), W_* = \#\mu(K_*)$. If $N \in \mathbb{Z}[G(K_*/K)]$ denotes the norm, we will use the fact that there is a Stark unit $\epsilon \in K^*$ associated to $(K/k, S, \mathfrak{B})$, such that $\epsilon^{\frac{W_*}{W}} \in \mu(K)\epsilon_*^N$ (see [13] or [3]). Let λ_* satisfy $\lambda_*^{W_*} = \epsilon_*$, take a lifting $N_* \in \mathbb{Z}[G(L_*(\lambda_*)/K)]$ of N, and set $\lambda = \lambda_*^{N_*}$. Then $\lambda^{WW_*} = \lambda_*^{WW_*N_*} = \epsilon^{W_*}$, so $\lambda^W = \zeta\epsilon$, where ζ is a root of unity. Take any $\epsilon_A \in V_A, \epsilon_A = \lambda^{A_0}$, where $A_0 \in Ann(L(\lambda)/k)$ restricts to A on L. Choose $A'_* \in Ann(L_*(\lambda_*)/k)$ which restricts to A_0 on $L(\lambda)$, and set $A_* = A'_*|_{L_*}$. We will be able to obtain the desired conclusion by applying $Gr_k(L_*/K_*, S, \mathfrak{p})$ to $\epsilon_{A_*} = \lambda_*^{A'_*} \in V_{A_*}$ in particular because its norm equals ϵ_A:

$$\epsilon_{A_*}^N = \lambda_*^{N_*A'_*} = \lambda^{A'_*} = \lambda^{A_0} = \epsilon_A.$$

Let $G = G(L/k), H = G(L/K), G_* = G(L_*/k), H_* = G(L_*/K_*)$. The restriction map $res : G_* \to G$ induces $H_* \to H$, $G_*/H_* \to G/H$, and $I_{H_*} \to I_H$. By $Gr_k(L_*/K_*, S, \mathfrak{p})$ applied to $A_* \in Ann(L_*/k)$ and $\epsilon_{A_*} \in V_{A_*}$, we have that

$$\mathrm{Exp}_{H_*}(A_*\theta_{L_*}) = [\tau_* \longmapsto r_{\mathfrak{B}_*^{\tau_*}}^{-1}(\epsilon_{A_*})]. \tag{4.1}$$

So, we now have to relate Exp_{H_*} to Exp_H. Consider the commutative diagram

$$\begin{array}{ccc} I_{H_*} & \xrightarrow{\mathrm{Exp}_{H_*}} & \mathrm{Fcts}(G_*/H_*, H_*) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle m} \\ I_H & \xrightarrow{\mathrm{Exp}_H} & \mathrm{Fcts}(G/H, H) \end{array}$$

where the homomorphism m is defined by

$$[m(f)](x) = \prod_{\substack{y \in G_*/H_* \\ res(y)=x}} res(f(y)).$$

Indeed, it suffices to check commutativity for the elements $\xi_*(\sigma_* - 1) \in I_{H_*}$, where $\xi_* \in G_*, \sigma_* \in H_*$. But in this case, clearly either composition sends $\xi_*(\sigma_* - 1)$ to the function which maps $res(\xi_*)H$ to $res(\sigma_*)$ and all other cosets to 1. Now, applying m to both sides of (4.1) and evaluating at $\tau \in G$ yields

$$\begin{aligned} [\mathrm{Exp}_H(A\theta_L)](\tau) &= [\mathrm{Exp}_H(res(A_*\theta_{L_*}))](\tau) \\ &= \prod_{\substack{\tau_* \in G_*/H_* \\ res(\tau_*)=\tau H}} res(r^{-1}_{\mathfrak{B}_*^{\tau_*}}(\epsilon_{A_*})) \\ &= \prod_{\alpha \in G(K_*/K)} res(r^{-1}_{\mathfrak{B}_*^{\tau\alpha}}(\epsilon_{A_*})) \\ &= res\ r^{-1}_{\mathfrak{B}_*^{\tau}}(\prod_{\alpha \in G(K_*/K)} \epsilon_{A_*}^{\alpha^{-1}}) \\ &= res\ r^{-1}_{\mathfrak{B}_*^{\tau}}(\epsilon_{A_*}^{N}) \\ &= res\ r^{-1}_{\mathfrak{B}_*^{\tau}}(\epsilon_A) \\ &= r^{-1}_{\mathfrak{B}^{\tau}}(\epsilon_A). \end{aligned}$$

□

Lemma 3. *$Gr_k(L/K, S, \mathfrak{p})$ holds if $\mathfrak{p}$ is unramified in L.*

Proof. By the previous Lemma, we can assume that K is the decomposition field of $\mathfrak{p}$ in L/k; in this case, $H = \langle \sigma_{\mathfrak{p}} \rangle$. Fix a prime $\mathfrak{B}$ of K above $\mathfrak{p}$, a Stark unit $\epsilon \in K$, $A \in Ann(L/k)$, take $\lambda^{W_K} = \epsilon$, and $\epsilon_A = \lambda^{A_0} \in V_A$.

The extension L/k is unramified outside $R = S - \{\mathfrak{p}\}$, and so $\tilde{\theta}_L = \theta_{L/k,R}(0)$ is well–defined. We know it is related to $\theta_L = \theta_{L/k,S}(0)$ via

$$\theta_L = (1 - \sigma_{\mathfrak{p}}^{-1})\tilde{\theta_L}.$$

If R is a set of coset representatives of H in G, write

$$A\tilde{\theta_L} = \sum_{\tau \in R} \gamma_\tau \tau, \quad \gamma_\tau \in \mathbb{Z}[H].$$

Then
$$A\theta_L = \sum_{\tau\in R} \left((1-\sigma_{\mathfrak{p}}^{-1})\gamma_\tau\right)\tau.$$

If $n(\tau)\in\mathbb{Z}$ is the sum of the coefficients in γ_τ, since Exp is constant on H-orbits, we obtain
$$[\mathrm{Exp}_H(A\theta_L)](\tau) = \sigma_{\mathfrak{p}}^{n(\tau)}.$$

On the other hand, since L/K is unramified at $\mathfrak{B}^\tau$, we know that
$$r_{\mathfrak{B}^\tau}^{-1}(\epsilon_A) = \sigma_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{B}^\tau}(\epsilon_A)}.$$

Since $\theta_{K/k,R} = \tilde{\theta_L}|_K$, we have that
$$(A|_K)\theta_{K/k,R}(0) = (A\tilde{\theta_L})|_K \sum_{\tau\in R} n(\tau)\tau|_K.$$

By definition,
$$(\epsilon_A)^{W_K} = (\lambda^{A_0 W_K}) = (\epsilon)^{A_0} = \mathfrak{B}^{W_K\theta_{K/k,R}(0)A_0|_K} = \prod_{\tau\in R}(\mathfrak{B}^\tau)^{W_K n(\tau)},$$

hence $\mathrm{ord}_{\mathfrak{B}^\tau}(\epsilon_A) = n(\tau)$, as desired. □

4.5 Gross's formulation

Gross has reformulated Conjecture 1 in a manner that will be more convenient in Chapter 6, so we now discuss Conjecture 2 from this point of view in its original formulation. This formulation of Conjecture 2 is slightly weaker than the one by Hayes that we discussed earlier.

As before, let L/k be an abelian extension and let S be a finite set of primes of k containing the archimedean primes as well as all ramified ones. Fix a finite prime $\mathfrak{p}$ in S, and let K be a subfield of L/k in which $\mathfrak{p}$ splits completely. Assume $|S|\geq 3$.

Let T be a finite set of primes of k disjoint from S, which either contains at least two primes of different residue characteristic, or a prime η of absolute ramification degree at most $l-2$, where l is the prime of $\mathbb{Q}$ below η. Under this assumption,
$$\prod_{\eta\in T}(\sigma_\eta - N\eta) \in Ann(L/k).$$

Indeed, if $\zeta\in\mu(L)$ and $e(\eta|\mathbb{Q})\leq l-2$, the root of unity $\zeta^{\sigma_\eta - N\eta}\equiv 1 \pmod{\eta}$ has to be 1 because $(1+\mathfrak{p}\mathcal{O}_{\mathfrak{p}})^*\simeq \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ (via the $\log_p$ map) is torsion–free in this case. If T contains primes η_1, η_2 of different residue characteristics l_1, l_2, write $e = \#\mu(L) = l_1^a l_2^b c$, with $(l_1 l_2, c) = 1$. Let $\alpha = \zeta^{(\sigma_{\eta_1}-N\eta_1)(\sigma_{\eta_2}-N\eta_2)}\in\mu(L)$, where $\zeta\in\mu(L)$. Then as in the proof of Proposition 1, we deduce that $\alpha^{l_1^a}=1, \alpha^{l_2^b}=1$, hence $\alpha=1$.

Consider the shifts $\zeta_{R,T}(\sigma,s)$ defined as follows in terms of the group ring of $G(K/k)$:
$$\sum_{\sigma\in G(K/k)}\zeta_{R,T}(\sigma,s)[\sigma] = \prod_{\eta\in T}(1-[\sigma_\eta]N\eta)\sum_{\sigma\in G(K/k)}\zeta_R(\sigma,s)[\sigma].$$

Explicitly, if we define the constants c_γ via

$$\prod_{\eta\in T}(1-[\sigma_\eta]N\eta) = \sum_{\gamma\in G(K/k)} c_\gamma[\gamma],$$

then

$$\zeta_{R,T}(\sigma,s) = \sum_\gamma c_\gamma \zeta_R(\gamma^{-1}\sigma,s).$$

Notice that if $\prod_{\eta\in T}(\sigma_\eta - N\eta)\sum\zeta_R(\sigma,s)\sigma^{-1} = \sum\overline{n}(\sigma,s)\sigma \in \mathbb{Z}[G(K/k)]$, then $\zeta_{R,T}(\sigma,0) = \overline{n}(\sigma_\eta\sigma^{-1},0)\in\mathbb{Z}$.

Recall the notation for the group of $\mathfrak{p}$-units of K,

$$U_\mathfrak{p} = \{x\in K^* \mid |x|_\mathfrak{B} = 1 \quad \text{for any (finite or infinite) prime} \quad \mathfrak{B}\nmid\mathfrak{p}\}.$$

Let

$$r_\mathfrak{p} : k_\mathfrak{p}^* \longrightarrow G(L/K)$$

be the reciprocity map of local class field theory. The ideal $\mathfrak{B}$ of K defines an embedding $K \hookrightarrow K_\mathfrak{B} \simeq k_\mathfrak{p}$, which allows us to evaluate $r_\mathfrak{p}$ on elements of K. The folllowing is Gross's original formulation of Conjecture 2:

Conjecture 4. *There exists an element $u_T\in U_\mathfrak{p}$ such that $u_T\equiv 1 \pmod T$ and*

$$ord_\mathfrak{B}(u_T^\sigma) = \zeta_{R,T}(K/k,\sigma,0).$$

Moreover, for each $\sigma\in G(K/k)$,

$$r_\mathfrak{p}(u_T^\sigma) = \prod_{\substack{\tau\in G(L/k)\\ \tau|_K=\sigma}} \tau^{\zeta_{S,T}(L/k,\tau,0)}.$$

Since u_T is specified up to a root of unity congruent to 1 modulo T, the condition on T implies u_T is unique, if it exists.

Chapter 5

The proof of Gross's conjecture over the rational field

Here we follow [8] to present a proof of Gross's conjecture over $\mathbb{Q}$. The main ingredient is the theorem by Gross–Koblitz ([7]), which in turn relies on a deep result of Katz. The minor mistake in [8] is easily fixed by constructing the appropriate Stark unit as in [5] and minor modifications of the argument.

5.1 The p-adic Gamma function

Let p be an odd prime. The p-adic Gamma function is defined as the unique continuous function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^*$ such that

$$\Gamma_p(k) = (-1)^k \prod_{\substack{1 \le j < k \\ p \nmid j}} j$$

for all positive integers k. From the definition,

$$\Gamma_p(z+1) = \begin{cases} -z\Gamma_p(z) & \text{if} \quad z \in \mathbb{Z}_p^* \\ -\Gamma_p(z) & \text{if} \quad z \in p\mathbb{Z}_p. \end{cases}$$

For $z \in \mathbb{Z}_p$, we denote by $\widehat{z}$ the unique integer such that $0 < \widehat{z} \le p$, $z \equiv \widehat{z} \pmod p$. For any $z \in \mathbb{Z}_p$, we have

$$\Gamma_p(z)\Gamma_p(1-z) = (-1)^{\widehat{z}}. \tag{5.1}$$

Fix a positive integer $m > 1$ with $p \nmid m$. Let f be the order of p in $(\mathbb{Z}/m\mathbb{Z})^*$, and let $q = p^f$. Consider the cyclotomic field $K = \mathbb{Q}(\mu_m)$, and fix a prime $\mathfrak{p}$ of K lying over p. The m-th roots of unity in $k = \mathcal{O}_K/\mathfrak{p}$ are distinct because $p \nmid m$, and so we can define a homomorphism t from the m-torsion subgroup of k^* to $\mu_m \subset \mathcal{O}_K$, which is inverse to reduction $\pmod{\mathfrak{p}}$. Fix a nontrivial p-th root of unity $\zeta = \zeta_p$, and let $L = K(\mu_p)$. For any $a = \frac{r}{m} \in (1/m)\mathbb{Z}/\mathbb{Z} - \{0\}$ (take $0 < r < m$), define the Gauss sum

$$g(a, \mathfrak{p}) = -\sum_{x \in k^*} t(x^{-a(q-1)})\zeta^{Tr(x)} \in L,$$

where Tr is the trace map $k \to \mathbb{Z}/p\mathbb{Z}$.

Note that if K' is the decomposition field of p in K, then $g(a, \mathfrak{p})$ belongs to $K'(\mu_p)$. This follows from Galois theory: if we take $b > 0, b \equiv p^i \pmod{m}, b \equiv 1 \pmod{p}$, then

$$g(a, \mathfrak{p})^{\sigma_b} = -\sum_{x \in k^*} t((x^{p^i})^{-a(q-1)})\zeta^{Tr(x)} = -\sum_{x \in k^*} t((x^p)^{-a(q-1)})\zeta^{Tr(x^p)} = g(a, \mathfrak{p}).$$

We will see below that $g(a, \mathfrak{p}) \neq 0$. We will use that if

$$(p^f - 1)\frac{r}{m} = z_f + z_1 p + \cdots + z_{f-1}p^{f-1}, \quad 0 \leq z_i \leq p-1 \tag{5.2}$$

is the p-adic expansion of $(p^f - 1)\frac{r}{m}$, then the following congruence holds (cf [11]):

$$\frac{g(a, \mathfrak{p})}{(\zeta - 1)^{\sum_{j=1}^f z_j}} \equiv \frac{1}{\prod_{j=1}^f z_j!} \pmod{(1 - \zeta)} \tag{5.3}$$

(this is due to Stickelberger). Gross explains that this is the motivation for what follows: (5.3) gives not only the valuation of $g(a, \mathfrak{p})$, but also the first digit in its $\mathfrak{p}$-adic expansion. One may ask for the entire $\mathfrak{p}$-adic expansion, and since the first digit involves factorials, one may expect the p-adic Gamma function to play a role.

Note that $\mathbb{Q}_p(\mu_p)$ contains a unique solution π of $x^{p-1} = -p$ with

$$\pi \equiv (\zeta - 1) \pmod{(\zeta_p - 1)^2}.$$

Indeed,

$$\begin{aligned} p &= (1 - \zeta_p)(1 - \zeta_p^2) \dots (1 - \zeta_p^{p-1}) \\ &= (1 - \zeta_p)^{p-1}(1 + \zeta_p)(1 + \zeta_p + \zeta_p^2) \dots (1 + \zeta_p + \cdots + \zeta_p^{p-2}), \end{aligned}$$

and so if $u = (1+\zeta_p)(1+\zeta_p+\zeta_p^2)\dots(1+\zeta_p+\cdots+\zeta_p^{p-2})$, we have to check that the equation $x^{p-1} = -u$ has a unique solution in the valuation ring $\mathcal{O}'$ of $\mathbb{Q}_p(\mu_p)$ with $x \equiv 1 \pmod{\zeta_p - 1}$. Observe that $u \equiv 2.3 \dots (p-1) \equiv -1 \pmod{\zeta_p - 1}$, and $x^{p-1} = 1$ has $x = 1$ as a solution of multiplicity one in the residue field $\mathcal{O}'/(\zeta_p - 1)$, hence the above claim follows from Hensel's lemma.

Let $\langle a \rangle$ denote the fractional part of a rational number a.

The goal of this section is to express the image of $g(a, \mathfrak{p})$ in $\mathbb{Q}_p(\mu_p) = \mathbb{Q}_p(\pi)$ as a special value of the p-adic Gamma function:

Theorem 1. *Let $a \in (1/m)\mathbb{Z}/\mathbb{Z} - \{0\}$. Then*

$$g(a, \mathfrak{p}) = \pi^{(p-1)\sum_{j=0}^{f-1}\langle p^j a \rangle} \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle).$$

To accomplish this, we need to introduce the free abelian group $A = \bigoplus_{a \in (1/m)\mathbb{Z}/\mathbb{Z} - \{0\}} \mathbb{Z}\delta_a$; for each $\mathfrak{a} = \sum m(a)\delta_a \in A$, define

$$\begin{aligned} \Gamma_p(\mathfrak{a}) &= \prod \Gamma_p(\langle a \rangle)^{m(a)}, \\ g(\mathfrak{a}, \mathfrak{p}) &= \prod g(a, \mathfrak{p})^{m(a)}, \\ n(\mathfrak{a}) &= \sum m(a)\langle a \rangle, \quad \text{and} \qquad n(\mathfrak{a}^{(p^j)}) = \sum m(a)\langle p^j a \rangle. \end{aligned}$$

If $\mathfrak{a} \in A$ and $n(\mathfrak{a}) \in \mathbb{Z}$, we note that $g(\mathfrak{a}, \mathfrak{p}) \in K$ by Galois theory: if $b \equiv 1 \pmod m$ and $p \nmid b$, the automorphism of L corresponding to $b \in (\mathbb{Z}/mp\mathbb{Z})^*$ acts on $g(a, \mathfrak{p})$ by multiplication by $t(b^{a(q-1)})$, hence acts trivially on $g(\mathfrak{a}, \mathfrak{p})$. So, $g(\mathfrak{a}, \mathfrak{p}) \in K'$. In particular, if $n(\mathfrak{a}) \in \mathbb{Z}$, the image of $g(\mathfrak{a}, \mathfrak{p})$ in $\mathbb{Q}_p(\mu_p)$ lies in $\mathbb{Q}_p$. Using the congruence satisfied by $g(a, \mathfrak{p})$ cited earlier, the above theorem will follow from the following

Proposition 2. *(Gross–Koblitz) Let $\mathfrak{a} \in A$ with $n(\mathfrak{a}) \in \mathbb{Z}$. Then*

$$g(\mathfrak{a}, \mathfrak{p}) = (-p)^{\sum_{j=0}^{f-1} n(\mathfrak{a}^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(\mathfrak{a}^{(p^j)}).$$

Indeed, assume the Proposition holds, and fix some $a \in (1/m)\mathbb{Z}/\mathbb{Z} - \{0\}$. Then $\mathfrak{a} = (q-1)\delta_a \in A$ has $n(\mathfrak{a}) \in \mathbb{Z}$, and so

$$\begin{aligned} g(a, \mathfrak{p})^{q-1} = g(\mathfrak{a}, \mathfrak{p}) &= (-p)^{\sum_{j=0}^{f-1} n(\mathfrak{a}^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(\mathfrak{a}^{(p^j)}) \\ &= \left(\pi^{(p-1)\sum_{j=0}^{f-1} \langle p^j a \rangle} \prod_{j=0}^{f-1} \Gamma_p(\langle p^j a \rangle) \right)^{q-1}. \end{aligned}$$

So, we need to prove that $g(a, \mathfrak{p})$ and $\pi^{(p-1)\sum \langle p^j a \rangle} \prod \Gamma_p(\langle p^j a \rangle)$ are congruent $\pmod{\zeta_p - 1}$; since they differ by a $(q-1)$-st root of unity in $\mathbb{Q}_p(\mu_p)$ and hence by a $(p-1)$-st root of unity, the conclusion will follow because the $(p-1)$-st roots of unity are distinct modulo $(\zeta_p - 1)$.

Since $(p) = (\zeta_p - 1)^{p-1}$ and $\zeta_p - 1 = \pi v$, where $v \equiv 1 \pmod{\zeta_p - 1}$, the congruence (5.3) wit $r = 1$ implies that

$$u = g(a, \mathfrak{p}) \pi^{-(p-1)\sum \langle p^j a \rangle} = g(a, \mathfrak{p}) \pi^{-\sum z_i} \equiv \frac{1}{\prod z_i!} \pmod{\zeta_p - 1}$$

(we used the elementary fact that $(p-1)\sum_{j=0}^{f-1} \langle p^j a \rangle = \sum_{j=1}^{f} z_j$). But $u \in \mathbb{Z}_p^*$ because it a product of a value of Γ_p and a $(p-1)$-st root of unity (which is in $\mathbb{Q}_p$), hence

$$u \equiv \frac{1}{\prod z_i!} \pmod p.$$

On the other hand, one can check that $\widehat{\langle p^j a \rangle} = p - z_{f-j}$, and so

$$\Gamma_p(\langle p^j a \rangle) \equiv (p - z_{f-j} - 1)!(-1)^{p - z_{f-j}} \equiv (z_{f-j}!)^{-1} \pmod p,$$

where we used that $a \equiv b \pmod p$ implies $\Gamma_p(a) \equiv \Gamma_p(b) \pmod p$, as well as Wilson's theorem. This gives the desired congruence.

To prove Proposition 2, note that if $\mathfrak{a} \in A$ and $n(\mathfrak{a}) \in \mathbb{Z}$, then $\mathfrak{a}$ is a $\mathbb{Z}$–linear combination of

$$\mathfrak{a}_0 = \delta_{\frac{1}{m}} + \delta_{\frac{m-1}{m}}$$

and

$$\mathfrak{a}_r = \delta_{\frac{r}{m}} + \delta_{\frac{1}{m}} - \delta_{\frac{r+1}{m}}, \quad \text{for} \quad 0 < r < m-1,$$

so it suffices to prove the Proposition only for $\mathfrak{a}_0$ and $\mathfrak{a}_r$.

By standard changes of variables in manipulating Gauss sums, we find

$$g(\mathfrak{a}_0, \mathfrak{p}) = q(-1)^{\frac{q-1}{m}}.$$

So,

$$\begin{aligned}
(-p)^{\sum n(\mathfrak{a}_0^{(p^j)})} \prod_{j=0}^{f-1} \Gamma_p(\mathfrak{a}_0^{(p^j)}) &= q(-1)^f \prod_{j=0}^{f-1} \Gamma_p(\langle \frac{p^j}{m} \rangle)\Gamma_p(1 - \langle \frac{p^j}{m} \rangle) \\
&= q(-1)^{f+\sum_{j=0}^{f-1} \widehat{\langle \frac{p^j}{m} \rangle}} \\
&= q(-1)^{f+\sum_{j=0}^{f-1}(p-z_{f-j})} \\
&= q(-1)^{\sum_{j=1}^{f} z_j} \\
&= g(\mathfrak{a}_0, \mathfrak{p})
\end{aligned}$$

because $\sum z_j \equiv \frac{q-1}{m} \pmod{p-1}$; here $\frac{q-1}{m} = z_f + z_1 p + \cdots + z_{f-1}p^{f-1}$.

Now consider $\mathfrak{a}_r$ for a fixed $0 < r < m-1$. By cross–multiplying, after a change of variables, we see that

$$g(\mathfrak{a}_r, \mathfrak{p}) = -\sum_{x \in k - \{0,1\}} t(x^{-\frac{r(q-1)}{m}}) t((1-x)^{-\frac{q-1}{m}}).$$

We will use the following deep result of Katz (cf. [6]): denote $t = m - r - 1 > 0$; then Katz proved that

$$g(\mathfrak{a}_r, \mathfrak{p}) = \prod_{j=0}^{f-1} \lim_{k \to -\langle \frac{p^j r}{m} \rangle} \frac{\binom{\langle \frac{p^{j+1}t}{m} \rangle - 1}{pk + (p\langle \frac{p^j r}{m} \rangle - \langle \frac{p^{j+1} r}{m} \rangle)}}{\binom{\langle \frac{p^j t}{m} \rangle - 1}{k}}.$$

The rest of the proof is by approximating the binomial coefficients and rearranging the factorials involved so that one can recognize special values of Γ_p. We now finish the proof assuming that $f = 1$; the general case is similar but notationally heavier.

If h, k are positive integers such that

$$\begin{aligned}
h &\equiv -\frac{t}{m} \pmod{p^a} \\
k &\equiv -\frac{r}{m} \pmod{p^a}
\end{aligned}$$

where a is large, then

$$\begin{aligned}
\langle \frac{pt}{m} \rangle &\equiv -ph - \lfloor \frac{pt}{m} \rfloor \pmod{p^{a+1}} \quad \text{and} \\
\langle \frac{pr}{m} \rangle &\equiv -pk - \lfloor \frac{pr}{m} \rfloor \pmod{p^{a+1}}.
\end{aligned}$$

Using $\binom{-n}{k} = (-1)^k \frac{(n+k-1)!}{k!(n-1)!}$, it follows that $g(\mathfrak{a}_r, \mathfrak{p})$ is the limit of

$$(-1)^{\lfloor \frac{pr}{m} \rfloor} \frac{(ph + pk + \lfloor \frac{pt}{m} \rfloor + \lfloor \frac{pr}{m} \rfloor)!}{p^{h+k}(h+k)!} \frac{p^k k!}{(pk + \lfloor \frac{pr}{m} \rfloor)!} \frac{p^h h!}{(ph + \lfloor \frac{pt}{m} \rfloor)!},$$

as a goes to infinity, hence, by the definition of Γ_p, we have

$$\begin{aligned} g(\mathfrak{a}, \mathfrak{p}) &= (-1)^{\lfloor \frac{pr}{m} \rfloor + 1} \frac{\Gamma_p(1 - \langle \frac{pr}{m} \rangle - \langle \frac{pt}{m} \rangle)}{\Gamma_p(1 - \langle \frac{pr}{m} \rangle)\Gamma_p(1 - \langle \frac{pt}{m} \rangle)} \\ &= (-1)^{\lfloor \frac{pr}{m} \rfloor + 1} \frac{\Gamma_p(1 - \frac{r}{m} - (1 - \frac{r+1}{m}))}{\Gamma_p(1 - \frac{r}{m})\Gamma_p(\frac{r+1}{m})} \\ &= \frac{\Gamma_p(\frac{r}{m})\Gamma_p(\frac{1}{m})}{\Gamma_p(\frac{r+1}{m})}, \end{aligned}$$

where we used that $p \equiv 1 \pmod m$ and (5.1).

5.2 Setup for the proof over $\mathbb{Q}$

Consider an abelian extension $L/\mathbb{Q}$, a subfield K, a prime p which splits completely in $K \subset L$, and a finite set S of primes of $\mathbb{Q}$ containing ∞, p, and all the primes which ramify in L. Let the finite part of the conductor of $L/\mathbb{Q}$ be $p^\nu n$, and let $m = n \prod_{v \in S - Supp(pn)} v$. We know that L is contained in $\mathbb{Q}(\mu_{p^\nu m})$ and $K \subset \mathbb{Q}(\mu_m)$. By the functorial lemmas, without loss of generality $L = \mathbb{Q}(\mu_{p^\nu m})$ and K is the decomposition field of p in $\mathbb{Q}(\mu_m)$. Also, since L is the compositum of $K(\mu_m)$ and $K(\mu_{p^\nu})$, and $K(\mu_m)/K$ is unramified over p, again by the functoriality lemmas, it suffices to assume that $L = K(\mu_{p^\nu})$. However, we cannot assume that p splits completely in $\mathbb{Q}(\mu_m)$ (as is assumed in [8]).

So, from now, fix a prime p, an integer $m \geq 1$, let $S = \{p, \infty\} \cup Supp(m)$, let K be the decomposition field of p in $\mathbb{Q}(\mu_m)$, and let $L = K(\mu_{p^\nu})$, where $\nu \geq 1$. We have to prove that $Gr_{\mathbb{Q}}(L/K, S, p)$ is true. Let f be the order of p in $(\mathbb{Z}/m\mathbb{Z})^*$.

Fix a system R of representatives for $(\mathbb{Z}/m\mathbb{Z})^*/\langle p \rangle$ that are positive and prime to p, and a system X_ν of representatives x for $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ which satisfy $x > 0$ and $x \equiv 1 \pmod m$. Then we know that

$$H = G(L/K) = \{\sigma_x \mid x \in X_\nu\} \quad \text{and} \quad G = G(L/\mathbb{Q}) = \{\sigma_{bx} \mid b \in R, x \in X_\nu\}.$$

Given $b \in R, x \in X_\nu, 0 \leq j < f$, let $t(p^j b, bx)$ be an integer congruent to $p^j b$ modulo m, and to $bx \pmod{p^\nu}$. Then, it will follow from Chapter 8 in particular (and it is also well–known) that $\zeta_{L/\mathbb{Q},S}(\sigma_{bx}, 0) = \sum_{j=0}^{f-1} \left(\frac{1}{2} - \langle \frac{t(p^j b, bx))}{mp^\nu} \rangle \right)$, so

$$\theta_\nu = \theta_{L/\mathbb{Q},S} = \sum_{b \in R} \sum_{x \in X_\nu} \left(\sum_{j=0}^{f-1} \left(\frac{1}{2} - \langle \frac{t(p^j b, bx)}{mp^\nu} \rangle \right) \right) \sigma_{bx}^{-1}.$$

If $a > 0$ is odd and prime to pm and $A = \sigma_a - a \in Ann(L/\mathbb{Q})$, we find

$$A\theta_\nu = \sum_{b \in R} \sum_{x \in X_\nu} \left(\sum_{j=0}^{f-1} \left(\frac{1-a}{2} + \lfloor \frac{t(p^j ab, abx)}{mp^\nu} \rfloor - a \lfloor \frac{t(p^j b, bx)}{mp^\nu} \rfloor \right) \right) \sigma_{bx}^{-1}. \tag{5.4}$$

Therefore,

$$\begin{aligned}\mathrm{Exp}_H(A\theta_\nu)(\sigma_b^{-1}) &= \prod_{x\in X_\nu} \sigma_x^{-\sum_{j=0}^{f-1}\left(\frac{1-a}{2}+\lfloor\frac{t(p^j ab,abx)}{mp^\nu}\rfloor - a\lfloor\frac{t(p^j b,bx)}{mp^\nu}\rfloor\right)} \\ &= r_p^{-1}\left(\prod_{x\in X_\nu} x^{\sum_{j=0}^{f-1}\left(\frac{1-a}{2}+\lfloor\frac{t(p^j ab,abx)}{mp^\nu}\rfloor - a\lfloor\frac{t(p^j b,bx)}{mp^\nu}\rfloor\right)}\right), \qquad (5.5)\end{aligned}$$

where

$$r_p : \mathbb{Q}_p^* \longrightarrow G$$

is the reciprocity map of local class field theory; we used that $\sigma_x = r_p(x)$ for $x > 0, x \equiv 1 \pmod m$, $p \nmid x$, which follows from the definition of r_p.

5.3 The case $m = 1$

Assume that $m = 1$, so $K = \mathbb{Q}$ and $H = G$. We will assume that $p > 2$, as the case $p = 2$ requires only small modifications.

It is easy to compute that the Stark unit $\epsilon \in \mathbb{Q}^*$ is $\epsilon = \pm\frac{1}{p}$ because $S = \{p, \infty\}$ and $\zeta_\infty(0) = -\frac{1}{2}$, so $\#\mu(\mathbb{Q})\zeta_\infty(0) = -1$. Let $\tau = \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^*}\left(\frac{b}{p}\right)\zeta^b \in \mathbb{Q}(\mu_p)$, with fixed $\zeta = \zeta_p \in \mathbb{Q}(\mu_p)$, so we can take $\lambda = \tau^{-1}$ because we know that $\tau^2 = (\frac{-1}{p})p$. Then $\lambda \in \mathbb{Q}(\mu_p)$, the set V_A is singleton, and so it suffices to prove Conjecture 3 for $A \in Ann(L/\mathbb{Q})$ of the form $A = \sigma_a - a$. We compute that

$$\epsilon_A = \lambda^{\sigma_a - a} = \frac{\tau^a}{\tau^{\sigma_a}} = \frac{\tau^a}{(\frac{a}{p})\tau} = (\frac{a}{p})(\frac{-1}{p})^{\frac{a-1}{2}}p^{\frac{a-1}{2}}.$$

Notice that $p \in \mathbb{Q}_p^*$ is a local norm from $L_{(1-\zeta_{p^\nu})}$ because $p = \prod_{j\in(\mathbb{Z}/p^\nu\mathbb{Z})^*}(1-\zeta_{p^\nu}^j)$. So, by local class field theory, if $j = \sigma_{-1} \in G$ denotes complex conjugation, we have that

$$r_p(\epsilon_A) = \begin{cases} 1, & \text{if } \ (\frac{a}{p})(\frac{-1}{p})^{\frac{a-1}{2}} = 1 \\ j, & \text{if } \ (\frac{a}{p})(\frac{-1}{p})^{\frac{a-1}{2}} = -1. \end{cases}$$

On the other hand, we can write $\mathrm{Exp}_H(A\theta_\nu)(1)$ more conveniently as follows. If

$$X = \{x \in \mathbb{Z} \mid 0 < x \le \frac{p^\nu - 1}{2}\},$$

then $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ can be realized as the disjoint union of X and $-X$, and thus

$$\theta_\nu = \sum_{x\in(\mathbb{Z}/p^\nu\mathbb{Z})^*}(\frac{1}{2} - \langle\frac{x}{p^\nu}\rangle)\sigma_x^{-1} = (1-j)\sum_{x\in X}(\frac{1}{2} - \langle\frac{x}{p^\nu}\rangle)\sigma_x^{-1}.$$

Hence,

$$A\theta_\nu = (1-j)\sum_{x\in X}\left(\frac{1-a}{2} + \lfloor\frac{ax}{p^\nu}\rfloor\right)\sigma_x^{-1}.$$

It follows that $\mathrm{Exp}_H(A\theta_\nu)(1)$ is a power of j. Since so is $r_p^{-1}(\epsilon_A)$, it suffices to prove that their restrictions to $\mathbb{Q}(\mu_p)$ are equal. In other words, we can assume without loss of generality that $\nu = 1$.

In this case,

$$\mathrm{Exp}_H(A\theta_\nu)(1) = j^{\sum_{x\in X}(\frac{a-1}{2}-\lfloor\frac{ax}{p}\rfloor)} = j^{\frac{a-1}{2}\frac{p-1}{2}-\sum_{x\in X}\lfloor\frac{ax}{p}\rfloor},$$

which completes the proof in the case $m = 1$ because of the known identity (cited in [8])

$$(-1)^{\sum_{x\in X}\lfloor\frac{ax}{p}\rfloor} = (\frac{a}{p}).$$

5.4 The Gross–Stark unit in the case $m > 1$

Now we assume $m > 1$ and find the Gross–Stark unit explicitly, following [5] (modifications from [8] are necessary). Fix a prime $\mathfrak{B}$ of K lying over p and $\zeta = \zeta_p \in \mathbb{Q}(\mu_p)$. Let $K' = \mathbb{Q}(\mu_m)$, and let $\mathfrak{B}'$ be the unique prime of K' over $\mathfrak{B}$. Consider the residue field $k = \mathcal{O}_{K'}/\mathfrak{B}'$, and let $q = |k| = p^f$. The m-th roots of unity are distinct in k because $p \nmid m$. Let t be the homomorphism from the m-torsion subgroup of k to $\mu_m \subset K'$, which is inverse to reduction (mod $\mathfrak{B}'$). For an odd positive integer a prime to $S = \{p, \infty\} \cup Supp(m)$, define

$$G(a) = -g(-\frac{a}{m}, \mathfrak{B}') = \sum_{x\in k^*} t(x^{\frac{a(q-1)}{m}})\zeta^{Tr(x)} \in \mathbb{Q}(\mu_m, \mu_p).$$

We saw earlier than $G(a) \in K(\mu_p)$. As before, let $g = \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^*}(\frac{b}{p})\zeta^b \in \mathbb{Q}(\mu_p)$ be the usual Gauss sum.

If $a > 0$ is odd and prime to pm, then

$$G(1)^{\sigma_a} = t(a)^{-\frac{a(q-1)}{m}} G(a).$$

So, if we let

$$\lambda = \frac{G(1)}{g^f} \in K(\mu_p)$$

and

$$\epsilon = \lambda^W,$$

where $W = \#\mu(K)$, we have that $\epsilon \in K$ by Galois theory. Indeed, if $a > 0, a \equiv p^j \pmod{m}$, for some j, and $a \equiv 1 \pmod{p}$ then $\epsilon^{\sigma_a} = \epsilon$ follows from the fact that $G(a) = G(1)$ (since $x \mapsto x^p$ is an automorphism of k) and $t(a) \in K$ (again by Galois theory, because $t(a)^p = t(a)$).

By the result of Gross–Koblitz, if π is a uniformizer in $\mathbb{Q}_p(\mu_p)$ such that $\pi^{p-1} = -p$ and $\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}$, then the image of $G(a) = -g(-\frac{a}{m}, \mathfrak{B}')$ in $\mathbb{Q}_p(\mu_p)$ equals

$$G(a) = -\pi^{(p-1)\sum_{j=0}^{f-1}(1-\langle\frac{p^j a}{m}\rangle)} \prod_{j=0}^{f-1} \Gamma_p(1 - \langle\frac{p^j a}{m}\rangle).$$

Comparing valuations, we see that

$$\nu_{\mathfrak{B}}(\epsilon^{\sigma_a}) = W\sum_{j=0}^{f-1}(\frac{1}{2} - \langle\frac{p^j a}{m}\rangle),$$

so ϵ is a Stark unit associated to $(K/\mathbb{Q}, S, \mathfrak{B})$. Since $\lambda \in K(\mu_p) \subset L$, it suffices to prove the conjecture for $A \in Ann(L/\mathbb{Q})$ of the form $A = \sigma_a - a$.

Also, we easily compute that for $A = \sigma_a - a$, the image of the Gross–Stark unit $\epsilon_A = \lambda^{\sigma_a - a}$ in $\mathbb{Q}_p$ under the embedding $K \hookrightarrow K_{\mathfrak{B}} \simeq \mathbb{Q}_p$ has p-adic unit part (with respect to the uniformizer p) equal to

$$\tau_p(\epsilon_A) = t(a^{-\frac{a(q-1)}{m}})(\frac{a}{p})^f(\frac{-1}{p})^{\frac{f(a-1)}{2}}(-1)^{\sum_{j=0}^{f-1}\lfloor\frac{p^j a}{m}\rfloor - a\lfloor\frac{p^j}{m}\rfloor}\frac{\prod_{j=0}^{f-1}\Gamma_p(1-\langle\frac{p^j a}{m}\rangle)}{\prod_{j=0}^{f-1}\Gamma_p(1-\langle\frac{p^j}{m}\rangle)^a}.$$

To find the image of ϵ_A under the local reciprocity map, we need to compute the above p-adic unit part up to a certain p-adic accuracy (in this case, modulo p^ν). But, we can use the functional equation of Γ_p to relate the values $\Gamma_p(1 - \langle\frac{p^j a}{m}\rangle)$ to a value of Γ_p that we understand better under approximations. Concretely, if for $z \in \mathbb{Z}_p$, we denote

$$\{z\} = \begin{cases} z & \text{if} \quad z \in \mathbb{Z}_p^* \\ -1 & \text{if} \quad z \in p\mathbb{Z}_p \end{cases}$$

then the functional equation for Γ_p implies that

$$\Gamma_p(1 - z - e) = \prod_{r=e}^{n}\{z + r\}\Gamma_p(-z - n)$$

for integers $e \leq n$. Let $z = \frac{p^j a}{m}$, $e = -\lfloor\frac{p^j a}{m}\rfloor$, and $n = \frac{p^j a(q^l - 1)}{m}$, where $l \geq 1$ is an integer, to obtain

$$\Gamma_p(1 - \langle\frac{p^j a}{m}\rangle) = \prod_{-\frac{p^j a}{m} < r \leq p^j a\frac{q^l-1}{m}} \left\{\frac{p^j a}{m} + r\right\}\Gamma_p(-\frac{p^j a q^l}{m}).$$

For $0 \leq j \leq f - 1$, denote

$$P_{l,j}(a) = \prod_{\substack{p\nmid rm + ap^j \\ -\frac{p^j a}{m} < r \leq ap^j\frac{q^l-1}{m}}} \left(p^j + \frac{rm}{a}\right).$$

Next, we simply count the number of integers r with $-\frac{p^j a}{m} < r \leq ap^j\frac{q^l-1}{m}$ such that $p \mid ap^j + rm$; this will allow us to write $\Gamma_p(1 - \langle\frac{p^j a}{m}\rangle)$ as a product involving -1 to an appropriate power, $\frac{a}{m}$ to an appropriate power, $P_{l,j}$, and the value $\Gamma_p(-\frac{p^j a q^l}{m})$ (we treat the cases $j = 0$ and $j > 0$ separately). Then we write the expression for $\tau_p(\epsilon_A)$ and after some cancellations (in particular, the exponent of m turns out to be zero), we let $l \to \infty$. We recall that $\Gamma_p(0) = 1$, so the leftover Γ_p-terms go to 1. Also, we note that $(a^{p^{f-1+f(l-1)}})^{\frac{a(q-1)}{m}}$

goes to $\chi_p(a)^{\frac{a(q-1)}{m}}$ as $l \to \infty$, where χ_p is the Teichmuller character. This cancels the term $t(a^{-\frac{a(q-1)}{m}})$ and we conclude that

$$\tau_p(\epsilon_A) = (\frac{p}{a})^f \lim_{l\to\infty} \left(\frac{\prod_{j=0}^{f-1} P_{l,j}(a)}{\prod_{j=0}^{f-1} P_{l,j}(1)^a} \right).$$

5.5 Comparing the actual value of $r_p(\epsilon_A)$ with the predicted one

We now finish the proof of $Gr_{\mathbb{Q}}(L_\nu/K_m, S, p)$. The p-local conductor of the extension $L/\mathbb{Q}$ is p^ν, and so r_p is trivial on $(1+p^\nu\mathbb{Z}_p)^*$ by class field theory. Also, $r_p(p) = 1$ because p is a local norm, as we remarked earlier. So, it suffices to prove that for l large enough, we have

$$(\frac{p}{a})^f \frac{\prod_{j=0}^{f-1} P_{l,j}(a)}{\prod_{j=0}^{f-1} P_{l,j}(1)^a} \equiv \prod_{x\in X_\nu} x^{\sum_{j=0}^{f-1}\left(\frac{1-a}{2} + \lfloor \frac{t(p^j a, ax)}{mp^\nu}\rfloor - a\lfloor \frac{t(p^j, x)}{mp^\nu}\rfloor\right)} \pmod{p^\nu}.$$

Let $X_{\nu,j}$ be a system of representatives x modulo $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ which satisfy $x > 0$ and $x \equiv p^j \pmod{m}$. We examine the product $P_{l,j}(a)$ more closely and count that the number of terms $(p^j + \frac{rm}{a})$ in it congruent to some fixed $x \in X_{\nu,j}$ modulo p^ν is equal to

$$\lfloor \frac{ax}{mp^\nu} \rfloor + \lfloor \frac{a(p^j q^l - x)}{mp^\nu} \rfloor + 1.$$

Therefore, if we let

$$E_1(x) = \frac{1-a}{2} + \lfloor \frac{ax}{mp^\nu} \rfloor - a\lfloor \frac{x}{mp^\nu} \rfloor$$

and

$$E_2(x) = \frac{1-a}{2} + \lfloor \frac{a(p^j q^l - x)}{mp^\nu} \rfloor - a\lfloor \frac{p^j q^l - x}{mp^\nu} \rfloor,$$

we obtain the congruence

$$\frac{P_{l,j}(a)}{P_{l,j}(1)^a} \equiv (\prod_{x\in X_{\nu,j}} x^{E_1(x)})(\prod_{x\in X_{\nu,j}} x^{E_2(x)}) \pmod{p^\nu}.$$

We claim that

$$(\frac{p}{a}) \equiv \prod_{x\in X_{\nu,j}} x^{E_2(x)} \pmod{p^\nu}. \tag{5.6}$$

Indeed, consider the change of variables $x = p^j q^l - my$ to reduce to the $m = 1$ case. Notice that as y runs over $(\mathbb{Z}/p^\nu\mathbb{Z})^*$, x runs over a system $X_{\nu,j}$ of representatives of $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ congruent to p^j modulo m. So, if we let

$$E_3(y) = \frac{1-a}{2} + \lfloor \frac{ay}{p^\nu} \rfloor - a\lfloor \frac{y}{p^\nu} \rfloor,$$

we have

$$\prod_{x\in X_{\nu,j}} x^{E_2(x)} \equiv \prod_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} (-my)^{E_3(y)}$$
$$= (-m)^{\sum_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} E_3(y)} \prod_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} y^{E_3(y)} \pmod{p^\nu}$$

We notice that if we restrict $A\theta_\nu$ from equation (5.4) in the case $m = 1$ to $\mathbb{Q}$, we obtain $\sum_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} E_3(y) = A\theta_\nu|_{\mathbb{Q}} = 0$. So, it now suffices to prove that

$$(\frac{p}{a}) \equiv \prod_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} y^{E_3(y)} \pmod{p^\nu}. \tag{5.7}$$

But, the proof in the case $m = 1$ yields

$$r_p\left((\frac{p}{a})\right) = r_p\left(\prod_{y\in(\mathbb{Z}/p^\nu\mathbb{Z})^*} y^{E_3(y)}\right);$$

since the kernel of $r_p|_{\mathbb{Z}_p^*}$ is precisely $1 + p^\nu\mathbb{Z}_p$, the desired congruence (5.7) follows. Now the proof of $Gr_{\mathbb{Q}}(L_\nu/K_m, S, p)$ is is reduced to the congruence

$$\prod_{j=0}^{f-1}\prod_{x\in X_{\nu,j}} x^{\frac{1-a}{2}+\lfloor\frac{ax}{mp^\nu}\rfloor - a\lfloor\frac{x}{mp^\nu}\rfloor} \equiv \prod_{j=0}^{f-1}\prod_{x\in X_\nu} x^{\frac{1-a}{2}+\lfloor\frac{t(p^j a,ax)}{mp^\nu}\rfloor - a\lfloor\frac{t(p^j,x)}{mp^\nu}\rfloor} \pmod{p^\nu},$$

which in turn follows from the definitions.

5.6 An observation

Notice that in the situation above, Gross's conjecture precisely gives the image of ϵ_A in $\mathbb{Q}_p$. Namely, consider a fixed m and let K be the decomposition field of p in $\mathbb{Q}(\mu_m)$. We now let ν vary and let $L = K(\mu_{p^\nu})$. Since the kernel of $r_p|_{\mathbb{Z}_p^*}$ is precisely $(1 + p^\nu\mathbb{Z}_p)^*$, equation (5.5) implies that if Gross's conjecture is true for all ν, since $p \in \mathbb{Q}_p$ is a local norm for all ν, there must exist $\epsilon_A \in \mathbb{Q}_p^*$ of the form $\epsilon_A = p^a\epsilon'$, where ϵ' is such that

$$\epsilon' \equiv \prod_{x\in X_\nu} x^{\sum_{j=0}^{f-1}\left(\frac{1-a}{2}+\lfloor\frac{t(p^j a,ax)}{mp^\nu}\rfloor - a\lfloor\frac{t(p^j,x)}{mp^\nu}\rfloor\right)} \pmod{p^\nu}$$

for all $\nu \geq 1$. Moreover, ϵ_A must come from the global field K and must satisfy $\epsilon_A \equiv 1 \pmod{a}$.

We can define a $\mathbb{Z}$-valued measure ν on $\mathbb{Z}_p^*$ as follows. Given $x \in \mathbb{Z}_p^*$ and $\nu \geq 1$, define

$$\nu(x + p^\nu\mathbb{Z}_p) = \sum_{j=0}^{f-1}\left(\frac{1-a}{2} + \lfloor\frac{t(p^j a, ax)}{mp^\nu}\rfloor - a\lfloor\frac{t(p^j, x)}{mp^\nu}\rfloor\right).$$

It is clear that this is a measure, because for a compact open $U \subset \mathbb{Z}_p^*$, $\nu(U)$ is simply the value at 0 of a shift of zeta functions which involve a sum over $n \geq 1$ in a certain fixed residue class modulo m, and with $n \in U$.

Therefore, Gross's conjecture implies in particular, on general grounds, that we must have

$$\epsilon' = \times\!\!\!\!\!\!\int_{\mathbb{Z}_p^*} x d\nu(x) \in \mathbb{Z}_p^*. \tag{5.8}$$

We will explore this approach of assuming Gross's conjecture and attempting to write down an explicit formula for the Gross–Stark unit in the remaining chapters.

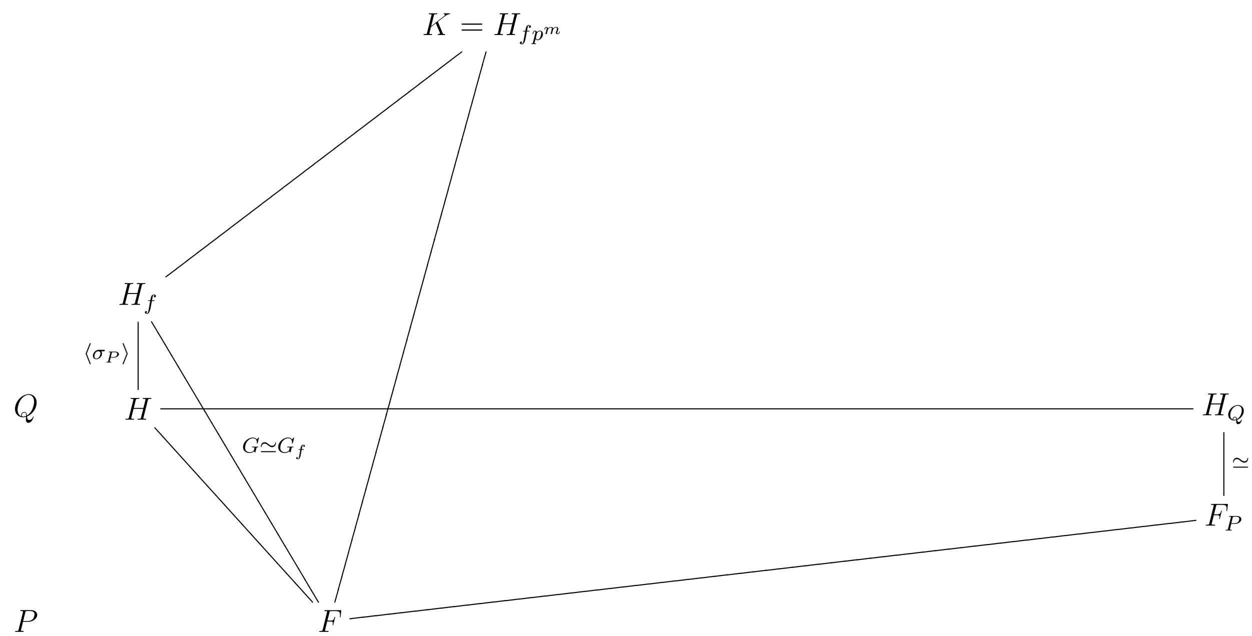
Chapter 6

Dasgupta's restatement

In this Chapter, we follow [4] to interpret Gross's conjecture as a formula generalizing (5.8) when $\mathbb{Q}$ is replaced by an arbitrary totally real number field. In general, there will be an obstruction and the image of the Gross–Stark unit will be determined only in a certain quotient.

We now consider the following setting. Let F be a totally real number field, and $\mathfrak{f}$ an integral ideal of F. Consider the narrow ray class field $H_{\mathfrak{f}}$ of F corresponding to the modulus $\mathfrak{f}$. The Artin map induces an isomorphism $G_{\mathfrak{f}} \simeq G(H_{\mathfrak{f}}/F)$, where $G_{\mathfrak{f}} = I_F^{\mathfrak{f}}/i(F_{f\infty,1})$ is the narrow ray class group corresponding to $\mathfrak{f}$. We fix a prime $\mathfrak{p}$ of F prime to $\mathfrak{f}$ and denote by H the decomposition field of $\mathfrak{p}$ in $H_{\mathfrak{f}}$. We fix a prime $\mathfrak{B}$ of H lying over $\mathfrak{p}$. We know the conductor of $H_{\mathfrak{f}}/F$ divides $\mathfrak{f}$, so we consider a set S containing at least the archimedean primes of F, the divisors of $\mathfrak{f}$, as well as $\mathfrak{p}$. The extension $H_{\mathfrak{f}}/F$ is then unramified outside S. We assume $|S| \geq 3$, excluding only the case $H_{\mathfrak{f}} = F = \mathbb{Q}$ (by the Kronecker–Weber theorem). Set $R = S - \{\mathfrak{p}\}$.

Let $K = H_{\mathfrak{f}\mathfrak{p}^m}$ be the narrow ray class field of F with respect to $\mathfrak{f}\mathfrak{p}^m$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$ and let $\mathfrak{p}^e = (\pi)$ with $\pi \gg 0$ and $\pi \equiv 1 \pmod{\mathfrak{f}}$. Let $E(\mathfrak{f})$ be the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$, and let $E_{\mathfrak{p}}(\mathfrak{f}) = \langle \pi \rangle \times E(\mathfrak{f})$ be the group of totally positive $\mathfrak{p}$-units of F congruent to 1 modulo $\mathfrak{f}$.



By the definition of the local reciprocity map, $E_{\mathfrak{p}}(\mathfrak{f}) \subset \ker r_{\mathfrak{p}}$, with $r_{\mathfrak{p}} : F_{\mathfrak{p}}^* \longrightarrow G(K/F)$, and since $r_{\mathfrak{p}}$ is trivial on $U_{\mathfrak{p}^m}$, also $U_{\mathfrak{p}^m} E_{\mathfrak{p}}(\mathfrak{f}) \subset \ker r_{\mathfrak{p}}$. Conversely, if $\alpha \in F_{\mathfrak{p}}^*$ belongs to $\ker r_{\mathfrak{p}}$, take $a \in F$ with $a \equiv \alpha \pmod{\mathfrak{p}^m}$, $a \gg 0$, and $a \equiv 1 \pmod{\mathfrak{f}}$. Then by definition $(a)\mathfrak{p}^{-\nu_{\mathfrak{p}}(a)} \in i(F_{\mathfrak{f}\mathfrak{p}^m\infty,1})$. This implies first that $e|\nu_{\mathfrak{p}}(a)$ and so $a = \pi^k a'$ for some k and a', with $(a') \in i(F_{\mathfrak{f}\mathfrak{p}^m\infty,1})$. So $a' = u\beta$ for a unit $u \in E(\mathfrak{f})$ and $\beta \equiv 1 \pmod{\mathfrak{p}^m}$. We conclude that $\alpha \in U_{\mathfrak{p}^m} E_{\mathfrak{p}}(\mathfrak{f})$. Also, if H' is the decomposition group of $\mathfrak{p}$ in K, then the conductor of H'/F has to divide $\mathfrak{f}\mathfrak{p}^m$, and so it must divide $\mathfrak{f}$. But then $H' \subset H_{\mathfrak{f}}$, and thus $H' = H$. We deduce that $r_{\mathfrak{p}}$ induces an isomorphism

$$r_{\mathfrak{p}} : F_{\mathfrak{p}}^* / U_{\mathfrak{p}^m} E_{\mathfrak{p}}(\mathfrak{f}) \simeq G(H_{\mathfrak{f}\mathfrak{p}^m}/H).$$

Let $\widehat{E(\mathfrak{f})} = \cap_{m \geq 1} U_{\mathfrak{p}^m} E(\mathfrak{f})$ be the closure of $E(\mathfrak{f})$ in $F_{\mathfrak{p}}^*$, and let $\widehat{E_{\mathfrak{p}}(\mathfrak{f})} = \langle \pi \rangle \times \widehat{E(\mathfrak{f})}$ be the closure of $E_{\mathfrak{p}}(\mathfrak{f})$ in $F_{\mathfrak{p}}^*$. Notice that $\widehat{E_{\mathfrak{p}}(\mathfrak{f})} = \cap_{m \geq 1} U_{\mathfrak{p}^m} E_{\mathfrak{p}}(\mathfrak{f})$, so Gross's conjecture applied to all fields $K = H_{\mathfrak{f}\mathfrak{p}^m}$ is a statement for the image of u_T^σ in $F_{\mathfrak{p}}^*/\widehat{E_{\mathfrak{p}}(\mathfrak{f})}$. It is now convenient to introduce the field $H_{\mathfrak{f}\mathfrak{p}^\infty} = \cup_{m \geq 1} H_{\mathfrak{f}\mathfrak{p}^m}$ and to note that the map

$$r_{\mathfrak{p}} : F_{\mathfrak{p}}^* \longrightarrow G(H_{\mathfrak{f}\mathfrak{p}^\infty}/F)$$

induces an isomorphism

$$r_{\mathfrak{p}} : F_{\mathfrak{p}}^* / \widehat{E_{\mathfrak{p}}(\mathfrak{f})} \longrightarrow G(H_{\mathfrak{f}\mathfrak{p}^\infty}/H).$$

We also have the Artin map

$$\begin{aligned} I_F^{\mathfrak{f}\mathfrak{p}} &\longrightarrow G(H_{\mathfrak{f}\mathfrak{p}^\infty}/F) \\ \mathfrak{a} &\longmapsto (\sigma_{\mathfrak{a}}), \end{aligned}$$

Denote $\mathbf{O} = \mathcal{O}_{\mathfrak{p}} - \pi\mathcal{O}_{\mathfrak{p}} \subset F_{\mathfrak{p}}^*$; it is a fundamental domain for the action of $\langle\pi\rangle$ on $F_{\mathfrak{p}}^*$, hence we have a bijection

$$\mathbf{O}/\widehat{E(\mathfrak{f})} \longrightarrow F_{\mathfrak{p}}^*/\widehat{E_{\mathfrak{p}}(\mathfrak{f})}$$

induced by inclusion.

Let $\mathfrak{b}$ be a fractional ideal of F prime to S and T. For a compact–open subset $U \subset \mathbf{O}/\widehat{E(\mathfrak{f})}$, let

$$\zeta_S(\mathfrak{b}, U, s) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F,(\mathfrak{a},S)=1\\ \sigma_{\mathfrak{a}}\in\sigma_{\mathfrak{b}}r_{\mathfrak{p}}(U)}} N\mathfrak{a}^{-s}$$

for $Re(s) > 1$. The condition is equivalent to $\sigma_{\mathfrak{a}\mathfrak{b}^{-1}} \in r_{\mathfrak{p}}(U)$; in particular, it must be that $\sigma_{\mathfrak{a}\mathfrak{b}^{-1}}$ is trivial on H. So, $\mathfrak{a}\mathfrak{b}^{-1} = (\alpha)\mathfrak{p}^k$ for some k and some $\alpha \in F_{\mathfrak{f}\infty,1}$, necessarily $k = -\nu_{\mathfrak{p}}(\alpha)$ because $\mathfrak{p}$ is prime to $\mathfrak{a}$ and $\mathfrak{b}$. Also, note that $\alpha \in \mathfrak{b}^{-1}$ because $\mathfrak{a}$ is integral. By definition, $r_{\mathfrak{p}}(\alpha) = \sigma_{(\alpha)\mathfrak{p}^{-\nu_{\mathfrak{p}}(\alpha)}} = \sigma_{\mathfrak{a}\mathfrak{b}^{-1}}$, which belongs to $r_{\mathfrak{p}}(U)$ if and only if $\alpha \in U$ (because $r_{\mathfrak{p}}$ is injective on $F_{\mathfrak{p}}^*/\widehat{E_{\mathfrak{p}}(\mathfrak{f})}$). We deduce that the sum above can be written also as

$$\zeta_S(\mathfrak{b}, U, s) = N\mathfrak{b}^{-s} \sum_{\alpha} N\mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha)s} N\alpha^{-s} \tag{6.1}$$

where the sum ranges over distinct representatives modulo $E(\mathfrak{f})$ of α satisfying $\alpha \in \mathfrak{b}^{-1}, \alpha \equiv 1 \pmod{\mathfrak{f}}, \alpha \gg 0, (\alpha, R) = 1$, and $\alpha \in U$ (when $\alpha \gg 0$, $N((\alpha)) = N\alpha$).

Define

$$\zeta_{S,T}(\mathfrak{b}, U, s) = \sum_{\mathfrak{a}} c_{\mathfrak{a}} \zeta_S(\mathfrak{a}^{-1}\mathfrak{b}, U, s)$$

where $\prod_{\eta\in T}(1 - [\eta]N\eta) = \sum_{\mathfrak{a}} c_{\mathfrak{a}}[\mathfrak{a}]$. It will follow from Chapter 8 that $\zeta_{S,T}(\mathfrak{b}, U, 0) \in \mathbb{Z}$ for all compact–open U, and hence

$$\mu(\mathfrak{b}, U) = \zeta_{S,T}(\mathfrak{b}, U, 0)$$

defines a $\mathbb{Z}$-valued measure on $\mathbf{O}/\widehat{E(\mathfrak{f})}$. We compute that

$$\mu(\mathfrak{b}, \mathbf{O}/\widehat{E(\mathfrak{f})}) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F,(\mathfrak{a},S)=1\\ \sigma_{H/F}(\mathfrak{a})=\sigma_{H/F}(\mathfrak{b})}} N\mathfrak{a}^{-s}|_{s=0} = \zeta_{S,T}(H/F, \sigma_{\mathfrak{b}}, 0) = 0$$

because S contains a prime $\mathfrak{p}$ which splits completely in H. Also, note that if $\alpha \in F_{\mathfrak{p}}^*/U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f})$ and U_α denotes its inverse image in $\mathbf{O}/\widehat{E(\mathfrak{f})}$ under $\mathbf{O}/\widehat{E(\mathfrak{f})} \to F_{\mathfrak{p}}^*/\widehat{E_{\mathfrak{p}}(\mathfrak{f})} \to F_{\mathfrak{p}}^*/U_{\mathfrak{p}^m}E_{\mathfrak{p}}(\mathfrak{f})$, then

$$\mu(\mathfrak{b}, U_\alpha) = \zeta_{S,T}(H_{\mathfrak{f}\mathfrak{p}^m}/F, \sigma_{\mathfrak{b}} r_{\mathfrak{p}}(\alpha), 0). \tag{6.2}$$

If $\rho \in \mathcal{O}_{\mathfrak{p}}$ is a local uniformizer, $\mathfrak{p} = (\rho)$, we compute that for any $i = 0, 1, ..., e-1$, we have

$$\zeta_{S,T}(\mathfrak{b}, \rho^i\mathcal{O}_{\mathfrak{p}}^*/\widehat{E(\mathfrak{f})}, 0) = \zeta_{S,T}(H_{\mathfrak{f}}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0). \tag{6.3}$$

To check this, we can drop the index T which only corresponds to a shift, and reduce to

$$\zeta_S(\mathfrak{b}, \rho^i\mathcal{O}_{\mathfrak{p}}^*/\widehat{E(\mathfrak{f})}, 0) = \zeta_S(H_{\mathfrak{f}}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0).$$

The right–hand–side above is the value at $s = 0$ of

$$\sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F,(\mathfrak{a},S)=1\\ \sigma_\mathfrak{a}=\sigma_{\mathfrak{b}\mathfrak{p}^{-i}} \text{ in } G(H_\mathfrak{f}/F)}} N\mathfrak{a}^{-s}.$$

We consider a change of variables $\mathfrak{a} = \mathfrak{b}\mathfrak{p}^{-i}(\alpha)$, where $\alpha \gg 0, \alpha \equiv 1 \pmod{\mathfrak{f}}, \alpha \in \mathfrak{b}^{-1}, \alpha \in \rho^i\mathcal{O}_\mathfrak{p}^*, (\alpha, R) = 1$, and α is defined up to $E(\mathfrak{f})$. So, we can rewrite the second sum in terms of a sum over such α:

$$N\mathfrak{b}^{-s}\sum_\alpha N\mathfrak{p}^{is}N((\alpha))^{-s}.$$

At $s = 0$, we obtain precisely the value of the left–handside above. As a matter of notation, for convenience, we will use for example $\zeta_S(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0)$ and $\zeta_S(H_\mathfrak{f}/F, \sigma_{\mathfrak{b}\mathfrak{p}^{-i}}, 0)$ interchangebly.

Next, we prove that for any i, we have

$$\zeta_{S,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0) = \zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0) - \zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i-1}, 0). \tag{6.4}$$

Again, after dropping the shift T, it suffices to note that

$$\zeta_R(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, s) - \zeta_S(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, s) = \sum_{\substack{(\mathfrak{a},R)=1,\mathfrak{p}|\mathfrak{a}\\ \sigma_\mathfrak{a}=\sigma_{\mathfrak{b}\mathfrak{p}^{-i}}}} N\mathfrak{a}^{-s}$$

and consider a change of variables $\mathfrak{a} = \mathfrak{p}\mathfrak{a}'$.

Finally, we also observe that

$$\sum_{i=1}^{e}\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0) = \zeta_{R,T}(H/F, \mathfrak{b}, 0). \tag{6.5}$$

This follows from the fact that $G(H/F) = G(H_\mathfrak{f}/F)/\langle\sigma_\mathfrak{p}\rangle$ and the functoriality of the Artin map.

Now we are ready for the following:

Proposition 3. *If Conjecture 4 is true, then*

$$u_T^{\sigma_\mathfrak{b}} = \pi^{\zeta_{R,T}(H_\mathfrak{f}/F,\mathfrak{b},0)} \times\!\!\!\!\!\!\int_{\mathbf{O}/\widehat{E(\mathfrak{f})}} x\ d\mu(\mathfrak{b}, x)$$

in $F_\mathfrak{p}^*/\widehat{E(\mathfrak{f})}$, *where the integrand* x *is the inclusion* $\mathbf{O}/\widehat{E(\mathfrak{f})} \hookrightarrow F_\mathfrak{p}^*/\widehat{E(\mathfrak{f})}$.

This is the generalization of the observation from section 5.6, which we obtained by applying Gross's conjecture to all fields $L = K(\mu_{p^\nu})$. Notice that when $F = \mathbb{Q}$, we have $\widehat{E(\mathfrak{f})} = \{1\}$, hence Gross's conjecture already gives an exact formula for u_T in $\mathbb{Q}_p^*$.

Proof. We compute the $\mathfrak{p}$-adic valuation of the multiplicative integral above. Note that this integral equals $\prod_{i=0}^{e-1} \times\!\!\!\!\!\!\int_{\rho^i\mathcal{O}_\mathfrak{p}^*/\widehat{E(\mathfrak{f})}} x\; d\mu(\mathfrak{b}, x)$ and so it has $\mathfrak{p}$-adic valuation

$$\begin{aligned}
\sum_{i=0}^{e-1} i\mu(\mathfrak{b}, \rho^i\mathcal{O}_\mathfrak{p}^*/\widehat{E(\mathfrak{f})}) &= \sum_{i=0}^{e-1} i\zeta_{S,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0) \quad \text{by (6.3)} \\
&= \sum_{i=0}^{e-1} i(\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0) - \zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i-1}, 0)) \quad \text{by (6.4)} \\
&= \left(\sum_{i=1}^{e} \zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-i}, 0)\right) - e\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}\mathfrak{p}^{-e}, 0) \\
&= \zeta_{R,T}(H/F, \mathfrak{b}, 0) - e\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0) \quad \text{by (6.5)}.
\end{aligned}$$

So, the $\mathfrak{p}$-adic valuations in the above formula match, and hence it suffices to prove the equality in

$$F_\mathfrak{p}^*/\widehat{E_\mathfrak{p}(\mathfrak{f})} = \lim_{\leftarrow} F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f}).$$

Concretely, we have to prove that for any fixed m, if $U_\alpha \subset \mathbf{O}/\widehat{E(\mathfrak{f})}$ denotes the inverse image of an element $\alpha \in F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})$ under $\mathbf{O}/\widehat{E(\mathfrak{f})} \to F_\mathfrak{p}^*/\widehat{E_\mathfrak{p}(\mathfrak{f})} \to F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})$, then

$$u_T^{\sigma_\mathfrak{b}} \equiv \prod_{\alpha\in F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})} \alpha^{\mu(b,U_\alpha)} \pmod{U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})}.$$

We now apply Gross's conjecture for $K = H_{\mathfrak{f}\mathfrak{p}^m}$. Namely, using the isomorphism $r_\mathfrak{p}$: $F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f}) \simeq G(K/H)$, we can write

$$\begin{aligned}
r_\mathfrak{p}(u_T^{\sigma_\mathfrak{b}}) &= \prod_{\substack{\tau\in G(K/F) \\ \tau\in\sigma_\mathfrak{b}G(K/H)}} \tau^{\zeta_{S,T}(K/F,\tau,0)} \\
&= \prod_{\substack{\tau=\sigma_\mathfrak{b}r_\mathfrak{p}(\alpha) \\ \alpha\in F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})}} \tau^{\zeta_{S,T}(K/F,\tau,0)} \\
&= \prod_{\alpha\in F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})} r_\mathfrak{p}(\alpha)^{\zeta_{S,T}(K/F,\sigma_\mathfrak{b}r_\mathfrak{p}(\alpha),0)}
\end{aligned}$$

and thus

$$u_T^{\sigma_\mathfrak{b}} \equiv \prod_{\alpha\in F_\mathfrak{p}^*/U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})} \alpha^{\zeta_{S,T}(K/F,\sigma_\mathfrak{b}r_\mathfrak{p}(\alpha),0)} \pmod{U_{\mathfrak{p}^m}E_\mathfrak{p}(\mathfrak{f})},$$

which finishes the proof, by (6.2). □

Notice that Proposition 3 is precisely the statement of Gross's conjecture applied to the fields $K = H_{\mathfrak{f}\mathfrak{p}^m}$ for all m. To obtain an exact reformulation of Gross's conjecture, we consider the compositum H_S of all fields K for which the conjecture can be applied. Namely, if $\mathfrak{g}$ is the product of the finite primes in S relatively prime to $\mathfrak{p}\mathfrak{f}$, we let $H_S = H_{\mathfrak{f}\mathfrak{p}\mathfrak{g}}^\infty$ be the compositum of all narrow ray class fields of F with respect to a modulus involving only primes in S. If v is a prime dividing $\mathfrak{fg}$, set

$$U_v = 1 + \mathfrak{f}\mathcal{O}_v = \begin{cases} 1 + v^t\mathcal{O}_v & \text{if} \quad v^t || \mathfrak{f} \\ \mathcal{O}_v^* & \text{if} \quad v \mid \mathfrak{g}. \end{cases}$$

Denote

$$\mathcal{U} = \prod_{v|\mathfrak{fg}} U_v.$$

Notice that $E_\mathfrak{p}(\mathfrak{f}) \subset F_\mathfrak{p}^* \times \mathcal{U}$ under the diagonal embedding. Let $\overline{E_\mathfrak{p}(\mathfrak{f})}$ be the closure of $E_\mathfrak{p}(\mathfrak{f})$ in $F_\mathfrak{p}^* \times \mathcal{U}$. The local reciprocity maps $r_\mathfrak{p}$ and r_v for $v \mid \mathfrak{fg}$ induce an isomorphism

$$r_S : (F_\mathfrak{p}^* \times \mathcal{U})/\overline{E_\mathfrak{p}(\mathfrak{f})} \simeq G(H_S/H),$$

and the choice of π from before gives a bijection $(\mathbf{O} \times \mathcal{U})/\overline{E(\mathfrak{f})} \to (F_\mathfrak{p}^* \times \mathcal{U})/\overline{E_\mathfrak{p}(\mathfrak{f})}$. We proceed as before and define, for each $\mathfrak{b}$ relatively prime to S, T a $\mathbb{Z}$-valued measure $\mu(\mathfrak{b}, U)$, given by $\mu(\mathfrak{b}, U) = \tilde{\zeta}_{S,T}(\mathfrak{b}, U, 0)$ on each compact open subset $U \subset (\mathbf{O} \times \mathcal{U})/\overline{E(\mathfrak{f})}$, where

$$\tilde{\zeta}_S(\mathfrak{b}, U, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_F, (\mathfrak{a}, S) = 1 \\ \sigma_\mathfrak{a} \in \sigma_\mathfrak{b} r_S(U) \text{ in } G(H_S/F)}} N\mathfrak{a}^{-s}.$$

We consider the statement that Conjecture 4 yields when applied to each $K = H_{\prod v_i^{a_i} \mathfrak{p}^m}$, where v_i are the divisors of $\mathfrak{fg}$. Thus, as above, we can restate it conveniently as follows:

Proposition 4. *Conjecture 4 is equivalent to the following statement: There exists an element $u_T \in U_\mathfrak{p}$ with $u_T \equiv 1 \pmod T$ such that for all fractional ideals $\mathfrak{b}$ of F prime to S, T we have*

$$(u_T^{\sigma_b}, 1) = \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{(\mathbf{O} \times \mathcal{U})/\overline{E(\mathfrak{f})}} x \; d\mu(\mathfrak{b}, x)$$

in $(F_\mathfrak{p}^* \times \mathcal{U})/\overline{E(\mathfrak{f})}$.

Chapter 7

Dasgupta's refinement

So far, Dasgupta's restatement of Gross's conjecture was a formula for u_T in a certain quotient. The goal however is to write an explicit formula for u_T in $F_{\mathfrak{p}}^*$ and not just in $F_{\mathfrak{p}}^*/\widehat{E(\mathfrak{f})}$. Such a formula can then be viewed as explicit $\mathfrak{p}$-adic class field theory for the extension H/F. The key is to refine the measure μ on $\mathbf{O}/\widehat{E(\mathfrak{f})}$ to a measure ν on $\mathcal{O}_{\mathfrak{p}}$ whose restriction to $\mathbf{O}$ pushes forward to μ under the projection $\mathbf{O} \to \mathbf{O}/\widehat{E(\mathfrak{f})}$. The natural idea is to refine the formula 6.1, where the summation is over certain elements α, but defined only modulo $E(\mathfrak{f})$. Writing an analogous formula for a compact–open subset U of $\mathcal{O}_{\mathfrak{p}}$, however, requires a choice of a fundamental domain for the action of $E(\mathfrak{f})$ on the positive quadrant $Q = \mathbb{R}^n_{>0}$. It turns out that there is a fundamental domain for that action which has a special geometric shape, and which is therefore a natural candidate for the definition of the refined measure ν.

7.1 Shintani domains

Fix a totally real number field F and let $n = [F : \mathbb{Q}]$. The n embeddings $F \to \mathbb{R}$ given by $x \mapsto x^i$ $(i = 1, ..., n)$ define an embedding $F \hookrightarrow \mathbb{R}^n$ and an action of F^* on $\mathbb{R}^n$ via $\alpha(x_1, ..., x_n) = (\alpha^1 x_1, ..., \alpha^n x_n)$. The totally positive elements in F^* act on $Q = \mathbb{R}^n_{>0}$. We now describe a fundamental domain for the action of $E(\mathfrak{f})$ on Q of special geometric shape.

For totally positive $v_1, ..., v_r \in F$, whose images in $\mathbb{R}^n$ are linearly independent over $\mathbb{R}$, define the *simplicial cone* generated by $v_1, ..., v_r$ as

$$C(v_1, ..., v_r) = \left\{ \sum_{i=1}^{r} c_i v_i \mid c_i > 0 \right\} \subset Q.$$

A *Shintani set* is a finite disjoint union of simplicial cones. The intersection of two Shintani sets is a Shintani set, and for Shintani sets $\mathcal{D}, \mathcal{D}'$, there are only finitely many $\epsilon \in E(\mathfrak{f})$ such that $\epsilon\mathcal{D} \cap \mathcal{D}' \neq \emptyset$ (see [4],[12]).

Shintani proved that there exists a Shintani set which is a fundamental domain for the action of $E(\mathfrak{f})$ on Q; such a set $\mathcal{D}$ is called a *Shintani domain.* For example, if $n = 2$ and $E(\mathfrak{f}) = \langle\epsilon\rangle$, a Shintani domain is $\mathcal{D} = C(1) \cup C(1, \epsilon)$. If $n = 3$ and $E(\mathfrak{f})$ has basis (ϵ_1, ϵ_2) as a free abelian group, Colmez proved ([2]) that we can take

$$\mathcal{D} = C(1) \cup C(1, \epsilon_1) \cup C(1, \epsilon_2) \cup C(1, \epsilon_1\epsilon_2) \cup C(1, \epsilon_1, \epsilon_1\epsilon_2) \cup C(1, \epsilon_2, \epsilon_1\epsilon_2)$$

as a Shintani domain, provided ϵ_1, ϵ_2 satisfy a mild sign condition

$$\det(1, \epsilon_1, \epsilon_1\epsilon_2)\det(1, \epsilon_2, \epsilon_1\epsilon_2) < 0,$$

where $\det(\alpha, \beta, \gamma) = \det \begin{pmatrix} \alpha^1 & \beta^1 & \gamma^1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{pmatrix}$ for $\alpha, \beta, \gamma \in F$.

A prime η of F is called *good* for a simplicial cone $C = C(v_1, ..., v_r)$ if $N\eta = l$ is a rational prime and the generators v_i can be chosen in $\mathcal{O} - \eta$. A set T is *good* for a simplicial cone C if it either contains two primes of different residue characteristic which are good for C, or a prime η which is good for C and $l = N\eta \geq n + 2$. Also, T is called *good* for a Shintani set $\mathcal{D}$ if $\mathcal{D}$ is a finite disjoint union $\mathcal{D} = \cup C$, with T good for each of the simplicial cones C.

Let $\mathcal{D} = \cup B, \mathcal{D}' = \cup B'$ be any two Shintani domains. For each B, B', there are finitely many $\epsilon \in E(\mathfrak{f})$ such that $B \cap \epsilon B' \neq \emptyset$, and we write each such nonempty Shintani set $B \cap \epsilon B'$ as a disjoint union $\cup C$ of simplicial cones. From here, we conclude that we can find simplicial cones $C_1, ..., C_d$ such that

$$\mathcal{D} = \bigcup_{i=1}^{d} C_i, \qquad \mathcal{D}' = \bigcup_{i=1}^{d} \gamma_i C_i \tag{7.1}$$

for some $\gamma_i \in E(\mathfrak{f})$. Such a decomposition is called a *simultaneous decomposition* for the pair $(\mathcal{D}, \mathcal{D}')$. A set T is called *good* for a pair $(\mathcal{D}, \mathcal{D}')$ of Shintani domains if there is a simultaneous decomposition as above with T good for each C_i. If $\beta \in F^*$ is totally positive, a set T is called β-*good* for a Shintani domain $\mathcal{D}$ if it is good for the pair $(\mathcal{D}, \beta^{-1}\mathcal{D})$. This property depends only on the coset of β in $F^*/E(\mathfrak{f})$. If $\mathfrak{p} = (p)$, then π can be chosen as $\pi = p^e$ for some e. In this case, since $p^e C_i = C_i$, the condition that T is π-good for $\mathcal{D}$ reduces to the condition that T is good for $\mathcal{D}$. Note that if T is β-good for a Shintani domain $\mathcal{D}$ and C_i are chosen as in (7.1) for $\mathcal{D}' = \beta^{-1}\mathcal{D}$, then for $\epsilon \in E(\mathfrak{f})$, we have that

$$\epsilon\mathcal{D} \cap \beta^{-1}\mathcal{D} = \bigcup_{\gamma_i = \epsilon} C_i$$

is a Shintani set for which T is good.

7.2 The refined measure $\nu(\mathfrak{b}, \mathcal{D})$

We assume no prime of T has the same residue characteristic as any prime of S. Consider a fractional ideal $\mathfrak{b}$ of F prime to S and the residue characteristic of any prime in T (write: $\mathfrak{b}$ prime to S, char T). Let $\mathcal{D}$ be a Shintani set.

For a compact open subset U of $\mathcal{O}_{\mathfrak{p}}$, define

$$\zeta_R(\mathfrak{b}, \mathcal{D}, U, s) = \sum_{\substack{(\alpha, R) = 1 \\ \alpha \in \mathfrak{b}^{-1} \\ \alpha \equiv 1 \pmod{\mathfrak{f}} \\ \alpha \in \mathcal{D} \\ \alpha \in U}} N\alpha^{-s}.$$

Next, define the shift $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s)$ as before. It will follow from Chapter 8 that when T is good for $\mathcal{D}$, the value at $s = 0$ of the analytic continuation of $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s)$ is in $\mathbb{Z}$. So, we can define a $\mathbb{Z}$-valued measure $\nu(\mathfrak{b}, \mathcal{D})$ on $\mathcal{O}_{\mathfrak{p}}$ by

$$\nu(\mathfrak{b}, \mathcal{D}, U) = \zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0)$$

for a compact open subset $U \subset \mathcal{O}_{\mathfrak{p}}$.

Equation (6.1) implies that if ν_* is the pushforward of $\nu|_{\mathbf{O}}$ to $\mathbf{O}/\widehat{E(\mathfrak{f})}$ under $\mathbf{O} \to \mathbf{O}/\widehat{E(\mathfrak{f})}$, then

$$\nu_*(\mathfrak{b}, \mathcal{D}, U) = \mu(\mathfrak{b}, U)$$

for any compact open $U \subset \mathbf{O}/\widehat{E(\mathfrak{f})}$. Thus we compute

$$\nu(\mathfrak{b}, \mathcal{D}, \mathbf{O}) = \nu_*(\mathfrak{b}, \mathcal{D}, \mathbf{O}/\widehat{E(\mathfrak{f})}) = \mu(\mathfrak{b}, \mathbf{O}/\widehat{E(\mathfrak{f})}) = 0.$$

Also, a change of variable $\mathfrak{a} = \mathfrak{b}(\alpha)$, with $\alpha \gg 0, \alpha \equiv 1 \pmod{\mathfrak{f}}$ in the definition of $\zeta_R(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)$ yields

$$\nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = \zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0).$$

7.3 The conjectural element $u_T(\mathfrak{b}, \mathcal{D})$. Naturality

Let $\mathcal{D}$ be a Shintani domain, and let T be π-good for $\mathcal{D}$. By above, for any $\epsilon \in E(\mathfrak{f})$, the intersection $\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}$ is either empty or (for a finite number of ϵ) a Shintani set for which T is good. So, we have that

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi) = \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})}$$

is a well-defined element in $E(\mathfrak{f})$ (the exponents above are integers).

Define

$$u_T(\mathfrak{b}, \mathcal{D}) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O}} x d\nu(\mathfrak{b}, \mathcal{D}, x) \in F_{\mathfrak{p}}^*.$$

Note that for any $\gamma \in E(\mathfrak{f})$, a change of variable $\alpha' = \alpha\gamma$ in the definition of ν implies

$$\nu(\mathfrak{b}, \gamma^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) = \nu(\mathfrak{b}, \mathcal{D}, \gamma\mathcal{O}_{\mathfrak{p}}) = \nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_{\mathfrak{p}}).$$

From here, using that $\mathcal{D}$ is a fundamental domain for the action of $E(\mathfrak{f})$ on Q, we readily find that

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi\gamma) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\gamma^{-\nu(\mathfrak{b}, \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\gamma^{-\zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)},$$

hence $u_T(\mathfrak{b}, \mathcal{D})$ is independent of the choice of π.

Proposition 5. *Assume T is π-good for a Shintani domain $\mathcal{D}$. If $\beta \in F^*$ is relatively prime to S, char T, totally positive, and $\beta \equiv 1 \pmod{\mathfrak{f}}$, then*

$$u_T(\mathfrak{b}(\beta), \mathcal{D}) = u_T(\mathfrak{b}, \beta\mathcal{D}).$$

Proof. The term $\pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)}$ is not affected by the change $\mathfrak{b} \mapsto \mathfrak{b}(\beta)$. To investigate the other two terms, we first note that a change of variables yields

$$\zeta_R(\mathfrak{b}(\beta), \mathcal{D}, U, s) = \zeta_R(\mathfrak{b}, \beta\mathcal{D}, \beta U, s),$$

and therefore

$$\nu(\mathfrak{b}(\beta), \mathcal{D}, U) = \nu(\mathfrak{b}, \beta\mathcal{D}, \beta U) \tag{7.2}$$

for any $U \subset \mathcal{O}_{\mathfrak{p}}$. Therefore, we compute

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}(\beta), \mathcal{D}, x) &= \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \beta\mathcal{D}, \beta x) \\ &= \beta^{-\nu(\mathfrak{b},\beta\mathcal{D},\mathbf{O})} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \beta\mathcal{D}, x) \\ &= \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \beta\mathcal{D}, x). \end{aligned}$$

On the other hand, again using (7.2), we find

$$\epsilon(\mathfrak{b}(\beta), \mathcal{D}, \pi) = \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}(\beta), \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} = \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\beta\mathcal{D} \cap \pi^{-1}\beta\mathcal{D}, \beta\mathcal{O}_{\mathfrak{p}})} = \epsilon(\mathfrak{b}, \beta\mathcal{D}, \pi),$$

since $\beta\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$. $\square$

Proposition 6. *Let $\mathcal{D}$ and $\mathcal{D}'$ be Shintani domains such that T is π-good for both $\mathcal{D}$ and $\mathcal{D}'$. Suppose that T is good for the pair $(\mathcal{D}, \mathcal{D}')$. Then*

$$u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}').$$

Proof. Consder a simultaneous decomposition

$$\mathcal{D} = \bigcup C_i \qquad \mathcal{D}' = \bigcup \gamma_i C_i, \quad \text{with} \quad \gamma_i \in E(\mathfrak{f}),$$

such that T is good for each C_i. To prove the conclusion, it suffices to show that if $\mathcal{D} = \mathcal{D}_0 \cup C$ for a Shintani set $\mathcal{D}_0$ and a cone C, such that T is good for $\mathcal{D}_0$ and C, then $u_T(\mathfrak{b}, \mathcal{D}) = u_T(\mathfrak{b}, \mathcal{D}_0 \cup \gamma C)$, for any $\gamma \in E(\mathfrak{f})$. This will allow us to start with the decomposition for $\mathcal{D}$ and replacing one of the cones C_i at a time with $\gamma_i C_i$, to obtain the domain $\mathcal{D}'$ without changing the value of u_T. So, set $\mathcal{D}' = \mathcal{D}_0 \cup \gamma C$.

A simple change of variable in the expression for $\zeta_R(\mathfrak{b}, \gamma C, U, s)$ shows that

$$\nu(\mathfrak{b}, \gamma C, U) = \nu(\mathfrak{b}, C, \gamma^{-1} U)$$

for any $U \subset \mathcal{O}_{\mathfrak{p}}$. Thus,

$$\times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \mathcal{D}', x) = \gamma^{\nu(\mathfrak{b},C,\mathbf{O})} \times\!\!\!\!\!\!\int_{\mathbf{O}} x \, d\nu(\mathfrak{b}, \mathcal{D}, x).$$

To examine the ϵ-term, suppose first that T is good for each intersection $\epsilon C_i \cap \pi^{-1}C_j$, where $\epsilon \in E(\mathfrak{f})$. We split $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$ and $\epsilon(\mathfrak{b}, \mathcal{D}', \pi)$ into four pieces as follows:

$$\begin{aligned}\epsilon(\mathfrak{b}, \mathcal{D}, \pi) = \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D}_0 \cap \pi^{-1}\mathcal{D}_0, \mathcal{O}_\mathfrak{p})} \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon C \cap \pi^{-1}C, \mathcal{O}_\mathfrak{p})} \\ \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D}_0 \cap \pi^{-1}C, \mathcal{O}_\mathfrak{p})} \times \prod_{\epsilon} \epsilon^{\nu(\mathfrak{b}, \epsilon C \cap \pi^{-1}\mathcal{D}_0, \mathcal{O}_\mathfrak{p})},\end{aligned}$$

and similarly for $\epsilon(\mathfrak{b}, \mathcal{D}', \pi)$.

The first two terms are invariant if C is replaced by γC, and changing variables $\epsilon \mapsto \epsilon\gamma$ and $\epsilon \mapsto \epsilon\gamma^{-1}$ in the second two terms yields

$$\epsilon(\mathfrak{b}, \mathcal{D}', \pi) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\gamma^{\sum_{\epsilon \in E(\mathfrak{f})} \nu(\mathfrak{b}, \epsilon\mathcal{D}_0 \cap \pi^{-1}C, \mathcal{O}_\mathfrak{p}) - \nu(\mathfrak{b}, C \cap \epsilon^{-1}\pi^{-1}\mathcal{D}_0, \mathcal{O}_\mathfrak{p})}. \tag{7.3}$$

The exponent of γ equals

$$\begin{aligned}&\sum_{\epsilon \in E(\mathfrak{f})} \left(\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}C, \mathcal{O}_\mathfrak{p}) - \nu(\mathfrak{b}, C \cap \epsilon^{-1}\pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})\right) \\ &= \nu(\mathfrak{b}, \pi^{-1}C, \mathcal{O}_\mathfrak{p}) - \nu(\mathfrak{b}, C, \mathcal{O}_\mathfrak{p}) \\ &= \nu(\mathfrak{b}, C, \pi\mathcal{O}_\mathfrak{p}) - \nu(\mathfrak{b}, C, \mathcal{O}_\mathfrak{p}) \\ &= -\nu(\mathfrak{b}, C, \mathbf{O}),\end{aligned}$$

which finishes the proof in the case when T is good for each intersection $\epsilon C_i \cap \pi^{-1}C_j$.

In general, the exponents in the decomposition above for $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$ and $\epsilon(\mathfrak{b}, \mathcal{D}', \pi)$ need not be integers, and the splitting into four parts is not possible. However, all exponents are rationals (as will follow from Chapter 8) and only finitely many of them are nonzero. So, we can look at their common denominator $M \in \mathbb{Z}$ and prove that the M-th powers of the two sides in

$$\epsilon(\mathfrak{b}, \mathcal{D}', \pi) = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\gamma^{-\nu(\mathfrak{b}, C, \mathbf{O})} \tag{7.4}$$

are equal. However, both sides of (7.4) belong to the torsion-free group $E(\mathfrak{f})$, hence in fact they must be equal. This completes the proof. $\square$

7.4 The refined conjecture

We are now ready to state Dasgupta's refinement of Conjecture 4. Consider a Shintani domain $\mathcal{D}$ and a set T which is π-good for $\mathcal{D}$. Let $\mathfrak{b}, \mathfrak{b}'$ be fractional ideals prime to S, char T. Also, fix a prime $\mathfrak{B}$ of H lying over $\mathfrak{p}$; it defines an embedding $H \hookrightarrow F_\mathfrak{p}$.

Conjecture 5. *1. The element $u_T(\mathfrak{b}, \mathcal{D}) \in F_\mathfrak{p}^*$ does not depend on the choice of Shintani domain $\mathcal{D}$ and depends only on the class of $\mathfrak{b}$ in $G_\mathfrak{f}/\langle\mathfrak{p}\rangle$. So, it can be denoted $u_T(\sigma_\mathfrak{b})$, for $\sigma_\mathfrak{b} \in G(H/F)$.*

2. $u_T(\sigma_\mathfrak{b}) \in U_\mathfrak{p}$ and $u_T(\sigma_b) \equiv 1 \pmod{T}$.

3. (Shimura reciprocity law) $u_T(\sigma_{\mathfrak{b}\mathfrak{b}'}) = u_T(\sigma_\mathfrak{b})^{\sigma_{\mathfrak{b}'}}$.

Proposition 7. *Conjecture (5) implies Conjecture (4).*

Proof. First, we extend the measure $\nu(\mathfrak{b}, \mathcal{D}, U)$ to a $\mathbb{Z}$-valued measure ν on $\mathcal{O}_\mathfrak{p} \times \mathcal{U}$ via $\nu(\mathfrak{b}, \mathcal{D}, U) = \zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0)$, where for a compact open $U \subset \mathcal{O}_\mathfrak{p} \times \mathcal{U}$, we define

$$\zeta_R(\mathfrak{b}, \mathcal{D}, U, s) = \sum_{\substack{(\alpha,R)=1 \\ \alpha \in \mathfrak{b}^{-1} \cap \mathcal{D} \\ \alpha \equiv 1 \pmod{\mathfrak{f}} \\ \alpha \in U}} N\alpha^{-s}.$$

The condition $\alpha \in U$ means the the image of α in $\mathcal{O}_\mathfrak{p} \times \mathcal{U}$ under the diagonal embedding lies in U. It follows from that

$$(\nu(\mathfrak{b}, \mathcal{D})|_{\mathbf{O} \times \mathcal{U}})_* = \mu(\mathfrak{b})$$

as measures on $\mathbf{O} \times \mathcal{U}/\overline{E(\mathfrak{f})}$ (the push–forward is with respect to the natural projection).

Consider the element

$$\Delta = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O} \times \mathcal{U}} x \; d\nu(\mathfrak{b}, \mathcal{D}, x) \in F_\mathfrak{p}^* \times \mathcal{U}.$$

For each finite prime $v \in S$, the projection of Δ onto the v-component of $F_\mathfrak{p}^* \times \mathcal{U}$ equals

$$\epsilon(\mathfrak{b}, \mathcal{D}, \pi)\pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{W_v} x \; d\nu_v(\mathfrak{b}, \mathcal{D}, x),$$

where $\nu_v(\mathfrak{b}, \mathcal{D})$ is the push–forward of $\nu(\mathfrak{b}, \mathcal{D})|_{\mathbf{O} \times \mathcal{U}}$ to the component W_v of $\mathbf{O} \times \mathcal{U}$ corresponding to v (so $W_\mathfrak{p} = \mathbf{O}$ and $W_v = U_v$ for $v|\mathfrak{f}\mathfrak{g}$). Thus, the projection of Δ onto $F_\mathfrak{p}^*$ equals precisely $u_T(\mathfrak{b}, \mathcal{D})$.

We now have to prove that for $v|\mathfrak{f}\mathfrak{g}$, the projection of Δ onto U_v equals 1. This will allow us to take $u_T = u_T((1), \mathcal{D}) \equiv 1 \pmod{T}$, $u_T \in U_\mathfrak{p}$, since in $F_\mathfrak{p}^* \times \mathcal{U}/\overline{E(\mathfrak{f})}$, we will have

$$\begin{aligned}
(u_T^{\sigma_\mathfrak{b}}, 1) &= (u_T(\mathfrak{b}, \mathcal{D}), 1) \\
&= \text{image of} \quad \Delta \quad \text{in} \quad F_\mathfrak{p}^* \times \mathcal{U}/\overline{E(\mathfrak{f})} \\
&= \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O} \times \mathcal{U}/\overline{E(\mathfrak{f})}} x \; d\nu_*(\mathfrak{b}, \mathcal{D}, x) \\
&= \pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{\mathbf{O} \times \mathcal{U}/\overline{E(\mathfrak{f})}} x \; d\mu(\mathfrak{b}, x).
\end{aligned}$$

Now, we fix a prime $v \in R$. Define measures $\nu_0(\mathfrak{b}, \mathcal{D})$ and $\nu_1(\mathfrak{b}, \mathcal{D})$ on U_v by

$$\begin{aligned}
\nu_0(\mathfrak{b}, \mathcal{D}, U) &= \nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_\mathfrak{p} \times U \times \prod_{w \in R - \{v\}} U_w) \\
\nu_1(\mathfrak{b}, \mathcal{D}, U) &= \nu(\mathfrak{b}, \mathcal{D}, \pi\mathcal{O}_\mathfrak{p} \times U \times \prod_{w \in R - \{v\}} U_w).
\end{aligned}$$

What we have to prove reduces now to

$$\frac{\times\!\!\!\!\!\!\int_{U_v} x \; d\nu_1(\mathfrak{b}, \mathcal{D}, x)}{\times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \mathcal{D}, x)} = \epsilon(\mathfrak{b}, \mathcal{D}, \pi)\pi^{\zeta_{R,T}(H_\mathfrak{f}/F, \mathfrak{b}, 0)}.$$

But an easy change of variable $\alpha' = \pi^{-1}\alpha$ yields

$$\nu_1(\mathfrak{b}, \mathcal{D}, U) = \nu_0(\mathfrak{b}, \pi^{-1}\mathcal{D}, \pi^{-1}U)$$

and so we can write

$$\begin{aligned}
\times\!\!\!\!\!\!\int_{U_v} x \; d\nu_1(\mathfrak{b}, \mathcal{D}, x) &= \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \pi^{-1}\mathcal{D}, \pi^{-1}x) \\
&= \pi^{\nu_0(\mathfrak{b}, \pi^{-1}\mathcal{D}, U_v)} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \pi^{-1}\mathcal{D}, x) \\
&= \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F, \mathfrak{b}, 0)} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \pi^{-1}\mathcal{D}, x).
\end{aligned}$$

Since $\mathcal{D}$ is a fundamental domain for the action of $E(\mathfrak{f})$ on Q, we have

$$\times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(b, \pi^{-1}\mathcal{D}, x) = \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, x)$$

and

$$\begin{aligned}
\times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \mathcal{D}, x) &= \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \mathcal{D} \cap \epsilon^{-1}\pi^{-1}\mathcal{D}, x) \\
&= \prod_{\epsilon \in E(\mathfrak{f})} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \epsilon x) \\
&= \prod_{\epsilon \in E(\mathfrak{f})} \left(\epsilon^{-\nu_0(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, U_v)} \times\!\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, x) \right),
\end{aligned}$$

hence

$$\frac{\times\!\!\!\!\!\int_{U_v} x \; d\nu_0(b, \pi^{-1}\mathcal{D}, x)}{\times\!\!\!\!\!\int_{U_v} x \; d\nu_0(\mathfrak{b}, \mathcal{D}, x)} = \epsilon(\mathfrak{b}, \mathcal{D}, \pi),$$

which finishes the proof. $\square$

Chapter 8

Computing the conjectural Gross–Stark unit

The goal of this chapter is to provide numerical evidence for Dasgupta's conjecture in the case of a quadratic and a cubic totally real fields. While it is easy to compute the Gross–Stark unit in practice, we now have to compute the conjectural element $u_T(\mathfrak{b}, \mathcal{D}) \in F_{\mathfrak{p}}^*$ from Dasgupta's formula, as well as its Galois conjugates, form its minimal polynomial and check that it agrees with the minimal polynomial of the Gross–Stark unit up to a desired $\mathfrak{p}$-adic accuracy.

The main term that we have to compute from Dasgupta's formula is the multiplicative integral; the naive approach by forming Riemann products is inefficient so we need an alternative formula. This computation can be easily reduced to the one of a certain additive integral (briefly, after taking $\log_p$ and then applying $\exp_p$). Thus, we have to compute a measure which is a generalization of the measure $\nu(\mathfrak{b}, \mathcal{D})$. We proceed by modifying the arguments in [4] to write the desired measure in terms of Shintani zeta functions, whose values at $s = 0$ are easy to obtain by modifying the analytic result from [12].

8.1 The analytic ingredient

Let $A = (a_{jk})$ $(1 \leq j \leq r, 1 \leq k \leq n)$ be an $r \times n$ matrix with positive entries. Consider the linear forms

$$L_j(t_1, ..., t_n) = \sum_{k=1}^{n} a_{jk} t_k, \quad 1 \leq j \leq r$$

and

$$L_k^*(z_1, ..., z_r) = \sum_{j=1}^{r} a_{jk} z_j, \quad 1 \leq k \leq n.$$

Let $x = (x_1, ..., x_r)$ with each $x_j > 0$ and let $\chi = (\chi_1, ..., \chi_r)$ be an r-tuple of complex numbers with $|\chi_j| \leq 1$ for all $j = 1, ..., r$. Let $a_1, ..., a_r$ be nonnegative integers. The Dirichlet series

$$\zeta_{a_1,...,a_r}(A, x, \chi, s) = \sum_{z_1,...,z_r=0}^{\infty} \frac{\chi_1^{z_1} \cdots \chi_r^{z_r} z_1^{a_1} \cdots z_r^{a_r}}{\prod_{k=1}^{n} (L_k^*(z+x))^s} \tag{8.1}$$

converges locally uniformly and absolutely for $Re(s) > \frac{r(1+max(a_1,...,a_r))}{n}$ and defines a holomorphic function for such s, called a Shintani zeta function.

For an integer $a \geq 0$, there exists a polynomial $Q_a(q) \in \mathbb{Z}[q]$ such that

$$\sum_{n=0}^{\infty} n^a q^n = \frac{Q_a(q)}{(1-q)^{a+1}} \quad \text{for} \quad |q| < 1.$$

We now mimic the proof Proposition 1 in [12] (where $a_i = 0$) to prove

Proposition 8. *The function $\zeta_{a_1,...,a_r}$ extends to a meromorphic function on $\mathbb{C}$. If $\chi_j \neq 1$ for all j, then*

$$\zeta_{a_1,...,a_r}(A, x, \chi, 0) = \frac{Q_{a_1}(\chi_1)}{(1-\chi_1)^{a_1+1}} \cdots \frac{Q_{a_r}(\chi_r)}{(1-\chi_r)^{a_r+1}}.$$

Proof. Let $\Gamma(s)$ be the classical Gamma function. For $b > 0$, we have $\Gamma(s)b^{-s} = \int_0^\infty e^{-bt}t^{s-1}dt$ and thus

$$\Gamma(s)^n \prod_{k=1}^{n} L_k^*(z+x)^{-s} = \int_0^\infty \cdots \int_0^\infty e^{-\sum_{k=1}^n t_k L_k^*(z+x)} (t_1 ... t_n)^{s-1} dt_1 \dots dt_n$$
$$= \int_0^\infty \cdots \int_0^\infty e^{-\sum_{j=1}^r (z_j+x_j)L_j(t)} (t_1 ... t_n)^{s-1} dt_1 \dots dt_n.$$

Therefore, since $|\chi_j e^{-L_j(t)}| < 1$, we can write

$$\Gamma(s)^n \zeta_{a_1,...,a_r}(A, x, \chi, s)$$
$$= \int_0^\infty \cdots \int_0^\infty \sum_{z_1,...,z_r=0}^{\infty} \chi_1^{z_1} \dots \chi_r^{z_r} z_1^{a_1} \dots z_r^{a_r} e^{-\sum_{j=1}^r (z_j+x_j)L_j(t)} (t_1 \dots t_n)^{s-1} dt_1 \dots dt_n$$
$$= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \left(\sum_{z_j=0}^{\infty} \chi_j^{z_j} z_j^{a_j} e^{-z_j L_j(t)} \right) e^{-\sum_{j=1}^r x_j L_j(t)} (t_1 ... t_n)^{s-1} dt_1 \dots dt_n$$
$$= \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{r} \left(\frac{Q_{a_j}(\chi_j e^{-L_j(t)})}{(1-\chi_j e^{-L_j(t)})^{a_j+1}} \right) e^{-\sum_{j=1}^r x_j L_j(t)} (t_1 ... t_n)^{s-1} dt_1 \dots dt_n$$
$$= \int_0^\infty \cdots \int_0^\infty g(t)(t_1 \dots t_n)^{s-1} dt_1 \dots dt_n,$$

where

$$g(t) = \prod_{j=1}^{r} \frac{Q_{a_j}(\chi_j e^{-L_j(t)}) e^{(a_j+1)L_j(t)}}{(e^{L_j(t)} - \chi_j)^{a_j+1}} e^{-x_j L_j(t)}.$$

For $1 \leq k \leq n$, consider the domain

$$D_k = \{t \in \mathbb{R}^n \mid 0 \leq t_l \leq t_k \quad \text{for all} \quad l = 1, ..., n\}.$$

Since the integral over a set of Lebesgue measure zero equals zero, we have that

$$\Gamma(s)^n \zeta_{a_1,...,a_r}(A, x, \chi, s) = \sum_{k=1}^{n} \int_{D_k} g(t)(t_1 ... t_n)^{s-1} dt_1 \dots dt_n.$$

Let

$$A_k = \Gamma(s)^{-n} \int_{D_k} g(t)(t_1...t_n)^{s-1} dt_1 \dots dt_n.$$

On D_k, consider a change of variables $t = uy$, with $u > 0, 0 \leq y_l \leq 1$ for $l \neq k$, and $y_k = 1$. Thus,

$$A_k = \Gamma(s)^{-n} \int_0^\infty \int_0^1 \dots \int_0^1 g(uy) u^{ns-1} (y_1...\widehat{y_k}...y_n)^{s-1} dy_1 \dots \widehat{dy_k} \dots dy_n du.$$

For $0 < \epsilon < 1$, let $I_\epsilon(1)$ (respectively $I_\epsilon(\infty)$) be the contour consisting of the interval $[1, \epsilon]$, (respectively $[\infty, \epsilon]$) followed by the counterclockwise circle of radius ϵ, followed by the interval $[\epsilon, 1]$ (respectively $[\epsilon, \infty]$).

If we fix the variables $u, y_2, ..., \widehat{y_k}, ..., y_n > 0$, also s with $Re(s)$ sufficiently big, and let $y_1 = y$ vary, we have to consider the integral

$$I = \int_0^1 h(y) y^{s-1} dy,$$

where

$$h(y) = \int_0^\infty \int_0^1 \dots \int_0^1 g(uy, uy_2, ..., u, ..., uy_n) u^{ns-1} (y_2...\widehat{y_s}...y_n)^{s-1} dy_2 \dots \widehat{dy_s} \dots dy_n du.$$

Since the denominator of $g(t)$ involves expressions of the form $(e^{L_j(t)} - \chi_j)^{a_j+1}$ and $L_j(t)$ is a linear form with positive coefficients, for $y_1 = y$ close enough to 0 ($y \in \mathbb{C}$), $e^{L_j(uy)}$ is close to a real number of absolute value greater than 1. So, there exists $\epsilon > 0$ such that $h(y)$ is holomorphic on a neighborhood of $\{z \mid 0 < |z| \leq \epsilon\}$. We claim that

$$\int_0^1 h(y) y^{s-1} dy = \frac{1}{e^{2\pi i s} - 1} \int_{I_\epsilon(1)} h(y) y^{s-1} dy,$$

or equivalently,

$$(e^{2\pi i s} - 1) \int_0^\epsilon h(y) y^{s-1} dy = \int_{|z|=\epsilon} h(y) y^{s-1} dy. \tag{8.2}$$

But, given ϵ, we can take δ sufficiently small and apply Cauchy's integral formula for the contour given by a counterclockwise circle of radius ϵ followed by the segment $[\epsilon, \delta]$, followed by the clockwise circle of radius δ, and finally by $[\delta, \epsilon]$. Then (8.2) reduces to

$$\lim_{\delta \to 0} \int_{|y|=\delta} h(y) y^{s-1} dy = 0,$$

which in turn follows from the expression for h, since $|h(\delta e^{i\theta}) \delta^s|$ approaches 0 as $\delta \to 0$. We argue similarly for the other variables (when we treat u, we recall that $Re(s)$ is sufficiently large) and deduce that

$$\begin{aligned} A_k &= \frac{\Gamma(s)^{-n}}{(e^{2n\pi i s} - 1)(e^{2\pi i s} - 1)^{n-1}} \int_{I_\epsilon(\infty)} \int_{I_\epsilon(1)^{n-1}} g(uy) u^{ns-1} (\prod_{l \neq k} y_l)^{s-1} (\prod_{l \neq k} dy_l) du \\ &= \frac{\Gamma(1-s)^n}{e^{n\pi i s}} \frac{(e^{2\pi i s} - 1)}{(e^{2n\pi i s} - 1)} \frac{1}{(2\pi i)^n} \int_{I_\epsilon(\infty)} \int_{I_\epsilon(1)^{n-1}} g(uy) u^{ns-1} (\prod_{l \neq k} y_l)^{s-1} (\prod_{l \neq k} dy_l) du \end{aligned}$$

This expression defines a meromorphic function on $\mathbb{C}$ whose value at $s = 0$ equals $\frac{1}{n}c_k$, where c_k is the constant term in the Taylor expansion of $g(uy_1, ..., uy_{k-1}, u, uy_{k+1}, \dots uy_n)$ around the origin. In particular, when $\chi_j \neq 1$ for all j, we obtain

$$\zeta_{a_1,\dots,a_r}(A, x, \chi, 0) = \prod_{j=1}^{r} \frac{Q_{a_j}(\chi_j)}{(1-\chi_j)^{a_j+1}},$$

as desired. $\square$

8.2 Reduction to the additive integral

Let F be a totally real number field of degree n over $\mathbb{Q}$, and let $p > 2$ be a prime of $\mathbb{Q}$ which is inert in F, with $\mathfrak{p} = (p)$. In this case, the map $\log = \log_p$ identifies

$$(1 + \mathfrak{p}\mathcal{O}_\mathfrak{p})^* \simeq \mathfrak{p}\mathcal{O}_\mathfrak{p},$$

and its inverse is given by $\exp = \exp_p$. Moreover, we know that if $x \in \mathfrak{p}$, then $\exp(x) = 1 + c$ where $\nu_\mathfrak{p}(c) = \nu_\mathfrak{p}(x)$, and $\log(1 + x) = c'$ where $\nu_\mathfrak{p}(c') = \nu_\mathfrak{p}(x)$. Also (cf [9]), if $n = a_0 + a_1 p + \cdots + a_k p^k$ with $0 \le a_i < p$, then

$$\nu_\mathfrak{p}\left(\frac{x^n}{n!}\right) = n\left(\nu_\mathfrak{p}(x) - \frac{1}{p-1}\right) + \frac{1}{p-1}(a_0 + a_1 + \cdots + a_k).$$

Finally, we know that

$$\mathcal{O}_\mathfrak{p}^* \simeq \mu_{F_\mathfrak{p}} \times (1 + \mathfrak{p}\mathcal{O}_\mathfrak{p})^* \simeq (\mathcal{O}_\mathfrak{p}/\mathfrak{p})^* \times (1 + \mathfrak{p}\mathcal{O}_\mathfrak{p})^*.$$

For a $\mathbb{Z}$-valued measure ν on $\mathcal{O}_\mathfrak{p}^*$, the goal is to evaluate the multiplicative integral

$$A = \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^*} x d\nu(x) \in \mathcal{O}_\mathfrak{p}^*$$

up to M $\mathfrak{p}$-adic digits. We reduce the calculation to a certain additive integral.

First, take

$$A_0 = \prod_{a \in (\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*} a^{\nu(a + \mathfrak{p}\mathcal{O}_\mathfrak{p})} \in (\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*$$

and find the root of unity $\gamma \in \mathcal{O}_\mathfrak{p}^*$ which reduces to A_0 modulo $\mathfrak{p}$, so that $A = \gamma \frac{A}{\gamma}$ and $\frac{A}{\gamma} \in (1 + \mathfrak{p}\mathcal{O}_\mathfrak{p})^*$. The problem now reduces to computing $\frac{A}{\gamma}$ up to M $\mathfrak{p}$-adic digits. But if $x, y \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$ and $x \equiv y \pmod{\mathfrak{p}^M}$, then $\exp(x) \equiv \exp(y) \pmod{\mathfrak{p}^M}$, and so we have to find $B \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$ such that

$$\log\left(\frac{A}{\gamma}\right) \equiv B \pmod{\mathfrak{p}^M}; \tag{8.3}$$

then we will have that $\frac{A}{\gamma} \equiv \exp(B) \pmod{\mathfrak{p}^M}$.

Moreover, given B, to compute $\exp(B)$ modulo $\mathfrak{p}^M$, it suffices to truncate the series for exp at the smallest index k such that $k + 1 \ge M\frac{p-1}{p-2}$ because for $m \ge k + 1$, we have $\nu_\mathfrak{p}(\frac{B^m}{m!}) > m(\nu_\mathfrak{p}(B) - \frac{1}{p-1}) \ge m\frac{p-2}{p-1} \ge M$.

To compute B, we note that if $a, b \in \mathcal{O}_{\mathfrak{p}}^*$ and $a \equiv b \pmod{\mathfrak{p}^M}$, then $\log(a) \equiv \log(b) \pmod{\mathfrak{p}^M}$. This holds because the same root of unity γ is congruent to both a and b modulo $\mathfrak{p}$, and $\frac{a}{\gamma} \equiv \frac{b}{\gamma} \pmod{\mathfrak{p}^M}$, so we reduce to the case $a, b \in 1 + \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$. Replacing a by $\frac{a}{b}$ further reduces to $b = 1$. But, we know that if $a \equiv 1 \pmod{\mathfrak{p}^M}$, then $\log(a) = \log(1 + (a-1))$ has valuation equal to $\nu_{\mathfrak{p}}(a-1) \geq M$.

For $a \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^*$, let

$$A_a = \times\!\!\!\!\!\!\int_{a+\mathfrak{p}\mathcal{O}_{\mathfrak{p}}} x d\nu(x) \in \mathcal{O}_{\mathfrak{p}}^*,$$

so $A = \prod A_a$ and $\log(A) = \sum \log(A_a)$. Modulo $\mathfrak{p}^M$, we have that

$$\begin{aligned}
\log A_a &\equiv \log \prod_{\substack{b \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^M)^* \\ b \equiv a \pmod{\mathfrak{p}}}} b^{\nu(b+\mathfrak{p}^M\mathcal{O}_{\mathfrak{p}})} \\
&= \sum_{\substack{b \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^M)^* \\ b \equiv a \pmod{\mathfrak{p}}}} \nu(b + \mathfrak{p}^M\mathcal{O}_{\mathfrak{p}}) \log(b) \\
&= \sum_{\substack{b \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^M)^* \\ b \equiv a \pmod{\mathfrak{p}}}} \nu(b + \mathfrak{p}^M\mathcal{O}_{\mathfrak{p}}) \left(\log\left(1 + (\frac{b}{a} - 1)\right) + \log(a)\right) \\
&= (\log a)\nu(a + \mathfrak{p}\mathcal{O}_{\mathfrak{p}}) \\
&+ \sum_{\substack{b \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^M)^* \\ b \equiv a \pmod{\mathfrak{p}}}} \nu(b + \mathfrak{p}^M\mathcal{O}_{\mathfrak{p}}) \left(\left(\frac{b}{a} - 1\right) - \frac{1}{2}\left(\frac{b}{a} - 1\right)^2 + \frac{1}{3}\left(\frac{b}{a} - 1\right)^3 - \dots\right)
\end{aligned}$$

We need to know where to truncate the series. For $y \in \mathfrak{p}\mathcal{O}_{\mathfrak{p}}$, we have that

$$\begin{aligned}
\nu_{\mathfrak{p}}\left(\frac{y^m}{m}\right) &= \nu_{\mathfrak{p}}\left(\frac{y^m}{m!}\right) + \nu_{\mathfrak{p}}((m-1)!) \\
&> m\frac{p-2}{p-1} + \nu_{\mathfrak{p}}((m-1)!) \\
&\geq M
\end{aligned}$$

provided $m \geq k+1$, where k is the smallest integer such that $(k+1)\frac{p-2}{p-1} + \nu_p(k!) \geq M$.

For this choice of k, write

$$\left(\frac{b}{a} - 1\right) - \frac{1}{2}\left(\frac{b}{a} - 1\right)^2 + \dots + \frac{(-1)^{k-1}}{k}\left(\frac{b}{a} - 1\right)^k = c_k(a)b^k + c_{k-1}(a)b^{k-1} + \dots + c_0(a).$$

Thus, again modulo $\mathfrak{p}^M$, we find

$$\begin{aligned}
\log A_a &\equiv (\log a)\nu(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) \\
&+ \sum_{\substack{b\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^M)^* \\ b\equiv a \pmod{\mathfrak{p}}}} \nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p})(c_k(a)b^k + c_{k-1}(a)b^{k-1} + \cdots + c_0(a)) \\
&= (\log a)\nu(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) \\
&+ c_k(a) \sum_{\substack{b\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^M)^* \\ b\equiv a \pmod{\mathfrak{p}}}} b^k\nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p}) + \cdots + c_0(a) \sum_{\substack{b\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^M)^* \\ b\equiv a \pmod{\mathfrak{p}}}} \nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p}) \\
&\equiv (\log a)\nu(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) + c_k(a)\int_{a+\mathfrak{p}\mathcal{O}_\mathfrak{p}} x^k d\nu(x) + \cdots + c_0(a)\int_{a+\mathfrak{p}\mathcal{O}_\mathfrak{p}} d\nu(x) \\
&= (\log a)\nu(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) + c_k(a)\mu_k(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) + \cdots + c_0(a)\mu_0(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}),
\end{aligned}$$

where μ_i is defined on compact open subsets of $\mathcal{O}_\mathfrak{p}$ via

$$\mu_i(U) = \int_U x^i d\nu(x).$$

Finally, the congruence

$$B \equiv \log A = \sum_{a\in(\mathcal{O}_\mathfrak{p}/\mathfrak{p})^*} \log A_a \pmod{\mathfrak{p}^M}$$

reduces the computation of A to the one of $\mu_i(a+\mathfrak{p}\mathcal{O}_\mathfrak{p})$.

8.3 The additive integral

Here we generalize the computation of the measure ν in [4] (which corresponds to $k=0$) with appropriate modifications of the arguments.

Fix a totally real number field F of degree n over $\mathbb{Q}$, a prime p of $\mathbb{Q}$ and a prime $\mathfrak{p}$ of F lying over p. Next, $\mathfrak{f}$ is an integral ideal of $\mathcal{O}_F$ prime to $\mathfrak{p}$, S is a finite set of primes of F containing the archimedean primes, the ones dividing $\mathfrak{f}$, and $\mathfrak{p}$. Recall the notation $R = S - \{\mathfrak{p}\}$. We take $T = \{\eta\}$, where $l = N\eta$ is prime (and of course, l is prime to S). Let $\mathfrak{b}$ be a fractional ideal of F prime to S and l. Consider a simplicial cone $C = C(v_1, ..., v_r)$ of dimension r with $v_i \in \mathcal{O}_F - \eta$ (i.e., T is good for C). The goal of this section is to compute the additive integral

$$\int_U x^k d\nu(\mathfrak{b}, C, x) \in \mathcal{O}_\mathfrak{p}$$

for a compact open subset $U \subset \mathcal{O}_\mathfrak{p}$.

8.3.1 Expression in terms of Shintani zeta functions

Let $K = F(\mu_l)$. Fix an embedding $\tau : K \hookrightarrow \mathbb{C}$. For a compact open $U \subset \mathcal{O}_\mathfrak{p}$, consider the series

$$\zeta_k(\mathfrak{b}, C, U, s) = \sum_{\substack{(\alpha,R)=1 \\ \alpha\in\mathfrak{b}^{-1}\cap C \\ \alpha\equiv 1 \pmod{\mathfrak{f}} \\ \alpha\in U}} \frac{\tau(\alpha)^k}{N\alpha^s} - l \sum_{\substack{(\alpha,R)=1 \\ \alpha\in\mathfrak{b}^{-1}\eta\cap C \\ \alpha\equiv 1 \pmod{\mathfrak{f}} \\ \alpha\in U}} \frac{\tau(\alpha)^k}{N\alpha^s}.$$

The series converges absolutely for $Re(s) > \frac{r}{n}(k+1)$, as will become evident from the form of ζ_k given below.

Given a compact open $U \subset \mathcal{O}_\mathfrak{p}$, choose e such that U is a finite disjoint union of translates of $\mathfrak{p}^e\mathcal{O}_\mathfrak{p}$. Set

$$\mathfrak{a} = \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{p}^e \prod_{\substack{v\in R,(v,\mathfrak{f})=1 \\ v \quad \text{finite}}} v,$$

so that by the Chinese Remainder Theorem, we can find $y_i \in \eta, i = 1, ..., d$, such that

$$\left\{\alpha \in \mathfrak{b}^{-1} \cap U \mid (\alpha, R) = 1, \alpha \equiv 1 \pmod{\mathfrak{f}}\right\} = \bigcup_{i=1}^{d} (\mathfrak{a} + y_i) \quad \text{(disjoint union)}.$$

So, if we define

$$Z_k(\mathfrak{a}, y, C, s) = \sum_{\alpha\in(\mathfrak{a}+y)\cap C} \frac{\tau(\alpha)^k}{N\alpha^s},$$

we have that

$$\zeta_k(\mathfrak{b}, C, U, s) = \sum_{i=1}^{d} (Z_k(\mathfrak{a}, y_i, C, s) - lZ_k(\mathfrak{a}\eta, y_i, C, s)).$$

Choose an integer in $\mathfrak{a}$ but not in η and multiply all generators v_i of the cone C by that integer, so $C = C(v_1, ..., v_r)$ with $v_i \in \mathfrak{a} - \mathfrak{a}\eta$. Denote

$$\Omega(\mathfrak{a}, y, v) = \left\{x \in \mathfrak{a} + y \mid x = \sum_{i=1}^{r} x_i v_i \quad \text{with} \quad 0 < x_i \leq 1\right\}.$$

This is a finite set because it is the intersection of a translate of a lattice and a compact. For $\alpha \in C$, we can write uniquely

$$\alpha = \sum_{i=1}^{r} (x_i + z_i) v_i,$$

where $0 < x_i \leq 1$ and $z_i \in \mathbb{Z}, z_i \geq 0$. So, $\alpha = x + \sum_{i=1}^{r} z_i v_i$ belongs to $\mathfrak{a} + y$ if and only if $x \in \Omega(\mathfrak{a}, y, v)$. Thus,

$$Z_k(\mathfrak{a}, y, C, s) = \sum_{x\in\Omega(\mathfrak{a},y,v)} \sum_{z_1,...,z_r=0}^{\infty} \frac{\tau(\sum_{i=1}^{r}(x_i + z_i)v_i)^k}{N(\sum_{i=1}^{r}(x_i + z_i)v_i)^s}.$$

Fix a nontrivial character $\chi_0 : \mathfrak{a}/\mathfrak{a}\eta \simeq \mathcal{O}_F/\eta \simeq \mathbb{Z}/l\mathbb{Z} \to F(\mu_l)$ and consider the composition $\chi = \tau \circ \chi_0$. To study the shift $Z_k(\mathfrak{a}, y, C, v) - lZ_k(\mathfrak{a}\eta, y, C, v)$, we will use the orthogonality relation: if $a \in \mathfrak{a}$, then

$$\sum_{t=0}^{l-1} \chi(a)^t = \begin{cases} l, & \text{if} \quad a \in \mathfrak{a}\eta \\ 0, & \text{if} \quad a \notin \mathfrak{a}\eta. \end{cases}$$

Namely, we can write

$$\begin{aligned} lZ_k(\mathfrak{a}\eta, y, C, s) &= l \sum_{x \in \Omega(\mathfrak{a}\eta, y, v)} \sum_{z_1, \ldots, z_r = 0}^{\infty} \frac{\tau(\sum_{i=1}^r (x_i + z_i) v_i)^k}{N(\sum_{i=1}^r (x_i + z_i) v_i)^s} \\ &= \sum_{x \in \Omega(\mathfrak{a}, y, v)} \sum_{t=0}^{l-1} \sum_{z_1, \ldots, z_r = 0}^{\infty} \chi(y - x - \sum_{i=1}^r z_i v_i)^t \frac{\tau(\sum_{i=1}^r (x_i + z_i) v_i)^k}{N(\sum_{i=1}^r (x_i + z_i) v_i)^s} \end{aligned}$$

and hence

$$\begin{aligned} &Z_k(\mathfrak{a}, y, C, s) - lZ_k(\mathfrak{a}\eta, y, C, s) = \\ &- \sum_{x \in \Omega(\mathfrak{a}, y, v)} \sum_{t=1}^{l-1} \sum_{z_1, \ldots, z_r = 0}^{\infty} \chi(y - x - \sum_{i=1}^r z_i v_i)^t \frac{\tau(\sum_{i=1}^r (x_i + z_i) v_i)^k}{N(\sum_{i=1}^r (x_i + z_i) v_i)^s} \\ &= - \sum_{x \in \Omega(a, y, v)} \sum_{t=1}^{l-1} \chi(y - x)^t \sum_{z_1, \ldots, z_r = 0}^{\infty} \frac{\left(\prod_{j=1}^r (\chi(-v_j)^t)^{z_j}\right) \left(\tau(x) + \sum_{j=1}^r z_j \tau(v_j)\right)^k}{N(\sum_{j=1}^r (x_j + z_j) v_j)^s}. \end{aligned} \tag{8.4}$$

To recognize this as a sum of Shintani zeta functions, consider the matrix

$$A_v = \begin{pmatrix} v_1^1 & v_1^2 & \ldots & v_1^n \\ \cdots & & & \\ v_r^1 & v_r^2 & \ldots & v_r^n \end{pmatrix}$$

and note that if we expand binomially the sum

$$\left(\tau(x) + \sum_{j=1}^r z_j \tau(v_j)\right)^k,$$

we obtain that $Z_k(\mathfrak{a}, y, C, s) - lZ_k(\mathfrak{a}\eta, y, C, s)$ is a finite sum of products of elements in $\tau(K)$ and sums of the form (8.1) with respect to the matrix A_v, and where $\chi_j = \chi(-v_j)^t$ are nontrivial roots of unity. Each χ_i then also belongs to $\tau(K)$ and by the explicit formula for the value at $s = 0$ of (8.1), we deduce that the value of $Z_k(\mathfrak{a}, y, C, s) - lZ_k(\mathfrak{a}\eta, y, C, s)$ at $s = 0$ belongs to $\tau(K)$ also.

Fix a prime $\mathfrak{B}$ of K lying over $\mathfrak{p}$. By above, we can define a $K_{\mathfrak{B}}$-valued distribution μ_k on $\mathcal{O}_{\mathfrak{p}}$ via

$$\mu_k(U) = \mu_k(\mathfrak{b}, C, U) = \tau^{-1} \zeta_k(\mathfrak{b}, C, U, 0).$$

8.3.2 Integrality of μ_k and the congruence it satisfies

By the explicit formula for the value of (8.1) at $s=0$, we see that this value in fact belongs to $\mathcal{O}_{\mathfrak{B}}$ because if ζ is a primitive l-th root of unity, $\frac{1}{1-\zeta}$ belongs to the valuation ring of $\mathbb{Q}_p(\mu_l)$, hence to $\mathcal{O}_{\mathfrak{B}}$. Taking into account that $v_j \in \mathfrak{a} \subset \mathcal{O}_{\mathfrak{p}}$ and that $x = \sum x_j v_j \in \mathfrak{a} + y_i$ for some i, with $y_i \in \mathcal{O}_F$, we deduce that in fact

$$\mu_k(U) \in \mathcal{O}_{\mathfrak{B}} \quad \text{for each compact open} \quad U \subset \mathcal{O}_{\mathfrak{p}}.$$

Now suppose that $U = a + \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}}$. Then we can take

$$\mathfrak{a} = \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{p}^N \prod_{\substack{v \text{ finite}, v \in R \\ (v,\mathfrak{f})=1}} v$$

and so

$$v_j \in \mathfrak{a} \subset \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}} \subset \mathfrak{B}^N.$$

Also, for $x = \sum x_j v_j \in \Omega(\mathfrak{a}, y_i, v)$, we have that $x \equiv a \pmod{\mathfrak{p}^N}$, hence also $x^k \equiv a^k \pmod{\mathfrak{B}^N}$. It is now clear that when we expand the sum (8.4) and set $s=0$, we obtain

$$x^k \nu(\mathfrak{b}, C, U) + \quad \text{terms of the form} \quad v_j T_j,$$

where $T_j \in \mathcal{O}_{\mathfrak{B}}$ (note that $\nu(\mathfrak{b}, C, U) = \mu_0(\mathfrak{b}, C, U)$). But this shows that

$$\mu_k(U) = \mu_k(a + \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}}) \equiv x^k \nu(\mathfrak{b}, C, U) \equiv a^k \nu(\mathfrak{b}, C, U) \pmod{\mathfrak{B}^N}$$

because we know that $\nu(\mathfrak{b}, C, U) \in \mathcal{O}_{\mathfrak{B}}$.

This congruence , together with the fact that μ_k is $\mathcal{O}_{\mathfrak{B}}$-valued allows us to compute the additive integral $\int_U x^k d\nu(\mathfrak{b}, C, x) \in \mathcal{O}_{\mathfrak{p}} \subset \mathcal{O}_{\mathfrak{B}}$ for a compact open $U \subset \mathcal{O}_{\mathfrak{p}}$. Indeed, let N be large enough so that U is a finite disjoint union

$$U = \bigcup_{i=1}^{d} (a_i + \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}}).$$

Then we have that

$$\begin{aligned} \int_U x^k d\nu(\mathfrak{b}, C, x) &\equiv \sum_{i=1}^{d} a_i^k \nu(\mathfrak{b}, C, a_i + \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}}) \pmod{\mathfrak{B}^N} \\ &\equiv \sum_{i=1}^{d} \mu_k(a_i + \mathfrak{p}^N \mathcal{O}_{\mathfrak{p}}) \pmod{\mathfrak{B}^N} \\ &= \mu_k(U). \end{aligned}$$

This holds for all sufficiently large N, which implies that

$$\int_U x^k d\nu(\mathfrak{b}, C, x) = \mu_k(U).$$

In particular, μ_k is $\mathcal{O}_{\mathfrak{p}}$-valued.

8.4 Integrality of ν

The case $k = 0$ above implies in particular that

$$
\begin{aligned}
Z_0(\mathfrak{a}, y, C, s) - lZ_0(\mathfrak{a}, y, C, s) &= -\sum_{x\in\Omega(\mathfrak{a},y,v)} \sum_{t=1}^{l-1} \chi(y-x)^t \zeta(A_v, x, (\chi(-v_i)^t), s) \\
&= -\sum_{x\in\Omega(\mathfrak{a},y,v)} \sum_{t=1}^{l-1} \chi(y-x)^t \prod_{i=1}^{r} \frac{1}{1-\chi(-v_i)^t} \\
&= -\sum_{x\in\Omega(\mathfrak{a},y,v)} Tr_{\mathbb{Q}(\mu_l)/\mathbb{Q}} \left(\frac{\chi(y-x)}{\prod_{i=1}^{r}(1-\chi(-v_i))} \right).
\end{aligned}
$$

This shows first that $\nu(\mathfrak{b}, C, U) \in \mathbb{Q}$ and in fact, since for a primitive l-th root of unity ζ, we have that $(1-\zeta)^{l-1} = (l)$ in $\mathbb{Z}[\zeta_l]$, the trace above is an l-integer. Thus, $\nu(\mathfrak{b}, C, U) \in \mathbb{Z}[\frac{1}{l}]$. Moreover, examining the valuation of l, we see that in fact $\nu(\mathfrak{b}, C, U)$ has denominator at most $l^{\frac{r}{l-1}}$.

Thus, if the set T contains two primes of different residue characteristics which are good for an r-dimensional cone C, or a prime η with $l = N\eta \geq r+2$, the value $\nu(\mathfrak{b}, C, U)$ is in $\mathbb{Z}$, for any compact open $U \subset \mathcal{O}_\mathfrak{p}$.

8.5 A quadratic example

Let $F = \mathbb{Q}(\sqrt{11})$ with $\mathcal{O}_F = \mathbb{Z}[\sqrt{11}]$. We take the prime $p = 3$, which is inert in F, $\mathfrak{p} = (3)$, and $l = 5$, which splits, and we fix η over l. Take $\mathfrak{f} = 1$, $S = \{\infty_1, \infty_2, \mathfrak{p}\}$, and $T = \{\eta\}$.

When we take $\mathfrak{b}_1 = 1$, $\mathcal{D} = C(1) \cup C(1, 10 - 3\sqrt{11})$ and compute the multiplicative integral

$$A = \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^*} x d\nu(\mathfrak{b}_1, \mathcal{D}, x) \in \mathcal{O}_\mathfrak{p}^*$$

up to, say, 9 $\mathfrak{p}$-adic digits (set $M = 9$), we obtain

$$A \equiv -118098 + 638972\sqrt{11} \pmod{3^9}.$$

Since $\nu(\mathfrak{b}_1, \mathcal{D}, \mathcal{O}_\mathfrak{p}) = \zeta_{R,T}(H/F, \mathfrak{b}_1, 0) = -1$ in this case, we have to take $u_T(\mathfrak{b}_1, \mathcal{D}) = \frac{A}{3}$. Next, we can work with, say, $\mathfrak{b}_2 = (\sqrt{11})$, which is a representative for the nontrivial coset in the narrow class group, and compute $\nu(\mathfrak{b}_2, \mathcal{D}, \mathcal{O}_\mathfrak{p}) = 1$, and also

$$A' = \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^*} x d\nu(\mathfrak{b}_2, \mathcal{D}, x) \equiv \frac{1}{A} \pmod{3^9}.$$

Thus, $u_T(\mathfrak{b}_1, \mathcal{D})$ and $u_T(\mathfrak{b}_2, \mathcal{D})$ are roots of the polynomial in $F_\mathfrak{p}[x]$, whose coefficients are as follows up to 9 $\mathfrak{p}$-adic digits:

$$x^2 - (\frac{A}{3} + \frac{3}{A})x + 1 \equiv x^2 + \frac{1}{3}\sqrt{11}x + 1 \pmod{3^9}.$$

On the other hand, the narrow Hilbert class field is $H = F(i)$, and we easily compute that the minimal polynomial for the Gross–Stark unit $u_T \in H^*$ is precisely

$$x^2 + \frac{1}{3}\sqrt{11}x + 1 \in F[x].$$

This example is strong computational evidence for Conjecture 5.

8.6 A cubic example

Consider the totally real number field $F = \mathbb{Q}(w)$, where $w^3 + 2w^2 - 3w - 2 = 0$, with ring of integers $\mathbb{Z}[w]$. The two fundamental units of F are $u_1 = w$ and $u_2 = -7w + 4w^2$. Under the three real embeddings of F, u_1 and u_2 have signs $(+--)$ and $(+++)$ respectively. So, the fundamental totally positive units of F are $v_1 = u_1^2 = w^2$ and $v_2 = u_2$.

We choose $\mathfrak{f} = I_1^2$, where $(2) = I_1 I_2$ with I_1, I_2 prime ideals and $N(I_1) = 2$. Then u_1, u_2 are congruent to 1 modulo $\mathfrak{f}$, and since -1 is not congruent to 1 modulo $\mathfrak{f}$, any unit congruent to 1 modulo $\mathfrak{f}$ has signs either $(+--)$ or $(+++)$ under the three embeddings. In particular, the degree of the narrow ray class field $H_\mathfrak{f}$ of F corresponding to the modulus $\mathfrak{f}$ over the wide ray class field $H'_\mathfrak{f}$ of F corresponding to the modulus $\mathfrak{f}$ equals 4. In fact, $H'_\mathfrak{f} = F$, and $H_\mathfrak{f}$ has degree 4 over F. The narrow ray class group G of F for the modulus $\mathfrak{f}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and can be realized as

$$G = \langle (3), I_2 \rangle.$$

We check that $\det(1, v_1, v_1 v_2) \det(1, v_2, v_1 v_2) < 0$, so we can take as a Shintani domain $\mathcal{D}$ for the action of $\langle v_1, v_2 \rangle$ on $\mathbb{R}^2_{>0}$ the union of the following cones:

$$C_1 = C(1), C_2 = C(1, v_1), C_3 = C(1, v_2), C_4 = C(1, v_1 v_2), C_5 = C(1, v_1, v_1 v_2), C_6 = C(1, v_2, v_1 v_2).$$

Fix $\mathfrak{b}_1 = I_2^2$; it is a representative for the trivial class in G. The reason we take $\mathfrak{b}_1 = I_2^2$ and not $\mathfrak{b}_1 = 1$ as a representative for the trivial class in the narrow ray class group corresponding to $\mathfrak{f}$ is that this choice makes the sets $\Omega(\mathfrak{a}, y, v)$ smaller. Indeed, regardless of whether $\mathfrak{b}_1 = 1$ or $\mathfrak{b}_1 = I_2^2$, we still have to multiply the generators of the cones by the same smallest possible choice 20, to ensure they lie in $\mathfrak{a}$ (and not in η). But, when $\mathfrak{b}_1 = I_2^2$, the lattice becomes strictly smaller, and the intersection with the compact set (which does not increase) decreases as well.

We take $p = 5$, which is inert in F, so $\mathfrak{p} = (p)$; then $\mathfrak{p}$ splits completely in $H = H_\mathfrak{f}$. Also, we take η with $(11) = \eta\eta'$ in F, with $N\eta = 11 = l$. We take $S = \{\infty_1, \infty_2, \infty_3, I_1, \mathfrak{p}\}$, $R = S - \{\mathfrak{p}\}$, and $T = \{\eta\}$. Since these will be fixed, we will not always include them in the subsequent notation.

8.6.1 The Gross–Stark unit

Since $\mathfrak{p}$ splits in $H = H_\mathfrak{f}$, we denote by $\mathfrak{B}_i, i = 1, ..., 4$ the primes of H over $\mathfrak{p}$ (i.e., we fix their order). Also, fix $\mathfrak{B} = \mathfrak{B}_1$. We now compute the Gross–Stark unit associated to $\mathfrak{B}$.

Namely, we compute

$$\zeta_{R,T}(H/F,\mathfrak{b},0)=\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})=\begin{cases}-10, & \text{if} \quad \mathfrak{b}=1,\\ 10, & \text{if} \quad \mathfrak{b}=(3),\\ -10, & \text{if} \quad \mathfrak{b}=I_2,\\ 10, & \text{if} \quad \mathfrak{b}=(3)I_2.\end{cases}$$

We check that if $\sigma_1=\sigma_{(3)}$ and $\sigma_2=\sigma_{I_2}$, then $\sigma_1(\mathfrak{B})=\mathfrak{B}_3$ and $\sigma_2(\mathfrak{B})=\mathfrak{B}_4$, so we form

$$\mathfrak{B}_1^{-10}\mathfrak{B}_2^{10}\mathfrak{B}_3^{10}\mathfrak{B}_4^{-10}=(u),$$

and we select u such that $u\equiv 1 \pmod{\eta}$ and $|u|_w=1$ for any infinite w. Then, we compute the minimal polynomial of u over F, and find that it is

$$x^2+\frac{1}{5^{10}}(-1154763w^2-6369741w+5739634)x+1. \tag{8.5}$$

The code for this computation is in the Appendix.

8.6.2 The minimal polynomial of $u_T(\mathfrak{b}_1,\mathcal{D})$

We set $M=5$, which will guarantee that our computation of the multiplicative integral in the formula for $u_T(\mathfrak{b}_1,\mathcal{D})$ will be an approximation up to $\mathfrak{p}^5$. We perform computations in the local field up to 10 p-adic digits, and get the following expression for

$$A=\times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}^*} x\ d\nu(\mathfrak{b}_1,\mathcal{D},x),$$

in the field $F_{\mathfrak{p}}=\mathbb{Q}_p(w)$, where $w^3+2w^2-3w-2=0$:

$$A=295388w^2+729116w+869741+O(5^9).$$

Now, since we know from the computation in the previous subsection that the norm of the Gross–Stark unit is 1 (a–priori, it is ±1), the refined conjecture would predict that $u_T(\mathfrak{b}_1,\mathcal{D})=5^{-10}A$ will satisfy the polynomial

$$x^2-\left(5^{-10}A+\frac{5^{10}}{A}\right)x+1\in F_{\mathfrak{p}}[x],$$

and that the coefficients of this polynomial actually lie in $F[x]$. But, the middle coefficient is

$$\frac{1}{5^{10}}\left(-A-\frac{5^{20}}{A}\right),$$

and we have that $-A-\frac{5^{20}}{A}$ and the numerator of the middle coefficient in (8.5) indeed agree up to 5^6.

8.6.3 The code

The Magma code computing this is the main one, and is included in Appendix B.2 and B.3. It follows the algorithm described in Sections 8.2 and 8.3. I thank Prof. Elkies for letting me know that the efficient way to compute the sets $\Omega(\mathfrak{a}, y, v)$ is through LLL reduction. We first compute all sets Ω that are needed, as well as all character values $\chi(y - x)$, and store them in arrays. Then we compute all the measures $\mu_k(\mathfrak{b}_1, \mathcal{D}, x)$ and store them in an external file. Finally, we just read the stored values and compute the multiplicative integral.

Appendix A

Class field theory

Let K be a number field. A modulus for K is a formal product $\mathfrak{m} = \mathfrak{m}_f \mathfrak{m}_\infty$ where $\mathfrak{m}_f$ is an integral ideal of K and $\mathfrak{m}_\infty$ is a subset of the set if real primes of K.

Let L/K be an abelian extension of number fields. Recall that an infinite prime v of K is ramified in L if v is real but admits an extension to a complex prime of L. If $\mathfrak{p}$ is a finite prime of K which is unramified in L, the Frobenius automorphism $\sigma_\mathfrak{p} \in G(L/K)$ is defined as the unique element in $G(L/K)$ with the property $\sigma_\mathfrak{p}(x) \equiv x^{N\mathfrak{p}} \pmod{\mathfrak{B}}$ for all $x \in \mathcal{O}_L$ and for any prime $\mathfrak{B}$ of L lying over $\mathfrak{p}$. If $\mathfrak{m}_f$ is an ideal of K divisible by all of the (finitely many) ramified finite primes in L, the Artin map

$$\sigma : I_K^{\mathfrak{m}_f} \longrightarrow G(L/K)$$

is defined via

$$I = \prod \mathfrak{p}_i^{a_i} \longmapsto \prod \sigma_{\mathfrak{p}_i}^{a_i} \in G(L/K).$$

The Artin map is surjective, and moreover, given any $\sigma \in G(L/K)$, the set of prime ideals $\mathfrak{p}$ of K with $\sigma_\mathfrak{p} = \sigma$ is infinite (weaker version of Chebotariov's density theorem).

If the exponents of the finite primes in $\mathfrak{m}$ are large enough,

$$i(K_{\mathfrak{m},1}) \subset \ker \sigma.$$

There exists a *smallest* modulus $\mathfrak{f}$ for K divisible precisely by the primes ramified in L (finite or infinite), such that $i(K_{\mathfrak{f},1}) \subset \ker \sigma|_{I_K^\mathfrak{f}}$. This modulus $\mathfrak{f}$ called the conductor of L/K.

If $\mathfrak{m}$ is any modulus for K and H is a subgroup of $I_K^\mathfrak{m}$ such that

$$i(K_{\mathfrak{m},1}) \subset H \subset I_K^\mathfrak{m},$$

there exists a unique abelian extension L/K which is unramified outside $\mathfrak{m}$ and such that the Artin map $I_K^\mathfrak{m} \longrightarrow G(L/K)$ has kernel precisely H and hence induces an isomorphism $I_K^\mathfrak{m}/H \simeq G(L/K)$. When $\mathfrak{f}$ is an integral ideal, $\mathfrak{m} = \mathfrak{f} \times \prod_{v \text{ real}} v$, and $H = i(K_{\mathfrak{m},1})$, the extension L is called the narrow ray class field of K with respect to $\mathfrak{f}$. Any abelian extension of conductor dividing $\mathfrak{m}$ is contained in this field L.

Let L/K be an abelian extension of conductor $\mathfrak{m} = \mathfrak{n}\mathfrak{p}^k$ for some $k \geq 0$, where $\mathfrak{p} \nmid \mathfrak{n}$. The local reciprocity map

$$r_\mathfrak{p} : K_\mathfrak{p}^* \longrightarrow G(L/K)$$

is defined as follows. Consider the composition

$$\theta : K_{\mathfrak{n},1} \xrightarrow{i} I_K^{\mathfrak{n}} \to I_K^{\mathfrak{m}} \xrightarrow{\sigma} G(L/K),$$

where the map $I_K^{\mathfrak{n}} \to I_K^{\mathfrak{m}}$ is given by dropping the factor of $\mathfrak{p}$, i.e., by $\mathfrak{a} \mapsto \mathfrak{a}\mathfrak{p}^{-\nu_{\mathfrak{p}}(\mathfrak{a})}$. If V is the kernel of the map $K_{\mathfrak{n},1} \to K_{\mathfrak{p}}^*/U_{\mathfrak{p}^k}$, then θ is trivial on V by definition. The map $r_{\mathfrak{p}}$ is defined via the composition

$$r_{\mathfrak{p}} : K_{\mathfrak{p}}^* \longrightarrow K_{\mathfrak{p}}^*/U_{\mathfrak{p}^k} \simeq K_{\mathfrak{n},1}/V \xrightarrow{\theta} G(L/K).$$

Concretely, if $\alpha \in K_{\mathfrak{p}}^*$, by the weak approximation theorem, we find $a \in K$ such that $a \equiv 1 \pmod{\mathfrak{n}}$ and $a \equiv \alpha \pmod{\mathfrak{p}^k}$. Then we take $\mathfrak{a} = (a)\mathfrak{p}^{-\nu_{\mathfrak{p}}(a)}$ and have that $r_{\mathfrak{p}}(\alpha) = \sigma_{\mathfrak{a}}$. In particular, if $\mathfrak{p}$ is unramified in L, then

$$r_{\mathfrak{p}}(x) = \sigma_{\mathfrak{p}}^{-\nu_{\mathfrak{p}}(x)}.$$

The image of $r_{\mathfrak{p}}$ is the decomposition group of $\mathfrak{p}$ in $G(L/K)$, and the kernel of $r_{\mathfrak{p}}$ is the group of local norms $N(L_{\mathfrak{B}}^*)$, where $\mathfrak{B}$ is a prime of L lying over $\mathfrak{p}$. Moreover, the kernel of $r_{\mathfrak{p}}|_{\mathcal{O}_{\mathfrak{p}}^*}$ is $U_{\mathfrak{p}^m}$.

Appendix B

The Magma code

B.1 The Gross–Stark unit

Here we compute the Gross–Stark unit in the cubic example from section 8.6.

```
n:=3;
P1<x>:=PolynomialRing(Integers()); f:=x^3+2*x^2-6*x-1;
F:=NumberField(f); O:=RingOfIntegers(F);
p:=5; P:=Factorization(p*O)[1][1];
eta:=Factorization(11*O)[1][1]; Norm(eta);
I:=Factorization(2*O); ff:=I[1][1]^2; q:=I[2][1];
G,m:=RayClassGroup(ff,[1,2,3]);
H:=AbelianExtension(m); HH:=NumberField(H); OO:=RingOfIntegers(HH);
b_temp:=
[
    OO![F![28, 176, -112], F![-2, 30, -16], F![0, -6, 4], F![-5, -1, 2]] / 4,
    OO![F![24, 176, -112], F![12, 26, -18], F![0, -6, 4], F![-5, -1, 2]] / 4,
    OO![F![32, 152, -100], F![-2, 30, -16], F![6, -6, 2], F![-5, -1, 2]] / 4,
    OO![F![-12, -156, 96], F![-12, -26, 18], F![-6, 6, -2], F![5, 1, -2]] / 4
];
beta:=[];
for i:=1 to 4 do beta[i]:=b_temp[i]*OO; end for;

x:=CRT([OO!0,OO!1,OO!1,OO!1],[beta[1],beta[2],beta[3],beta[4]]);

sigma1:=ArtinMap(H)(3*O); sigma2:=ArtinMap(H)(q); sigma3:=ArtinMap(H)(q*(3*O));

OO!sigma1(x) in beta[3]; OO!sigma2(x) in beta[4]; OO!sigma3(x) in beta[2];

b,u0:=IsPrincipal(beta[1]^(-10)*beta[2]^(10)*beta[3]^(10)*beta[4]^(-10)); u0;
T:=ideal<OO|eta>;
```

```
function check(u0,T);
ans:=true;
for i:=1 to #Factorization(T) do
ans:=ans and (Valuation(u0-1,Factorization(T)[i][1]) ge Factorization(T)[i][2]);
end for; return ans; end function;

u0:=
(80557852/9765625*O.1 + 421888581/9765625*O.2 - 52265052/1953125*O.3)*OO.1 +
(23049723/19531250*O.1 + 67281819/9765625*O.2 - 86137569/19531250*O.3)*OO.2 +
(-21364623/19531250*O.1 - 489156/390625*O.2 + 19315023/19531250*O.3)*OO.3 +
(17629923/19531250*O.1 - 15580323/19531250*O.2 + 74694/390625*O.3)*OO.4;

check(u0,T);
Habs:=AbsoluteField(HH); u00:=Habs!u0; AbsoluteValues(u00);

MinimalPolynomial(HH!u0);
```

B.2 Computing the measures μ_k

This is the main part of the code. The purpose is to compute the measure $\mu_k(\mathfrak{b}_1, C, a+\mathfrak{p}\mathcal{O}_\mathfrak{p})$. We include the code from the cubic example. We then store the measures in an array "seqalpha," which we read in the code from the next subsection.

```
n:=3; P1<x>:=PolynomialRing(Integers()); f:=x^3+2*x^2-6*x-1;
F:=NumberField(f); O:=RingOfIntegers(F);
l:=11;eta:=Factorization(l*O)[1][1];
p:=5;P:=Factorization(p*O)[1][1];
I:=Factorization(2*O);ff:=I[1][1]^2;
v1:=F.1^2;v2:=-7*F.1+4*F.1^2;

function multiply_a_cone(v,a);
new_v:=[];
for i:=1 to #v do
Append(~new_v,a*v[i]);
end for;
return new_v;
end function;

C:=[];C[1]:=[O!1];C[2]:=[O!1,v1];C[3]:=[O!1,v2];C[4]:=[O!1,v1*v2];
C[5]:=[O!1,v1,v1*v2];C[6]:=[O!1,v2,v1*v2];
b1:=I[2][1]^2; a:=b1*ff*P; h_e:=20;
D:=[]; for i:=1 to 6 do D[i]:=multiply_a_cone(C[i],h_e); end for;
```

```
//the function below returns a set of representatives for (O/P)^*

function Nonzero_reps();
PP,proj:=quo<O|P>;
G,g:=MultiplicativeGroup(PP);
G1:={};
for x in G do G1:=G1 join {g(x)}; end for;
answ:={};
for x in G1 do answ:=answ join {x@@proj}; end for;
return answ;
end function;


//TT:=SetToSequence(Nonzero_reps() join {O!0});

TT:=
[O![0, 0, 0],O![1, 0, 0],O![2, 0, 0],O![-2, 0, 0],O![-1, 0, 0],
O![-2, 1, 0],O![-1, 1, 0],O![0, 1, 0],O![1, 1, 0],O![2, 1, 0],
O![-2, -2, -1],O![-2, 2, 0],O![-1, -2, -1],O![-1, 2, 0],O![0, -2, -1],
O![0, 2, 0],O![1, -2, -1],O![1, 2, 0],O![2, -2, -1],O![2, 2, 0],
O![-2, -2, 2],O![-1, -2, 2],O![0, -2, 2],O![1, -2, 2],O![2, -2, 2],
O![-2, -1, -1],O![-1, -1, -1],O![0, -1, -1],O![1, -1, -1],O![2, -1, -1],
O![-2, -1, 2],O![-1, -1, 2],O![0, -1, 2],O![1, -1, 2],O![2, -1, 2],
O![-2, 0, -1],O![-1, 0, -1],O![0, 0, -1],O![1, 0, -1],O![2, 0, -1],
O![-2, 0, 2],O![-1, 0, 2],O![0, 0, 2],O![1, 0, 2],O![2, 0, 2],O![-2, 1, -1],
O![-1, 1, -1],O![0, 1, -1],O![1, 1, -1],O![2, 1, -1],O![-2, 1, 2],O![-2, -2, -2],
O![-1, 1, 2],O![-1, -2, -2],O![0, 1, 2],O![0, -2, -2],O![1, 1, 2],O![1, -2, -2],
O![2, 1, 2],O![2, -2, -2],O![-2, -2, 1], O![-2, 2, -1],O![-1, -2, 1],O![-1, 2, -1],
O![0, -2, 1],O![0, 2, -1],O![1, -2, 1],O![1, 2, -1],O![2, -2, 1],O![2, 2, -1],
O![-2, 2, 2],O![-1, 2, 2],O![-2, -1, -2],O![0, 2, 2],O![-1, -1, -2],O![1, 2, 2],
O![0, -1, -2],O![2, 2, 2],O![1, -1, -2],O![2, -1, -2],O![-2, -1, 1],O![-1, -1, 1],
O![0, -1, 1],O![1, -1, 1],O![2, -1, 1],O![-2, 0, -2],O![-1, 0, -2],O![0, 0, -2],
O![1, 0, -2],O![2, 0, -2],O![-2, 0, 1],O![-1, 0, 1],O![0, 0, 1],O![1, 0, 1],
O![2, 0, 1],O![-2, 1, -2],O![-1, 1, -2],O![0, 1, -2],O![1, 1, -2],O![2, 1, -2],
O![-2, 1, 1],O![-1, 1, 1],O![0, 1, 1],O![1, 1, 1],O![2, 1, 1],O![-2, 2, -2],
O![-1, 2, -2],O![0, 2, -2],O![1, 2, -2],O![2, 2, -2],O![-2, 2, 1],O![-2, -2, 0],
O![-1, -2, 0],O![-1, 2, 1],O![0, -2, 0],O![0, 2, 1],O![1, -2, 0],O![1, 2, 1],
O![2, -2, 0],O![2, 2, 1],O![-2, -1, 0], O![-1, -1, 0],O![0, -1, 0],
O![1, -1, 0],O![2, -1, 0]];

function yyy(j);
return CRT([O!0,O!0,TT[j],O!1],[b1,eta,P,ff]);
end function;
```

```
yyyy:=[]; for j:=1 to 125 do yyyy[j]:=yyy(j); end for;
function yy(j);return yyyy[j];end function;

//y is in F, and B is a Q-basis of F; we want to express y in terms of B.

function CoeffWRTBasis(y,B);
BB:=[];
for i:=1 to n do Append(~BB, Eltseq(B[i])); end for;
M:=Matrix(RationalField(), n,n, BB);
y00:=Vector(RationalField(), Eltseq(y));
answ:=Solution(M,y00); return answ;
end function;

//gives the upper bound for tw, w in [-s,1-s]
function upp(t,s); return Maximum(-t*s, t*(1-s)); end function;
//gives the lower bound for tw, w in [-s,1-s]
function low(t,s); return Minimum(-t*s, t*(1-s)); end function;

//a is a fractional ideal, y is in F, v=[v_1,...,v_r] with v_i\in O is a cone.
//Now works only for r=3. (!!!)
function OmegaFast3(a,y,v);
Set_that_we_need:={};
P:=Basis(a);r:=#v;
//r is 3 here.
AA:=CoeffWRTBasis(v[1],P);BB:=CoeffWRTBasis(v[2],P);CC:=CoeffWRTBasis(v[3],P);
DD:=CoeffWRTBasis(y,P);
A:=Matrix(RationalField(), 3,3, [[AA[1],BB[1],CC[1]],[AA[2],BB[2],CC[2]],
[AA[3],BB[3],CC[3]]]);
A_good,L:=LLL(A);
transl:=A^(-1)*Matrix(RationalField(), 3,1,[[DD[1]],[DD[2]],[DD[3]]]);
Range1_l:=Ceiling( low(A_good[1,1],transl[1,1])+low(A_good[1,2],transl[2,1]   )
+low(A_good[1,3],transl[3,1]) );
Range1_r:=Floor( upp(A_good[1,1],transl[1,1])+upp(A_good[1,2],transl[2,1]   )+
upp(A_good[1,3],transl[3,1]) );
Range2_l:=Ceiling( low(A_good[2,1],transl[1,1])+low(A_good[2,2],transl[2,1]   )+
low(A_good[2,3],transl[3,1]) );
Range2_r:=Floor( upp(A_good[2,1],transl[1,1])+upp(A_good[2,2],transl[2,1]   )+
upp(A_good[2,3],transl[3,1]) );
Range3_l:=Ceiling( low(A_good[3,1],transl[1,1])+low(A_good[3,2],transl[2,1]   )+
low(A_good[3,3],transl[3,1]) );
Range3_r:=Floor( upp(A_good[3,1],transl[1,1])+upp(A_good[3,2],transl[2,1]   )+
upp(A_good[3,3],transl[3,1]) );
for w1:=Range1_l to Range1_r do
for w2:=Range2_l to Range2_r do
for w3:=Range3_l to Range3_r do
```

```
p_s:=Solution(Transpose(A_good), Vector(RationalField(),3, [w1,w2,w3]));
if (-transl[1,1] lt p_s[1]) and (p_s[1] le 1-transl[1,1]) and (-transl[2,1] lt p_s[2])
 and (p_s[2] le 1-transl[2,1])
and (-transl[3,1] lt p_s[3]) and (p_s[3] le 1-transl[3,1]) then
z:=L^(-1)*Matrix(Integers(),3,1,[[w1],[w2],[w3]]);
z1:=z[1,1];
z2:=z[2,1];
z3:=z[3,1];
Set_that_we_need:=Set_that_we_need join {z1*P[1]+z2*P[2]+z3*P[3]+y};
end if;end for; end for; end for;
return Set_that_we_need; end function;

function OmegaFast2(a,y,v);
Set_that_we_need:={};P:=Basis(a);r:=#v;
//r is 2 here.
//I am using that for each v in {v1,v2,v1*v2}, (1,v,F.1) is a Q-basis of F !!!
AA:=CoeffWRTBasis(v[1],P);BB:=CoeffWRTBasis(v[2],P);CC:=CoeffWRTBasis(F.1,P);
DD:=CoeffWRTBasis(y,P);
A:=Matrix(RationalField(), 3,3, [[AA[1],BB[1],CC[1]],[AA[2],BB[2],CC[2]],
[AA[3],BB[3],CC[3]]]);
A_good,L:=LLL(A);
transl:=A^(-1)*Matrix(RationalField(), 3,1,[[DD[1]],[DD[2]],[DD[3]]]);
Range1_l:=Ceiling( low(A_good[1,1],transl[1,1])+low(A_good[1,2],transl[2,1]   )
-A_good[1,3]*transl[3,1] );
Range1_r:=Floor( upp(A_good[1,1],transl[1,1])+upp(A_good[1,2],transl[2,1]   )
-A_good[1,3]*transl[3,1] );
Range2_l:=Ceiling( low(A_good[2,1],transl[1,1])+low(A_good[2,2],transl[2,1]   )
-A_good[2,3]*transl[3,1] );
Range2_r:=Floor( upp(A_good[2,1],transl[1,1])+upp(A_good[2,2],transl[2,1]   )
-A_good[2,3]*transl[3,1] );
Range3_l:=Ceiling( low(A_good[3,1],transl[1,1])+low(A_good[3,2],transl[2,1]   )
-A_good[3,3]*transl[3,1] );
Range3_r:=Floor( upp(A_good[3,1],transl[1,1])+upp(A_good[3,2],transl[2,1]   )
-A_good[3,3]*transl[3,1] );
for w1:=Range1_l to Range1_r do for w2:=Range2_l to Range2_r do
for w3:=Range3_l to Range3_r do
p_s:=Solution(Transpose(A_good), Vector(RationalField(),3, [w1,w2,w3]));
if (-transl[1,1] lt p_s[1]) and (p_s[1] le 1-transl[1,1]) and (-transl[2,1] lt p_s[2])
 and (p_s[2] le 1-transl[2,1])
and (-transl[3,1] eq p_s[3]) then
z:=L^(-1)*Matrix(Integers(),3,1,[[w1],[w2],[w3]]); z1:=z[1,1]; z2:=z[2,1]; z3:=z[3,1];
Set_that_we_need:=Set_that_we_need join {z1*P[1]+z2*P[2]+z3*P[3]+y};
end if; end for; end for; end for;
return Set_that_we_need; end function;
```

```
//a is a fractional ideal, y is in F, v=[v_1,...,v_r] with v_i\in O is a cone.
function Omega(a,y,v);
r:=#v;
if (r eq 3) then return OmegaFast3(a,y,v); else
if (r eq 2) then return OmegaFast2(a,y,v); else
Set_that_we_need:={};P:=Basis(a);
A:=[]; for i:=1 to r do Append(~A, CoeffWRTBasis(v[i], P)); end for;
q:=CoeffWRTBasis(y,P); lbound:=[]; ubound:=[];
for i:=1 to n do
lbound[i]:=-q[i]+  &+[ ((1-Sign(A[j,i]))/2)*A[j,i]   : j in [1..r]];
ubound[i]:=-q[i]+  &+[ ((1+Sign(A[j,i]))/2)*A[j,i]   : j in [1..r]];
lbound[i]:=Ceiling(lbound[i]);
ubound[i]:=Floor(ubound[i]);
end for;
//below, nn is an n-tuple of integers.
function add_or_no(nn);
for i:=1 to n do nn[i]:=RationalField()!nn[i]; end for;
ch,L:=IsConsistent(Matrix(RationalField(), r,n, A),
 q+Vector(RationalField(), n, nn));
 if (not ch) then return {}; else
bool:=true; for i:=1 to r do
bool:=bool and (L[i] gt 0) and (L[i] le 1);
end for;
if bool then return
  {&+[nn[j]*P[j]: j in [1..n]]+y};
  else return {};
  end if; end if;
end function;
//n is geq 1, and we use the cartesian product construction.
cart:=[Floor(lbound[1])..Floor(ubound[1])];
for i:=2 to n do
cart:=car<cart,[Floor(lbound[i])..Floor(ubound[i])]>;
end for;
cart:=Flat(cart);
for nnn in cart do
nn:=[]; for i:=1 to n do nn[i]:=nnn[i]; end for;
Set_that_we_need:=Set_that_we_need join add_or_no(nn);end for;
return Set_that_we_need;
end if; end if;
end function;


//*************************************************************

//this function returns the set of all sequences of t nonnegative integers with sum k.
```

```
function Partition(k,t);
answ:={}; if t eq 1 then return {[k]}; else
for x:=0 to k do for p1 in Partition(k-x,t-1) do answ:=answ join {Append(p1,x)};
end for; end for; return answ; end if;  end function;

//given a geq 0, returns a polynomial Q(q) such that sum n^a q^n=Q_a(q)/(1-q)^{a+1}.

function Poly_expansion_compute(a);
PP<q>:=PolynomialRing(RationalField());
if a eq 0 then return PP!1; else
return q*( Derivative(Poly_expansion_compute(a-1))*(1-q)+Poly_expansion_compute(a-1)*a);
end if; end function;

Polys:=[]; for i:=1 to 8 do Polys[i]:=Poly_expansion_compute(i-1); end for;

function Poly_expansion(a);
if (a le 7) then return Polys[a+1]; else return Poly_expansion_compute(a);
end if; end function;

//given an integer sequence c=[c[1],...,c[r]] of length r,
//nontrivial roots of unity chi=[chi[1],...,chi[r]],
//computes the value of the Shinatani zeta function
//zeta_{c}(A,x,chi,s) at s=0 --- which does not depend on A or x.

function Shint(c,chi2);
return
&*[ (Evaluate( Poly_expansion(c[i]) , chi2[i] ) )/(1-chi2[i])^(c[i]+1) : i in [1..#c]];
end function;


//given x in F, v=[v[1],...,v[r]] --- a sequence of elements of F,
//chi=[chi[1],...,chi[r]] -- a sequence of nontrivial roots of unity in K.
//k is a nonnegative integer
//the function below returns the value at 0 of
//sum_{z_1,...,z_r\geq 0} \frac{chi_1^z_1...(x+\sum z_jv_j)^k}{N(x+\sum z_jv_j)^s}
//The result will belong to K.

function Shint_sum(chi1, v11, x, kk);
s:=K!0; r:=#v11; Partit:=Partition(kk, r+1);
for PP in Partit do
y:=(Factorial(kk)/&*[ Factorial(PP[i]): i in [1..r+1] ])*x^(PP[1])*
(&*[  v11[j]^(PP[j+1]) : j in [1..r]   ])*Shint( Remove(PP,1) ,chi1);
s:=s+y; end for; return s; end function;
```

```
X:=[]; for i:=1 to 6 do X[i]:=[];
for j:=1 to 125 do X[i][j]:=SetToSequence(Omega(a,yy(j),D[i])); end for; end for;

function chi_0(c);
i:=0; while not (c-i*h_e in a*eta) do i:=i+1; end while; return K.1^i;
end function;

character_values:=[];
for ii:=1 to 6 do character_values[ii]:=[]; for j:=1 to #C[ii] do
character_values[ii][j]:=chi_0(-D[ii][j]);
end for; end for;

value:=[]; for i:=1 to 6 do value[i]:=[]; for j:=1 to 125 do value[i][j]:=[];
for ind:=1 to #X[i][j] do value[i][j][ind]:=chi_0(yy(j)-X[i][j][ind]);
end for; end for; end for;

masiv:=[];
for i:=1 to 6 do masiv[i]:=[]; for t:=1 to l-1 do masiv[i][t]:=[]; for j:=1 to #C[i] do
masiv[i][t][j]:=character_values[i][j]^t; end for; end for; end for;

function MU1(ii,jj,kk);
r:=#C[ii];y:=yy(jj);
answer:=0;
for ind:=1 to #X[ii][jj] do x:=X[ii][jj][ind];
for t:=1 to l-1 do
answer:=answer-(( value[ii][jj][ind]  )^t )*Shint_sum(masiv[ii][t], D[ii], x , kk);
end for;end for; return answer; end function;
```

B.3 Computing the multiplicative integral

Here is the code which computes the multiplicative integral, given the measures μ_k for the additive integrals. We first read the measures from the files where we stored them.

```
n:=3;P1<x>:=PolynomialRing(Integers());
f:=x^3+2*x^2-6*x-1;F:=NumberField(f);O:=RingOfIntegers(F);
l:=11;eta:=Factorization(l*O)[1][1];
p:=5;P:=Factorization(p*O)[1][1];

//TT:=SetToSequence(Nonzero_reps() join {O!0});

TT:=
```

```
[
O![0, 0, 0],O![1, 0, 0],O![2, 0, 0],O![-2, 0, 0],O![-1, 0, 0],O![-2, 1, 0],
O![-1, 1, 0],O![0, 1, 0],O![1, 1, 0],O![2, 1, 0],O![-2, -2, -1],O![-2, 2, 0],
O![-1, -2, -1],O![-1, 2, 0],O![0, -2, -1],O![0, 2, 0],O![1, -2, -1],O![1, 2, 0],
O![2, -2, -1],O![2, 2, 0],O![-2, -2, 2],O![-1, -2, 2],O![0, -2, 2],O![1, -2, 2],
O![2, -2, 2],O![-2, -1, -1],O![-1, -1, -1],O![0, -1, -1],O![1, -1, -1],O![2, -1, -1],
O![-2, -1, 2],O![-1, -1, 2],O![0, -1, 2],O![1, -1, 2],O![2, -1, 2],O![-2, 0, -1],
O![-1, 0, -1],O![0, 0, -1],O![1, 0, -1],O![2, 0, -1],O![-2, 0, 2],O![-1, 0, 2],
O![0, 0, 2],O![1, 0, 2],O![2, 0, 2],O![-2, 1, -1],O![-1, 1, -1],O![0, 1, -1],
O![1, 1, -1],O![2, 1, -1],O![-2, 1, 2],O![-2, -2, -2],O![-1, 1, 2],O![-1, -2, -2],
O![0, 1, 2],O![0, -2, -2],O![1, 1, 2],O![1, -2, -2],O![2, 1, 2],O![2, -2, -2],
O![-2, -2, 1], O![-2, 2, -1],O![-1, -2, 1],O![-1, 2, -1],O![0, -2, 1],O![0, 2, -1],
O![1, -2, 1],O![1, 2, -1],O![2, -2, 1],O![2, 2, -1],O![-2, 2, 2],O![-1, 2, 2],
O![-2, -1, -2],O![0, 2, 2],O![-1, -1, -2],O![1, 2, 2],O![0, -1, -2],O![2, 2, 2],
O![1, -1, -2],O![2, -1, -2],O![-2, -1, 1],O![-1, -1, 1],O![0, -1, 1],O![1, -1, 1],
O![2, -1, 1],O![-2, 0, -2],O![-1, 0, -2],O![0, 0, -2],O![1, 0, -2],O![2, 0, -2],
O![-2, 0, 1],O![-1, 0, 1],O![0, 0, 1],O![1, 0, 1],O![2, 0, 1],O![-2, 1, -2],
O![-1, 1, -2],O![0, 1, -2],O![1, 1, -2],O![2, 1, -2],O![-2, 1, 1],O![-1, 1, 1],
O![0, 1, 1],O![1, 1, 1],O![2, 1, 1],O![-2, 2, -2],O![-1, 2, -2],O![0, 2, -2],
O![1, 2, -2],O![2, 2, -2],O![-2, 2, 1],O![-2, -2, 0],O![-1, -2, 0],O![-1, 2, 1],
O![0, -2, 0],O![0, 2, 1],O![1, -2, 0],O![1, 2, 1],O![2, -2, 0],O![2, 2, 1],
O![-2, -1, 0], O![-1, -1, 0],O![0, -1, 0],O![1, -1, 0],O![2, -1, 0]];

M:=5;
Q_p := pAdicField(p,10); helpmap := map<Integers() -> P1 | k :-> f>;
F_P := ext<Q_p | helpmap>; O_P:=RingOfIntegers(F_P);
P2<y>:=PolynomialRing(O_P);
P3:=PolynomialRing(F); f:=P3!CyclotomicPolynomial(l);
f1:=Factorization(f)[1][1]; K:=ext<F|f1>;

load "seqfirstfive.txt";
load "asix.txt";

mu:=[]; for i:=1 to 6 do mu[i]:=[]; for j:=1 to 125 do mu[i][j]:=F!seqalpha[i][j];
end for; end for;

//given k and a, returns the d_i such that
//(b/a-1)-1/2(b/a-1)^2+...+((-1)^{k-1}/k)(b/a-1)^k=d_k(a)b^k+...+d_0(a)

function coeff(k,d1,i);
P3<x>:=PolynomialRing(F_P); g:=0;
for i:=1 to k do g:=g+((-1)^(i-1)/i)*(x-1)^i; end for;
g1:=Eltseq(g); return g1[i+1]/d1^i; end function;

//the function below finds where we have to truncate
```

```
//the series for the expansion of the log

function truncate_log();
i:=0; while (i+1)*(p-2)/(p-1)+Valuation(Factorial(i),p) lt M do
i:=i+1; end while;  return i; end function;

//given an element a in (O_P)^*,
//the function below returns log(a).

function arb_log(a);
hensel_poly:=y^(Norm(P)-1)-1; a1:= HenselLift(hensel_poly, a);
return Log(a/a1); end function;

//Computes log(A_d)

function logA_d(j,kcut); d1:=O_P!(F_P!Eltseq(F!TT[j]));
return arb_log(d1)*   (F_P!Eltseq(F!mu[1][j]  ))+
(&+[ coeff(kcut,d1,i)* (F_P!Eltseq(F!mu[i+1][j] )) : i in [0..kcut]]);
end function;

//the function below is the main one here. It returns the multiplicative integral
//over O_P^* of x w.r.t. the measure mu(b,v,x,0).

function multipl_int();
A0:=&*[ TT[j]^(Integers()!mu[1][j]  )    :j in [2..125]  ]; A00:=F_P!Eltseq(F!A0);
A00:=O_P!A00; hensel_poly:=y^(Norm(P)-1)-1; gamma:=HenselLift(hensel_poly, A00);
kcut:=truncate_log(); B:=&+[ logA_d(j,kcut): j in [2..125]]; return gamma*Exp(B);
end function;
```

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