Oriented cycles in digraphs of large outdegree

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Abstract

In 1985, Mader conjectured that for every acyclic digraph F there exists $K = K(F)$ such that every digraph D with minimum out-degree at least K contains a subdivision of F. This conjecture remains widely open, even for digraphs F on five vertices. Recently, Aboulker, Cohen, Havet, Lochet, Moura and Thomassé studied special cases of Mader's problem and made the following conjecture: for every $\ell \geq 2$ there exists $K = K(\ell)$ such that every digraph D with minimum out-degree at least K contains a subdivision of every orientation of a cycle of length ℓ.

We prove this conjecture and answer further open questions raised by Aboulker et al.

1 Introduction

A *subdivision* of a graph F is a graph obtained from F by replacing its edges with internally vertex-disjoint paths. This notion appears in some of the most fundamental results of graph theory, such as Kuratowski's characterization of planar graphs, as well as many classical results in the structure theory of sparse graphs. Because of these applications, it is desirable to understand by which means a given graph G can be forced to contain a subdivision of a given graph F. One such direction of study that has received a great amount of attention in the literature is the question of how "dense" G should be to guarantee a subdivided F. For undirected graphs, this problem has been solved with great precision. Mader [12] was the first to prove that for every fixed $k \in \mathbb{N}$, every graph of sufficiently large average degree contains a subdivision of K_k, and hence also of any other graph on at most k vertices. The precise asymptotic dependence of the average degree on k, that is required to force K_k as a subdivision, was independently determined by Bollobás and Thomason [4] and by Komlós and Szemerédi [11].

Theorem 1 ([4, 11])**.** *There is an absolute constant $C > 0$ such that every graph with average degree at least Ck^2 contains a subdivision of K_k. This bound is best-possible up to the value of C.*

There is a natural analogue of subdivisions in directed graphs. Given a digraph F, a *subdivision* of F is a digraph obtained by replacing every arc (x, y) in F by a directed path from x to y, such that subdivision-paths corresponding to different arcs are internally vertex-disjoint.

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It is natural to ask to what extent the above phenomenon, that every "sufficiently dense" graph contains a subdivision of a fixed graph F, extends to digraphs.

Aboulker et al. [1] introduced the following handy terminology for the study of forcing subdivisions of digraphs through various digraph parameters. Given a digraph parameter γ ranging in $\mathbb{N}$, a digraph F is called *γ-maderian* if there exists a (smallest) number $\mathrm{mader}_\gamma(F) \in \mathbb{N}$ such that every digraph D with $\gamma(D) \geq \mathrm{mader}_\gamma(F)$ contains a subdivision of F as a subdigraph. We call $\mathrm{mader}_\gamma(F)$ the *Mader number* of F (with respect to γ).

For example, using the natural analogue of these notions for undirected graphs, Theorem 1 states that the Mader number of K_k with respect to the graph parameter $\overline{d}$, namely the average degree, is quadratic in k, and in particular every graph F is $\overline{d}$-maderian.

The *average out-degree* (or, equivalently, *average in-degree*) of a digraph D is $\overline{d}(D) := \frac{|A(D)|}{|V(D)|}$. As the transitive tournament is a digraph of very high average out-degree which does not even contain a subdivision of a directed cycle, it should be clear that an analogue of Theorem 1 for digraphs cannot hold in its full generality. It turns out that the family of $\overline{d}$-maderian digraphs is limited to the so-called *anti-directed forests*: forests in which every vertex is a sink or a source. The positive direction of this result is the consequence of a theorem of Burr [7], who proved that every digraph of sufficiently large average degree contains every anti-directed forest as a subgraph (and hence also as a subdivision). The negative direction, as pointed out by Aboulker et al. [1], follows by considering dense bipartite graphs of large girth and orienting all their edges from one side of the bipartition to the other.

The above constructions of dense digraphs without certain subdivisions all contain sinks (i.e. vertices of out-degree zero); this motivates the study of subdivisions in digraphs with large minimum out-degree. The *minimum out-degree* (*minimum in-degree*) of a digraph D is denoted by $\delta^+(D)$ (respectively, $\delta^-(D)$).

Since $\delta^+ \leq \overline{d}$, every $\overline{d}$-maderian digraph is obviously also δ^+-maderian. However, a characterization of δ^+-maderian digraphs is still widely unknown. Thomassen [21], answering a question of Seymour in the negative, constructed digraphs of arbitrarily large minimum out-degree not containing directed cycles of even length. As a consequence, if a digraph F has the property that each of its subdivisions contains a directed cycle of even length, then F is not δ^+-maderian. Digraphs with this property are known in the literature as *even digraphs*, and have been extensively studied due to their relation to the so-called *even cycle problem.* We refer the reader to [17, 18, 21, 22, 24] for a selection of relevant literature. As can easily be verified by hand, the smallest even digraph is the bioriented clique $\overleftrightarrow{K}_3$ of order 3. This is also the smallest non-δ^+-maderian digraph; indeed, the following theorem states that $\overleftrightarrow{K}_3 - e$, the digraph obtained from $\overleftrightarrow{K}_3$ by removing a single arc, is δ^+-maderian. The proof of this theorem appears in Section 4.

Theorem 2. *Every digraph D with $\delta^+(D) \geq 2$ contains a subdivision of $\overleftrightarrow{K}_3 - e$.*

Observe that for every digraph F it holds that $\mathrm{mader}_{\delta^+}(F) \geq |V(F)| - 1$, since the bioriented clique on $|V(F)| - 1$ vertices has minimum out-degree $|V(F)| - 2$ but no subdivision of F. Hence, the bound in Theorem 2 is optimal.

Theorem 2 strengthens an earlier result by Thomassen (cf. [20], Theorem 6.2), who proved that every digraph of minimum out-degree 2 contains two directed cycles sharing precisely one vertex (this configuration is present in every subdivision of $\overleftrightarrow{K}_3 - e$). On the negative side, another construction by Thomassen [21] shows that there are digraphs of arbitrarily high minimum out-degree having no three directed cycles which share exactly one common vertex (and are otherwise disjoint). In other words, the bioriented 3-star $\overleftrightarrow{S}_3$ is not δ^+-maderian. This result is somewhat surprising when compared to another positive result of Thomassen [19], which shows that for every $k \in \mathbb{N}$ the digraph $k\overleftrightarrow{K}_2$ (i.e., the disjoint union of k digons) is δ^+-maderian. More concretely, Thomassen proved that for every $k \in \mathbb{N}$ we have $\text{mader}_{\delta^+}(k\overleftrightarrow{K}_2) \leq (k+1)!$. The first linear bound on $\text{mader}_{\delta^+}(k\overleftrightarrow{K}_2)$ was proven by Alon [2], and then further improved by Bucić [6]. The famous Bermond-Thomassen conjecture states that in fact $\text{mader}_{\delta^+}(k\overleftrightarrow{K}_2) = 2k-1$, but this remains widely open.

A further negative result was established by DeVos et al. [8]. Building on previous work of Mader [13], they constructed digraphs of arbitrarily high minimum out-degree having no pair of vertices x, y with two arc-disjoint dipaths from x to y as well as two from y to x (see [8, Observation 8]). This result shows that every δ^+-maderian digraph F has arc-connectivity $\kappa'(F) \leq 1$. On the positive side, Aboulker et al. [1] proved that if F is a digraph consisting of two vertices x and y and three internally vertex-disjoint dipaths between x and y – two from x to y and one from y to x – then F is δ^+-maderian.

The negative results discussed so far show that digraphs F with a sufficiently rich directed cycle structure are not δ^+-maderian. However, to this date, no *acyclic* digraph is known that is not δ^+-maderian. This lead Mader [13] to the following conjecture.

Conjecture 3 (Mader, 1985)**.** *Every acyclic digraph is δ^+-maderian.*

Clearly, it would suffice to prove Mader's conjecture for the transitive tournaments $\vec{K}_k$. Mader [15] proved that $\text{mader}_{\delta^+}(\vec{K}_4) = 3$, but the existence of $\text{mader}_{\delta^+}(\vec{K}_k)$ remains unknown for any $k \geq 5$. In view of the apparent difficulty of Mader's question, it is natural to try and verify Mader's conjecture for subclasses of acyclic digraphs. Mader himself [14] considered the digraph consisting of two vertices x and y and k dipaths of length two from x to y, and showed that it is δ^+-maderian for all $k \in \mathbb{N}$. Aboulker et al. [1] proposed to study the following two special cases of Mader's conjecture:

Conjecture 4 ([1])**.** *Every orientation of a forest is δ^+-maderian.*

Conjecture 5 ([1])**.** *Every orientation of a cycle is δ^+-maderian.*

Aboulker et al. [1] proved two special cases of Conjecture 4, showing that every orientation of a path and every in-arborescence (an oriented tree with all edges directed towards a specified root) is δ^+-maderian. They also proved Conjecture 5 for oriented cycles consisting of two *blocks*[1], i.e., oriented cycles having exactly one source and one sink.

Our main contribution in this paper is to verify Conjecture 5 in its full generality. Moreover, we show that the Mader number mader_{δ^+} of an oriented cycle grows (only) polynomially with the cycle length. Let C_ℓ denote the undirected cycle of length ℓ.

Theorem 6. *There exists a polynomial function $K : \mathbb{N} \to \mathbb{N}$ such that for every $\ell \geq 2$, every digraph D with $\delta^+(D) \geq K(\ell)$ contains a subdivision of every orientation of C_ℓ.*

The proof of Theorem 6 is presented in Section 2.

Let $k_1, k_2 \in \mathbb{N}$. Following the notation in [1], we denote by $C(k_1, k_2)$ the two-block cycle consisting of two vertices x, y and two internally vertex-disjoint dipaths from x to y of length k_1 and k_2, respectively. As mentioned above, we have the trivial lower bound $\text{mader}_{\delta^+}(C(k_1, k_2)) \geq k_1 + k_2 - 1$. Aboulker et al. (see [1, Theorem 24]) proved the upper bound $\text{mader}_{\delta^+}(C(k_1, k_2)) \leq 2(k_1 + k_2) - 1$. They also observed that the trivial lower bound gives the truth if $k_2 = 1$, showing that $\text{mader}_{\delta^+}(C(k, 1)) = k$ for every $k \geq 1$. They then asked whether or not their aforementioned bound on $\text{mader}_{\delta^+}(C(k_1, k_2))$ is tight.

Problem 7 ([1], Problem 25)**.** *For $k_1, k_2 \geq 1$, what is the value of $\text{mader}_{\delta^+}(C(k_1, k_2))$?*

[1] By a *block* in an oriented cycle we mean a maximal directed subpath.

Our next result improves upon the bound given by Aboulker et al. [1].

Theorem 8. *Let $k_1 \geq k_2 \geq 2$ be integers. Then $\text{mader}_{\delta^+}(C(k_1,k_2)) \leq k_1 + 3k_2 - 5$.*

Theorem 8 improves upon the result of [1] for all values of $k_1, k_2 \geq 2$, and is asymptotically better if $k_1 \gg k_2$. Furthermore, if $k_2 = 2$ then the bound in Theorem 8 is optimal, as it matches the aforementioned trivial lower bound, thus showing that $\text{mader}_{\delta^+}(C(k,2)) = k+1$ for every $k \geq 1$. The proof of Theorem 8 appears in Section 3.

To conclude, let us mention that in contrast to the aforementioned negative results for general directed graphs, if we restrict our attention to the class of tournaments, which have an inherent density property, then it can be proved that every digraph is forcible as a subdivision by means of large minimum out-degree. This is a recent result by Girão, Popielarz and Snyder [10], which in addition gives a best-possible asymptotic bound of Ck^2 on the minimum out-degree of a tournament required to guarantee the existence of a subdivision of the bioriented k-clique.

As the family of δ^+-maderian digraphs is still somewhat limited, Aboulker et al. [1] initiated the study of the effect of even stronger density conditions, involving the *strong vertex-connectivity* κ, and the *strong arc-connectivity* κ' of digraphs. Since $\kappa \leq \kappa' \leq \delta^+$, every δ^+-maderian digraph is obviously κ'- and κ-maderian. Not much is known however concerning how much richer the families of κ- and κ'-maderian digraphs are. The following interesting questions were posed in [1]:

Problem 9 ([1], Problem 16)**.** *Is every digraph κ-maderian? Is every digraph κ'-maderian?*

While the first question remains open, we can resolve the second question in the negative by proving that neither the bioriented 4-clique $\overleftrightarrow{K}_4$ nor the bioriented 4-star $\overleftrightarrow{S}_4$ is κ'-maderian:

Proposition 10. *For every $k \in \mathbb{N}$, there exists a digraph G_k with $\kappa'(G_k) \geq k$ such that G_k contains no subdivision of $\overleftrightarrow{K}_4$.*

Proposition 11. *For every $k \in \mathbb{N}$, there exists a digraph H_k with $\kappa'(H_k) \geq k$ such that H_k contains no subdivision of $\overleftrightarrow{S}_4$.*

The proofs of Propositions 10 and 11 are presented in Section 5.

We note that a main difficulty arising when studying subdivisions in digraphs (as opposed to undirected graphs) is that digraphs of large (strong) vertex-connectivity may not be linked. Recall that a digraph is called *k-linked* if for every $2k$-tuple of distinct vertices $x_1, \ldots, x_k, y_1, \ldots, y_k$, there are vertex-disjoint dipaths $P_1, \ldots, P_k$ such that P_i goes from x_i to y_i. In undirected graphs, it is known that a graph with sufficiently large vertex-connectivity is k-linked (see [5]), and linkedness has proven very useful for embedding subdivisions. In stark contrast, a construction of Thomassen [23] shows that for every $k \in \mathbb{N}$ there is a strongly k-vertex-connected digraph which is not 2-linked. This makes subdivision questions for digraphs significantly more challenging.

Notation. Digraphs in this paper are considered loopless, have no parallel edges, but are allowed to have anti-parallel pairs of edges (*digons*). A directed edge (also called arc) with tail u and head v is denoted by (u,v). For a graph G, we denote by $V(G)$, $E(G)$ its vertex- and edge-set, respectively. Similarly, the vertex-set (resp. arc-set) of a digraph D is denoted by $V(D)$ (resp. $A(D)$). For $X \subseteq V(D)$, we denote by $D[X]$ the subdigraph of D induced by X. For a set X of vertices or arcs in D, we denote by $D - X$ the subdigraph obtained by deleting the objects in X from D. Given an undirected simple graph G, an *orientation* of G is any digraph obtained by replacing each edge $\{u,v\}$ of G with (exactly) one of the arcs (u,v) or (v,u). Evidently, any orientation is digon-free. The *biorientation* $\overleftrightarrow{G}$ is defined as the digraph obtained from G by replacing every edge with a digon, i.e. $A(\overleftrightarrow{G}) := \{(u,v),(v,u) \mid \{u,v\} \in E(G)\}$.

We use K_k to denote the k-clique, C_k to denote the cycle of length k, and S_k to denote the star with k edges (all of these notations are for undirected graphs). For a digraph D and a vertex $v \in V(D)$, we let $N^+(v)$, $N^-(v)$ denote the out- and in-neighborhood of v in D and $d^+(v)$, $d^-(v)$ their respective sizes. We denote by $\delta^+(D)$, $\delta^-(D)$, $\Delta^+(D)$, $\Delta^-(D)$ the minimum or maximum out- or in-degree of D, respectively. A *directed walk* in a digraph is an alternating sequence $v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$ of vertices and arcs such that $e_i = (v_i, v_{i+1})$ for all $1 \leq i \leq k-1$. The walk is called *closed* if $v_k = v_1$. We use the words "path" and "cycle" to mean an orientation of a path or a cycle (respectively). For example, a *path* P in a digraph D is an alternating sequence $v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$ of pairwise distinct vertices $v_1, \ldots, v_k \in V(D)$ and arcs $e_1, \ldots, e_{k-1} \in A(D)$, such that e_i connects v_i and v_{i+1} (i.e., either $e_i = (v_i, v_{i+1})$ or $e_i = (v_{i+1}, v_i)$). If $e_i = (v_i, v_{i+1})$ for every $i = 1, \ldots, k-1$, then we say that P is a *directed path* or *dipath* from v_1 to v_k (*v_1-v_k-dipath* for short). We will call v_1 the first vertex of P, v_2 the second vertex of P, v_k the last vertex of P, etc. We denote by $|P|$ the length of P (i.e. its number of arcs). Given two distinct vertices $x \neq y$ on a path P, we denote by $P[x, y] = P[y, x]$ the subpath of P with endpoints x and y. A vertex v in a digraph D is said to be *reachable* from a vertex u if there exists a u-v-dipath. In this case, the *distance* from u to v in D is defined as the length of a shortest u-v-dipath. For a pair of dipaths P, Q such that the first vertex of Q is the last vertex of P, we denote by $P \circ Q$ the concatenation of P and Q, i.e. the directed walk obtained by first traversing P and then traversing Q. When P (resp. Q) consists of a single arc (x, y), we will sometimes write $(x, y) \circ Q$ (resp. $P \circ (x, y)$) instead of $P \circ Q$. A *directed cycle* is a cycle with all arcs oriented consistently in one direction. The *directed girth* of D, i.e. the minimum length of a directed cycle in D, is denoted by $\vec{g}(D)$. For a *directed* cycle C and two distinct vertices $x, y \in V(C)$, we denote by $C[x, y]$ the segment of C which forms a dipath from x to y. A digraph D is called *weakly connected* (or just *connected*) if every two vertices can be connected by a path (i.e., if the underlying undirected graph is connected), and is called *strongly connected* if for every ordered pair of vertices $(x, y) \in V(D) \times V(D)$, x can reach y in D. The maximal strongly connected subgraphs of a digraph D are called *(strong) components* and induce a partition of $V(D)$. For a natural number $k \in \mathbb{N}$, a digraph D is called *strongly k-vertex (arc)-connected* if for every set K of at most $k-1$ vertices (arcs) of D, the digraph $D - K$ is strongly connected. An in- (resp., out-) arborescence is a directed rooted tree in which all arcs are directed towards (resp., away from) the root.

Preliminaries. We now quickly recall Menger's Theorem, which will use in the course of the article. The following is a well-known variant of Menger's Theorem for directed graphs.

Theorem 12 (see [16])**.** *Let D be a digraph and $u, v \in V(D)$ be distinct vertices such that $(u, v) \notin A(D)$. Then for every $k \in \mathbb{N}$, either there are k internally vertex-disjoint u-v-dipaths in D, or there is a set $K \subseteq V(D) \setminus \{u, v\}$ such that $|K| < k$ and $D - K$ contains no u-v-dipath.*

Given a digraph D and two (not necessarily disjoint) subsets $A, B \subseteq V(D)$, an *A-B-dipath* is a directed path in D which starts at a vertex of A, ends at a vertex of B, and is internally vertex-disjoint from $A \cup B$ (an A-B-dipath is allowed to consist of a single vertex in $A \cap B$). If A or B are of size one, say $A = \{u\}$ or $B = \{u\}$, then we will simply write "u-B-dipath" or "A-u-dipath", respectively. The following is a well-known consequence of Theorem 12.

Theorem 13. *Let D be a digraph, let $v \in V(D)$ and let $A \subseteq V(D) \setminus \{v\}$. Then either there are k different v-A-dipaths which pairwise only intersect at v, or there is a subset $K \subseteq V(D) \setminus \{v\}$ such that $|K| < k$ and such that there is no dipath in $D - K$ starting in v and ending in A.*

Proof. Consider the digraph H obtained from D by adding an artificial vertex $v_A \notin V(D)$ and adding the arc (y, v_A) for every $y \in A$. The claim now follows by applying Theorem 12 to the vertices v and v_A in H. Indeed, if there are k internally vertex-disjoint v-v_A-dipaths in H, then by deleting all successors of the first vertex in A from each of these dipaths (i.e. by cutting each of the dipaths as soon as it reaches A), we obtain k distinct v-A-dipaths in D which pairwise only share the vertex v. And if we can hit all v-v_A-dipaths in H with a subset $K \subseteq V(H) \setminus \{v, v_A\} = V(D) \setminus \{v\}$ such that $|K| < k$, then there are no dipaths in $D - K$ starting in v and ending in A. This proves the claim. $\square$

2 Subdivisions of oriented cycles

In this section, we prove Theorem 6, which we restate here for convenience.

Theorem 14. *For every $\ell \geq 2$ there is a polynomially bounded $K = K(\ell)$ such that every digraph D with $\delta^+(D) \geq K$ contains a subdivision of every oriented cycle of length ℓ.*

It is well-known and easy to show that every digraph with minimum out-degree k contains a directed cycle of length at least $k+1$. Thus, in what follows we restrict our attention to acyclic oriented cycles. For integers $a, b \geq 1$, let $C_{a,b}$ be the oriented cycle consisting of $2a$ vertices $s_1, \dots, s_a, t_1, \dots, t_a$ and $2a$ internally-disjoint length-b dipaths: one from s_i to t_i and one from s_i to t_{i+1} for each $1 \leq i \leq a$ (with indices taken modulo a). See Figure 1 for an illustration of $C_{2,3}$. It is easy to see that for every acyclic oriented cycle C, there are $a, b \geq 1$ such that every subdivision of $C_{a,b}$ is also a subdivision of C (specifically, a is the number of sources (or, equivalently, sinks) in C, and b is the largest length of a dipath contained in C). Therefore, it is sufficient to show that digraphs with minimum out-degree at least $k(a,b)$ contain a subdivision of $C_{a,b}$ (for some suitable choice of $k(a,b) = \mathsf{poly}(a,b)$). For $a = 1$, this statement was proven in [1], and we also give a new proof in Section 3. Consequently, it is sufficient to consider the case $a \geq 2$ (and, in fact, the assumption $a \geq 2$ is required by our method).

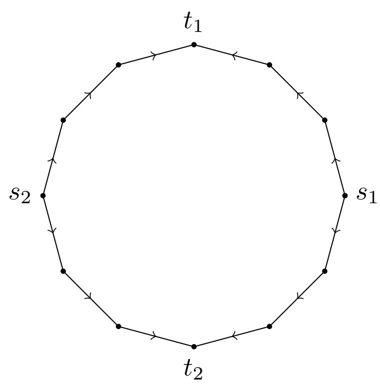

Figure 1: The oriented cycle $C_{2,3}$

When trying to construct a subdivision of $C_{a,b}$, one of course needs to construct long directed paths (i.e. of size at least b). To this end, it is useful to know that walks up to a certain length do not intersect themselves, namely, that the digraph has large directed girth. And indeed, our argument crucially relies on this guarantee. Luckily, the following result of Dellamonica, Koubek, Martin and Rödl [9] allows us to reduce the general case of Theorem 14 to the case that D has large girth:

Theorem 15 ([9])**.** *For every $k \geq 1$ and $g \geq 3$ there exists $K = K(k, g)$ such that every digraph D with $\delta^+(D) \geq K$ contains a subdigraph D' with $\delta^+(D') \geq k$ and with directed girth $\vec{g}(D')$ at least g.*

Given Theorem 15, we see that in order to prove Theorem 14, it suffices to establish the following:

Theorem 16. *There is an absolute constant C such that for every pair of integers $a \geq 2$, $b \geq 1$, every digraph D with $\delta^+(D) \geq Cab^7$ and $\vec{g}(D) \geq g := 4b^2$ contains a subdivision of $C_{a,b}$.*

A quantitative version[2] of Theorem 15 is that $K(k, g) \leq O(kg^2 \log g)$. It follows that having minimum out-degree at least $a \cdot \text{poly}(b)$ is enough to force a subdivision of $C_{a,b}$, and that the conclusion of Theorem 14 holds with $K(\ell) = \mathsf{poly}(\ell)$.

Before delving into the details, we present an overview of the proof in the following subsection. We then proceed to give the precise definitions of the gadgets and to establish the main lemmas needed in the proof of Theorem 16; this is done in Subsections 2.2, 2.3 and 2.4. Finally, in Subsection 2.5 we combine all ingredients to prove Theorem 16.

2.1 Overview of the Proof

Our general strategy for embedding a $C_{a,b}$-subdivision is similar in spirit to the basic argument for the existence of a long directed cycle in a digraph with large out-degree: consider a longest directed path, so that each edge emanating from its last vertex yields a directed cycle. If the out-degree of the last vertex is at least k, then one of these directed cycles has length at least $k+1$. The first issue in creating a $C_{a,b}$-subdivision (using the same approach) is that, instead of a directed path, we need to construct an oriented path in which many "forward"- and "backward"-oriented segments of length at least b follow each other alternatingly. The second issue comes up when one would like to close such an oriented path into a $C_{a,b}$-subdivision, as there is usually no control over where exactly potential closing edges enter some "forward"- or "backward"-oriented segment, and hence, whether they create a segment that is shorter than b, or a cycle with too few/too many blocks (different from the desired number of precisely $2a$ blocks).

The central concept in resolving these issues is a certain structure, which we call a *chain.* A chain will consist of a directed path, called its *spine*, together with suitable *gadgets* attached to some of the arcs of the spine. Each gadget intersects the spine in a single arc, different gadgets do not intersect outside of the spine, and every arc of the spine has at most one gadget on it.

In the proof we proceed along the spine of a chain to create "forward"-segments of length at least b. The primary purpose of gadgets is that besides containing a "forward" edge of the spine, they also provide the *option* to take a "backwards" detour of length at least b, and then continue forward on the spine. In order to start creating a $C_{a,b}$-subdivision this way, we aim to have at least a such detours, separated by at least b edges of the spine from each other. Finally, when a chain is maximal, so every extension of it would intersect it, we would like to find our $C_{a,b}$-subdivision using these intersections. Ideally, we of course would hope that the last segment of length at least b at the end of our chain connects back smoothly through an intersection point to the beginning of a segment of length at least b, of opposite direction, which is "far enough" from the end, so that a $C_{a,b}$-subdivision is found. Various issues might arise, which we also resolve with an appropriate choice of gadgets.

We will use three types of gadgets, each having the crucial "detour-property". The set of gadgets has to be carefully chosen, so that they (1) exist in abundance along our spine and

[2] In fact, the bound on $K(k, g)$ appearing in [9] was slightly weaker — in that the logarithmic factor depended on k — but it is easy to see that by using the argument of [9] and replacing a union bound used there with a tighter concentration inequality (say, Chernoff's bound), one obtains the stronger estimate stated here.

(2) provide rich enough structure to resolve any issues we might face when closing our oriented path into a cycle. The simplest gadget with the detour-property is a directed cycle (of length at least $g \geq b+1$); this is what we call a gadget of type I. These however do not necessarily appear frequently enough for (1). Hence come gadgets of type II into the picture. Usefully, we can show that every arc (in a minimal counterexample to Theorem 16) has either a type I or a type II gadget on it, providing great flexibility.

Gadgets of type I and II can both handle the situation when one of the extensions of a maximal chain intersects it "far away" from its end, but some of the forward- or backward-segments of the chain around the self-intersection points might not be long enough anymore. It turns out that the union of two intersecting gadgets is already rich enough in edges, so that it is possible to reroute and obtain the necessary directed paths of length at least b of the appropriate directions and hence complete a $C_{a,b}$-subdivision. In the other case, when *all* the gadgets sitting on the extensions of the spine intersect the chain "close" to its end, then a relatively dense structure attached to the end of the chain arises. In the latter we can find yet another type of structure with the detour-property of gadgets and extend the chain with it. We will call these structures gadgets of type III. Being able to extend the chain leads to our final contradiction (to the maximality of our chain).

We now explain in more detail the proof steps outlined above, as well as how we arrived at the definitions of the various gadgets. The first key observation is that if D is a minimal counterexample to Theorem 16 (i.e., a smallest digraph with $\delta^+(D) \geq k := Cab^7$, $\vec{g}(D) \geq g := 4b^2$, but no $C_{a,b}$-subdivision), then for every arc (x, y) of D it holds that either (x, y) is on a directed cycle of length exactly g, or there is a vertex z which *dominates* both x and y, i.e. $(z, x), (z, y) \in A(D)$. This will be argued in full detail in Claim 2 in Subsection 2.5. The minimality assumption on D also easily implies that every vertex in D has out-degree *exactly* k.

Now, using the above property of D, one can show (see Lemma 2.6) that for every arc (p, q), either there is a directed cycle of length between g and $2g$ through (p, q) (hence, a *gadget of type I* of bounded size), or (p, q) is contained in a *basic gadget of type II* of bounded size, see the left-hand side of Figure 2. Both of these gadgets have the useful property that they have an arc from p to q, as well as a "detour" from p to q which consists of a (possibly empty) forward segment, followed by a backward segment of length at least b, followed by another (possibly empty) forward segment. It is now easy to see that in a chain with sufficiently many such gadgets, we can find an oriented path with a changes of direction and directed segments of length at least b between the endpoints of the spine, for any prescribed $a \geq 1$. (To do so, from each gadget we either pick the arc (p, q) or the above detour.) This will be argued in detail in Lemma 2.4.

For the proof we take a longest possible chain ending with a gadget, in which consecutive gadgets are not that far (closer than $(4g+3)(2b-1)$). This in particular means that any gadget on any out-edge of the last vertex v of the spine must intersect the chain. (Recall that all these out-edges have at least one gadget (of type I or II) sitting on them.) More generally, any gadget sitting on any edge at the end of a directed path of length at most $(4g+3)(2b-1)$ exiting v (and internally disjoint from the chain) must also intersect the chain.

If one of these paths intersects the chain only "far away" from v, it is not difficult to use the chain together with this back-path to close a $C_{a,b}$-subdivision. This will be done in Lemma 2.3(1). Lemma 2.3(2) resolves the same issue when not the base but the body of a gadget of type II sitting on the end of one of these paths intersects the chain far away from v. In this case serious additional technical difficulties could occur, since when trying to close the cycle naively, one of the blocks of the cycle may become too short. We overcome this difficulty by making use of more robust structures on top of a basic gadget of type II. These "extensions" are called extended gadgets of type II, they are depicted in Figure 3. Using the additional paths provided by these extensions, we can complete the technical task of routing a $C_{a,b}$-subdivision through the intersection of the gadget and the chain successfully (the technical details of this step are part of the proof of Lemma 2.5). Actually, the extended gadgets are only used for the purpose of completing this very step of the proof. Of course, we also need to show that extended gadgets of type II exist. And indeed, we will be able to show that in a minimal counterexample, every arc is contained in a gadget of type I or an extended (and not just a basic) gadget of type II (this is the actual statement of Lemma 2.6).

Otherwise, *all* gadgets of type I or II sitting on edges in the iterated out-neighborhood of v of radius $(4g+3)(2b-1)$ outside the chain (i.e., edges contained in dipaths of length at most $(4g+3)(2b-1)$ that start at v and are internally-disjoint from the chain) must intersect the chain "close" to v. Note that in this case every vertex in this iterated out-neighborhood must have many out-neighbors outside the chain, for otherwise we would be in the first case. (To be precise, here we will need to use a property that we will (and can) require from our gadgets, that they themselves are not too large. Because of this, any vertex having a large number of out-neighbors in the chain necessarily also sees a vertex in the chain which is far from v.)

Hence there are many (not too long) directed paths $v, u_1, ..., u_t$, with $u_1, \ldots, u_t$ outside of the chain, and for each such path any gadget of type I or II sitting on (u_{t-1}, u_t) intersects the chain close to v. The structure of these gadgets guarantees that there is a short path starting in a vertex on the chain close to v and ending at u_t. Therefore, if we are unable to extend the chain or close a $C_{a,b}$-subdivision, then every such vertex u_t must be reachable by a short path from one of only a small number of vertices, namely the vertices on the chain close to v. This in turn allows us to upper bound the number of vertices u_t which are at distance t from v after removing the rest of the chain. With a careful choice of the parameters, the bound we will get will be much smaller than k^t (which is the situation in a k-out-regular tree rooted at v). This should mean, at least intuitively, that the branches emanating from v must have many intersections. And indeed, we will be able to turn this intuition into a concrete statement, by showing that there exist two long dipaths P_1,P_2 which only intersect in their first and last endpoints, and whose first endpoint is reachable from v (see Lemma 2.7). Crucially, the union of these paths then forms exactly a gadget of type III; compare the right-hand side of Figure 2 (here p is the starting point of P_1 and P_2). Since we initially allowed also for gadgets of type III to be incorporated in our chain, the existence of this gadget then allows us to extend the chain by adding in this new gadget of type III (with a connecting directed path). Thereby we obtain a contradiction to the assumed maximality of the chain, which completes the proof.

2.2 The Gadgets

We will use three types of gadgets. Each of the gadgets will have a special pair of vertices p, q with an arc from p to q. The gadgets are defined as follows:

(I) A gadget of type I is a directed cycle of length at least g through the arc (p, q).

(II) A *basic gadget of type II* is a digraph consisting of vertices p, q, r and a dipath P_1 from r to p, such that P_1 has length at least $2b^2 + b - 2$, $q \notin V(P_1)$, and every vertex of P_1 has an arc to q (so in particular, (p, q) is an arc). An *extended gadget of type II* consists of a basic gadget of type II, comprised of vertices p, q, r and a dipath P_1 as above, as well as an additional dipath P_2 of length at least b having the following properties:

  (a) The last vertex of P_2 is r, $V(P_1) \cap V(P_2) = \{r\}$, and $q \notin V(P_2)$.

  (b) Either there is an arc from the first vertex of P_2 to the second vertex of P_1, or there is an arc from some vertex in $V(P_1) \setminus \{r\}$ to the first vertex of P_2.

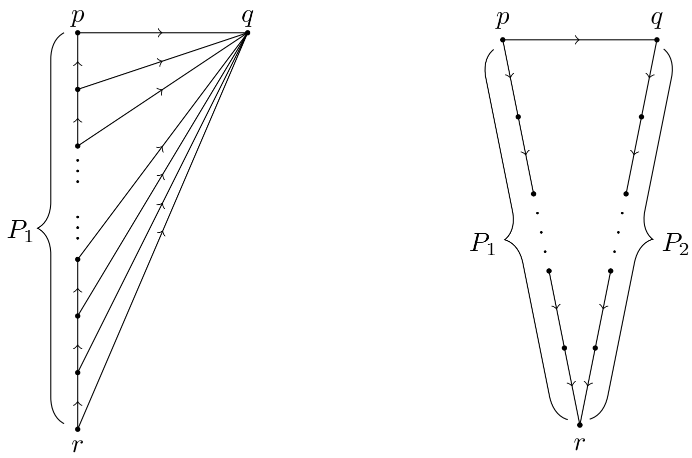

Figure 2: A basic gadget of type II (left) and a gadget of type III (right)

For an extended type-II gadget G, the *basic part* of G is the corresponding basic type-II gadget, namely the subgraph of G induced by $V(P_1) \cup \{q\}$.

(III) A gadget of type III is a digraph consisting of vertices p, q, r, the arc (p, q), and two internally disjoint dipaths P_1, P_2 from p and q, respectively, to r, such that P_1 and P_2 have length at least $2b - 1$ each.

The various types of gadgets are depicted in Figures 2-3. For convenience, we also introduce the notion of a *trivial gadget*: a trivial gadget simply consists of vertices p, q and the arc (p, q) (and no other vertices).

We now introduce another useful definition. Recall that for integers $a, b \geq 1$, an (a, b)-*alternating-path* is an oriented path R containing special vertices $s_1, t_1, \dots, s_a, t_a$ (appearing in this order along R), such that R is the union of pairwise internally-disjoint dipaths $Q_1, \dots, Q_a$, $Q'_1, \dots, Q'_{a-1}$, with the property that Q_i is a dipath from s_i to t_i (for each $1 \leq i \leq a$), Q'_i is a dipath from s_{i+1} to t_i (for each $1 \leq i \leq a-1$), and $Q_2, \dots, Q_{a-1}, Q'_1, \dots, Q'_{a-1}$ have length at least b each. We note that Q_1 or Q_a may have length zero (in which case $s_1 = t_1$ or $s_a = t_a$, respectively). In particular, for vertices u, v, any dipath from u to v is a $(1, b)$-alternating-path with $s_1 = u$ and $t_1 = v$ (for any value of b); and any dipath of length at least b from u to v is a $(2, b)$-alternating-path with $s_2 = t_2 = u$ and $s_1 = t_1 = v$. The path R is called *strong* if Q_1 and Q_a also have length at least b. When several paths are considered at the same time, we will write $s_i(R), t_i(R), Q_i(R), Q'_i(R)$ (instead of s_i, t_i, Q_i, Q'_i) so as to prevent confusion. The following observation follows immediately from the definitions of $C_{a,b}$ and alternating-paths.

Observation 17. *Let $a_1, a_2, b \geq 1$ be integers, and for each $i = 1, 2$, let R_i be a strong (a_i, b)-alternating path. Suppose that $s_1(R_1) = t_{a_2}(R_2)$, $s_1(R_2) = t_{a_1}(R_1)$ and that R_1 and R_2 do not share any other vertices. Then $R_1 \cup R_2$ spans a subdivision of $C_{a_1+a_2-2,b}$.*

Let us now prove some simple facts about type–I and type–II gadgets.

Lemma 2.1. *Let G be a gadget of type I or II (either basic or extended). Then:*

1. *G contains a $(2, b)$-alternating path R_0 with $s_1(R_0) = t_1(R_0) = p$ and $t_2(R_0) = q$.*

2. *$\{p, q\}$ is reachable from every vertex of G.*

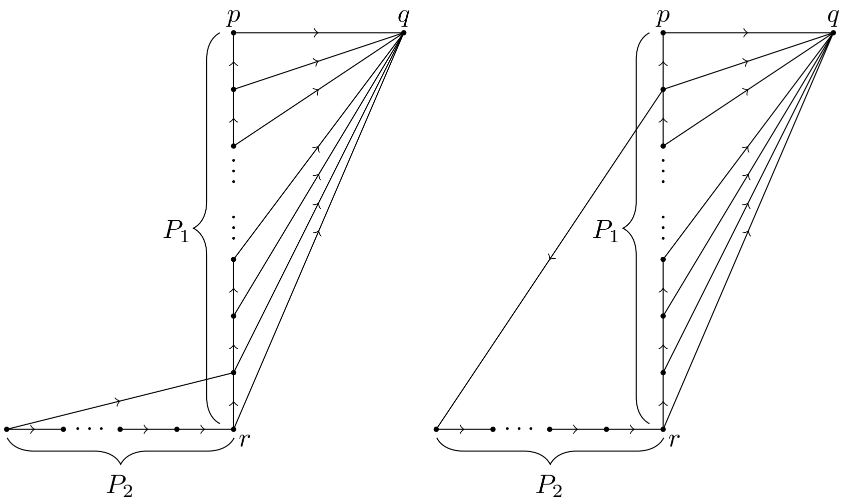

Figure 3: The two options for an extended gadget of type II: either there is an arc from the first vertex of P_2 to the second vertex of P_1 (left), or there is an arc from some vertex in $V(P_1) \setminus \{r\}$ to the first vertex of P_2 (right).

Proof. Item 2 follows immediately from the definitions of these gadgets. Let us prove Item 1. If G is of type I, i.e. a directed cycle of length at least $g > b$ through (p, q), then define R_0 by letting $s_1(R_0) = t_1(R_0) = p$, $s_2(R_0) = t_2(R_0) = q$ and $Q'_1(R_0) = G[q, p]$ (i.e., $Q'_1(R_0)$ is simply the q-p-dipath obtained from the cycle by removing the arc (p, q)). If G is of type II then define R_0 by letting $s_1(R_0) = t_1(R_0) = p$, $s_2(R_0) = r$, $t_2(R_0) = q$, $Q'_1(R_0) = P_1$ and $Q_2(R_0) = (r, q)$. $\square$

Lemma 2.2. *Let G be an extended gadget of type II, and let p, q, r and P_1, P_2 be as in the definition of such a gadget. Then:*

1. *For every $x \in V(G) \setminus \{p, q\}$, there exists $1 \leq a \leq 2$ and an (a, b)-alternating-path R with $t_a(R) = x$, $s_1(R) \in \{p, q\}$ and $|V(R) \cap \{p, q\}| = 1$.*
2. *For every non-empty set $X \subseteq V(P_1) \setminus \{p, r\}$, there exists $1 \leq a \leq 2$ and an (a, b)-alternating-path R with $t_a(R) \in X$, $s_1(R) \in \{p, q\}$ and $|V(R) \cap \{p, q\}| = |V(R) \cap X| = 1$.*

Proof. We start by proving Item 2, from which Item 1 will then easily follow. So let $\emptyset \neq X \subseteq V(P_1) \setminus \{p, r\}$. Denote by z the first vertex of P_2, and by y the second vertex of P_1. By the definition of an extended type-II gadget, either $(z, y) \in A(G)$ or there is some $w \in V(P_1) \setminus \{r\}$ such that $(w, z) \in A(G)$. Suppose first that $(z, y) \in A(G)$. Traverse the dipath P_1 starting from y until the first vertex of X is reached, and denote this vertex by x. Evidently, we have $X \cap V(P_1[y, x]) = \{x\}$. Now define R by setting $s_1(R) = t_1(R) = q$, $s_2(R) = z$, $t_2(R) = x$, $Q'_1(R) = P_2 \circ (r, q)$ and $Q_2(R) = (z, y) \circ P_1[y, x]$. Then R is indeed a $(2, b)$-alternating-path (since $|P_2| \geq b$), and we have $V(R) \cap \{p, q\} = \{q\}$ (since $x \neq p$) and $V(R) \cap X = \{x\}$, as required.

Suppose now that there is $w \in V(P_1) \setminus \{r\}$ such that $(w, z) \in A(G)$. If $w = p$ then, as before, we let x be the first vertex of X reached when traversing $P_1[y, p]$. Observe that $(w, z) \circ P_2 \circ P_1[r, x]$ is a dipath from $p = w$ to x, and thus also a $(1, b)$-alternating-path R with $s_1(R) = p$ and $t_1(R) = x$. Moreover, our choice of x implies that $V(R) \cap X = \{x\}$, as required.

So from now we assume that $w \neq p$. In this case, choose an element $x' \in X$, which is closest to w in the *undirected* path underlying P_1. In other words, we choose $x' \in X$ such that the subpath of P_1 between w and x' contains no vertex of X other than x' itself. We consider two cases, based on the relative position of x' and w along P_1. Assume first that when traversing the dipath P_1 (starting from r), w is reached before x' is (here we allow $w = x'$). In this case, define R by setting $s_1(R) = t_1(R) = q$, $s_2(R) = w$, $t_2(R) = x'$, $Q'_1(R) = (w, z) \circ P_2 \circ (r, q)$ and $Q_2(R) = P_1[w, x']$. Assume now that x' is reached before w when traversing P_1. In this case, define a $(2, b)$-alternating-path R by setting $s_1(R) = t_1(R) = q$, $s_2(R) = t_2(R) = x'$ and $Q'_1(R) = P_1[x', w] \circ (w, z) \circ P_2 \circ (r, q)$. Observe that in both cases, R is indeed a $(2, b)$-alternating-path (because $|P_2| \geq b$), $V(R) \cap \{p, q\} = \{q\}$ (because $w, x' \neq p$), and $V(R) \cap X = \{x'\}$ (by our choice of x'). This concludes the proof of Item 2.

It remains to prove Item 1. So let $x \in V(G) \setminus \{p, q\}$. If $x \in V(P_2)$, then we define a $(2, b)$-alternating-path R by setting $s_1(R) = t_1(R) = p$, $s_2(R) = t_2(R) = x$ and $Q'_1(R) = P_2[x, r] \circ P_1$. And if $x \in V(G) \setminus (V(P_2) \cup \{p, q\}) = V(P_1) \setminus \{p, r\}$, then we obtain the required alternating-path R by applying Item 2 with $X := \{x\}$. It is easy to see that in both cases, R satisfies the assertion of Item 1. This completes the proof of the lemma. $\square$

2.3 Gadget Chains

We now define the notion of a *chain* of gadgets, a structure which will be instrumental to our proof of Theorem 16. In what follows, for a gadget G, we will denote by $p(G)$ and $q(G)$ the designated vertices p and q of G.

Definition 18. *A* chain $\mathcal{C}$ *consists of a directed path $P = v_0, \dots, v_m$, a partition $A_1 \cup A_2 = A(P) = \{(v_0, v_1), \dots, (v_{m-1}, v_m)\}$ of the arc-set of P, and a collection of (non-trivial) gadgets $(G_e : e \in A_2)$ having the following four properties:*

1. *For every $e \in A_2$, the gadget G_e is either of type I, type III, or basic type-II.*
2. *For every $e = (v_i, v_{i+1}) \in A_2$, $p(G_e) = v_i$ and $q(G_e) = v_{i+1}$.*
3. *$V(G_{(v_i,v_{i+1})}) \cap \{v_0, \dots, v_m\} = \{v_i, v_{i+1}\}$ for every $(v_i, v_{i+1}) \in A_2$.*
4. *$V(G_e) \cap V(G_f) \subseteq \{v_0, \dots, v_m\}$ for every pair of distinct $e, f \in A_2$.*

We will use the following terminology and notation:

- *With a slight abuse of notation, we identify the chain $\mathcal{C}$ and the digraph consisting of the union of P and the gadgets $G_e, e \in A_2$.*
- *For convenience, for $(v_i, v_{i+1}) \in A_1$ we denote by $G_{(v_i,v_{i+1})}$ the trivial gadget with vertices v_i, v_{i+1} and arc (v_i, v_{i+1}).*
- *In cases where several chains are considered at the same time, we will write $A_1(\mathcal{C})$, $A_2(\mathcal{C})$ and $G_e(\mathcal{C})$ to indicate that we are considering the chain $\mathcal{C}$.*
- *The dipath P is called the* spine *of the chain, and $|P| = m$ is the* length *of the chain.*
- *The vertex set of $\mathcal{C}$, denoted $V(\mathcal{C})$, is defined as $V(\mathcal{C}) = V(P) \cup \bigcup_{e \in A_2} V(G_e)$.*
- *For integers $0 \leq i < j \leq m$, we denote by $\mathcal{C}[v_i, v_j]$ the* subchain *of $\mathcal{C}$ whose spine is $P[v_i, v_j] = v_i, v_{i+1}, \dots, v_j$; so $A_\ell(\mathcal{C}[v_i, v_j]) = A_\ell(\mathcal{C}) \cap A(P[v_i, v_j])$ for $\ell = 1, 2$, and $\mathcal{C}[v_i, v_j]$ inherits the gadgets of $\mathcal{C}$.*

The main result of this subsection is the following lemma, stating that if a sufficiently "rich" chain (i.e., a chain $\mathcal{C}$ with $|A_2(\mathcal{C})|$ large enough) "self-intersects" in some well-defined way, then it contains a subdivision of $C_{a,b}$.

Lemma 2.3. *Let $a \geq 2$ and $b \geq 1$ be integers, let $\mathcal{C}$ be a chain contained in a digraph D and let $P = z_0, \dots, z_\ell$ and A_1, A_2 be as in Definition 18. Suppose that $|A_2| \geq (a+3)(b+1) - 2$ and that at least one of the following two conditions is satisfied:*

1. *There exists $x \in V(G_{(z_0,z_1)})$ such that $(z_\ell, x) \in A(D)$.*
2. *There exists a vertex $z^* \in V(D) \setminus V(\mathcal{C})$ such that $(z_\ell, z^*) \in A(D)$, and there exists an extended type-II gadget G^* such that $p(G^*) = z_\ell$, $q(G^*) = z^*$, $V(G_{(z_0,z_1)}) \cap V(G^*) \neq \emptyset$ and $V(\mathcal{C}) \cap V(G^*) \subseteq V(G_{(z_0,z_1)}) \cup \{z_\ell\}$.*

Then D contains a subdivision of $C_{a,b}$.

The proof of Lemma 2.3 consists of two steps, namely Lemmas 2.4 and 2.5. First, in Lemma 2.4 we shall find a suitable alternating-path. Then, we will close this path into a cycle (i.e., a $C_{a,b}$-subdivision), using Lemma 2.5.

Lemma 2.4. *Let $a, b \geq 1$ be integers. Let $\mathcal{C}$ be a chain, let $P = v_0, \dots, v_m$ and A_1, A_2 be as in Definition 18, and suppose that $|A_2| \geq a(b+1) - 1$. Then $\mathcal{C}$ contains a strong (a,b)-alternating-path R with $s_1(R) = v_0$ and $t_a(R) = v_m$.*

Proof. The proof is by induction on a. In the base case $a = 1$, the condition in the lemma states that $|A_2| \geq b$. This implies that $m = |A(P)| \geq b$, meaning that P is a dipath of length at least b from v_0 to v_m, and hence also a strong $(1,b)$-alternating-path with $s_1(P) = v_0$ and $t_1(P) = v_m$.

We now move on to the induction step. So let $a \geq 2$. Let j be the largest integer in the set $\{0, \dots, m-b-1\}$ satisfying $(v_j, v_{j+1}) \in A_2$. Set $\mathcal{C}' := \mathcal{C}[v_0, v_j]$. Then $|A_2(\mathcal{C}')| \geq |A_2(\mathcal{C})| - (b+1) \geq (a-1)(b+1) - 1$. By the induction hypothesis, $\mathcal{C}'$ contains a strong $(a-1,b)$-alternating-path R' with $s_1(R') = v_0$ and $t_{a-1}(R') = v_j$.

Setting $e := (v_j, v_{j+1})$, suppose first that G_e is either of type I or a basic gadget of type II. By Item 1 of Lemma 2.1, G_e contains a $(2,b)$-alternating path R_0 with $s_1(R_0) = t_1(R_0) = v_j$ and $t_2(R_0) = v_{j+1}$. Now let R be the (a,b)-alternating-path obtained by attaching to R' the dipaths $Q'_1(R_0)$ and $Q_2(R_0) \circ P[v_{j+1}, v_m]$. Formally, R is defined by setting $s_i(R) = s_i(R')$ and $t_i(R) = t_i(R')$ for every $1 \leq i \leq a-1$ (so in particular, $s_1(R) = v_0$), $s_a(R) = s_2(R_0)$, $t_a(R) = v_m$, $Q_i(R) = Q_i(R')$ for every $1 \leq i \leq a-1$, $Q'_i(R) = Q'_i(R')$ for every $1 \leq i \leq a-2$, $Q'_{a-1}(R) = Q'_1(R_0)$ and $Q_a(R) = Q_2(R_0) \circ P[v_{j+1}, v_m]$. Note that $|Q_a(R)| \geq b$ because $j \leq m - b - 1$. It follows that R is indeed a strong (a,b)-alternating-path, as required.

Suppose now that G_e is of type III. Then G_e consists of the vertices v_j, v_{j+1}, a vertex r, and two internally vertex-disjoint dipaths P_1, P_2 from v_j and v_{j+1}, respectively, to r, such that P_1 and P_2 have length at least $2b - 1 \geq b$ each. Now let R be the (a,b)-alternating-path obtained by attaching to R' the dipaths P_1, P_2 and $P[v_{j+1}, v_m]$. Formally, R is defined by setting $s_i(R) = s_i(R')$ for every $1 \leq i \leq a-1$ (so in particular, $s_1(R) = v_0$), $t_i(R) = t_i(R')$ for every $1 \leq i \leq a-2$, $t_{a-1}(R) = r$, $s_a(R) = v_{j+1}$, $t_a(R) = v_m$, $Q_i(R) = Q_i(R')$ and $Q'_i(R) = Q'_i(R')$ for every $1 \leq i \leq a-2$, $Q_{a-1}(R) = Q_{a-1}(R') \circ P_1$, $Q'_{a-1}(R) = P_2$ and $Q_a(R) = P[v_{j+1}, v_m]$. Again, it is easy to check that R is a strong (a,b)-alternating-path, as required. □

Lemma 2.5. *Let G, G^* be gadgets such that $V(G) \cap V(G^*) \neq \emptyset$, $p(G^*), q(G^*) \notin V(G)$, and G^* is an extended gadget of type II. Then there exists $1 \leq a \leq 3$ such that $G \cup G^*$ contains an (a,b)-alternating-path R with $t_a(R) \in \{p(G), q(G)\}$, $s_1(R) \in \{p(G^*), q(G^*)\}$ and $|V(R) \cap \{p(G), q(G)\}| = |V(R) \cap \{p(G^*), q(G^*)\}| = 1$.*

Note that the gadget G in Lemma 2.5 is allowed to be trivial.

Proof. For convenience, let us put $p := p(G)$, $q := q(G)$, $p^* := p(G^*)$ and $q^* := q(G^*)$. The assumption $p^*, q^* \notin V(G)$ will be used implicitly throughout the proof. We proceed by a case analysis over the types of G and G^*.

Case 1. G is trivial, a gadget of type I, or a gadget of type II. Recall that $V(G) \cap V(G^*) \neq \emptyset$ by assumption. By Item 2 of Lemma 2.1, $\{p, q\}$ is reachable from every vertex of $V(G)$ via a dipath inside G (this is evident if G is trivial). In particular, G contains a dipath from $V(G) \cap V(G^*)$ to $\{p, q\}$. Fix a shortest such dipath $P \subseteq G$, and let $x \in V(G) \cap V(G^*)$ be the first vertex of P. The minimality of P implies that $V(P) \cap V(G^*) = \{x\}$ and $|V(P) \cap \{p, q\}| = 1$. By Item 1 of Lemma 2.2, there is $1 \leq a \leq 2$ such that G^* contains an (a, b)-alternating-path R^* with $t_a(R^*) = x$, $s_1(R^*) \in \{p^*, q^*\}$ and $|V(R^*) \cap \{p^*, q^*\}| = 1$. (The condition $x \notin \{p^*, q^*\}$ appearing in Item 1 of Lemma 2.2 is satisfied here because $x \in V(G) \cap V(G^*)$ whereas $p^*, q^* \notin V(G)$ by assumption.) Note that $V(P) \cap V(R^*) = \{x\}$ because $V(P) \cap V(G^*) = \{x\}$ and $V(R^*) \subseteq V(G^*)$. Now it is easy to see that by combining P and R^* we obtain an (a, b)-alternating-path R with $t_a(R) \in \{p, q\}$, $s_1(R) \in \{p^*, q^*\}$ and $|V(R) \cap \{p, q\}| = |V(R) \cap \{p^*, q^*\}| = 1$. Formally, R is defined by setting $Q_a(R) := Q_a(R^*) \circ P$ (so $t_a(R) \in \{p, q\}$ is the last vertex of P); $s_i(R) = s_i(R^*)$ for every $1 \leq i \leq a$; and $t_i(R) = t_i(R^*)$, $Q_i(R) = Q_i(R^*)$ and $Q'_i(R) = Q'_i(R^*)$ for every $1 \leq i \leq a - 1$. This completes the proof in Case 1.

Case 2. G is a gadget of type III. In this case G consists of the arc (p, q), a vertex r, and two internally vertex-disjoint dipaths P_1, P_2 from p and q, respectively, to r, such that P_1 and P_2 have length at least $2b - 1$ each.

As G^* is an extended gadget of type II, it consists of the vertices p^*, q^*, a vertex r^* and dipaths P_1^*, P_2^*, all satisfying the properties stated in the definition of a type-II gadget. We start by handling the case that there is some $x \in V(P_1^*) \cap V(G)$ such that the distance from $\{p, q\}$ to x in G is at least $b - 1$. Since $V(G) = V(P_1) \cup V(P_2)$, we have either $x \in V(P_1)$ or $x \in V(P_2)$. Suppose without loss of generality that $x \in V(P_1)$ (the case $x \in V(P_2)$ is symmetric). Our assumption on x then means that $|P_1[p, x]| \geq b - 1$. Now, $R := P_1[p, x] \circ (x, q^*)$ is a dipath of length at least b from $\{p, q\}$ to q^* (note that $(x, q^*) \in A(G^*)$ by the definition of a type-II gadget). Hence, R constitutes a $(2, b)$-alternating-path with $s_1(R) = t_1(R) = q^*$ and $s_2(R) = t_2(R) \in \{p, q\}$. Moreover, $|V(R) \cap \{p, q\}| = 1$ and $V(R) \cap \{p^*, q^*\} = \{q^*\}$ (since $p^* \notin V(G)$), as required.

So from now on we assume that every $x \in V(P_1^*) \cap V(G)$ is at distance at most $b-2$ from $\{p, q\}$ in G (in particular, if $b = 1$ then $V(P_1^*) \cap V(G) = \emptyset$). It follows that $|V(G) \cap V(P_1^*)| \leq 2(b - 1)$. Moving forward, we will consider two cases, based on the intersection of $V(G)$ with $V(P_2^*)$.

Case 2.1. $V(G) \cap V(P_2^*) = \emptyset$. Set $X := V(G) \cap V(G^*)$, noting that $X \neq \emptyset$ by assumption. As $V(G) \cap V(P_2^*) = \emptyset$ and $p^*, q^* \notin V(G)$, we must have that $X \subseteq V(G^*) \setminus (V(P_2^*) \cup \{p^*, q^*\}) = V(P_1^*) \setminus \{p^*, r^*\}$. By Item 2 of Lemma 2.2, there exists $1 \leq a \leq 2$ and an (a, b)-alternating-path R^* contained in G^*, such that $t_a(R^*) \in X$, $s_1(R^*) \in \{p^*, q^*\}$ and $|V(R^*) \cap \{p^*, q^*\}| = |V(R^*) \cap X| = 1$. For convenience, put $x := t_a(R^*)$. Note that $V(R^*) \cap V(G) = \{x\}$ by our choice of X and R^*. We now see that if $x \in \{p, q\}$, then $R := R^*$ satisfies the requirements of the lemma. Suppose then that $x \notin \{p, q\}$. Since $x \in V(G)$, we have either $x \in V(P_1)$ or $x \in V(P_2)$. Without loss of generality, we assume that $x \in V(P_1)$ (the case that $x \in V(P_2)$ is symmetric). Recall that by our assumption, x is at distance at most $b - 2$ from $\{p, q\}$ in G; in other words, the length of the dipath $P_1[p, x]$ is at most $b - 2$. As $|P_1| \geq 2b - 1 \geq 2b - 2$, we get that $|P_1[x, r]| = |P_1| - |P_1[p, x]| \geq b$. Now let R be the $(a + 1, b)$-alternating-path obtained by combining R^* with the dipaths $P_1[x, r]$ and P_2. Formally, R is defined by setting $s_i(R) = s_i(R^*)$ for every $1 \leq i \leq a$; $t_i(R) = t_i(R^*)$, $Q_i(R) = Q_i(R^*)$ and $Q'_i(R) = Q'_i(R^*)$ for every $1 \leq i \leq a-1$; $t_a(R) = r$; $s_{a+1}(R) = t_{a+1}(R) = q$; $Q_a(R) = Q_a(R^*) \circ P_1[x, r]$; and $Q'_a(R) = P_2$. Note that R is indeed an $(a + 1, b)$-alternating-path; this follows from our choice of R^*, the fact that $V(R^*) \cap V(G) = \{x\}$, and the bounds $|Q'_a(R)| = |P_2| \geq 2b - 1 \geq b$ and $|Q_a(R)| \geq |P_1[x, r]| \geq b$. We also have $|V(R) \cap \{p^*, q^*\}| = 1$ (as $V(R) \cap \{p^*, q^*\} = V(R^*) \cap \{p^*, q^*\}$) and $V(R) \cap \{p, q\} = \{q\}$

(by our definition of R and as $x \notin \{p, q\}$). Since $a + 1 \leq 3$, we see that the assertion of the lemma holds in Case 2.1.

Case 2.2. $V(G) \cap V(P_2^*) \neq \emptyset$. In this case, we traverse the dipath P_2^* *backwards* (i.e., starting from its last vertex, r^*), until the first time a vertex of $V(G)$ is reached, and denote this vertex by w. Evidently, $V(G) \cap V(P_2^*[w, r^*]) = \{w\}$. For convenience, let us set $P^* := P_2^*[w, r^*] \circ P_1^*$, noting that P^* starts at w, ends at p^*, and has length at least $|P_1^*| \geq 2b^2 + b - 2$ (by the definition of a type-II gadget). For every $u \in V(G) \cap (V(P^*) \setminus \{w\})$, denote by e_u the (unique) arc of P^* whose head is u. Let $R_1, \dots, R_m$ be the connected components of the digraph obtained from P^* by deleting the arc e_u for every $u \in V(G) \cap (V(P^*) \setminus \{w\})$ (this digraph is a dipath forest). Then for each $1 \leq i \leq m$, R_i is a dipath whose first vertex is in $V(G)$ and all of whose other vertices are not in $V(G)$. Recall that by our assumption, $|V(G) \cap V(P_1^*)| \leq 2(b-1)$. Now, our choice of w implies that $V(G) \cap V(P^*) = (V(G) \cap V(P_1^*)) \cup \{w\}$. The number of edges we deleted from P^* to obtain $R_1, \dots, R_m$ is, one the one hand, equal to $m - 1$, and on the other hand equal to $|V(G) \cap (V(P^*) \setminus \{w\})| \leq |V(G) \cap V(P_1^*)| \leq 2(b-1)$. It follows that $m \leq 2(b-1) + 1 = 2b - 1$ and $|R_1| + \dots + |R_m| \geq |P^*| - 2(b-1) \geq 2b^2 + b - 2 - 2(b-1) = 2b^2 - b$. By averaging, there is some $1 \leq i \leq m$ such that $|R_i| \geq \frac{2b^2 - b}{m} \geq \frac{2b^2 - b}{2b-1} \geq b$.

Let u (resp. v) be the first (resp. last) vertex of R_i. Note that $v \in V(P_1^*)$ due to our choice of w. Define a dipath R^* as follows: if $v = p^*$ then set $R^* := R_i$, and otherwise set $R^* := R_i \circ (v, q^*)$. (That $(v, q^*) \in A(G^*)$ follows from the definition of a type-II gadget and the fact that $v \in V(P_1^*)$.) Then $|V(R^*) \cap \{p^*, q^*\}| = 1$ and $V(G) \cap V(R^*) = \{u\}$ (because $V(G) \cap V(R_i) = \{u\}$ and $q^* \notin V(G)$). In particular, $u \in V(G) = V(P_1) \cup V(P_2)$. Suppose, without loss of generality, that $u \in V(P_1)$ (the case $u \in V(P_2)$ is symmetric). Now define a $(2, b)$-alternating-path R as follows: $s_1(R) = t_1(R) = q^*$, $s_2(R) = t_2(R) = p$, $Q_1'(R) = P_1[p, u] \circ R^*$. Note that $Q_1'(R)$ is indeed a dipath (because $V(P_1) \cap V(R^*) \subseteq V(G) \cap V(R^*) = \{u\}$), and that $|Q_1'(R)| \geq |V(R^*)| \geq |V(R_i)| \geq b$. Furthermore, $q \notin V(R)$ and $V(R) \cap \{p^*, q^*\} = V(R^*) \cap \{p^*, q^*\}$. Thus, R satisfies the requirements of the lemma. This completes the proof. $\square$

By combining Lemmas 2.4 and 2.5, we can now derive Lemma 2.3.

Proof of Lemma 2.3. For convenience, put $G := G_{(z_0, z_1)}$. We start by showing that for some $1 \leq a_1 \leq 3$, $G \cup G^*$ contains an (a_1, b)-alternating-path R^* satisfying $t_{a_1}(R^*) \in \{z_0, z_1\}$, $s_1(R^*) \in \{z_\ell, z^*\}$ and $|V(R^*) \cap \{z_0, z_1\}| = |V(R^*) \cap \{z_\ell, z^*\}| = 1$. If Condition 2 in the lemma holds, then this assertion follows immediately from Lemma 2.5. Note that the conditions of Lemma 2.5 are indeed satisfied in our setting: we have $V(G) \cap V(G^*) \neq \emptyset$ and $z^* \notin V(G)$ by assumption, and $z_\ell \notin V(G)$ by the definition of a chain and as $m \geq |A_2| \geq 2$.

Suppose now that Condition 1 in the lemma holds. Let $x \in V(G)$ be such that $(z_\ell, x) \in A(D)$. If $x \in \{z_0, z_1\}$ then the arc (z_ℓ, x) itself constitutes a $(1, b)$-alternating-path R^* with the required properties. Suppose from now on that $x \notin \{z_0, z_1\}$. So in particular, G is not a trivial gadget. Assume first that G is of type I or II. By Item 2 of Lemma 2.1, $\{z_0, z_1\}$ is reachable from x inside G. Fix a shortest path P_0 from x to $\{z_0, z_1\}$ contained in G. Then $|V(P_0) \cap \{z_0, z_1\}| = 1$. Now $R^* := (z_\ell, x) \circ P_0$ is a dipath from z_ℓ to $\{z_0, z_1\}$, and hence also a $(1, b)$-alternating-path with $t_1(R^*) \in \{z_0, z_1\}$ and $s_1(R^*) = z_\ell$, as required. Assume now that G is of type III. Let $r \in V(G)$ and $P_1, P_2 \subseteq V(G)$ be as in the definition of a type-III gadget (so P_1, P_2 are dipaths from z_0, z_1, respectively, to r, each having length at least $2b - 1$). Suppose without loss of generality that $x \in V(P_1)$ (the case that $x \in V(P_2)$ is symmetric). Now define a $(2, b)$-alternating-path R^* by setting $s_1(R^*) = z_\ell$, $t_1(R^*) = r$, $s_2(R^*) = t_2(R^*) = z_1$, $Q_1(R^*) = (z_\ell, x) \circ P_1[x, r]$ and $Q_1'(R^*) = P_2$, noting that $|Q_1'(R^*)| = |P_2| \geq 2b - 1 \geq b$ by the definition of a type-III gadget. Note that $z_0 \notin V(R^*)$ because $x \notin \{z_0, z_1\}$ by assumption. Thus, R^* satisfies our requirements.

We have thus shown that for some $1 \leq a_1 \leq 3$, $G \cup G^*$ contains an (a_1, b)-alternating-path R^* satisfying $t_{a_1}(R^*) \in \{z_0, z_1\}$, $s_1(R^*) \in \{z_\ell, z^*\}$ and $|V(R^*) \cap \{z_0, z_1\}| = |V(R^*) \cap \{z_\ell, z^*\}| = 1$. Let R_1 be the (a_1, b)-alternating-path obtained by combining R^* with the dipaths $P[t_{a_1}(R^*), z_{b+1}]$ and $(P \circ (z_\ell, z^*))[z_{\ell-b}, s_1(R^*)]$. Formally, we set $s_1(R_1) = z_{\ell-b}$; $t_{a_1}(R_1) = z_{b+1}$; $Q_1(R_1) = (P \circ (z_\ell, z^*))[z_{\ell-b}, s_1(R^*)] \circ Q_1(R^*)$; $Q_{a_1}(R_1) = Q_{a_1}(R^*) \circ P[t_{a_1}(R^*), z_{b+1}]$; $s_i(R_1) = s_i(R^*)$ for each $2 \leq i \leq a_1$; and $t_i(R_1) = t_i(R^*)$, $Q_i(R_1) = Q_i(R^*)$ and $Q'_i(R_1) = Q'_i(R^*)$ for each $1 \leq i \leq a_1 - 1$. Note that R_1 is strong, i.e., that $Q_1(R_1)$ and $Q_{a_1}(R_1)$ have length at least b each.

Put $a_2 := a + 2 - a_1$. Then $1 \leq a_2 \leq a + 1$ because $a \geq 2$ and $1 \leq a_1 \leq 3$. Now set $\mathcal{C}' := \mathcal{C}[z_{b+1}, z_{\ell-b}]$, noting that $|A_2(\mathcal{C}')| \geq |A_2(\mathcal{C})| - (2b+1) \geq (a+1)(b+1) - 1 \geq a_2(b+1) - 1$. By Lemma 2.4, applied with parameter a_2, the chain $\mathcal{C}'$ contains a strong (a_2, b)-alternating-path R_2 with $s_1(R_2) = z_{b+1} = t_{a_1}(R_1)$ and $t_{a_2}(R_2) = z_{\ell-b} = s_1(R_1)$. Note that $V(R_1) \cap V(R_2) = \{z_{b+1}, z_{\ell-b}\}$ because $V(R_1) \subseteq V(G) \cup V(G^*) \cup \{z_0, \ldots, z_{b+1}\} \cup \{z_{\ell-b}, \ldots, z_\ell\}$, $V(R_2) \subseteq V(\mathcal{C}') = V(\mathcal{C}[z_{b+1}, z_{\ell-b}])$ and $V(\mathcal{C}) \cap V(G^*) \subseteq V(G) \cup \{z_\ell\}$, and by the definition of a chain (see Items 3-4 in Definition 18). By Observation 17, $R_1 \cup R_2$ spans a subdivision of $C_{a,b}$, as required. □

2.4 Embedding Gadgets

In this section we prove two lemmas on finding certain gadgets in digraphs D possessing some suitable properties. Recall that g is chosen as $g = 4b^2$.

As mentioned before (see Subsection 2.1), a minimal counterexample to Theorem 16 has the property that every arc either lies on a directed cycle of length g or is dominated by a vertex. The following lemma shows that under these conditions, we can find on every given arc either a gadget of type I or an extended gadget of type II. The proof of this fact proceeds as follows. Starting from an arc $e = (p, q)$, we repeatedly take a vertex dominating the current arc. If the process carries through, then we find an extended gadget of type II as on the left-hand side of Figure 3. Otherwise, at some point we find a directed cycle of length g through one of the arcs. We then use this to either find directed cycle through (p, q) or an extended gadget of type II as on the right-hand side of Figure 3.

Lemma 2.6. *Let D be a digraph of directed girth at least g, and assume that for every $(x, y) \in A(D)$, either D contains a directed cycle of length exactly g through (x, y), or there is $z \in V(D) \setminus \{x, y\}$ such that $(z, x), (z, y) \in A(D)$. Then for every $(p, q) \in A(D)$, there is a type I or extended type-II gadget G contained in D such that $p(G) = p$, $q(G) = q$ and $|V(G)| \leq 2g$.*

Proof. Let $(p, q) \in A(D)$. We inductively define a sequence of vertices r_i, $i \geq 0$, with the property that $(r_i, q) \in A(D)$ for every $i \geq 0$ and $(r_i, r_{i-1}) \in A(D)$ for every $i \geq 1$. Set $r_0 := p$. Let $i \geq 1$, and suppose we have already defined $r_0, \ldots, r_{i-1}$. By assumption, either D contains a directed cycle C of length exactly g through (r_{i-1}, q), or there is $z \in V(D) \setminus \{r_{i-1}, q\}$ such that $(z, r_{i-1}), (z, q) \in A(D)$. In the latter case, we set $r_i := z$. In the former case, we stop, noting that $r_{i-1}, \ldots, r_1, r_0 = p, q, C[q, r_{i-1}]$ is a closed directed walk of length $i + (|C| - 1) = g + i - 1$ containing the arc (p, q). It follows that if we stop at step i, then there is a directed cycle of length at most $g + i - 1$ containing (p, q). Therefore, if the process stopped at step i for some $0 \leq i \leq 2b^2 + b - 2$, then D must contain a directed cycle of length at most $g + 2b^2 + b - 3 \leq 2g$ through (p, q). Moreover, this cycle must have length at least g since the directed girth of D is at least g. So we see that in this case, D contains a gadget G of type I with $p(G) = p$, $q(G) = q$ and $|V(G)| \leq 2g$, as required.

Suppose then that the process carried through to step $2b^2 + b - 2$ (inclusive), and let $r_0 = p, r_1, \ldots, r_{2b^2+b-2}$ be the vertices produced by the process. Recall that $r_{2b^2+b-2}, \ldots, r_1, p$ is a directed walk in D, all of whose vertices have an arc to q. Let $r := r_{2b^2+b-2}$ and $u := r_{2b^2+b-3}$ denote the first and second vertex of this dipath, respectively. We now inductively define a sequence of vertices w_i, $i \geq 0$, with the property that $(w_i, u) \in A(D)$ for every $i \geq 0$ and $(w_i, w_{i-1}) \in A(D)$ for every $i \geq 1$. Set $w_0 := r$. Let $i \geq 1$, and suppose we have already defined $w_0, \dots, w_{i-1}$. By assumption, either D contains a directed cycle C of length exactly g through (w_{i-1}, u), or there is $z \in V(D) \setminus \{w_{i-1}, u\}$ such that $(z, w_{i-1}), (z, u) \in A(D)$. In the latter case, we set $w_i := z$. In the former case, we stop and output the directed cycle C.

Suppose first that the process carried through to step b, and let $w_0 = r, w_1, \dots, w_b$ be the vertices produced by the process. Note that

$$P := (w_b, w_{b-1}, \dots, w_0 = r = r_{2b^2+b-2}, r_{2b^2+b-3}, \dots, r_1, r_0 = p, q)$$

is a directed walk of length at most $2b^2 + 2b - 1$ in D. Since $\vec{g}(D) \geq g = 4b^2 > 2b^2 + 2b - 1$, the vertices of P must be pairwise distinct. Now set $P_1 := (r = r_{2b^2+b-2}, \dots, r_1, r_0 = p)$ and $P_2 := (w_b, \dots, w_1, w_0 = r)$, and observe that P_1 and P_2 satisfy all the requirements in the definition of an extended type-II gadget (note that there is an arc from the first vertex of P_2, namely w_b, to the second vertex of P_1, namely u, by our choice of the vertices w_i, $i \geq 0$). Moreover, the resulting gadget has $2b^2 + 2b \leq 2g$ vertices, as required.

Suppose now that the process stopped at step i for some $1 \leq i \leq b$, and let C be the outputted directed cycle of length (exactly) g through the arc (w_{i-1}, u). As before, the $2b^2+b+i-1$ vertices $w_{i-1}, \dots, w_0 = r = r_{2b^2+b-2}, \dots, r_1, r_0 = p, q$ are pairwise distinct because $\vec{g}(D) \geq g = 4b^2 > 2b^2 + b + i - 1$ (as $i \leq b$). Traverse the directed cycle C *backwards*, starting from w_{i-1}, until the first time a vertex $v \in V := \{w_{i-2}, \dots, w_0 = r_{2b^2+b-2}, \dots, r_1, p, q\}$ is hit (this will surely happen because $u = r_{2b^2+b-3} \in V(C) \cap V$). By our choice of v we have $V(C[v, w_{i-1}]) \cap V = \{v\}$. We now rule out the possibility that $v \in \{w_0, \dots, w_{i-2}\}$. To this end, suppose by contradiction that $v = w_j$ for some $0 \leq j \leq i-2$. Since $w_{i-1}, w_{i-2}, \dots, w_j = v, C[v, w_{i-1}], w_{i-1}$ is a directed cycle and $\vec{g}(D) \geq g$, it must be the case that $|C[v, w_{i-1}]| \geq g-(i-1-j) \geq g-b+1$. Now, as C consists of the arc (w_{i-1}, u) and the dipaths $C[v, w_{i-1}]$ and $C[u, v]$, we have $|C[u, v]| = |C|-1-|C[v, w_{i-1}]| = g-1-|C[v, w_{i-1}]| \leq b-2$. Finally, we get that $v = w_j, (w_j, u), u, C[u, v], v = w_j$ is a (non-trivial) directed closed walk of length at most $b - 1 < g$, a contradiction.

We have thus shown that $v \notin \{w_0, \dots, w_{i-2}\}$. If $v = q$ then $w_{i-1}, u = r_{2b^2+b-3}, \dots, r_1, r_0 = p, q = v, C[v, w_{i-1}]$ is a directed cycle which goes through the arc (p, q) and has length at most $g + 2b^2 + b - 1 \leq 2g$ and at least g. Hence, in this case D contains a gadget of type I with the required properties. It remains to handle the case that $v \in \{u = r_{2b^2+b-3}, \dots, r_1, r_0 = p\}$. In this case, let s be the second vertex of $C[v, w_{i-1}]$ (so in particular, $(v, s) \in A(D)$). Now define the dipaths $P_1 := (r_{2b^2+b-2} = r, \dots, r_1, r_0 = p)$ and $P_2 := C[s, w_{i-1}] \circ (w_{i-1}, \dots, w_1, w_0 = r)$. We claim that $|P_2| \geq b$. Indeed, since $(v, s) \circ P_2 \circ P_1[r, v]$ is a directed cycle in D and $\vec{g}(D) \geq g$, it must be the case that $|P_2| \geq g - 1 - |P_1| = g - 1 - (2b^2 + b - 2) \geq b$. We now see that all of the requirements in the definition of an extended type-II gadget are met (note that the vertex $v \in V(P_1) \setminus \{r\}$ has an arc to the first vertex of P_2, namely s). Finally, observe that the resulting type-II gadget has $2b^2 + b + (i-1) + |C[s, w_{i-1}]| \leq 2b^2 + 2b + g \leq 2g$ vertices, as required. □

As explained in Subsection 2.1, if at some point in the proof of Theorem 16 we can neither extend the chain nor close a $C_{a,b}$-subdivision, then for the last vertex v of the spine of the chain, the directed paths emanating from v must have many intersections. In the following lemma we show that in this situation, we can find a gadget of type III.

Lemma 2.7. *Let $b, h, d \geq 1$ be integers, let D' be a digraph and let $v \in V(D')$. Suppose that the following two conditions hold.*

1. *Every vertex of D' reachable from v has out-degree at least $(h+1) \cdot (d(2b-2)+1) + d$;*
2. *The number of vertices of D' at distance at most $(h+1)(2b-1)$ from v is less than d^h.*

Then D' contains a type-III gadget G and a dipath P_0 from v to $p(G)$ such that $V(P_0) \cap V(G) = \{p(G)\}$, $|V(G)| \leq (2h+2)(2b-1)$ and $|V(P_0)| \leq h(2b-1)$.

Proof. We describe a process for producing a (specific) out-arborescence $T \subseteq D$ with root v. The idea is as follows: going level by level (in a breadth-first manner), we will try to attach to each vertex u of the (current) lowest level a collection of d dipaths of length $2b-1$ each, which intersect only at u and do not intersect the (current) tree in any other vertex. In this manner, we will construct a $(2b-1)$-*subdivision* of a *d-ary out-arborescence*, where an s-subdivision of a digraph F is a subdivision of F in which every arc is replaced with a dipath of length (exactly) s, and a d-ary out-arborescence is an out-arborescence in which every non-leaf vertex has exactly d children. We will then use Item 2 to argue that rather soon in this process, intersections of branches must occur. Such an intersection will give rise to the desired type-III gadget. The details follow.

Throughout the process, we will maintain and update an out-arborescence T and sets L_i, $i \geq 0$. We start by setting $L_0 = \{v\}$ and initializing T to be the one-vertex tree with root v. Let $i \geq 0$, and suppose that we have already defined $L_0, \dots, L_i$. If $L_i = \emptyset$ then we stop and say that the process terminated at step i. Otherwise, initialize L_{i+1} to be the empty set and proceed as follows. Let $u_1, \dots, u_t$ be an enumeration of the vertices in L_i. Going over $j = 1, \dots, t$ in increasing order, we let $\mathcal{P}(u_j)$ be the set of all dipaths of length $2b-1$ which start at u_j and are otherwise disjoint from $V(T)$. If $\mathcal{P}(u_j)$ contains d dipaths $Q_1, \dots, Q_d$ with $V(Q_k) \cap V(Q_\ell) = \{u_j\}$ for all $1 \leq k < \ell \leq d$, then attach these dipaths to T and add their endpoints to L_{i+1}. Otherwise, i.e. if $\mathcal{P}_j$ does not contain d dipaths $Q_1, \dots, Q_d$ which pairwise intersect only at u_j, then do nothing; in this case u_j will remain a leaf of T throughout the process.

Consider the out-arborescence T at the end of the process. It is easy to see that T is indeed the $(2b-1)$-subdivision of some d-ary out-arborescence T_0, and that the branch vertices of this subdivision are precisely the elements of $\bigcup_{i \geq 0} L_i$. It follows that $|L_i| \leq d^i$ for every $i \geq 0$.

We claim that there is $0 \leq i \leq h$ such that L_i contains a leaf of T. Indeed, suppose by contradiction that for every $0 \leq i \leq h$, no vertex of L_i is a leaf. Then $|L_i| = d^i$ for every $0 \leq i \leq h+1$. Observe that the number of vertices of T which are at distance at most $(h+1)(2b-1)$ from v (in T) is exactly $|L_{h+1}| + \sum_{i=0}^{h} \big(d(2b-2)+1\big) \cdot |L_i|$. Hence, the number of such vertices is at least

$$\sum_{i=0}^{h} \big(d(2b-2)+1\big) \cdot |L_i| = \big(d(2b-2)+1\big) \cdot \sum_{i=0}^{h} d^i \geq d^h,$$

in contradiction to the assumption in Item 2 of the lemma.

So let $0 \leq i \leq h$ be such that L_i contains a leaf of T, and let $u \in L_i$ be such a leaf. Let X be the set of vertices of T which are at distance at most $(i+1) \cdot (2b-1)$ from the root v. In other words, X consists of the sets $L_0, \dots, L_{i+1}$ and the (vertices of the) subdivision dipaths connecting L_j to L_{j+1} for $j = 0, \dots, i$. Say that a d-tuple of dipaths $(Q_1, \dots, Q_d)$ is *good* if

(a) For every $k = 1, \dots, d$, it holds that $|Q_k| \leq 2b-1$, $V(Q_k) \cap X = \{u\}$, and u is the first vertex of Q_k.

(b) $V(Q_k) \cap V(Q_\ell) = \{u\}$ for all $1 \leq k < \ell \leq d$.

Among all good d-tuples of dipaths, let $(Q_1, \dots, Q_d)$ be one which maximizes $|Q_1| + \cdots + |Q_d|$ (note that taking $Q_1, \dots, Q_d$ to be empty dipaths (starting at u) gives a good d-tuple, so the set of good d-tuples is non-empty). Observe that if we had $|Q_1| = \cdots = |Q_d| = 2b-1$, then the algorithm would have attached $Q_1, \dots, Q_d$ to T, in contradiction to our assumption that u is a leaf. Thus, there must be some $1 \leq k \leq d$ such that $|Q_k| \leq 2b-2$. Suppose without loss of generality that $|Q_1| \leq 2b-2$, and let w be the last vertex of Q_1. Evidently, w is reachable from u and hence also from v, implying that $d^+(w) \geq (h+1) \cdot (d(2b-2)+1) + d$ by Item 1. If w had an out-neighbour in $V(D') \setminus (X \cup V(Q_1) \cup \cdots \cup V(Q_d))$, then we could extend Q_1 and thus obtain a longer good d-tuple of dipaths, in contradiction to the maximality of $(Q_1, \ldots, Q_d)$. Thus, $N^+(w) \subseteq X \cup V(Q_1) \cup \cdots \cup V(Q_d)$. As $|N^+(w) \cap (V(Q_1) \cup \cdots \cup V(Q_d))| \leq |V(Q_1) \cup \cdots \cup V(Q_d)| - 1 = |V(Q_1)| + \cdots + |V(Q_d)| - (d-1) - 1 \leq d \cdot 2b - 1 - d = d(2b-1) - 1$, we must have $|N^+(w) \cap X| \geq d^+(w) - d(2b-1) + 1 \geq hd(2b-2) + h + 2$.

For each vertex $x \in X$, let $y(x)$ denote the lowest common ancestor of u and x in the tree T. Let X' be the set of all vertices $x \in X$ such that (at least) one of the vertices u, x is at distance at most $2b-2$ from $y(x)$ in T. We now show that $|X'| \leq hd(2b-2)+h+1$. Let P be the unique dipath (in T) from v to u. For each $0 \leq j \leq i-1$, let y_j be the unique element of $V(P) \cap L_j$. Observe that if $x \in X'$, then either $x \in V(P)$, or there is $0 \leq j \leq i-1$ such that x is an internal vertex of one of the $d-1$ subdivision dipaths which start at y_j and are not subpaths of P. (Recall that every non-leaf branching vertex of T is the first vertex of exactly d subdivision dipaths. It is evident that for every $0 \leq j \leq i-1$, exactly one of the d subdivision dipaths starting at y_j is a subpath of P, while the other $d-1$ only intersect P at y_j.) It follows that $|X'| = |V(P)| + i \cdot (d-1) \cdot (2b-2) = i \cdot (2b-1) + 1 + i \cdot (d-1) \cdot (2b-2) = i \cdot d \cdot (2b-2) + i + 1 \leq hd(2b-2) + h + 1$, as claimed.

As $|N^+(w) \cap X| \geq hd(2b-2) + h + 2 > |X'|$, there exists $x \in N^+(w) \cap (X \setminus X')$. Setting $y = y(x)$, let P_1' (resp. P_2') be the unique dipath (in T) from y to x (resp. u). Since $x \notin X'$, we have $|P_1'|, |P_2'| \geq 2b-1$. As $u \in L_i$, we have $|P_2'| \leq i(2b-1) \leq h(2b-1)$, and as $x \in X$ we have $|P_1'| \leq (h+1)(2b-1)$. Let z be the second vertex of P_2'. Now set $P_0 := P[v,y]$, $P_1 := P_1'$ and $P_2 := P_2'[z,u] \circ Q_1 \circ (w,x)$. Observe that P_0, P_1, P_2 are internally vertex-disjoint (as y is the lowest common ancestor of x and u), and that P_1 and P_2 have length at least $2b-1$ each (indeed, we have $|P_1| = |P_1'| \geq 2b-1$ and $|P_2| \geq |P_2'| - 1 + |Q_1| + 1 \geq |P_2'| \geq 2b-1$). So we see that P_1, P_2 form a type-III gadget G with $p(G) = y$ and $q(G) = z$, and that this gadget satisfies $V(P_0) \cap V(G) = \{y\} = \{p(G)\}$. Finally, observe that $|P_0| \leq (i-1)(2b-1) \leq h(2b-1)$, $|P_1| = |P_1'| \leq (h+1)(2b-1)$ and $|P_2| = (|P_2'|-1) + |Q_1| + 1 \leq h(2b-1) + 2b-2 \leq (h+1)(2b-1)$. It follows that $|V(G)| = |V(P_1)| + |V(P_2)| - 1 \leq (2h+2)(2b-1)$. This completes the proof. ☐

2.5 Putting It All Together

Proof of Theorem 16. Let $a \geq 2$ and $b \geq 1$. Recall that we set $g := 4b^2$. Suppose, for the sake of contradiction, that the theorem is false, and let D be a counterexample to the theorem which minimizes $|V(D)| + |A(D)|$. Namely, we assume that $\delta^+(D) \geq k$, $\vec{g}(D) \geq g$ and D does not contain a subdivision of $C_{a,b}$, but every digraph D' with $|V(D')| + |A(D')| < |V(D)| + |A(D)|$, $\delta^+(D') \geq k$ and $\vec{g}(D') \geq g$ does contain a subdivision of $C_{a,b}$. Here, $k = k(a,b)$ is an integer depending on a and b chosen such that $k = O(ab^7)$, for a precise value compare page 22, after the proof of Claim 4.

Claim 1. $d^+(v) = k$ for every $v \in V(D)$.

Proof. Suppose, by contradiction, that $d^+(v) \geq k+1$ for some $v \in V(D)$. Let D' be the digraph obtained from D by deleting an (arbitrary) arc whose tail is v. Then $\delta^+(D') \geq k$ and $\vec{g}(D') \geq g$, but D' does not contain a subdivision of $C_{a,b}$ (as D' is a subgraph of D). This contradicts the minimality of D. ☐

Claim 2. For every $(x,y) \in A(D)$, either D contains a directed cycle of length exactly g through (x,y), or there is $z \in V(D) \setminus \{x,y\}$ such that $(z,x), (z,y) \in A(D)$.

Proof. Let $(x,y) \in A(D)$. Suppose by contradiction that the assertion of the claim is false. Let D' be the digraph obtained from D by deleting x and adding the arc (z,y) for every $z \in N_D^-(x)$. Evidently, $|V(D')| + |A(D')| < |V(D)| + |A(D)|$. We claim that $\delta^+(D') \geq k$ and $\vec{g}(D') \geq g$. First, note that $d_{D'}^+(y) = d_D^+(y) = k$ because $(y,x) \notin A(D)$ (as $\vec{g}(D) \geq g > 2$). Next, observe that for every $z \in V(D') \setminus \{y\} = V(D) \setminus \{x,y\}$ we also have $d_{D'}^+(z) = d_D^+(z) = k$, because z does not have both x and y as out-neighbors (by our assumption). It follows that $\delta^+(D') \geq k$. Now suppose, for the sake of contradiction, that D' contains a directed cycle C' of length at most $g-1$. If there is no $z \in N_D^-(x)$ such that $(z,y) \in A(C')$, then C' is also contained in D, which is impossible as $\vec{g}(D) \geq g$. So let $z \in N_D^-(x)$ be such that $(z,y) \in A(C')$, and let C be the directed cycle obtained from C' by deleting the arc (z,y) and adding the arcs $(z,x),(x,y)$. Then C is contained in D and has length $|C'|+1 \leq g$, implying that $|C| = g$. But this is impossible as we assumed that D contains no directed cycle of length g through the arc (x,y). We conclude that $\vec{g}(D') \geq g$, as claimed.

The minimality of D implies that D' contains a subdivision S' of $C_{a,b}$. If there is no $z \in N_D^-(x)$ such that $(z,y) \in A(S')$, then S' is also contained in D, contradicting our assumption that D contains no subdivision of $C_{a,b}$. Suppose then that the set $Z := \{z \in N_D^-(x) : (z,y) \in A(S')\}$ is non-empty. Since the maximum in-degree of $C_{a,b}$ is 2, we have $|Z| \leq 2$. Assume first that $|Z| = 1$, and write $Z = \{z\}$. By replacing the edge (z,y) of S' with the path $(z,x),(x,y)$ (which is present in D), we obtain a subdivision of $C_{a,b}$ contained in D, a contradiction. Suppose now that $|Z| = 2$, and write $Z = \{z_1, z_2\}$. Then y must be a branch vertex in S', and we must have $d_{S'}^+(y) = 0$ (since every branch vertex of $C_{a,b}$ is either a source or a sink). Let S be the subgraph of D obtained from S' by deleting the edges $(z_1,y),(z_2,y)$ and adding the edges $(z_1,x),(z_2,x)$. Then S is a subdivision of $C_{a,b}$ in which x plays the branch-vertex role played in S' by y. Again, we have arrived at a contradiction to our assumption that D contains no subdivision of $C_{a,b}$. $\square$

Let $\mathcal{C}$ be a chain with spine $P = v_0, \dots, v_m$ and partition $A(P) = A_1 \cup A_2$ (as in Definition 18). We say that $\mathcal{C}$ is *good* if the following conditions are satisfied:

(a) Every gadget in $\mathcal{C}$ has at most $(8g+6)(2b-1)$ vertices;

(b) $(v_{m-1}, v_m) \in A_2$;

(c) Among any $(4g+3)(2b-1)$ consecutive arcs of P, there is an arc belonging to A_2.

Among all good chains contained in D, let $\mathcal{C}$ be one of maximal length, and let $P = v_0, \dots, v_m$, A_1, A_2 and $(G_e)_{e \in A_2}$ be as in Definition 18. Define $i_0 := \max\left\{0, m-(4g+3)(2b-1)(a+3)(b+1)\right\}$ and $\mathcal{C}' := \mathcal{C}[v_{i_0}, v_m]$. Item (a) implies that

$$\begin{aligned} |V(\mathcal{C}')| &\leq (8g+6)(2b-1) \cdot (m-i_0) \leq 2(4g+3)^2(2b-1)^2(a+3)(b+1) \\ &\leq 8b^2(4g+3)^2(a+3)(b+1). \end{aligned} \tag{1}$$

Our choice of i_0 implies that either $i_0 = 0$ and $\mathcal{C}' = \mathcal{C}$, or $i_0 = m-(4g+3)(2b-1)(a+3)(b+1)$, in which case we have by Item (c) that

$$|A_2(\mathcal{C}')| \geq \left\lfloor \frac{m-i_0}{(4g+3)(2b-1)} \right\rfloor = (a+3)(b+1).$$

Let D' be the digraph obtained from D by deleting the vertex-set $V(\mathcal{C}) \setminus \{v_m\}$.

Claim 3. Every $u \in V(D')$ which is reachable from v_m in D' satisfies $d^+_{D'}(u) \geq k - |V(\mathcal{C}')|$.

Proof. If $i_0 = 0$, then $\mathcal{C}' = \mathcal{C}$ and the claim follows directly by definition of D'. So assume now that $i_0 > 0$ and hence $|A_2(\mathcal{C}')| \geq (a+3)(b+1) \geq (a+3)(b+1) - 2$. Let Q be a dipath from v_m to u in D'. Suppose by contradiction that $d^+_{D'}(u) < k - |V(\mathcal{C}')|$. Since $d^+_D(u) \geq k$ and $V(D) \setminus V(D') \subseteq V(\mathcal{C})$, we must have $|N^+_D(u) \cap V(\mathcal{C})| > |V(\mathcal{C}')|$. Hence, there must be some $x \in V(\mathcal{C}) \setminus V(\mathcal{C}')$ such that $(u, x) \in A(D)$. Let $0 \leq j \leq m$ be such that $x \in G_{(v_j, v_{j+1})}$, and note that $j < i_0$ because $x \notin V(\mathcal{C}')$. Now let $\mathcal{C}''$ be the chain formed by concatenating $\mathcal{C}[v_j, v_m]$ with the dipath Q (in this chain, all arcs of Q belong to $A_1(\mathcal{C}''))$. This is indeed a chain because $V(Q) \cap V(\mathcal{C}) = \{v_m\}$. As $\mathcal{C}'$ is contained in $\mathcal{C}''$, we have $|A_2(\mathcal{C}'')| \geq |A_2(\mathcal{C}')| \geq (a+3)(b+1) - 2$. Observe that we are precisely in the setting of Item 1 of Lemma 2.3 with respect to the chain $\mathcal{C}''$. Indeed, the last vertex of the spine of $\mathcal{C}''$, namely u, sends an arc to $x \in V(G_{(v_j, v_{j+1})})$, and (v_j, v_{j+1}) is the first arc of the spine of $\mathcal{C}''$. So we may apply Item 1 of Lemma 2.3 to deduce that D contains a subdivision of $C_{a,b}$, a contradiction. $\square$

Claim 4. Let $u \in V(D')$ be a vertex whose distance from v_m in D' is at most $(4g+3)(2b-1)$. Then D contains a dipath of length at most $2g$ from $V(\mathcal{C}')$ to u.

Proof. Let $Q = (w_0 = v_m, w_1, \ldots, w_{t-1}, w_t = u)$ be a shortest dipath from v_m to u in D'. Then $t = |Q| \leq (4g+3)(2b-1)$. Claim 2 states that D satisfies the condition of Lemma 2.6. By applying Lemma 2.6 to the arc (w_{t-1}, u), we infer that D contains a gadget G^* which is either of type I or extended type-II, such that $p(G^*) = w_{t-1}$, $q(G^*) = u$ and $|V(G^*)| \leq 2g$.

We now show that if $V(G^*) \cap V(\mathcal{C}') \neq \emptyset$, then the assertion of the claim holds. So suppose that $V(G^*) \cap V(\mathcal{C}') \neq \emptyset$, and let $x \in V(G^*) \cap V(\mathcal{C}')$. By Item 2 of Lemma 2.1, G^* contains a dipath from x to $\{w_{t-1}, u\}$, and hence also to u, as $(w_{t-1}, u) \in A(G^*)$. Evidently, this dipath has length at most $|V(G^*)| \leq 2g$. So we see that D contains a dipath of length at most $2g$ from $V(\mathcal{C}')$ to u, as required. To complete the proof, it hence suffices to show that $V(G^*) \cap V(\mathcal{C}') \neq \emptyset$. For the rest of the proof we assume, for the sake of contradiction, that $V(G^*) \cap V(\mathcal{C}') = \emptyset$. We proceed by a case analysis over the type of G^*.

Case 1. G^* is an extended gadget of type II. Let G^*_0 be the basic part of G^*. We claim that $V(G^*_0) \cap V(Q) = \{w_{t-1}, u\}$. Suppose otherwise, and let $0 \leq j \leq t-2$ be such that $w_j \in V(G^*_0)$. By the definition of a basic type-II gadget, every vertex in $V(G^*_0) \setminus \{u\}$ has an arc to $q(G^*_0) = u$. In particular, $(w_j, u) \in A(D)$, and hence also $(w_j, u) \in A(D')$ (as $w_j, u \in V(D')$). It follows that $w_0, \ldots, w_{j-1}, w_j, u$ is a dipath from $w_0 = v_m$ to u in D' which is shorter than Q, in contradiction to our choice of Q. So indeed we have $V(G^*_0) \cap V(Q) = \{w_{t-1}, u\}$.

We claim that $V(G^*) \cap V(\mathcal{C}) \neq \emptyset$. So suppose by contradiction that $V(G^*) \cap V(\mathcal{C}) = \emptyset$. Then one can extend the chain $\mathcal{C}$ into a longer good chain $\mathcal{C}_1$ by adding the dipath Q and the gadget G^*_0; the definition of $\mathcal{C}_1$ includes setting $(w_{t-1}, u) \in A_2(\mathcal{C}_1)$, $G_{(w_{t-1},u)}(\mathcal{C}_1) = G^*_0$, and $(w_j, w_{j+1}) \in A_1(\mathcal{C}_1)$ for every $0 \leq j \leq t-2$. Then $\mathcal{C}_1$ is indeed a chain because $V(G^*_0) \cap V(Q) = \{w_{t-1}, u\}$ and due to our assumption that $V(G^*) \cap V(\mathcal{C}) = \emptyset$. The goodness of $\mathcal{C}_1$ (i.e. that $\mathcal{C}_1$ satisfies Items (a)-(c) above) follows from the goodness of $\mathcal{C}$ and the fact that $|Q| \leq (4g+3)(2b-1)$ and $|V(G^*_0)| \leq 2g \leq (8g+6)(2b-1)$. As the existence of $\mathcal{C}_1$ stands in contradiction to the maximality of $\mathcal{C}$, our assumption $V(G^*) \cap V(\mathcal{C}) = \emptyset$ must have been wrong, as required.

We have thus shown that $V(G^*) \cap V(\mathcal{C}) \neq \emptyset$. Since $V(G^*) \cap V(\mathcal{C}') = \emptyset$ by assumption, we must have $V(G^*) \cap (V(\mathcal{C}) \setminus V(\mathcal{C}')) \neq \emptyset$. This means that $V(G^*) \cap V(G_{(v_i, v_{i+1})}) \neq \emptyset$ for some $0 \leq i < i_0$ (as $V(\mathcal{C}) \setminus V(\mathcal{C}')$ is contained in the union of $V(G_{(v_i, v_{i+1})})$ over all $0 \leq i < i_0$). Let i_1 be the largest such $0 \leq i_1 < i_0$, and set $G := G_{(v_{i_1}, v_{i_1+1})}$. Now let $\mathcal{C}_1$ be the chain obtained by attaching to $\mathcal{C}[v_{i_1}, v_m]$ the dipath $Q - u = (w_0 = v_m, w_1, \ldots, w_{t-1})$. This is indeed a chain because $V(Q) \cap V(\mathcal{C}) = \{v_m\}$ (as $V(Q) \subseteq V(D')$ and $(V(\mathcal{C}) \setminus \{v_m\}) \cap V(D') = \emptyset$). Then $|A_2(\mathcal{C}_1)| \geq |A_2(\mathcal{C}')| \geq (a+3)(b+1) - 2$ because $\mathcal{C}_1$ contains $\mathcal{C}'$ and $i_0 > 0$. Observe that Condition 2 in Lemma 2.3 holds for the chain $\mathcal{C}_1$ with respect to the vertex $z^* := u$ (and with $z_\ell = w_{t-1}$, $z_0 = v_{i_1}$ and $z_1 = v_{i_1+1}$). Indeed, there is an arc from the last vertex of the spine of $\mathcal{C}_1$, namely w_{t-1}, to $u \notin V(\mathcal{C}_1)$, and there is an extended type-II gadget G^* such that $p(G^*) = w_{t-1}$, $q(G^*) = u$, $V(G) \cap V(G^*) \neq \emptyset$ and $V(\mathcal{C}_1) \cap V(G^*) \subseteq V(G) \cup \{w_{t-1}\}$ (here we use our choice of i_1). By Lemma 2.3, D contains a subdivision of $C_{a,b}$, a contradiction.

Case 2. G^* is of type I, i.e., a directed cycle of length at least g through (w_{t-1}, u). Let j be the smallest integer in $\{0, \ldots, t-1\}$ satisfying $w_j \in V(G^*)$; note that j is well-defined because $w_{t-1} \in V(G^*)$. Let w' be the vertex of the directed cycle G^* immediately following w_j, and consider the dipath $Q' := (w_0 = v_m, w_1, \ldots, w_j, w')$. Our choice of j implies that $w' \notin \{w_0, \ldots, w_{j-1}\}$ (so Q' is indeed a path) and that $V(G^*) \cap V(Q') = \{w_j, w'\}$. Note also that $j \geq 1$ because $w_0 = v_m \in V(\mathcal{C}')$ and $V(G^*) \cap V(\mathcal{C}') = \emptyset$ by assumption. Hence, $w_j \notin V(\mathcal{C})$.

Similarly to the previous case, if $V(G^*) \cap V(\mathcal{C}) = \emptyset$, then one can extend $\mathcal{C}$ into a longer good chain $\mathcal{C}_1$ by adding the dipath Q' and the gadget G^*; the definition of $\mathcal{C}_1$ includes setting $(w_j, w') \in A_2(\mathcal{C}_1)$, $G_{(w_j,w')}(\mathcal{C}_1) = G^*$, and $(w_i, w_{i+1}) \in A_1(\mathcal{C}_1)$ for every $0 \leq i \leq j-1$. Then $\mathcal{C}_1$ is indeed a chain because $V(G^*) \cap V(Q') = \{w_j, w'\}$ and $V(G^*) \cap V(\mathcal{C}) = \emptyset$, and the goodness of $\mathcal{C}_1$ follows from the goodness of $\mathcal{C}$ and the fact that $|Q| \leq (4g+3)(2b-1)$ and $|V(G^*)| \leq 2g \leq (8g+6)(2b-1)$. So we see that having $V(G^*) \cap V(\mathcal{C}) = \emptyset$ contradicts the maximality of $\mathcal{C}$, and hence $V(G^*) \cap V(\mathcal{C}) \neq \emptyset$.

Walk along the directed cycle G^*, starting from w_j, until the first time that a vertex of $V(\mathcal{C})$ is met. Denote this vertex by x, and the preceding vertex on G^* by y. Consider the dipath $Q'' := (w_0 = v_m, w_1, \ldots, w_j) \circ G^*[w_j, y]$, and observe that $V(Q'') \cap V(\mathcal{C}) = \{v_m\}$ because $V(Q) \cap V(\mathcal{C}) = \{v_m\}$ and by our choice of x. Since $V(G^*) \cap V(\mathcal{C}') = \emptyset$, we must have $x \in V(G^*) \cap (V(\mathcal{C}) \setminus V(\mathcal{C}')) \neq \emptyset$. This means that $x \in V(G_{(v_i,v_{i+1})})$ for some $0 \leq i < i_0$. Now let $\mathcal{C}_1$ be the chain obtained by concatenating $\mathcal{C}[v_i, v_m]$ with the dipath Q''. This is indeed a chain because $V(Q'') \cap V(\mathcal{C}) = \{v_m\}$. Then $|A_2(\mathcal{C}_1)| \geq |A_2(\mathcal{C}')| \geq (a+3)(b+1) - 2$ because $\mathcal{C}_1$ contains $\mathcal{C}'$ and $i_0 > 0$. Observe that Condition 1 in Lemma 2.3 holds for the chain $\mathcal{C}_1$ (with y playing the role of z_ℓ). Indeed, there is an arc from the last vertex of the spine of $\mathcal{C}_1$, namely y, to $x \in V(G_{(i,i+1)})$, and (v_i, v_{i+1}) is the first arc of the spine of $\mathcal{C}_1$. By Lemma 2.3, D contains a subdivision of $C_{a,b}$, a contradiction. This completes the proof of Claim 4. $\square$

With Claims 3-4 at hand, we can complete the proof of the theorem. To this end, we will apply Lemma 2.7. By combining Claim 4 with the fact that $\Delta^+(D) = k$, we conclude that the number of vertices of D' at distance at most $(4g+3)(2b-1)$ from v_m (in D') is at most $|V(\mathcal{C}')| \cdot k^{2g}$. We will apply Lemma 2.7 with parameters $h := 4g+2$ and $d := 2b(4g+3)(a+3)(b+1)$. To this end, we will need to verify that

$$k - |V(\mathcal{C}')| \geq (4g+3) \cdot (d(2b-2)+1) + d \text{ and } |V(\mathcal{C}')| \cdot k^{2g} < d^{4g+2}. \tag{2}$$

This is the point where we choose the value of k; set $k := 12b^2(4g+3)^2(a+3)(b+1)$, noting that $k = O(ab^7)$ because $g = 4b^2$. Both inequalities in (2) follow from (1) and our choice of k and d. Indeed, we have:

$$\begin{aligned} |V(\mathcal{C}')| + (4g+3) \cdot (d(2b-2)+1) + d &\leq |V(\mathcal{C}')| + (4g+3) \cdot 2db \\ &\leq 8b^2(4g+3)^2(a+3)(b+1) + (4g+3) \cdot 2db \\ &\leq 12b^2(4g+3)^2(a+3)(b+1) = k, \end{aligned}$$

and

$$\begin{aligned}|V(\mathcal{C}')|\cdot k^{2g} &\le 8b^2(4g+3)^2(a+3)(b+1)\cdot k^{2g}\\ &= 8\cdot 12^{2g}\cdot b^{4g+2}(4g+3)^{4g+2}(a+3)^{2g+1}(b+1)^{2g+1}\\ &= 8\cdot 12^{2g}\cdot 2^{-4g-2}\cdot (a+3)^{-2g-1}(b+1)^{-2g-1}\cdot d^{4g+2}\\ &= 2\cdot 3^{2g}\cdot (a+3)^{-2g-1}(b+1)^{-2g-1}\cdot d^{4g+2} < d^{4g+2}.\end{aligned}$$

Claims 3 and 4 imply that D' satisfies Conditions 1 and 2 in Lemma 2.7, respectively, with the role of v played by v_m, and with the parameters h and d chosen above. By Lemma 2.7, D' contains a type-III gadget G and a dipath P_0 from v_m to $p(G)$ such that $V(P_0)\cap V(G) = \{p(G)\}$, $|V(G)| \le (2h+2)(2b-1) = (8g+6)(2b-1)$ and $|V(P_0)| \le h(2b-1) = (4g+2)(2b-1) \le (4g+3)(2b-1)-1$. Now, let $\mathcal{C}_1$ be the chain formed by appending to $\mathcal{C}$ the dipath P_0 and the gadget G; so the spine of $\mathcal{C}_1$ is $P \circ P_0 \circ (p(G), q(G))$, $A_1(\mathcal{C}_1) = A_1(\mathcal{C}) \cup A(P_0)$ and $A_2(\mathcal{C}_1) = A_2(\mathcal{C}) \cup \{(p(G), q(G))\}$. It is easy to see that $\mathcal{C}_1$ is indeed a chain and that it satisfies Conditions (a)-(c) above. But this contradicts the maximality of $\mathcal{C}$. This final contradiction means that our initial assumption, that D is a counterexample to Theorem 16, was false. This completes the proof of the theorem. □

3 Oriented cycles with two blocks

In this section, we prove Theorem 8. We will repeatedly use the following observation:

Lemma 3.1. *Let $\ell_1, \ell_2 \in \mathbb{N}$, and D a digraph with $\delta^+(D) \ge \ell_1 + \ell_2$. Then for every $v \in V(D)$, there are dipaths P_1 and P_2 in D of length ℓ_1 and ℓ_2, respectively, which start in v and satisfy $V(P_1) \cap V(P_2) = \{v\}$.*

Proof. Greedily build two disjoint dipaths starting at v by attaching out-neighbors at their ends until they have lengths ℓ_1 and ℓ_2, respectively. □

Proof of Theorem 8. Let D be an arbitrary digraph such that $\delta^+(D) \ge k_1 + 3k_2 - 5$. We have to show that there exist two internally vertex-disjoint dipaths in D which start and end in the same vertices, one of length at least k_1, the other of length at least k_2. Throughout the proof, we will say that a dipath P in D with terminal vertex x is *k_2-good* if there exist dipaths P_1 and P_2 of length $k_2 - 1$ starting at x such that $V(P_1) \cap V(P_2) = \{x\}$ and $V(P_i) \cap V(P) = \{x\}$ for $i \in \{1, 2\}$. Note that D contains a k_2-good dipath of positive length. Indeed, choose some arbitrary vertex $u \in V(D)$ and some out-neighbor v of u. Since $\delta^+(D-u) \ge \delta^+(D) - 1 \ge k_1 + 3k_2 - 6 \ge (k_2 - 1) + (k_2 - 1)$, we can apply Lemma 3.1 with $\ell_1 := \ell_2 := k_2 - 1$ to the vertex v in the digraph $D - u$ to infer that $P := (u, v)$ is a k_2-good dipath. Let P_0 be a longest k_2-good dipath in D. We have just shown that $|P_0| > 0$. Denote by x the end-vertex of P_0 and by P_1, P_2 two dipaths of length $k_2 - 1$ starting in x such that $V(P_i) \cap V(P_j) = \{x\}$ for $i \ne j \in \{0, 1, 2\}$. Let a be the terminus of P_1 and b the terminus of P_2.

Claim 1. There exist dipaths P_a, P_b starting in a, b and ending in vertices $a', b' \in V(P_0) \setminus \{x\}$, respectively, such that P_a and P_b are internally vertex-disjoint from $V(P_0) \cup V(P_1) \cup V(P_2)$.

Proof. We prove the existence of P_a and a'; the proof for the existence of P_b and b' is completely analogous. Let $D' := D - ((V(P_1) \cup V(P_2)) \setminus \{a\})$. Note that since $|(V(P_1) \cup V(P_2)) \setminus \{a\}| = 2k_2 - 2$, we have $\delta^+(D') \ge \delta^+(D) - (2k_2 - 2) \ge k_1 + k_2 - 3 \ge 2k_2 - 3$. Let $R \subseteq V(D')$ be the set of vertices reachable from a by a dipath in D'. We claim that $R \cap (V(P_0) \setminus \{x\}) \ne \emptyset$. Suppose towards a contradiction that $R \cap (V(P_0) \setminus \{x\}) = \emptyset$. Since for every vertex $r \in R$ we have

$N^+_{D'}(r) \subseteq R$, we see that $\delta^+(D'[R]) \geq \delta^+(D') \geq 2k_2 - 3$. We can now apply Lemma 3.1 to the vertex a of $D'[R]$ with $\ell_1 := k_2 - 1, \ell_2 := k_2 - 2$ and find that $D'[R]$ contains dipaths P_1' and P_2' of lengths ℓ_1 and ℓ_2, respectively, which start at a and satisfy $V(P_1') \cap V(P_2') = \{a\}$. Let w be the end-vertex of the path P_2'. We have $N^+_D(w) \cap (V(P_0) \setminus \{x\}) \subseteq R \cap (V(P_0) \setminus \{x\}) = \emptyset$. Since $d^+_D(w) \geq k_1 + 3k_2 - 5 > 3k_2 - 4 = |V(P_1) \cup V(P_1') \cup (V(P_2') \setminus \{w\})|$, there must exist $w' \in N^+_D(w) \setminus (V(P_1) \cup V(P_1') \cup V(P_2'))$. Now the dipaths P_1' and $P_2' \circ (w, w')$ are of length $k_2 - 1$, have only the starting vertex a in common and are disjoint from the set $(V(P_0) \cup V(P_1)) \setminus \{a\}$. Hence, $P_0 \circ P_1$ is a k_2-good dipath in D which is longer than P_0, a contradiction. This shows that indeed $R \cap (V(P_0) \setminus \{x\}) \neq \emptyset$. Hence, by the definition of R, there is a shortest dipath P_a from a to $R \cap (V(P_0) \setminus \{x\})$ in $D'[R]$. Write $V(P_a) \cap (V(P_0) \setminus \{x\}) =: \{a'\}$. Now P_a and a' satisfy the claimed properties. □

Let $A, B \subseteq V(P_0) \setminus \{x\}$ be the sets of vertices on $P_0 - x$ reachable from a, b, respectively, by a dipath which is internally vertex-disjoint from $V(P_0) \cup V(P_1) \cup V(P_2)$. By the previous claim we have $A, B \neq \emptyset$. Let a^* respectively b^* denote the vertex in A respectively B whose distance from x on P_0 is maximum. By symmetry, we may assume without loss of generality that $\mathrm{dist}_{P_0}(a^*, x) \geq \mathrm{dist}_{P_0}(b^*, x)$. Hence, $B \subseteq V(P_0[a^*, x])$. Fix some dipath P_{a^*} from a to a^* in D which is internally disjoint from $V(P_0) \cup V(P_1) \cup V(P_2)$. Set $Q := P_1 \circ P_{a^*}$, and note that $|Q| = |P_1| + |P_{a^*}| = k_2 - 1 + |P_{a^*}| \geq k_2$. Let $Q' \subseteq Q$ be defined as follows: if the length of Q is at most k_1 then $Q' := Q$, and otherwise Q' is the unique subpath of Q which starts at x and has length exactly k_1. In the following, let r denote the length of Q'. Observe that $r = |Q'| = \min\{|Q|, k_1\}$, and hence $k_2 \leq r \leq k_1$. Moreover, $P_1 \subseteq Q'$ because P_1 consists of the first k_2 vertices of Q. Let $y \in V(Q)$ be the terminus of Q', and let us define B^* as the subset of B consisting of those vertices in $B \subseteq V(P_0 - x)$ which are reachable from b by a dipath which is internally vertex-disjoint from $V(P_0) \cup V(Q) \cup V(P_2)$.

Claim 2. We either have $|B^*| \geq k_1 - r + 1$, or there exists a dipath starting at b and ending in $V(Q)$ which is vertex-disjoint from $(V(P_0) \cup V(Q') \cup V(P_2)) \setminus \{b, y\}$.

Proof. Suppose towards a contradiction that $|B^*| \leq k_1 - r$ but there exists no dipath starting at b and ending in $V(Q)$ which is vertex-disjoint from $(V(P_0) \cup V(Q') \cup V(P_2)) \setminus \{b, y\}$. Let us consider the digraph $D'' := D - ((V(P_0) \cup V(Q') \cup V(P_2)) \setminus \{b, y\})$. Let $R \subseteq V(D'')$ denote the set of vertices reachable from b in D''. By our assumption, we have $R \cap V(Q) = \emptyset$, and hence $R \cap (V(P_0) \cup V(Q) \cup V(P_2)) = \{b\}$ (since $R \subseteq V(D'')$ and by the definition of D''). We claim that $N^+_D(u) \cap (V(P_0) \setminus \{x\}) \subseteq B^*$ for all $u \in R$. Indeed, let $u \in R$ and $v \in N^+_D(u) \cap (V(P_0) \setminus \{x\})$. By definition, there exists a b-u-dipath P_u in D'', and $V(P_u) \subseteq R$. Then the dipath $P_u \circ (u, v)$ starts at b, ends in $V(P_0) \setminus \{x\}$ and is internally vertex-disjoint from $V(P_0) \cup V(Q) \cup V(P_2)$, certifying that $v \in B^*$.

Since $|(V(Q') \cup V(P_2)) \setminus \{y, b\}| = r + k_2 - 2$, for every $u \in R$ we have:

$$d^+_{D''}(u) \geq d^+_D(u) - |N^+_D(u) \cap (V(P_0) \setminus \{x\})| - |(V(Q') \cup V(P_2)) \setminus \{y, b\}|$$

$$\geq k_1 + 3k_2 - 5 - |B^*| - (r + k_2 - 2) \geq 2k_2 - 3,$$

where in the last inequality we used our assumption that $|B^*| \leq k_1 - r$. As $N^+_{D''}(u) \subseteq R$ for every $u \in R$ (by the definition of R), we get that $\delta^+(D''[R]) \geq 2k_2 - 3$. Applying Lemma 3.1 to the vertex b in $D''[R]$ with $\ell_1 := k_2 - 1, \ell_2 := k_2 - 2$, we find dipaths P_1'' and P_2'' in $D''[R]$ starting at b of lengths ℓ_1 and ℓ_2, respectively, such that $V(P_1'') \cap V(P_2'') = \{b\}$. By the definition of D'' we have $V(P_i'') \cap (V(P_0) \cup V(P_2)) = \{b\}$ for every $i = 1, 2$. Let z denote the end-vertex of P_2''. We have

$|V(P_2) \cup V(P_1'') \cup (V(P_2'') \setminus \{z\})| = 3k_2 - 4$ and $|N_D^+(z) \cap (V(P_0) \setminus \{x\})| \leq |B^*| \leq k_1 - r \leq k_1 - k_2$. Here we used the fact that $z \in R$ and hence $N_D^+(z) \cap (V(P_0) \setminus \{x\}) \subseteq B^*$. So we see that

$$|N_D^+(z) \setminus (V(P_0) \cup V(P_2) \cup V(P_1'') \cup V(P_2''))| \geq k_1 + 3k_2 - 5 - (k_1 - k_2) - (3k_2 - 4) = k_2 - 1 > 0.$$

Let $z' \notin V(P_0) \cup V(P_2) \cup V(P_1'') \cup V(P_2'')$ be an out-neighbor of z. The two dipaths P_1'' and $P_2'' \circ (z, z')$ start at b and have length $k_2 - 1$ each. Moreover, the three dipaths P_1'', $P_2'' \circ (z, z')$ and $P_0 \circ P_2$ intersect each other only in the vertex b. Hence, $P_0 \circ P_2$ is a k_2-good dipath in D which is strictly longer than P_0, a contradiction. This contradiction shows that our initial assumption was wrong, concluding the proof of Claim 2. □

We will now show how to find a subdivision of $C(k_1, k_2)$ in D using Claim 2. Consider the two alternatives in the conclusion of this claim. The first case is that $|B^*| \geq k_1 - r + 1$. Since $B^* \subseteq B \subseteq V(P_0[a^*, x])$, this clearly implies that there exists a vertex $b^* \in B$ whose distance from a^* on the dipath P_0 is at least $k_1 - r$. By definition of B^*, there exists a dipath P_{b^*} in D starting in b and ending at b^* which is internally disjoint from $V(P_0) \cup V(Q) \cup V(P_2)$. Now the two dipaths $Q \circ P_0[a^*, b^*]$ and $P_2 \circ P_{b^*}$ in D both start at x and end at b^*, are internally vertex-disjoint, and have lengths $|Q| + |P_0[a^*, b^*]| \geq r + k_1 - r = k_1$ and $|P_2| + |P_{b^*}| \geq k_2 - 1 + 1 = k_2$, respectively. Hence, they form a subdivision of $C(k_1, k_2)$.

The second case is that there exists a dipath in D starting at b and ending in $V(Q)$, which is vertex-disjoint from $(V(P_0) \cup V(Q') \cup V(P_2)) \setminus \{b, y\}$. Let P^* be a shortest such dipath, and let $q \in V(Q)$ denote its end-vertex. Then clearly $V(P^*) \cap V(Q) = \{q\}$, as well as $q \notin V(Q') \setminus \{y\}$ and $q \neq a^*$ (as $a^* \in V(P_0)$). This readily implies that $Q' \neq Q$, and hence by definition of Q' we conclude that Q' has length exactly k_1. Let us consider the two dipaths $Q[x, q]$ and $P_2 \circ P^*$ in D, which both start in x and end in q. These two dipaths are internally vertex-disjoint, and have lengths $|Q[x, q]| \geq |Q'| = k_1$ and $|P_2| + |P^*| \geq k_2 - 1 + 1 = k_2$, respectively. Hence, they form a subdivision of $C(k_1, k_2)$ in D.

Summarizing, we have shown that D contains a subdivision of $C(k_1, k_2)$ in all the cases, which concludes the proof of the theorem. □

4 Subdivisions of $\overleftrightarrow{K}_3 - e$

In this section we give a proof of Theorem 2. As it turns out, it is convenient to prove the following slightly stronger result, which clearly implies that $\text{mader}_{\delta^+}(\overleftrightarrow{K}_3 - e) = 2$.

Proposition 19. *Let D be a digraph and $v_0 \in V(D)$ such that $d^+(v_0) \geq 1$ and $d^+(v) \geq 2$ for every $v \in V(D) \setminus \{v_0\}$. Then D contains a subdivision of $\overleftrightarrow{K}_3 - e$.*

Proof. Suppose towards a contradiction that the claim is false, and let D be a counterexample which minimizes $|V(D)|$ with first priority and $|A(D)|$ with second priority. Let $v_0 \in V(D)$ be a vertex such that $d^+(v_0) \geq 1$ and $d^+(v) \geq 2$ for all $v \in V(D) \setminus \{v_0\}$.

Claim 1. We have $d^+(v_0) = 1$ and $d^+(v) = 2$ for all $v \in V(D) \setminus \{v_0\}$.

Proof. If $d^+(v_0) > 1$ or $d^+(v) > 2$ for some $v \in V(D) \setminus \{v_0\}$, then we may delete an arc of D and be left with a digraph D' which still satisfies $d_{D'}^+(v_0) \geq 1$ and $d_{D'}^+(v) \geq 2$ for every $v \in V(D) \setminus \{v_0\}$. This contradicts the assumed minimality of D (as D' evidently contains no subdivision of $\overleftrightarrow{K}_3 - e$ either). □

Claim 2. D is strongly connected.

Proof. If not, then there is $\emptyset \neq X \subsetneq V(D)$ such that no arc of D leaves X. Then clearly $d^+_{D[X]}(x) = d^+(x)$ for all $x \in X$, and hence $D[X]$ meets the conditions of the Lemma. But as $D[X]$ contains no subdivision of $\overleftrightarrow{K}_3 - e$ and is smaller than D, we get a contradiction to the minimality of D. $\square$

Claim 3. There exists no partition (W, K, Z) of $V(D)$ such that $W, Z \neq \emptyset$, $v_0 \in K \cup Z$, $|K| \leq 1$ and there is no arc in D with tail in W and head in Z.

Proof. Suppose towards a contradiction that a partition (W, K, Z) with the described properties exists. Since D is strong, we must have $|K| = 1$; say $K = \{s_0\}$ for some vertex $s_0 \in V(D)$. Since $v_0 \notin W$ and since no arc of D goes from W to Z, every vertex in W has out-degree 2 in $D[W \cup \{s_0\}]$. Since D is strongly connected, there must be an s_0-W-dipath P in D (this path is allowed to consist of a single arc pointing from s_0 to a vertex of W). Denoting the last vertex of P by $w \in W$, we note that $V(P) \setminus \{s_0, w\} \subseteq Z$. Let D' be the digraph obtained from $D[W \cup \{s_0\}]$ by adding the arc (s_0, w) (if it does not exist yet). We clearly have $d^+_{D'}(s_0) \geq 1$, as well as $d^+_{D'}(v) = 2$ for every $v \in W$ by the above. Since $|V(D')| < |V(D)|$, the minimality of D implies that D' contains a subdivision S' of $\overleftrightarrow{K}_3 - e$. If S' does not use the arc (s_0, w) then $S' \subseteq D$. And otherwise, the subdigraph $S \subseteq D$ of D defined by $V(S) := V(S') \cup V(P)$, $A(S) := (A(S') \setminus \{(s_0, w)\}) \cup A(P)$ forms a subdivision of $\overleftrightarrow{K}_3 - e$ in D. In both cases we obtain a contradiction to our assumption that D does not contain a subdivision of $\overleftrightarrow{K}_3 - e$. This concludes the proof of the claim. $\square$

In the following, let $v_1 \in V(D)$ denote the unique out-neighbor of v_0. The rest of the proof is divided into two cases depending on whether v_0 and v_1 have common in-neighbors.

Case 1. $N^-(v_0) \cap N^-(v_1) = \emptyset$. Since $d^+(v_1) = 2$, there exists $v_2 \in N^+(v_1) \setminus \{v_0\}$. Let D' be the digraph obtained from $D - v_1$ by adding the arc (v_0, v_2) and the arcs (x, v_0) for all $x \in N^-_D(v_1) \setminus \{v_0\}$. We clearly have $d^+_{D'}(v_0) = 1$ and $d^+_{D'}(v) = 2$ for all $v \in V(D') \setminus \{v_0\}$, since no vertex in D has arcs to both v_0 and v_1. Since $|V(D')| < |V(D)|$, there must be a subdivision S' of $\overleftrightarrow{K}_3 - e$ contained in D'. If $v_0 \notin V(S')$, then S' is a subdigraph of D, which contradicts our assumption that D contains no $(\overleftrightarrow{K}_3 - e)$-subdivision. Hence we must have $v_0 \in S'$. Since v_2 is the only out-neighbor of v_0 in D', we must have $d^+_{S'}(v_0) = 1$ and $(v_0, v_2) \in A(S')$. We now distinguish between two subcases depending on the in-degree of v_0 in S'. Note that every vertex of $\overleftrightarrow{K}_3 - e$ has in-degree either 1 or 2. Hence, $d^-_{S'}(v_0) \in \{1, 2\}$.

Case 1(a). $d^-_{S'}(v_0) = 1$. Let $x_0 \in N^-_{D'}(v_0)$ be the unique in-neighbor of v_0 in S'. By definition of D', we must have either $x_0 \in N^-_D(v_0) \setminus \{v_1\}$ or $x_0 \in N^-_D(v_1) \setminus \{v_0\}$. Define a subdigraph $S \subseteq D$ of D as follows: If $x_0 \in N^-_D(v_0) \setminus \{v_1\}$, then we put $V(S) := V(S') \cup \{v_1\}$ and $A(S) := (A(S') \setminus \{(v_0, v_2)\}) \cup \{(v_0, v_1), (v_1, v_2)\}$, and if $x_0 \in N^-_D(v_1) \setminus \{v_0\}$, then we put $V(S) := (V(S') \setminus \{v_0\}) \cup \{v_1\}$, $A(S) := (A(S') \setminus \{(x_0, v_0), (v_0, v_2)\}) \cup \{(x_0, v_1), (v_1, v_2)\}$. It is easy to see that in each case S is isomorphic to a subdivision of S', and hence forms a subdivision of $\overleftrightarrow{K}_3 - e$ contained in D, a contradiction to our assumption on D.

Case 1(b). $d^-_{S'}(v_0) = 2$. Let $x_1, x_2 \in N^-_{D'}(v_0)$ be the two in-neighbors of v_0 in S'. By definition of D', we have $x_i \in N^-_D(v_0) \setminus \{v_1\}$ or $x_i \in N^-_D(v_1) \setminus \{v_0\}$ for each $i = 1, 2$. Let us define a subdigraph $S \subseteq D$ of D as follows. Firstly, if $x_1, x_2 \in N^-_D(v_0) \setminus \{v_1\}$, then we set $V(S) := V(S') \cup \{v_1\}$ and $A(S) := (A(S') \setminus \{(v_0, v_2)\}) \cup \{(v_0, v_1), (v_1, v_2)\}$. Secondly, if $x_i \in N^-_D(v_0) \setminus \{v_1\}$ and $x_{3-i} \in N^-_D(v_1) \setminus \{v_0\}$ for some $i \in \{1, 2\}$, then we set $V(S) := V(S') \cup \{v_1\}$ and $A(S) := (A(S') \setminus \{(v_0, v_2), (x_{3-i}, v_0)\}) \cup \{(v_0, v_1), (v_1, v_2), (x_{3-i}, v_1)\}$. Lastly, if $x_1, x_2 \in N^-_D(v_1) \setminus \{v_0\}$ then we set $V(S) := (V(S') \setminus \{v_0\}) \cup \{v_1\}$ and $A(S) := (A(S') \setminus \{(v_0, v_2), (x_1, v_0), (x_2, v_0)\}) \cup$

$\{((v_1, v_2), (x_1, v_1), (x_2, v_1)\}$. It is easy to check that in each of the three cases, S is isomorphic to a subdivision of S', and hence forms a subdivision of $\overleftrightarrow{K}_3 - e$ which is contained in D. This contradiction to our initial assumption on D rules out Case 1.

Case 2. There exists a vertex $z_0 \in N^-(v_0) \cap N^-(v_1)$. Let now $A := \{v_0, z_0\}$ and apply Theorem 13 to the vertex v_1 versus the set A in D. We conclude that either there are two v_1-A-dipaths intersecting only at v_1, or there is a set $K \subseteq V(D) \setminus \{v_1\}$ such that $|K| \leq 1$ and there is no dipath in $D - K$ starting in v_1 and ending in A.

In the first case, let P_1 and P_2 be dipaths such that $V(P_1) \cap V(P_2) = \{v_1\}$ and such that P_1 ends in v_0, while P_2 ends in z_0. Now the subdigraph $S \subseteq D$ with vertex set $V(S) := V(P_1) \cup V(P_2)$ and arc-set $A(S) := A(P_1) \cup A(P_2) \cup \{(v_0, v_1), (z_0, v_0), (z_0, v_1)\}$ forms a subdivision of $\overleftrightarrow{K}_3 - e$ with branch vertices v_0, v_1, z_0. This is a contradiction to our initial assumption on D.

In the second case, let $W \subseteq V(D) - K$ be the subset of vertices reachable from v_1 by a dipath in $D - K$ and let $Z := V(D) \setminus (W \cup K)$. Since there is no v_1-A-dipath in $D - K$, we must have $v_0 \in A \subseteq K \cup Z$. We further have $v_1 \in W$ and $A \setminus K \subseteq Z$, hence $W, Z \neq \emptyset$. Moreover, by definition of W, no arc in D starts in W and ends in Z. All in all, this shows that the partition (W, K, Z) of $V(D)$ yields a contradiction to Claim 3.

Since we arrived at contradictions in all possible cases, we conclude that our initial assumption about the existence of D was wrong. This completes the proof of Proposition 19. $\square$

5 Subdivisions and arc-connectivity

In this section we prove Propositions 10 and 11, showing that $\overleftrightarrow{K}_4$ and $\overleftrightarrow{S}_4$ are not κ'-maderian.

Proposition 10. *For every $k \in \mathbb{N}$, there exists a digraph G_k with $\kappa'(G_k) \geq k$ such that G_k contains no subdivision of $\overleftrightarrow{K}_4$.*

Proof. A construction of Thomassen [21] shows that for every integer $k \geq 1$, there exists a digraph D_k such that $\delta^+(D_k) = k$ and D_k contains no directed cycle of even length. For every $k \geq 1$ let $\overleftarrow{D}_k$ denote the digraph obtained from D_k by reversing all its arcs. Then clearly we have $\delta^-(\overleftarrow{D}_k) = k$. Let G'_k be the digraph obtained from the vertex-disjoint union of a copy of D_k with vertex-set A and a copy of $\overleftarrow{D}_k$ with vertex-set B by adding all the arcs in $B \times A$ (i.e., all arcs from B to A). Note that since $|A| = |B| = |V(D_k)| > k$, we have $\delta^+(G'_k) = \delta^-(G'_k) = k$. Finally, we define G_k as the digraph obtained from G'_k by adding a vertex $v \notin V(G'_k)$ as well as all arcs $(v, x), (x, v)$ for $x \in V(G'_k)$. We claim that G_k is strongly k-arc-connected. Indeed, let $E \subseteq A(G_k)$ be a set of arcs such that $|E| < k$. We claim that in $G_k - E$, every vertex $x \in V(G'_k)$ can reach and is reachable from v via a dipath. This will show that $G_k - E$ is strongly connected, as required. Let $x \in V(G'_k)$ be given arbitrarily, and let $x_1, \ldots, x_k \in V(G'_k)$ be k pairwise distinct out-neighbors of x in G'_k. Consider the k arc-disjoint dipaths $P_i := (x, x_i) \circ (x_i, v)$, $i = 1, \ldots, k$. At least one of these dipaths must be disjoint from E and hence constitute an x-v-dipath in $D - E$. With a symmetric argument considering k distinct in-neighbors of x, we also obtain that there is a v-x-dipath in $G_k - E$, as required. We further claim that G_k contains no subdivision of $\overleftrightarrow{K}_4$. Indeed, suppose this was the case, then clearly there would be $S \subseteq G_k - v = G'_k$ such that S is a subdivision of $\overleftrightarrow{K}_3$. As is easy to see, S must contain an even directed cycle. Since there is no arc in G'_k from A to B, we find that this cycle must be entirely contained in either $G'_k[A] \simeq D_k$ or $G'_k[B] \simeq \overleftarrow{D}_k$. This however means that D_k contains an even directed cycle, a contradiction. This contradiction shows that G_k contains no subdivision of $\overleftrightarrow{K}_4$, and this concludes the proof. $\square$

Proposition 11. *For every $k \in \mathbb{N}$, there exists a digraph H_k with $\kappa'(H_k) \geq k$ such that H_k contains no subdivision of $\overleftrightarrow{S}_4$.*

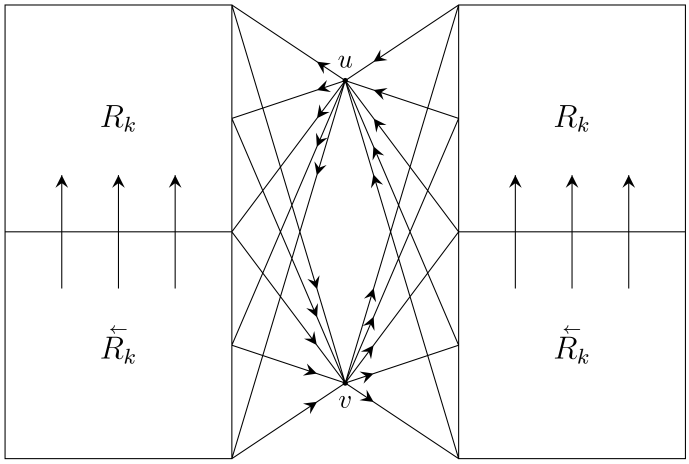

Figure 4: The digraph H_k

Proof. A construction of Thomassen [21] shows that for every integer $k \geq 1$, there exists a digraph R_k such that $\delta^+(R_k) = k$ and R_k contains no subdivision of the bioriented 3-star $\overleftrightarrow{S}_3$. For $k \geq 1$, let us denote by $\overleftarrow{R}_k$ the digraph obtained from R_k by reversing all its arcs. Let H'_k be the digraph obtained from the disjoint union of a copy of R_k with vertex-set A and a copy of $\overleftarrow{R}_k$ with vertex-set B by adding all the arcs in $B \times A$. Since R_k and $\overleftarrow{R}_k$ have at least k vertices, we obtain that $\delta^+(H'_k) = \delta^-(H'_k) = k$. We now define H_k to be the digraph obtained from two disjoint copies of H'_k with vertex-sets X and Y by adding two distinct new vertices u and v as well as the following arcs: (u, x) and (x, v) for every $x \in X$, and (y, u) and (v, y) for every $y \in Y$. See Figure 4 for an illustration. We claim that H_k is strongly k-arc-connected. Indeed, let $E \subseteq A(H_k)$ be an arbitrarily given set of arcs such that $|E| < k$. We must prove that $H_k - E$ is strongly connected. For this, it clearly suffices to show that in $H_k - E$, every vertex in X can reach v and is reachable from u, and every vertex in Y can reach u and is reachable from v. Let $x \in X$ be any given vertex, and let $x_1^-, \ldots, x_k^- \in X$ denote k distinct in-neighbors of x in $H_k[X] \simeq H'_k$. Among the k arc-disjoint u-x-dipaths $(u, x_i^-) \circ (x_i^-, x), i = 1, \ldots, k$ in H_k, at least one must also exist in $H_k - E$, and hence x is reachable from u in $H_k - E$. Similarly, considering k distinct out-neighbors $x_1^+, \ldots, x_k^+ \in X$ of x in $H_k[X]$, and considering the arc-disjoint x-v-dipaths $(x, x_i^+), (x_i^+, v), i = 1, \ldots, k$, we find that there is an x-v-dipath in $H_k - E$. With a symmetric argument for the vertices in Y, we can verify the above claim, showing that $H_k - E$ is strongly connected. This shows that indeed $\kappa'(H_k) \geq k$.

Next we claim that H_k does not contain a subdivision of $\overleftrightarrow{S}_4$. Suppose otherwise. Then there exists a vertex $w \in V(H_k)$ and directed cycles C_1, C_2, C_3, C_4 in H_k such that $w \in V(C_i)$ for $i = 1, \ldots, 4$, and such that the sets $V(C_i) \setminus \{w\}, 1 \leq i \leq 4$, are pairwise disjoint. Suppose first that $w \in \{u, v\}$. Without loss of generality, we may assume that $w = u$ (the case $w = v$ is symmetric). Then for each $1 \leq i \leq 4$, $C_i - w$ is a dipath which starts in X and ends in Y (since the vertex of C_i preceding $w = u$ must be in Y, and the vertex of C_i succeeding $w = u$ must be in X). It follows that $C_i - w, 1 \leq i \leq 4$, are pairwise vertex-disjoint dipaths from X to Y, contradicting the fact that X and Y can be disconnected in H_k by deleting only two vertices, namely u and v. Suppose now that $w \in X \cup Y$. Note that every directed cycle in H_k is either contained in $H_k[X]$, or contained in $H_k[Y]$, or contains both u and v. Hence, if $w \in X$, then at least three of the cycles $C_i, 1 \leq i \leq 4$, are contained in $H_k[X] \simeq H'_k$, and if $w \in Y$ then at least three of the cycles $C_i, 1 \leq i \leq 4$, are contained in $H_k[Y] \simeq H'_k$. So we see that in each case, H'_k

must contain a subdivision of $\overleftrightarrow{S}_3$. Since every subdivision of $\overleftrightarrow{S}_3$ is a strongly connected digraph, and since there are no arcs from A to B in H'_k, we find that this subdivision must be entirely contained in either $H'_k[A] \simeq R_k$ or $H'_k[B] \simeq \overleftarrow{R}_k$. Since $\overleftrightarrow{S}_3$ is invariant under the reversal of all arcs, we obtain that in each case R_k must contain a subdivision of $\overleftrightarrow{S}_3$. This contradicts our initial assumptions on the sequence $(R_k)_{k \geq 1}$. This contradiction proves the claim of the proposition; namely, H_k is indeed a k-strongly arc connected digraph not containing $\overleftrightarrow{S}_4$ as a subdivision. □

6 Open Problems

In this concluding section, we would like to mention further open problems related to subdivisions in digraphs of large minimum out-degree, which we discovered during the work on this paper.

Theorem 6 shows that orientations of cycles are δ^+-maderian, and that for an orientation C of a cycle, $\text{mader}_{\delta^+}(C)$ grows polynomially in $|C|$. Aboulker et al. actually conjectured the very explicit bound of $\text{mader}_{\delta^+}(C) \leq 2|C| - 1$ (cf. [1], Conjecture 27). However, it is even unclear to us whether $\text{mader}_{\delta^+}(C)$ should be linear in $|C|$ at all.

Problem 20. *Does it hold that $\text{mader}_{\delta^+}(C) = O(|V(C)|)$ for every orientation C of a cycle?*

We remark that Theorem 16 gives a positive answer to this question when the size of a longest block in C is bounded by a constant.

Disjoint union is a basic graph operation under which one might naturally anticipate the δ^+-maderian property to be preserved. Yet, despite quite a bit of effort, this intuition is only known to hold in a few special cases. Thomassen's Theorem for example states that the disjoint union of k digons is δ^+-maderian for all k. A common generalization of this result and Theorem 6 would be the following.

Conjecture 21. *Any disjoint union of orientations of cycles is δ^+-maderian.*

Digraph subdivision is another graph operation under which it is plausible to expect that the δ^+-maderian property is preserved.

Conjecture 22. *If a digraph F is δ^+-maderian, all subdivisions of F are δ^+-maderian as well.*

Conjecture 22 would follow if we could show that every digraph of large enough out-degree contains a subdivision of some digraph of out-degree k in which every subdivision path is long.

Conjecture 23. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and for every digraph D with $\delta^+(D) \geq f(k)$, there exists a digraph D' such that $\delta^+(D') \geq k$ and D contains a subdivision of D' in which every subdivision-path has length at least two.*

An important step towards Conjecture 4 would be to show that attaching an out-leaf to any vertex of a δ^+-maderian digraph yields still a δ^+-maderian digraph.

Conjecture 24. *If F is a δ^+-maderian digraph, $v_0 \in V(F)$ and F^* is the digraph obtained from F by adding a new vertex v_1 and the arc (v_0, v_1), then F^* is δ^+-maderian as well.*

Conjecture 24 would follow directly from the following natural statement. We call a set of vertices X in a digraph D *in-dominating set* if every $y \in V(D) \setminus X$ has an out-neigbor in X.

Conjecture 25. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds for every $k \geq 1$. If D is a digraph with $\delta^+(D) \geq f(k)$, then there exists an in-dominating set $X \subsetneq V(D)$ such that $\delta^+(D - X) \geq k$.*

Another interesting direction is to characterize the undirected graphs F for which the biorientation $\overleftrightarrow{F}$ of F is δ^+-maderian. If $\overleftrightarrow{F}$ is δ^+-maderian, then F must be a forest, since every bioriented cycle has arc-connectivity two and hence is not δ^+-maderian (see the necessary properties of δ^+-maderian digraphs mentioned in the introduction). Furthermore, it is known that $\overleftrightarrow{S_3}$ is not δ^+-maderian [21]. Thus, if $\overleftrightarrow{F}$ is δ^+-maderian then F must be a path-forest. Thomassen's result [19] shows that a biorientation of any matching is δ^+-maderian. By Theorem 2, $\overleftrightarrow{S_2} = \overleftrightarrow{P_3}$ is δ^+-maderian (where P_ℓ denotes the path on ℓ vertices). The first open case is that of $\overleftrightarrow{P_4}$.

Problem 26. *Is $\overleftrightarrow{P_4}$ δ^+-maderian?*

Finally, several open problems arise from the questions considered in Section 5. Given that $\overleftrightarrow{K_4}$ and $\overleftrightarrow{S_4}$ are not κ'-maderian (see Propositions 10-11), it is natural to ask whether $\overleftrightarrow{K_3}$ and $\overleftrightarrow{S_3}$ are.

Problem 27. *Is $\overleftrightarrow{K_3}$ κ'-maderian? Is $\overleftrightarrow{S_3}$ κ'-maderian?*

As mentioned in the introduction, every subdivision of $\overleftrightarrow{K_3}$ contains an even dicycle, and one cannot force an even dicycle by means of minimum out-degree [21]. Thus, even dicycles can be thought of as an obstacle to forcing subdivisions of $\overleftrightarrow{K_3}$. Interestingly, this obstacle disappears when considering arc-connectivity (rather than out-degree), as a theorem of Thomassen [24] shows that every digraph D with $\kappa'(D) \geq 3$ contains an even dicycle. This can be thought of as a hint that $\overleftrightarrow{K_3}$ could in fact be κ'-maderian.

A critical first step towards the resolution of Problem 9 for vertex-connectivity is the following.

Problem 28. *Is there a constant $K \in \mathbb{N}$ such that every K-strongly-vertex connected digraph contains two vertices $x \neq y$ and four pairwise internally vertex-disjoint dipaths, two from x to y and two from y to x?*

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