Constructing Dense Grid-Free Linear 3-Graphs

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Abstract

We show that there exist linear 3-uniform hypergraphs with n vertices and $\Omega(n^2)$ edges which contain no copy of the 3×3 grid. This makes significant progress on a conjecture of Füredi and Ruszinkó. We also discuss connections to proving lower bounds for the $(9,6)$ Brown-Erdős-Sós problem and to a problem of Solymosi and Solymosi.

1 Introduction

In recent years there has been some interest in Turán-type results for linear hypergraphs [4, 5, 6]. In this paper, all hypergraphs are 3-uniform. For a family $\mathcal{H}$ of 3-uniform hypergraphs, we let $\text{ex}_{\text{lin}}(n,\mathcal{H})$ denote the maximum number of edges in a linear 3-uniform $\mathcal{H}$-free hypergraph on n vertices. When $\mathcal{H}$ has a single element H, we will write $\text{ex}_{\text{lin}}(n,H)$. Arguably, the interest in problems of this type is motivated by the famous Brown-Erdős-Sós conjecture [1, 2], which states that, for every $k \geq 3$, if $\mathcal{H}_{k+3,k}$ is the set of all 3-uniform hypergraphs with k edges and at most $k+3$ vertices (such hypergraphs are called $(k+3,k)$-*configurations*), then[1] $\text{ex}_{\text{lin}}(n,\mathcal{H}_{k+3,k}) = o(n^2)$. So far, this conjecture has only been proven in the case $k = 3$. This is a celebrated result of Ruzsa and Szemerédi [7], which became known as the $(6,3)$ theorem. Ruzsa and Szemerédi [7] have also given a construction which shows that $\text{ex}_{\text{lin}}(n,\mathcal{H}_{6,3}) \geq n^{2-o(1)}$, implying that the exponent 2 in the $(6,3)$ theorem cannot be improved. For $k \geq 4$, the Brown-Erdős-Sós conjecture remains widely open despite considerable effort, with the best approximate result recently obtained in [3] (see also [8, 10]).

It is easy to check that $\mathcal{H}_{6,3}$ contains only one linear hypergraph: the triangle $\mathbb{T}$, which is the hypergraph with vertices $1,2,3,4,5,6$ and edges $\{1,2,3\},\{3,4,5\},\{5,6,1\}$. Thus, the aforementioned results of Ruzsa and Szemerédi [7] are equivalent to the statement $n^{2-o(1)} \leq \text{ex}_{\text{lin}}(n,\mathbb{T}) \leq o(n^2)$.

It is natural to try and prove that $\text{ex}_{\text{lin}}(n,\mathcal{H}_{k+3,k}) \geq n^{2-o(1)}$ for every $k \geq 3$, which would mean that, in a sense, the Brown-Erdős-Sós conjecture is optimal. For $k = 4,5$, such a lower bound follows from the simple observation that every $(7,4)$- or $(8,5)$-configuration contains a $(6,3)$-configuration. Similar considerations were used in [5] to handle the cases $k = 7,8$. For $k = 6$, however, such arguments could not be used, since there exists a $(9,6)$-configuration which contains no $(6,3)$-configuration; this is the 3×3 *grid* $\mathbb{G}_{3\times 3}$, which is the 3-uniform hypergraph whose vertices are the nine points in a 3×3 point array, and whose edges correspond to the 6 horizontal and vertical lines of this array. It

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[1]The Brown-Erdős-Sós conjecture is usually stated about general (i.e., not necessarily linear) hypergraphs, but it is well-known that it suffices to consider linear hypergraphs. Indeed, if a hypergraph H contains no $(k+3,k)$-configuration, then every pair of vertices is contained in at most $k-1$ edges, which implies that for every edge e, there are at most $3k-6$ other edges f with $|e\cap f| \geq 2$. Hence, H has a linear subhypergraph with at least $e(H)/(3k-5) = \Omega(e(H))$ edges.

is not hard to verify[2] (see also [5]) that every linear $(9,6)$-configuration either contains a triangle $\mathbb{T}$ or is isomorphic to $\mathbb{G}_{3\times 3}$. Hence, $\mathrm{ex}_{\mathrm{lin}}(n,\mathcal{H}_{9,6}) \geq \mathrm{ex}_{\mathrm{lin}}(n,\{\mathbb{T},\mathbb{G}_{3\times 3}\})$. This relation has led Füredi and Ruszinkó [4] to study extremal problems related to the grid. In particular, they conjectured that $\mathrm{ex}_{\mathrm{lin}}(n,\mathbb{G}_{3\times 3}) = \left(\frac{1}{6} - o(1)\right) n^2$, and, more strongly, that for every large enough admissible n, there exists a Steiner triple system of order n which is $\mathbb{G}_{3\times 3}$-free. Using a standard probabilistic alterations argument, Füredi and Ruszinkó [4] showed that $\mathrm{ex}_{\mathrm{lin}}(n,\mathbb{G}_{3\times 3}) = \Omega(n^{1.8})$. This was then slightly improved (as a special case of a more general result) to $\Omega(n^{1.8}\log^{1/5} n)$ by Shangguan and Tamo [9]. Here we make significant progress on the conjecture of Füredi and Ruszinkó [4], by showing that $\mathrm{ex}_{\mathrm{lin}}(n,\mathbb{G}_{3\times 3}) = \Omega(n^2)$.

Theorem 1. *For infinitely many n, there exists a linear $\mathbb{G}_{3\times 3}$-free 3-uniform hypergraph with n vertices and $(\frac{1}{16} - o(1))n^2$ edges.*

Theorem 1 is proved in the following section. Then, in Section 3, we discuss some related open problems.

2 The Construction

Construction 2.1. *Let $\mathbb{F}$ be a field and let $X, A \subseteq \mathbb{F}$. Define $H(X,A)$ to be the 3-partite 3-uniform hypergraph with sides X, $Y := \{x + a : x \in X, a \in A\}$ and $Z := \{x \cdot a : x \in X, a \in A\}$, and with an edge $(x, x+a, x\cdot a) \in X \times Y \times Z$ for every $x \in X$ and $a \in A$.*

We now prove that the hypergraph $H(X,A)$ defined in Construction 2.1 is always $\mathbb{G}_{3\times 3}$-free. We will then show that it contains a dense linear subhypergraph. We denote the vertices of $\mathbb{G}_{3\times 3}$ by $\{p_i, q_i, r_i : 1 \leq i \leq 3\}$ and its edges by $\{\{p_i, q_i, r_i\}, \{p_{i+1}, q_{i+2}, r_i\} : 1 \leq i \leq 3\}$, where (here and later on) indices are taken modulo 3. A 3-*partition* of a 3-uniform hypergraph F is a partition $V(F) = P \cup Q \cup R$ such that every edge of F contains one element from each of the sets P, Q, R. Observe that $\{p_1,p_2,p_3\}, \{q_1,q_2,q_3\}, \{r_1,r_2,r_3\}$ is a 3-partition of $\mathbb{G}_{3\times 3}$. It can be verified[3] that every two 3-partitions of $\mathbb{G}_{3\times 3}$ are equivalent, in the sense that there is an automorphism of $\mathbb{G}_{3\times 3}$ which maps every class of one to a class of the other.

Lemma 2.2. *Let $\mathbb{F}$ be a field and let $X, A \subseteq \mathbb{F}$. Then $H(X,A)$ is $\mathbb{G}_{3\times 3}$-free.*

Proof. Suppose, for the sake of contradiction, that $H(X,A)$ contains a copy of $\mathbb{G}_{3\times 3}$. Since all 3-partitions of $\mathbb{G}_{3\times 3}$ are equivalent (as explained above), we may assume, without loss of generality, that $p_1,p_2,p_3 \in X$, $q_1,q_2,q_3 \in Y = \{x + a : x \in X, a \in A\}$ and $r_1,r_2,r_3 \in Z = \{x \cdot a : x \in X, a \in A\}$. By definition of $H(X,A)$, for every edge $\{x,y,z\} \in E(H)$ (with $x \in X$, $y \in Y$ and $z \in Z$) there is $a \in A$ such that $y = x + a$ and $z = x \cdot a$; hence, $z = x \cdot (y - x)$. It follows that for every $1 \leq i \leq 3$, we must have $r_i = p_i \cdot (q_i - p_i)$ and $r_i = p_{i+1} \cdot (q_{i+2} - p_{i+1})$. Here and throughout the proof, indices are taken modulo 3. By comparing these two expressions for r_i, we see that

$$p_i \cdot (q_i - p_i) = p_{i+1} \cdot (q_{i+2} - p_{i+1}). \tag{1}$$

[2]Indeed, let H be a linear $(9,6)$-configuration avoinding $\mathbb{T}$. First, observe that H has maximum degree 2, for if $\{a,b,c\}, \{a,d,e\}, \{a,f,g\}$ are three edges containing a, then there can be only one edge containing the remaining two vertices (as H is linear), so there must be an edge which contains two vertices from $\{b,c,d,e,f,g\}$, which gives a $\mathbb{T}$. Now, as $e(H) = 6$, all degrees in H must be 2. Consider the two edges $\{a,b,c\}, \{a,d,e\}$ containing some vertex a. Let f,g,h,i be the four remaining vertices. Each of the four remaining edges must contain two vertices from $\{f,g,h,i\}$ and one from $\{b,c,d,e\}$. Every vertex from $\{b,c,d,e\}$ must be covered once by these edges, and every vertex from $\{f,g,h,i\}$ twice. Hence, the pairs from $\{f,g,h,i\}$ which are covered by these edges must form a C_4. Since H is $\mathbb{T}$-free, b and c must be contained in opposite edges of this C_4, and the same for d and e. This gives a $\mathbb{G}_{3,3}$.

[3]Indeed, every 3-partition of $\mathbb{G}_{3\times 3}$ is either obtained from the 3-partition (P,Q,R) by permuting its classes, or equals $(\{p_1,q_3,r_2\}, \{p_2,q_1,r_3\}, \{p_3,q_2,r_1\})$ or one of its permutations.

for every $1 \le i \le 3$. Multiplying (1) by p_{i+2} and then summing over $1 \le i \le 3$, we obtain

$$\sum_{i=1}^{3} p_i p_{i+2} \cdot (q_i - p_i) = \sum_{i=1}^{3} p_{i+1} p_{i+2} \cdot (q_{i+2} - p_{i+1}).$$

It is easy to see that for every $1 \le i \le 3$, both sides have the term $p_i p_{i+2} q_i$. Cancelling out these terms and rearranging, we get

$$0 = \sum_{i=1}^{3} p_i^2 p_{i+2} - \sum_{i=1}^{3} p_{i+1}^2 p_{i+2} = (p_1 - p_2)(p_2 - p_3)(p_3 - p_1).$$

Hence, there must be $1 \le i \le 3$ such that $p_{i+1} = p_i$. However, this is impossible as $p_1, p_2, p_3 \in X$ must correspond to distinct vertices of a copy of $\mathbb{G}_{3\times 3}$. This contradiction completes the proof. ∎

Proof of Theorem 1. We first prove Theorem 1 with a slightly worse bound, namely, with the fraction $\frac{1}{16}$ replaced by $\frac{1}{18}$. We then explain how our argument can be modified to give $\frac{1}{16}$.

Let p be a prime power, and set $X := \mathbb{F}_p \backslash \{0\}$ and $A := \mathbb{F}_p$. Let $H = H(X, A)$ be the hypergraph from Construction 2.1. By Lemma 2.2, H is $\mathbb{G}_{3\times 3}$-free. We claim that for each edge $e = (x, x + a, x \cdot a) \in E(H) \subseteq X \times Y \times Z$, if $f \in E(H) \setminus \{e\}$ satisfies that $|e \cap f| = 2$ then $f = (a, x + a, x \cdot a)$. So let $f = (y, y + b, y \cdot b) \in E(H) \setminus \{e\}$ be such that $|e \cap f| = 2$. We cannot have $(x, x + a) = (y, y + b)$ or $(x, x \cdot a) = (y, y \cdot b)$, for otherwise we would have $x = y, a = b$ and hence $e = f$. Therefore, we must have $(x+a, x \cdot a) = (y+b, y \cdot b)$, which gives $y(x+a-y) = x \cdot a$. Solving this quadratic equation for y, we get that $y = x$ or $y = a$, and hence $(y, b) = (x, a)$ or $(y, b) = (a, x)$. In the former case, $f = e$, and in the latter case $f = (a, x + a, x \cdot a)$. This proves our claim. It follows that for each $e \in E(H)$ there is at most one other edge $f \in E(H)$ such that $|e \cap f| = 2$. By deleting one edge from each such pair (e, f), we obtain a linear sub-hypergraph H' of H with $e(H') \ge \frac{e(H)}{2} = |X||A|/2 = (\frac{1}{2} - o(1))p^2 = (\frac{1}{18} - o(1))v(H)^2$, where in the last equality we used the fact that $v(H) = 3p - 1$ as $|X| = p - 1$ and $|Y| = |Z| = p$. This shows that $\text{ex}_{\text{lin}}(n, \mathbb{G}_{3\times 3}) \ge (\frac{1}{18} - o(1))n^2$.

To improve the constant, we assume that p is odd and choose X and A differently: let X be the set of (non-zero) quadratic residues and A be the set of (non-zero) quadratic non-residues in $\mathbb{F}_p$. Evidently, $|X| = |A| = \frac{p-1}{2}$ and $|Y| \le p$. As $Z = \{x \cdot a : x \in X, a \in A\} = A$, one also has $|Z| = \frac{p-1}{2}$. Altogether we get $v(H) = |X| + |Y| + |Z| \le 2p - 1$. Moreover, $e(H) = |X||A| = (\frac{1}{4} - o(1))p^2 = (\frac{1}{16} - o(1))v(H)^2$. Crucially, we observe that H is linear, because for every $e = (x, x + a, x \cdot a) \in E(H)$, the edge $f = (a, x+a, x \cdot a)$ is not in H, as a is not a quadratic residue (and so $a \notin X$). This completes the proof. ∎

3 Concluding Remarks And Open Problems

- Another problem raised in [4] is to prove that $\text{ex}_{\text{lin}}(n, \mathcal{H}_{9,6}) \ge n^{2-o(1)}$. This problem remains open. Recalling that $\text{ex}_{\text{lin}}(n, \mathcal{H}_{9,6}) \ge \text{ex}_{\text{lin}}(n, \{\mathbb{T}, \mathbb{G}_{3\times 3}\})$, we see, in light of Lemma 2.2, that it suffices to find a choice of sets $X, A \subseteq \mathbb{F}_p$, $|X|, |A| \ge p^{1-o(1)}$, such that the hypergraph $H(X, A)$ has no triangles (i.e., no copies of $\mathbb{T}$). For this, one needs that there are no $x \in X$ and distinct $a, b, c \in A$ such that $(x + a - b) \cdot b = x \cdot c$.

- There is another construction of a linear 3-uniform grid-free hypergraph with $\Omega(n^2)$ edges. As in the previous construction, we first construct a grid-free hypergraph which is not linear, and then show that it contains a linear subhypergraph with a constant fraction of all edges. For sets $X, A \subseteq \mathbb{F}_p$, define a 3-partite hypergraph with sides X, Y, Z by placing the edge $(x, x + a, x + a^2) \in X \times Y \times Z$ for every $x \in X, a \in A$ (where $Y, Z = \mathbb{F}_p$). Here one needs to be more careful: unlike Construction 2.1, this hypergraph can contain a copy of $\mathbb{G}_{3\times 3}$, but only if there are $x_1, x_2 \in X$ and $a \in A$ satisfying $4x_1 + 4a = 4x_2 + 1$. Let us prove this. Consider a copy of $\mathbb{G}_{3,3}$ with vertices $\{p_i, q_i, r_i : 1 \le i \le 3\}$, as described before Lemma 2.2. Here, this copy corresponds to the equations $r_i - p_i = (q_i - p_i)^2$ and $r_i - p_{i+1} = (q_{i+2} - p_{i+1})^2$ for $i = 1, 2, 3$. Hence, $p_i + (q_i - p_i)^2 = p_{i+1} + (q_{i+2} - p_{i+1})^2$. Substituting $u_i := p_{i+1} - p_i$ and $v_i := q_i - p_{i+1}$ ($i = 1, 2, 3$), we get $(v_i + u_i)^2 = u_i + (v_{i+2} - u_i)^2$, and, after rearranging,

$$(2v_i + 2v_{i+2} - 1)u_i = v_{i+2}^2 - v_i^2. \tag{2}$$

Now, if $2v_i + 2v_{i+2} \neq 1$ for all $1 \le i \le 3$, then in equation (2) we can divide and get $u_i = (v_{i+2}^2 - v_i^2)/(2v_i + 2v_{i+2} - 1)$ for all $1 \le i \le 3$. Summing this over i and using the fact that $u_1 + u_2 + u_3 = (p_2 - p_1) + (p_3 - p_2) + (p_1 - p_3) = 0$, we get

$$0 = \sum_{i=1}^{3} u_i = \sum_{i=1}^{3} \frac{v_{i+2}^2 - v_i^2}{2v_i + 2v_{i+2} - 1} = \frac{-2(v_3 - v_1)(v_1 - v_2)(v_2 - v_3)}{(2v_1 + 2v_3 - 1)(2v_2 + 2v_1 - 1)(2v_3 + 2v_2 - 1)} .$$

Hence, there must be $1 \le i \le 3$ such that $v_{i+2} = v_i$. Plugging this into (2) and using that $2v_i + 2v_{i+2} \neq 1$, we get that $u_i = p_{i+1} - p_i = 0$, which is impossible as p_i, p_{i+1} are distinct vertices. Therefore, there must be $1 \le i \le 3$ such that $2v_i + 2v_{i+2} = 1$, hence also $v_{i+2}^2 - v_i^2 = 0$ by (2). Plugging $v_{i+2} = 1/2 - v_i$ into $v_{i+2}^2 - v_i^2 = 0$, we get that $v_i = 1/4$, hence $q_i - p_{i+1} = 1/4$. Now, recall that by construction, $p_i, p_{i+1} \in X$ and $q_i = p_i + a$ for some $a \in A$. Hence, we have our desired solution to $4x_1 + 4a = 4x_2 + 1$ with $x_1, x_2 \in X$, $a \in A$. So in order for the hypergraph to be $\mathbb{G}_{3\times 3}$-free, it suffices to choose X, A that avoid such solutions; for example, one can take $X = A = \{1, \ldots, \lfloor p/8 \rfloor\}$. What remains is to show that the constructed hypergraph has a large linear subhypergraph. For this, we show that each pair of vertices of the hypergraph is contained in at most two edges. Recall that each edge is of the form $e_{x,a} = (x, x + a, x + a^2) \in X \times Y \times Z$. Now, for $(x, y) \in X \times Y$, the only possible edge containing (x, y) is $e_{x,y-x}$; for $(x, z) \in X \times Z$, if $e_{x,a}$ contains (x, z) then $a^2 = z - x$; and for $(y, z) \in Y \times Z$, if $e_{x,a}$ contains (y, z) then $y = x + a, z = x + a^2$. In each of the three cases there are at most two solutions for x, a.

This construction can also be a candidate for showing that $\text{ex}_{\text{lin}}(n, \mathcal{H}_{9,6}) \ge n^{2-o(1)}$. Again, the issue is choosing X, A so as to avoid triangles, which in this case correspond to solutions to the equation $a + c^2 - c = b^2$ with distinct $a, b, c \in A$. Thus, in order to show that $\text{ex}_{\text{lin}}(n, \mathcal{H}_{9,6}) \ge n^{2-o(1)}$, it suffices to show that there exists $A \subseteq \mathbb{F}_p$, $|A| = p^{1-o(1)}$, with no non-trivial solution to this equation.

- A related conjecture of Solymosi and Solymosi [10] states that every (large enough) 3-uniform hypergraph with n vertices and $\Omega(n^2)$ edges contains a 2-*core* on at most 9 vertices, where a 2-core is a hypergraph with minimum degree 2. This conjecture is closely related[4] to the case $k = 6$ of the Brown-Erdős-Sós conjecture, since a 2-core on 9 vertices has at least 6 edges.

  Let H be the 3-partite hypergraph with sides X, Y, Z, all equal to $\mathbb{F}_p$, and with edge-set $\{(x, x + a, x + 2a) \in X \times Y \times Z : x, a \in \mathbb{F}_p\}$. Alternatively, this is the hypergraph whose edges are all triples $(x, y, z) \in X \times Y \times Z$ satisfying $y = (x + z)/2$. By a somewhat lengthy case analysis, one can show that H avoids all 2-cores on at most 9 vertices except for the grid $\mathbb{G}_{3\times 3}$. Thus, the hypergraph corresponding to a linear relation (namely, the relation $y = (x + z)/2$) avoids all but one of the 2-cores on at most 9 vertices, whereas in order to avoid $\mathbb{G}_{3\times 3}$ one needs a non-linear relation (as in Construction 2.1 or in the construction described in the previous item).

[4]Strictly speaking, the Solymosi-Solymosi conjecture does not imply the case $k = 6$ of the Brown-Erdős-Sós conjecture, since the former allows the 2-core to have less than 9 vertices, and hence less than 6 edges.

It would be interesting to understand the connection between the structure of a configuration F and the relation which can be used to define a hypergraph which avoids F.

We note that inspite of the above construction, it is plausible that the Solymosi-Solymosi conjecture is true; namely, that while there exist dense linear hypergraphs which avoid any individual 2-core on at most 9 vertices (and even hypergraphs which avoid all but one of them), avoiding all such 2-cores in a dense linear hypergraph is impossible.

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