On the Slice Genus of Twist Knots

Master thesis
Faculty of Science, University of Bern

handed in by

Damian Iltgen

2019

Supervisor

Dr. L. P. Lewark

Contents

Abstract

The present work deals with the study of the slice genus of the twist knots K_n. More precisely, we present another proof of Casson and Gordon's famous result that the only slice knots among the twist knots are the unknot K_0 and the Stevedore knot K_2, based on the work of Patrick Gilmer, discuss to what extent the methods used can be applied to find the order of the twist knots in the knot concordance group $\mathcal{C}$, and provide an upper bound for the stable 4-genus $g_{st}(K_n)$ which was recently introduced by Charles Livingston. Moreover, our methods can possibly be used to derive a lower bound for $g_{st}(K_n)$ as well. In addition, we provide an overview of classical knot theory, slice knots, knot concordance, and Casson-Gordon invariants.

0 Introduction

Historical Remarks

Knots have fascinated humanity for a very long time. Tying knots dates back to prehistoric times, and beside the many practical uses of knots, their aesthetics and spiritual symbolism has interested mankind ever since. It is thus not surprising that it was just a matter of time until knots were studied from a scientific point of view. Indeed, in the 1860's, Lord Kelvin's theory of atoms said that properties of chemical elements are connected to knottings of atoms, indicating that one might gain insight to chemistry by understanding knots. This led P. G. Tait in the 1870's to the first attempt of classifying knots up to a small crossing number. While he was first concentrating on knots up to five or six crossings, his tabulation already included at the beginning of the 20th century, together with the work of C. N. Little, almost all knots up to ten crossings. Tait considered two knots as equivalent if one could be deformed into the other, and he was focusing on having each type of knot once in his tabulation. However, the problem of deciding whether two knots are equivalent or not can be remarkably difficult, and back at his time, there was no exact formalism available for justifying his classification. Tait's arguments were reasonable, but they were entirely of empirical nature.

The mathematical study of knots began in the 19th century with Carl Friedrich Gauss, who developed the linking integral for computing the linking number of two knots. However, it was not until the early 20th century where advances in topology laid the foundations for a proper mathematical treatment of knots. Indeed, the formalism of topology developed in the early 1900's by famous mathematicians such as Henri Poincaré or Felix Hausdorff allowed for precise definitions of the objects arising from the theory of knots, and it became possible to formulate and prove theorems about them. At this time, deciding whether two knots are equivalent or not was still one of the main problems, and the introduction of algebraic methods to topology led to first advances to this problem. Max Dehn, for example, succeeded in 1914 to prove that the right-handed and left-handed trefoil are not equivalent, while J. W. Alexander used homology theory to derive an invariant of equivalent knots, known as the *Alexander polynomial*. In 1932, Kurt Reidemeister published the first book about knot theory, and he presented tools, known as the *Reidemeister moves*, which allowed to check the equivalence of two knots by simple modifications in a two-dimensional illustration (known as a *knot diagram*) of each individual knot. However, the computations involved were too lengthy for complicated knots in order to be of full benefit. In 1961, Wolfgang Haken discovered an algorithm that can determine whether or not a knot is nontrivial, but the complexity in computation restricted the algorithm from being of general use as well. Thus, the problem of detecting equivalent knots in useful time was still not fully solved.

The main approach to knots continued to be the study of algebraic objects associated to the complement of a knot, until William P. Thurston introduced in the 1970's hyperbolic geometry to knot theory, bringing new wind into the subject. Thurston's advances in hyperbolic geometry led to the definition of new, powerful knot invariants, and after Vaughan Jones' discovery of the Jones polynomial in 1984, knot theory became an established subject with interest throughout the entire mathematical community. Contributions from Edward Witten, Maxim Kontsevich, and others in the 1980's revealed a deep connection between the theory of knots and mathematical methods in statistical mechanics and quantum field theory, showing further the omnipresence of knots in mathematical and physical science.

Since then, many breakthroughs have been made in knot theory. The discovery of Khovanov homology and knot Floer homology in the 1990's greatly generalize the Jones and Alexander polynomial and serve as a new source for knot invariants that is investigated and studied up to this date. At the same time, scientists started to find applications of knot theory to biology and chemistry, leading to new understandings in knotting phenomena in DNA and other polymers. Although knot theory has been studied extensively throughout the 20th century, it is still one of the most active fields of research in post-modern mathematics and bears many open questions, one of them still being to decide whether or not a knot is trivial in useful time.

Motivation

There is one method of study that has been common practice in the history of knot theory: acquiring insight to the general theory by studying specific families of knots. In the present work, we follow this practice and study one specific type of knots, the *twist knots* K_n, which are obtained by repeatedly twisting a closed loop and linking its ends together (see Figure 1 below).

Twist knots are considered, next to the torus knots, as the most simple families of knots and have been studied extensively already in classical knot theory. However, connections between knot theory and 4-dimensional topology have been made in the past, and there are still open questions about the behavior of twist knots in relation to 4-dimensional topology. Thus, the aim of the present work is to review existing results and study further the properties of twist knots.

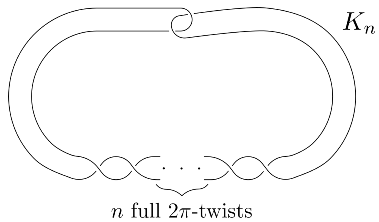


Figure 1: The twist knots K_n

Our motivating questions arise from the notion of *knot concordance*; an equivalence relation on the set of knots in the 3-sphere S^3 which turns them into an abelian group under the operation of connected sum, the *knot concordance group* $\mathcal{C}$. More precisely, two knots $J_1, J_2 \subset S^3$ are called *concordant* if the connected sum $J_1 \# - J_2^*$, where $-J_2^*$ denotes the mirror image of J_2 with reversed orientation, bounds a properly and smoothly embedded 2-disk D in the 4-dimensional unit ball $B^4 \subset \mathbb{R}^4$. In general, a knot $K \subset S^3$ that bounds a disk as described above is called *slice*. Knot concordance defines a well-defined equivalence relation, and the associated set of equivalence classes form the aforementioned knot concordance group $\mathcal{C}$. The identity element is formed by the class of the unknot, which contains all slice knots.

The knot concordance group was first introduced in 1966 by Fox and Milnor [14] and has been subject of extensive study ever since. Although there has been a lot of progress in the study of its structure, some of the most basic questions about it remain unsolved. For example, it is still an open problem to fully describe torsion in $\mathcal{C}$. Fox [13] could show that the figure-eight knot is of order two in $\mathcal{C}$ by using the Alexander polynomial, while Murasugi [41] used the signature of a knot to obstruct sliceness, showing that the trefoil is of infinite order in $\mathcal{C}$. In general, all that is known about its structure as an abstract group is that it is countable, and that it splits off an infinitely generated free summand and an infinite summand consisting of 2-torsion.

The order of the twist knots K_n in $\mathcal{C}$ is difficult to describe. Casson and Gordon [5, 6] defined in the late 1970's invariants of knots based on the Atiyah-Singer G-signature which could be used to prove that the only slice knots among the twist knots are the unknot K_0 and the knot K_2, also known as *Stevedore knot*, showing that K_0 and K_2 represent zero in $\mathcal{C}$. Furthermore, Livingston and Naik [36] could show that if $4n+1 = pm$, where p is a prime congruent to 3 mod 4 and $\gcd(p, m) = 1$, then K_n is of infinite order in $\mathcal{C}$. Recent results [23] have shown that all twist knots are of infinite order in $\mathcal{C}$. However, the methods used are based on sophisticated tools such as Heegaard Floer homology that exclusively work in the smooth setting. In the present work, we provide another proof of Casson and Gordon's result that the only slice knots among the twist knots are K_0 and K_2 based on the work of Patrick Gilmer [18, 19] about the Casson-Gordon invariants, and describe a way that could possibly lead to the order of any twist knot K_n in $\mathcal{C}$ by using tools that also work in the topological setting.

The classical genus $g(K)$ of a knot K is defined as the minimal genus of a Seifert surface for the knot, i.e. a compact, connected, orientable surface that bounds K in S^3. Such a surface always exists [47], and $g(K)$ is thus well-defined. Asking whether or not a knot bounds a properly and smoothly embedded disk in B^4 leads to an extension of the classical genus, the *slice* or 4-*ball* genus g_4, which is defined as the minimal genus of a compact, connected, orientable surface smoothly and properly embedded in B^4 that bounds the knot in S^3. Since a Seifert surface can be pushed into the 4-ball while leaving its boundary fixed in S^3, we have the bound $g_4 \leq g$. Clearly, if K is slice, then $g_4(K) = 0$.

While there are techniques to compute the classical genus, there are no known methods to compute the slice genus of an arbitrary knot. Since all twist knots K_n bound a genus one Seifert surface and only K_0 and K_2 are slice, we know that $g_4(K_n) = 1$ for $n \neq 0, 2$. It is thus an interesting question to ask how the slice genus behaves for a connected sum of twist knots rK_n. We will give a partial answer to this question as well by providing an upper bound for the slice genus of any connected sum rK_n.

In 2010, Charles Livingston [34] defined a variation of the slice genus, which is called the *stable* 4-*genus* g_{st}, which is defined as

$$g_{st}(K) = \lim_{n\to\infty} \frac{g_4(nK)}{n},$$

where nK denotes the n-fold connected sum of K. The stable 4-genus defines a semi-norm on the rationalized concordance group $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes \mathbb{Q}$, and there are many open questions related to this semi-norm. In particular, the value of g_{st} is unknown for many knots, especially for the twist knots K_n. In the present work, we will provide an upper bound for the stable 4-genus of the twist knots K_n.

Organization

The present work is structured as follows. There are two parts, Part I: General Knot Theory and Part II: The Slice Genus of Twist Knots. The first part consists of Sections 1 to 3 and provide an overview of topics from general knot theory that will be relevant to us. More precisely, in Section 1, we give a brief survey of classical knot theory, presenting and discussing those objects that will be used throughout the text. In Section 2, we discuss slice knots and the knot concordance group, and in Section 3, we provide an introduction to the Casson-Gordon invariants $\sigma(K, \chi)$ and $\tau(K, \chi)$, which will serve as a main tool for the second part of this text. Furthermore, we discuss Patrick Gilmer's results about the invariant $\tau(K, \chi)$, which will be used for computations later on.

The second part consists of Sections 4 to 7 and deals with the study of the slice genus of the twist knots K_n. More precisely, in Section 4, we turn our interest to the main actors of this text, namely the twist knots K_n. We discuss basic properties and prove first (of course well-known) results about them. In Section 5, we reprove Casson and Gordon's famous result that the only slice knots among the twist knots are the unknot K_0 and the Stevedore knot K_2 by using the results of Gilmer. Section 6 is devoted to the discussion of the sliceness of the connected sum rK_n, and Section 7 concludes the work by presenting the upper bound for the (stable) 4-genus of the twist knots K_n. References will be included throughout the text.

Part I

General Knot Theory

1 Elements from Classical Knot Theory

The aim of this section is to give a short survey of classical knot theory and the elements arising from it that will be used throughout this text. Since our survey has more the character of a summary rather than a full treatment of knot theory (which would be far beyond the scope of this text), we assume that the reader has already been exposed to the basics of knot theory. A little more emphasis will be given on advanced topics, such as the construction of finite cyclic branched coverings. Proofs will be mostly omitted, but references where proofs can be found will be included whenever a result is cited. The survey is based on classical texts about knot theory, such as [25], [31], or [45]; additional references will be included separately throughout the text when needed.

As it is in the nature of the topic, knot theory makes strong use of topology and its subfields differential and algebraic topology, and we assume that the reader has already been exposed to these subjects. In particular, we assume that the reader is familiar with (smooth) manifolds and maps between them, covering spaces, homology theory, and so on. References for these topics include [24], [27], and [28]. Throughout this section, homology will always be understood with coefficients in $\mathbb{Z}$ if not mentioned otherwise.

1.1 Knots in S^3

Knots and Knot Equivalence

Before we start our discussion about knots we will fix some notation and conventions. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, equipped with the standard topology induced by the Euclidean metric $\|x\| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and oriented positively by choosing the canonical basis. We will denote the n-dimensional ball by

$$B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\| \leq 1\},$$

and the n-dimensional sphere by

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

We have the relation $\partial B^{n+1} = S^n$, where $\partial B^{n+1} \subset \mathbb{R}^{n+1}$ denotes the boundary of B^{n+1}. As B^n is a (smooth) n-dimensional submanifold of $\mathbb{R}^n$, it inherits a natural orientation from $\mathbb{R}^n$. An orientation on $S^{n-1} = \partial B^n$ is induced from B^n by the standard convention "outward pointing normal first".

Let us turn our attention now to the heart of this text, knots. Intuitively, a mathematical knot can be understood as a knot in a rope with its ends glued together. Formally, this is realized as an embedding of the circle S^1 into three-dimensional space.

Definition 1.1 (Knot). A *knot* K is a smooth embedding $f : S^1 \to S^3$.

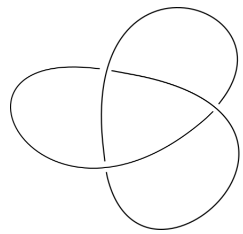


Figure 1.1: A trefoil

Figure 1.1 above shows a knot called trefoil. Knots will be usually denoted by K or J. A knot can naturally be given an orientation by specifying a string orientation of the knot. An *oriented knot* is a knot with a given orientation. If it is clear from the context that we are speaking of oriented knots, we will sometimes omit the word 'oriented' and simply speak of knots.

It is customary to identify the knot with the image of the embedding $f : S^1 \to S^3$. While this is convenient for visualizing a knot, it is important to keep in mind that the definition of a knot as the image of an embedding would not yield any interesting objects as all one-dimensional subspaces of S^3 diffeomorphic to S^1 are isotopic to each other (though not ambient isotopic, see below). Thus, the embedding itself is important and carries useful information.

The requirement of the embedding being smooth is necessary in order to exclude so called *wild knots*, in which we are not particularly interested.

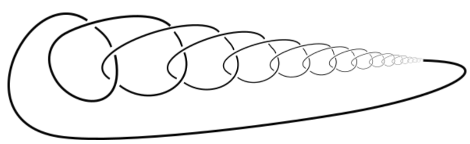


Figure 1.2: A depiction of a wild knot (picture taken from the Wikipedia article "Wild knots")

We could have also defined a knot as an embedding $S^1 \to \mathbb{R}^3$ rather than $S^1 \to S^3$, and the two definitions yield in fact equivalent classes of knots (see below). However, it is very convenient for various applications to work in a compact space rather than a non-compact one, which is why knots are usually defined as embeddings of the circle into the 3-sphere.

More generally, one can consider knotted objects in the 3-sphere that consist of multiple components. A *link* L of n components is a subset of S^3 (or $\mathbb{R}^3$) that is diffeomorphic to a disjoint union of n knots $K_1, \dots, K_n$, denoted by $L = K_1 \cup \dots \cup K_n$. A link with one component is just a knot as defined above, and a link whose components are unknots that can be arranged such that each of them is contained in disjoint 3-balls is called the *unlink*. Links are an interesting subject on their own, and in fact most of classical knot theory is treated in terms of links. However, as we are mostly interested in knots themselves, we will continue our discussion in terms of ordinary knots. Note though that most of the definitions and statements made in this and following sections admit a straightforward generalization to links.

We will define now what it means for knots to be equivalent.

Definition 1.2 (Ambient Isotopy). Let X and Y be topological spaces. Two smooth embeddings $f_1, f_2 : X \to Y$ are called *ambient isotopic* if there is a smooth isotopy

$$H : Y \times [0,1] \to Y, \quad (x,t) \mapsto H(x,t) = H_t(x)$$

such that $H_0 = \mathrm{id}_Y$ and $H_1 = g$, where $g : Y \to Y$ is a diffeomorphism such that $g \circ f_1 = f_2$.

In our case, $X = S^1$ and $Y = S^3$, so the definition above reads as follows: two knots K_1 and K_2 are ambient isotopic if there is an isotopy $H : S^3 \times [0,1] \to S^3$ from the identity on S^3 to a diffeomorphism g such that $g(K_1) = K_2$. It is not difficult to see that ambient isotopy defines an equivalence relation on the class of embeddings $f : S^1 \to S^3$.

Definition 1.3 (Knot Equivalence). Two knots K_1 and K_2 are called *equivalent* (or *isotopic*) if they are ambient isotopic.

Definition 1.3 makes the idea precise that two knots are to be considered as equivalent if they can be continuously transformed into each other without cutting any of the strands of the knots. Note that there are also other notions of knot equivalence, compare e.g. [3], [36], or [45].

We have mentioned above that knots in $\mathbb{R}^3$ and S^3 define the same equivalence classes. This can be seen as follows: Any knot $K \subset S^3$ can be moved by means of an isotopy such that it misses the north-pole in S^3. Stereographic projection then defines a knot in $\mathbb{R}^3$, and we see that equivalent knots in S^3 define equivalent knots in $\mathbb{R}^3$.

Knot Diagrams

Knots are usually visualized by means of a *knot diagram*, i.e. the result of a regular projection $p : \mathbb{R}^3 \to E$ applied to a knot $K \subset \mathbb{R}^3$, where $E \subset \mathbb{R}^3$ is a plane. Here, "regular" means that there are no occurrences of singularities, i.e. no triple points, tangential intersections or cusps, in the image $p(K)$. Below is a knot diagram of the figure-eight knot.

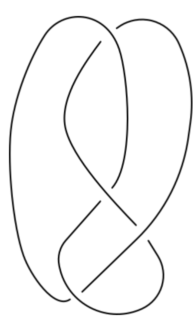


Figure 1.3: The figure-eight knot

It can be shown that such a regular projection $p : \mathbb{R}^3 \to E$ always exists (cf. [3]). Note that in the knot diagram above, we have indicated the over- and undercrossings of the strands. This is a necessary condition in order to reconstruct the knot from a knot diagram, and is in fact sometimes included in its definition. Of course, a knot can have many different knot diagrams, but the diagrams of equivalent knots are all related by simple operations called the *Reidemeister moves.*

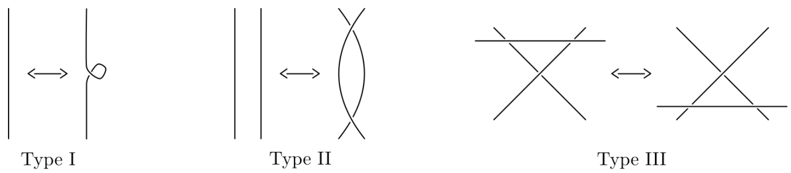


Figure 1.4: The three Reidemeister moves

It can be shown that two knots are equivalent if and only if their diagrams are related by a finite sequence of Reidemeister moves (cf. [3]). The minimal number of crossings in any knot diagram of a knot K is called the *crossing number* $u(K)$ of K. The orientation of the knot can be indicated in a knot diagram by adding a string orientation in the knot diagram. However, this is usually omitted and only indicated when needed.

The crossing number of a knot together with an index is commonly used as a reference for a knot in a traditional knot tabulation based on the number of crossings. For example, 3_1 denotes the first (and only) knot with 3 crossings, the trefoil, and 8_5 denotes the fifth knot with 8 crossings. The ordering of the knots in such a tabulation (i.e. the index of the crossing number) has no particular meaning and is traditionally coined. Knot tabulations up to 10 or fewer crossings can be found in most standard text books about knot theory, such as [3, 25, 31, 45]. Note that these tabulations usually do not take into account orientations or symmetries and only list prime knots (we will describe symmetries and prime knots shortly). Below is a table of the number of knots up to 16 or fewer crossings (with the aforementioned conventions), due to the work M. B. Thistlewaite, J. Hoste and J. Weeks [53].

Crossing number	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Number of knots	1	1	2	3	7	21	49	165	552	2'176	9'988	46'972	253'293	1'388'795

Table 1.1: Number of knots up to 16 or fewer crossings

1.2 Knot Operations

Knot Symmetries

Given an oriented knot $K \subset \mathbb{R}^3$, there are three natural geometric operations that can be performed on K, resulting in three new knots.

Definition 1.4. Let $K \subset \mathbb{R}^3$ be an oriented knot.

1.) The *reverse* $-K$ of K is defined as K with reversed orientation;

2.) The *mirror image* K^* of K is defined as K reflected in the plane of a regular projection $p : \mathbb{R}^3 \to E$;

3.) The *inverse* of K is defined as $-K^*$.

K is called *reversible* if $K = -K$, *amphicheiral* if $K = K^*$, and *invertible* (or *negative amphicheiral*) if $K = -K^*$.

The reverse, mirror image, and inverse of the trefoil is depicted below.

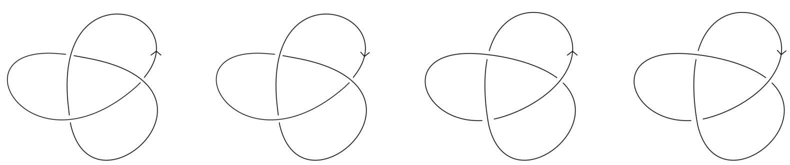


Figure 1.5: The reverse, mirror image, and inverse of the trefoil

Note that there is no consistent usage in the literature of the terms in Definition 1.4; for example, the reverse is often called the inverse. We are adopting here the convention of [36]. Examples of reversible knots include the trefoil and the figure-eight knot. While the figure-eight knot is amphicheiral, the trefoil is not, as first proved by Max Dehn in 1914 [11]. This leads to the notion of the left-handed and right-handed trefoil (K respectively K^* in Figure 1.5). As the figure-eight knot is reversible and amphicheiral, it is an example of an invertible knot.

While the operations defined above are an interesting object of study on their own, we are going to use them to define an equivalence relation on the set of isotopy classes of knots which is known as knot concordance (cf. Section 2.2).

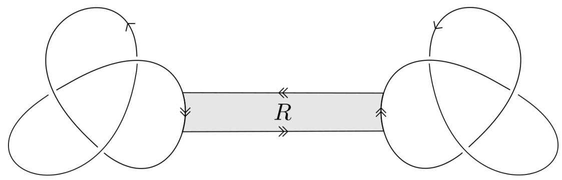
Connected Sum of Knots

Another geometric operation that arises naturally is the connected sum of two knots, defined as follows (compare the connected sum of two manifolds of same dimension).

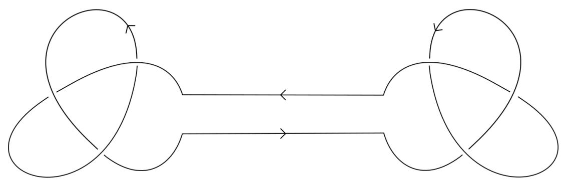
Let $J_1, J_2 \subset S^3$ be two oriented knots with a regular projection to a plane such that the images of both knots are disjoint. Perform the following steps (see Figure 1.6 below):

1.) Find a rectangle in the plane that meets J_1 and J_2 along one pair of arcs of the rectangle and that is disjoint otherwise, and so that the orientation of the knots induced on the pair of arcs that meet J_1 and J_2 is opposed to the orientation of the boundary of the rectangle (after possibly applying a suitable (orientation-preserving) transformation to one of the knots in S^3, such a rectangle always exists).

2.) Join the two knots together by deleting the pair of arcs that meet J_1 and J_2 and adding the other pair of arcs of the rectangle.



Step 1



Step 2

Figure 1.6: Forming the connected sum of two knots

The resulting knot is called the *connected sum* of J_1 and J_2 and is denoted by $J_1\#J_2$. The connected sum $J_1\#J_2$ inherits an orientation from the knots J_1 and J_2 which is consistent with the original orientation of J_1 and J_2. In particular, the connected sum is well-defined up to (oriented) ambient isotopy of knots. Note that it is essential to take orientations into account when forming the connected sum of knots in order to get a well-defined result.

The operation of connected sum turns the set of equivalence classes of oriented knots in S^3 into a commutative monoid with a unique prime factorization, leading to the notion of a *prime knot*, i.e. a knot that can not be further decomposed into a connected sum of non-trivial knots. The unit is formed by the unknot, and examples of prime knots include the left- and right-handed trefoil. We will see below that the unknot can not be written as the connected sum of two non-trivial knots. Unfortunately, there are in general no inverses under the connected sum, which turns the monoid not into a group. However, we will encounter in Section 2.2 an equivalence relation on knots called knot concordance, which turns the corresponding set of equivalence classes into a countable, abelian group, the knot concordance group $\mathcal{C}$.

1.3 Knot Complement, Knot Exterior, and Linking Numbers

Knot Complement and Knot Exterior

In order to gain information about the isotopy type of a knot, it can be very useful to study not the knot itself, but rather its surroundings in

S^3. This leads to the notion of the knot complement and knot exterior, which are defined as follows. Let $K \subset S^3$ be a knot. Considering K as a submanifold of S^3, it has a tubular neighborhood $N(K) \subset S^3$, which is the image of an embedding $f : S^1 \times D^2 \to S^3$ such that $f(S^1 \times (0,0)) = K$. Note that $N(K)$ is unique up to isotopy that fixes K.

Definition 1.5. Let $K \subset S^3$ be a knot.

1.) The *knot complement* is defined as $S^3 \setminus K$.

2.) The *knot exterior* $X(K)$ of K is defined as

$$X(K) := S^3 \setminus (N(K))^\circ.$$

The knot complement is simply the topological complement of K in S^3, and the knot exterior $X(K)$ is a 3-manifold with boundary $\partial X(K) = \partial N(K) \cong T^2$, where T^2 denotes the standard torus. It is easy to see that knots with diffeomorphic exteriors have diffeomorphic complements. The converse is true as well, but it is non-trivial. In fact, a much stronger result holds: if the complements of two knots are diffeomorphic, then the corresponding knots are isotopic (see [22]).

A powerful invariant that arises from the knot complement (respectively the knot exterior) is the *knot group* $\pi_1(S^3 \setminus K) = \pi_1(X(K))$. It can be shown that prime knots with isomorphic knot groups have diffeomorphic complements (see [57]). The downside of the knot group is that (non-abelian) groups are in general difficult to handle, and its abelianization $H_1(S^3 \setminus K)$ is the infinite cyclic group $\mathbb{Z}$ for any knot K:

Proposition 1.6. Let $K \subset S^3$ be a knot. Then

$$H_1(X(K)) \cong \mathbb{Z}.$$

Proposition 1.6 is a direct consequence of Alexander duality. For a proof without Alexander duality, see [36]. Since the knot complement $S^3 \setminus K$ deformation retracts to the knot exterior $X(K)$, the inclusion $i : X(K) \to S^3 \setminus K$ induces an isomorphism on the homology groups, showing that $H_1(S^3 \setminus K) \cong \mathbb{Z}$ as well. The fact that the first homology group of the knot exterior is infinite cyclic is essential and will be used in order to construct other objects based on the knot complement (for example the finite cyclic branched coverings, see Section 1.7).

Linking Numbers in S^3

Given a knot $J \subset S^3$, its exterior $X(J)$ and the fact that $H_1(X(J)) \cong \mathbb{Z}$ can be used to define a quantity that measures how two knots are linked in S^3.

Definition 1.7. Let J_1 and J_2 be two oriented knots in S^3. Then the *linking number* $\mathrm{lk}(J_1, J_2)$ of J_1 and J_2 is defined as

$$\mathrm{lk}(J_1, J_2) := [J_2] \in H_1(X(J_1)) \cong \mathbb{Z}.$$

The linking number can be computed algorithmically by counting how many times J_2 passes over J_1 in a regular projection of J_1 and J_2, where a crossing is counted positive or negative if J_1 passes from right to left or left to right under J_2, respectively (see Figure 1.7 below).

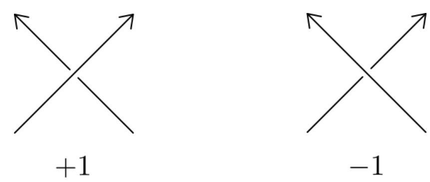


Figure 1.7: Sign convention at crossings

Note that the linking number doesn't change if we count at undercrossings instead of overcrossing. It follows that the linking number is symmetric, i.e. $\mathrm{lk}(J_1, J_2) = \mathrm{lk}(J_2, J_1)$.

The concept of linking numbers in S^3 can be used to define an important bilinear pairing in knot theory, which will be described in the next section.

1.4 Seifert Surfaces, Seifert Pairing, and Seifert Matrices

Seifert Surfaces

One of the most important objects in knot theory is the Seifert surface of a knot.

Definition 1.8 (Seifert Surface). Let $K \subset S^3$ be an oriented knot. A compact, connected, orientable surface F embedded in S^3 that has K as its oriented boundary is called a *Seifert surface* for K.

Seifert surfaces can be used to study knots in different ways, and they form the starting point for many other constructions in knot theory. Note that it is not obvious why an arbitrary knot should have a Seifert surface; the justification in the definition above is given by the following proposition, first proved by Frankl and Pontrjagin in 1930 [15].

Proposition 1.9. Any oriented knot $K \subset S^3$ has a Seifert surface.

A proof can be found in any textbook about knot theory. The proof is usually done by giving an explicit algorithm, known as *Seifert algorithm*, for constructing a Seifert surface from a knot diagram, which is due to Herbert Seifert [47]. We omit the description of the algorithm at this point as we are not going to need it and refer the interested reader to [31] instead. In Figure 1.8 below is an example of a Seifert surface for the trefoil.

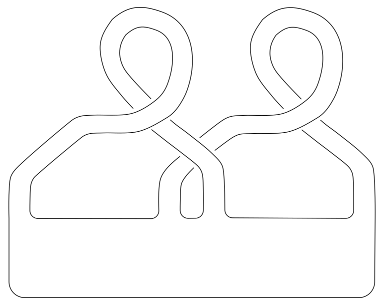


Figure 1.8: A Seifert surface for the trefoil

It is not difficult to see that the boundary of this surface is actually a trefoil since a curled band can be transformed into a twisted band (cf. Figure 4.5 in Section 6).

Of course, a Seifert surface for a knot K need not be unique. However, two Seifert surfaces for a knot are related by two simple operations, which consist, roughly speaking, of adding and removing embedded cylinders from the Seifert surfaces in S^3 (this is known as 0- and 1-surgery, and leads to the notion of S-equivalence of Seifert surfaces, cf. [25]). More generally, it can be shown that any two Seifert surfaces of isotopic knots J_1 and J_2 are related by the moves described above (cf. [25]).

Knot Genus

Recall from the theory of surfaces that any closed and orientable surface F is homeomorphic to a connected sum of a sphere and $n \in \mathbb{N}_0$ tori T^2. The number of tori in the connected sum is called the genus of the surface, and is denoted by $g(F)$. This leads to the following definition:

Definition 1.10. The *genus* $g(K)$ of a knot $K \subset S^3$ is defined as

$$g(K) := \min\{g(F) : F \text{ is a Seifert surface for } K\}.$$

The genus of a knot is sometimes also called classical genus or 3-genus in order to distinguish it from other notions of knot genus (such as the slice genus, which will be described below). For the same reasion, it is sometimes also denoted by $g_3(K)$.

A Seifert surface F is an orientable surface with one boundary component, so it is topologically equivalent to a disk with an even number of twisted, knotted and linked bands attached to it (see for example [3] for a proof). Figure 1.8 above shows a Seifert surface for the trefoil in this form. The number of bands divided by two is the genus of F. Alternatively, the Euler characteristic of a Seifert surface F is $\chi(F) = 1 - 2g(F)$, so

$$g(F) = \frac{1}{2}(1 - \chi(F)).$$

It follows that a knot K is the unknot if and only if $g(K) = 0$. This implies that if K has a genus one Seifert surface and it is known that K is not the unknot, then $g(K) = 1$. In general, it is difficult to explicitly compute the genus of an arbitrary knot, but possible, as recent results of Friedl and Vidussi demonstrate [17].

There is a very useful property of the knot genus: it is additive under connected sum of knots.

Theorem 1.11. Let $K_1, K_2 \subset S^3$ be two knots. Then

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

For a proof, see [31]. The additivity of the knot genus has several implications:

Corollary 1.12. There is no additive inverse for a (non-trivial) knot. That is, if $K_1 \# K_2$ is the unknot, then K_1 and K_2 are unknotted as well.

Corollary 1.13. If K is a non-trivial knot and if nK denotes the n-fold connected sum of K, then $nK \neq mK$ whenever $n \neq m$. In particular, there are infinitely many distinct knots.

Corollary 1.14. If $g(K) = 1$, then K is prime.

Corollary 1.15. Any knot can be expressed as a finite sum of prime knots.

The last corollary implies in particular that there is indeed a prime factorization in the monoid of isotopy classes of knots as mentioned in Section 1.2.

Seifert Pairing and Seifert Matrices

Given a Seifert surface F for a knot K, there is a bilinear pairing that captures how F is embedded in S^3, defined as follows. Let K be a knot with Seifert surface F, and let $i_+ : H_1(F) \to H_1(S^3 \setminus F)$ be the homomorphism induced by the positive push-off $F \to S^3 \setminus F$. Then there is a bilinear form θ defined by

$$\theta : H_1(F) \times H_1(F) \to \mathbb{Z}, \quad ([\alpha], [\beta]) \mapsto \mathrm{lk}(\alpha, i_+(\beta)).$$

Of course, one has to check that θ is indeed a well-defined bilinear form. A proof can be found in [31]. Note that this bilinear form need not necessarily be non-singular.

Definition 1.16 (Seifert Pairing). Let $K \subset S^3$ be a knot with Seifert surface F. Then the bilinear form θ described above is called *Seifert form* (or *Seifert pairing*) associated to F.

From now on, we will start to omit brackets for homology classes to shorten notation. The first homology $H_1(F)$ of a Seifert surface F is isomorphic to the free abelian group in $2g$ generators, where g denotes the genus of F. If we fix a basis $x_1, \ldots, x_{2g}$ of $H_1(F)$, we can express the Seifert form θ in terms of a matrix

$$A = (\theta(x_i, x_j))_{i,j=1}^{2g}.$$

We have the following definition.

Definition 1.17 (Seifert Matrix). A matrix A associated to the Seifert form θ of a knot K as described above is called *Seifert matrix* for K.

It is customary to slightly abuse notation and write θ for both the Seifert pairing and the associated Seifert matrix, and we will start doing so in the following.

We have remarked earlier that a Seifert surface F is topologically equivalent to a disk with a certain number of twisted, linked and knotted bands attached to it. Given such a representation of F, it is possible to find a basis of $H_1(F)$ in terms of simple closed curves, making it easy to compute the Seifert matrix A. Note that for some given element $x \in H_1(F)$, the *self-linking number* or *framing* $\theta(x,x) = \mathrm{lk}(x, i_+(x))$ can be computed by first representing $i_+(x)$ by a parallel copy $\widetilde{x}$ of x in F, and then counting the number of curls with sign in the band that gets traced out by $\widetilde{x}$ and x in F. As an example, consider the Seifert surface for the trefoil with the indicated basis for the first homology in Figure 1.9.

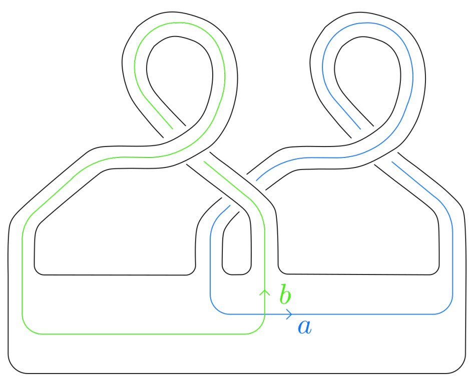


Figure 1.9: A Seifert surface for the trefoil with first homology basis

In this basis, the Seifert matrix for the trefoil is given as

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Seifert matrices are not only important for computing the Seifert pairing, but also in the sense that there are a lot of knot invariants arising from them. However, there is a lot of choice involved in the definition of a Seifert matrix, and it is not obvious at all why a knot invariant that is derived from a Seifert matrix is left unchanged when choosing another Seifert matrix. We have elaborated above how Seifert surfaces of isotopic knots are related to each other. Fortunately, this relation descends to Seifert matrices, leading to the notion of S-equivalence of Seifert matrices. Similarly as for Seifert surfaces, Seifert matrices of isotopic knots are related by a finite sequence of certain matrix operations. We are not going into further detail at this point since the exact relations are not of particular interest to us. For a full description of the terms and relations mentioned above, see [25]. For our purposes, it will always be sufficient to specify one particular Seifert surface and a corresponding Seifert matrix for a knot K.

1.5 Alexander Polynomial

An important family of knot invariants is formed by certain polynomials defined from geometric observations about knots. One of them is the Alexander polynomial, defined as follows.

Definition 1.18. Let $K \subset S^3$ be a knot with Seifert matrix A. Then the *Alexander polynomial* $\Delta_K(t)$ of K is defined as

$$\Delta_K(t) = \det(A - tA^T) \in \mathbb{Z}[t, t^{-1}].$$

The Alexander polynomial is a well-defined knot invariant up to multiplication by $\pm t^{\pm n}$, meaning that if two different Seifert surfaces (resp. Seifert matrices) are used to compute $\Delta_K(t)$, then the resulting polynomials only differ by $\pm t^{\pm n}$. The Alexander polynomial has some interesting properties, such as

$$\Delta_K(t) \doteq \Delta_{rK}(t) \doteq \Delta_{mK}(t),$$

and

$$\Delta_{K_1 \# K_2}(t) \doteq \Delta_{K_1}(t) \cdot \Delta_{K_2}(t),$$

for any knots $K, K_1, K_2 \subset S^3$. Here, $\doteq$ denotes equality up to multiplication of $\pm t^k$ for some $k \in \mathbb{Z}$. In particular, we have the inequality

$$\deg(\Delta_K(t)) \leq 2g(K).$$

For a proof of these results, see [3] or [31]. The Alexander polynomial will be important for us because its zeros mark the points where the so-called Tristram-Levine signature of a knot can change. The description of this and other knot signatures will be the topic of the next section.

1.6 Knot Signatures

Let $K \subset S^3$ be a knot with Seifert surface F and Seifert pairing θ. Further, let A be a Seifert matrix associated to θ. Note that the matrix A is in general neither symmetric nor non-singular (in fact, A is symmetric if and only if A is the empty matrix [36]). However, the *symmetrized Seifert matrix* $A+A^T$ is a non-singular and symmetric matrix with integer entries, and thus diagonalizable (over $\mathbb{R}$). Applying Sylvester's law of intertia, we know that the signature of $A + A^T$, i.e. the number of positive minus the number of negative entries in a diagonalization of $A + A^T$, is an invariant of the Seifert form. This leads to the following definition:

Definition 1.19 (Classical Knot Signature). Let $K \subset S^3$ be a knot with Seifert matrix A. The *signature* $\sigma(K)$ of K is defined as the number of positive entries minus the number of negative entries in a diagonalization of $A + A^T$.

The signature of a knot is sometimes referred to as the *classical signature*. The knot signature has some interesting properties, which are summarized in the following theorem.

Theorem 1.20. Let $K \subset S^3$ be a knot. Then the signature $\sigma(K)$ is a well-defined knot invariant, satisfying the following properties:

1.) $\sigma(K)$ is an even number for any knot K;

2.) $\sigma(rK) = \sigma(K)$;

3.) $\sigma(mK) = -\sigma(K)$;

4.) $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$.

A proof can be found in [45]. Sylvester's law of inertia is also valid for Hermitian matrices, leading to another definition of knot signature.

Definition 1.21 (Tristram-Levine Signature). Let $K \subset S^3$ be an oriented knot with Seifert matrix A, and let $\omega \in S^1 \subset \mathbb{C}$, $\omega \neq 1$. Then the ω-signature $\sigma_\omega(K)$ is defined as the signature of the Hermitian matrix

$$A_\omega := (1-\omega)A + (1-\overline{\omega})A^T.$$

In case that $\omega = e^{\frac{2\pi i s}{m}}$ with $s, m \in \mathbb{Z}$, $m \neq 0$, and $\omega \neq 1$, the ω-signature is also denoted by $\sigma_{\frac{s}{m}}(K)$.

The ω-signature is also known as the *Tristram-Levine signature*, and was first introduced and studied by A. G. Tristram [54] and J. Levine [29], generalizing the work of Murasugi [42]. It follows directly from the definition that $\sigma_{\frac{1}{2}}$ resp. σ_π is the classical signature σ as defined above. The Tristram-Levine signature is an integer and defines thus a *signature function*

$$f : S^1 \setminus \{1\} \to \mathbb{Z}, \quad \omega \mapsto \sigma_\omega(K).$$

Since

$$A_\omega = (1-\omega)A + (1-\overline{\omega})A^T = (1-\omega)(A - \overline{\omega}A^T) = (1-\omega)\Delta_A(\overline{\omega}),$$

the matrix A_ω is non-singular except at zeros of the Alexander polynomial $\Delta_A(\omega)$. Moreover, the diagonal entries in a diagonalization of A_ω are (real-valued) polynomials in ω, so the signature of A_ω has a constant value in a neighborhood to the left and in a neighborhood to the right of a zero of $\Delta_A(\omega)$ on S^1. It follows that the signature function is a continuous and piecewise-constant function with a finite set of discontinuities, which are a subset of the zeros of the Alexander polynomial. Moreover, the function is even valued outside of the discontinuities.

The Tristram-Levine signature shares many properties with the classical signature. For example, for any knot K we have the symmetries $\sigma_\omega(rK) = \sigma_\omega(K)$ and $\sigma_\omega(mK) = -\sigma_\omega(K)$, and σ_ω is additive under connected sum of knots:

$$\sigma_\omega(K_1 \# K_2) = \sigma_\omega(K_1) + \sigma_\omega(K_2).$$

These properties can be proved in essentially the same way as for the classical signature $\sigma(K)$, but with the matrix A_ω instead of $A + A^T$. For more on knot signatures and their properties, see [21].

1.7 Finite Cyclic Branched Coverings of a Knot

In this section we are going to describe finite cyclic branched covering spaces of a knot $K \subset S^3$. These coverings, in particular their homology, capture a lot of information about the knot exterior and find applications throughout knot theory. Our description follows mostly [7]. For a more detailed description, see [25, 31, 36, 45].

Recall from covering space theory that for a sufficiently nice space X (that is, connected, locally connected, semi-locally simply connected), covering spaces are classified up to covering isomorphisms by conjugacy classes of subgroups of the fundamental group $\pi_1(X)$. In particular, specifying a surjective homomorphism $\varphi : \pi_1(X) \to G$ to some group G gives rise to a regular covering $\widehat{X} \to X$ that corresponds to the normal subgroup $\ker(\varphi) \subset \pi_1(X)$ with group of deck transformations isomorphic to G.

Returning to knots, let $K \subset S^3$ be a knot with exterior $X(K)$, and consider the composition

$$\pi_1(X(K)) \xrightarrow{\varphi} H_1(X(K)) \cong \mathbb{Z} \xrightarrow{p} \mathbb{Z}/n\mathbb{Z},$$

where φ denotes the abelianization map and p denotes the projection. Then there is an associated covering $X_n(K) \to X(K)$ with group of deck transformations isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Definition 1.22. The covering space $X_n(K)$ is called the *n-fold cyclic covering* of the knot exterior $X(K)$.

Our aim is now to extend the n-fold covering $X_n(K)$ (almost) to a covering of the 3-sphere S^3. For this, note that the boundary $\partial X(K)$ of the knot exterior is a torus $K \times S^1$. Since the Euler characteristic is multiplicative under finite covers, the n-fold covering $X_n(K) \to X(K)$ restricts on the boundary to a map $\xi : K \times S^1 \to K \times S^1$. In other words, $\partial X_n(K)$ has to be a torus as well. Let ℓ be the preferred longitude (i.e. the longitude that is null-homologous in $X(K)$, unique up to ambient isotopy of $\partial X(K)$) and μ a meridian of $\partial X(K)$. Now, the map φ can be expressed in terms of the linking number, that is,

$$\varphi(\alpha) = \mathrm{lk}(\alpha, K) \in \mathbb{Z}$$

(for a proof, see [45]). Then $\varphi(\ell) = 0$ and $\varphi(\mu) = 1$, so we see that the preferred longitude ℓ lifts to a closed curve $\widetilde{\ell}$ in $\partial X_n(K)$, while the meridian μ in general does not. However, the n-fold composition μ^n does lift to a closed curve $\widetilde{\mu^n}$ in $\partial X_n(K)$. In other words, the boundary map ξ is given by

$$\xi(z_1, z_2) = (z_1, z_2^n).$$

If we now glue a solid torus $K \times D^2$ to the boundary of $X_n(K)$ by a diffeomorphism that maps the meridian μ to $\widetilde{\mu^n}$, we get a closed 3-manifold L_n. If we further glue a solid torus to the boundary of $X(K)$ in the canonical way, we get a map

$$L_n := X_n(K) \cup_\partial (K \times D^2) \to X(K) \cup_\partial (K \times D^2) \cong S^3$$

that is an extension of the n-fold cyclic covering $X_n(K) \to X(K)$ except at $K \times \{0\} \cong K$. Indeed, the map ξ extends to $K \times D^2$ except at $K \times \{0\}$, showing that $L_n \to S^3$ is a well-defined covering except at $K \subset S^3$. This leads to the following definition.

Definition 1.23 (Finite Cyclic Branched Covering). Let $K \subset S^3$ be a knot and X_n the n-fold covering of $X(K)$. Then the associated closed 3-manifold L_n as constructed above is called the *n-fold cyclic cover of S^3 branched along K.*

If the choice of the knot K is not ambiguous, L_n is sometimes just called the n-fold cyclic branched covering of S^3. It might seem that the definition of L_n depends on the gluing homomorphism that is used. However, the diffeomorphism type of L_n is entirely determined by the image of the meridian in $H_1(\partial X_n(K))$, showing that it is enough to specify where the meridian is mapped to (cf. [45]).

Our approach to the cyclic branched coverings L_n is rather algebraic, but they can also be constructed purely geometrical: take a knot $K \subset S^3$ with Seifert surface F, and cut S^3 along F. Denote the resulting space X. Now take n copies of X and glue them cyclically together along their boundaries. The result is a closed 3-manifold that is diffeomorphic to the branched covering L_n. The group of deck transformations of this covering acts by shifting indices, showing the the group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. For a detailed description of this construction, see [25] or [45].

A characteristic property of the n-fold branched coverings is that their homology is not too difficult to handle.

Proposition 1.24. Let p be a prime and $r > 0$. Then the p^r-fold cyclic branched covering L_{p^r} is a rational homology 3-sphere, i.e. $H_*(L_{p^r}; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$.

A proof can be found in [7]. Our interest will mainly lie in the 2-fold (or double) branched covering L_2. More precisely, we are going to use its first cohomology with $\mathbb{Q}/\mathbb{Z}$-coefficients to derive information about the slice genus of the twist knots K_n. For this, we need to understand how to compute $H^1(L_2; \mathbb{Q}/\mathbb{Z})$. Fortunately, there is a very convenient way to do so.

Proposition 1.25. If $K \subset S^3$ is a knot with Seifert matrix A and double branched covering L_2, then $A + A^T$ is a presentation matrix for $H_1(L_2; \mathbb{Z})$. In particular,

$$H_1(L_2; \mathbb{Z}) \cong \operatorname{coker}(A + A^T).$$

A proof can be found in [45]. We have the following corollary.

Corollary 1.26. Under the assumptions of Proposition 1.25,

$$H^1(L_2; \mathbb{Q}/\mathbb{Z}) \cong \operatorname{coker}(A + A^T).$$

Proof. Since $A + A^T$ is non-singular, it follows that $H_1(L_2; \mathbb{Z})$ is torsion (of course, this also follows from Proposition 1.24). The universal coefficient theorem implies that

$$H^1(L_2; \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(L_2; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}),$$

and since $H_1(L_2;\mathbb{Z})$ is torsion, we have

$$H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong H_1(L_2;\mathbb{Z}) \cong \operatorname{coker}(A + A^T),$$

where the last isomorphism is given by Proposition 1.25. $\square$

Thus, in order to compute $H^1(L_2;\mathbb{Q}/\mathbb{Z})$, all we have to do is to compute the Smith normal form of $A + A^T$ and read-off the elementary divisors of $H^1(L_2;\mathbb{Q}/\mathbb{Z})$. We will do this computation for the twist knots in Section 4.

2 Slice Knots and Knot Concordance

In this section we introduce slice knots, the knot concordance group and related notions. More precisely, we start by discussing slice knots and their properties, continue with the definition of knot concordance and the knot concordance group, and discuss the slice genus and stable 4-genus of a knot. This section is mainly based on [25] and [36]. As in the previous section, proofs will be mostly omitted, but references where proofs can be found will be included whenever a result is cited.

Before we start, a last remark has to be made about the distinction between the topological and the smooth setting. Slice knots and knot concordance can be studied from the topological and from the smooth point of view, and the two approaches differ substantially. Throughout this work, we will work exclusively in the smooth setting. Some essential differences will be mentioned, but for a more extensive comparison, see [35].

2.1 Slice Knots

It is a standard fact that a knot $K \subset S^3$ bounds an embedded 2-disk D in S^3 if and only if K is the unknot (cf. [45]). Thus, one might ask if the situation changes if we go one dimension up. That is, are there non-trivial knots $K \subset S^3$ that bound an embedded disk D in the 4-ball D^4? The answer is yes, and we will encounter an example shortly.

Definition 2.1 (Slice Knot). A knot $K \subset S^3$ is called *slice* if it bounds a smoothly and properly embedded 2-disk D in the 4-ball B^4. Such a disk is called a *slice disk* for the knot K.

Here, proper means that $D \cap S^3 = \partial D$. In some sense, slice knots can be considered as the next best thing to the unknot (a phrase coined by Teichner [51]). The term slice knot first appeared in the work of Fox and Milnor [14], but it was Artin [1] that described in 1926 the construction of certain smooth, knotted 2-spheres in $\mathbb{R}^4$ such that their intersection with the standard $\mathbb{R}^3 \subset \mathbb{R}^4$ yields a non-trivial knot in $\mathbb{R}^3$ that bounds a smooth embedded disk in the upper half-space. At first it seemed possible that every knot is a slice knot, and it was not until the early 1960's that Murasugi [41] and Fox and Milnor [14] showed that some knots are not slice.

Note that it is essential to require that the slice disk is smoothly embedded, because otherwise we could just take the cone of a knot K to show that any knot is slice.

Since we work in the smooth setting, a slice knot is sometimes also called smoothly slice in order to distinguish it from the topological counterpart. In the topological setting, a slice disk is required to be locally flat instead of smooth, leading to topological slice knots. The two notions are not equivalent: several mathematicians have constructed knots that are topologically, but not smoothly slice (see for example [20] or [46]), so smoothly slice is a stronger notion than topologically slice. We will continue to work in the smooth setting. For an overview of the differences between the smooth and topological setting, see [35].

It is in general difficult to decide whether a knot is slice or not. One method to do so is to try to find a "movie" that describes the slice disk of a knot (cf. [36]). More precisely, let $D \subset B^4$ be a slice disk for a knot $K \subset S^3$, and let $S_t^3 \subset B^4$ denote the 3-sphere of radius $0 < t \leq 1$. The standard norm on $\mathbb{R}^4$ induces a height function h on B^4 that restricts to a height function of the slice disk D. After a sufficiently large perturbation, we can assume that h is a Morse function, with critical points arranged according to the order of their index. Then it is possible to arrange the slice disk D such that the cross-sections $K_t = S_t^3 \cap D$ consist of either

1.) an ordinary knot or link without any singularities; or

2.) a knot or link with singularities corresponding to a local maximum, local minimum, or a saddle point of D.

By increasing the parameter t of the cross-sections K_t, we get a series of finitely many diagrams that describe the slice disk D. Such a series of diagrams is called a *(slice) movie* of the slice disk D. Figure 2.1 below shows an example of such a movie for the square-knot, which is the connected sum of a trefoil K with its inverse $-K^*$ (note that this proves that the square-knot is slice).

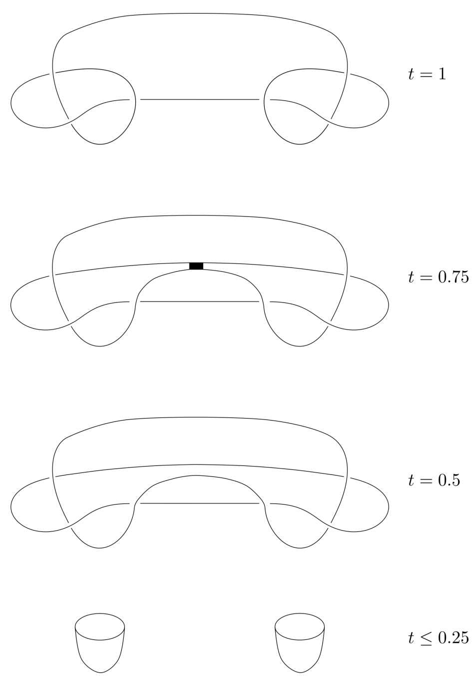


Figure 2.1: Slice movie for the square knot (picture as in [36], page 42)

Looking at Figure 2.1 top down, we can see that at $t = 1$ we have the square knot, followed by a saddle-point (index 1) at $t = 0.75$, where the circle pinches together by a band, resulting in a link with two components at $t = 0.5$ that is isotopic to the unlink with two components at

$t = 0.25$, each bounding a disk that corresponds to one of the two local minima (index 0) of the slice disk. Observe that the number of local minima and maxima minus the number of saddle-points equals one; an Euler characteristic argument shows that this is a necessary condition for the cross-sections K_t to trace out a disk in B^4.

The existence of such a slice movie follows from Morse theory [13]. For a more detailed description of the procedure described above, see [36]. In general, it is difficult to find a slice movie for an arbitrary knot, especially when it is not known beforehand whether or not the knot is slice. Thus, other methods to detect sliceness are needed. But first, we discuss a special type of knots that is related to slice knots.

Definition 2.2 (Ribbon Knot). A knot $K \subset S^3$ is called a *ribbon knot* if it is the boundary of of a *ribbon disk*, i.e. the image $\alpha(D^2)$ of an immersion $\alpha : D^2 \to \mathbb{R}^3 \subset S^3$ such that

1.) the only singularities of α are transverse double points (i.e. double points belonging to a transverse self-intersection of $\alpha(D^2)$);
2.) the set of double points forms a collection of arcs;
3.) the preimage of each of those arcs is a pair of arcs on D^2, one with endpoints on the boundary of the disk and one in an open neighborhood in the interior of the disk.

Figure 2.2 below shows an example of a ribbon knot with ribbon disk.

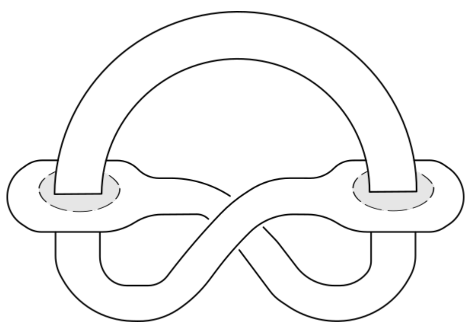


Figure 2.2: A ribbon knot (picture as in [3], page 27)

It is not difficult to see that a ribbon knot is slice: simply push an open neighborhood of the one-dimensional self-intersections of the ribbon disk into $\mathbb{R}^4$ (compare Figure 2.2 above).

Theorem 2.3. A ribbon knot is slice.

In fact, it can be shown that a knot is ribbon if it bounds a slice disk with no index 2 critical points [36]. An interesting open question is the following: if K is a slice knot, is K ribbon? This question was first posed by Fox in 1966, and is still unsolved at the time of writing. Note that this question is only relevant in the smooth setting: a ribbon knot is smoothly slice, but since there are topological slice knots that are not smoothly slice, there exist topological slice knots which are not ribbon.

Let's return to the problem of detecting sliceness. It is possible to find an obstruction to sliceness based on a general result that arises from studying the inclusion $\partial W \to W$ of a compact orientable manifold with boundary W. Proofs of the following results can be found in [36].

Theorem 2.4. Let W be a compact, orientable, $(2n+1)$-dimensional manifold with boundary. Then, for any coefficient field $\mathbb{F}$, we have

$$2 \cdot \dim(\ker(H_n(\partial W; \mathbb{F}) \xrightarrow{i_*} H_n(W; \mathbb{F}))) = \dim H_n(\partial W; \mathbb{F}),$$

where i_* denotes the map induced by the inclusion $i : \partial W \hookrightarrow W$. Moreover, the intersection pairing $H_n(\partial W; \mathbb{F}) \times H_n(\partial W; \mathbb{F}) \to \mathbb{F}$ vanishes on this kernel.

Theorem 2.4 is known as the "half lives, half dies" principle. The connection to slice knots is made by the following theorem.

Theorem 2.5. Let $K \subset S^3$ be a slice knot with Seifert surface F and slice disk $D \subset B^4$. Then there exists a compact orientable 3-manifold with boundary $M \subset B^4$ such that $\partial M = D \cup_K F$.

Combining Theorem 2.4 and 2.5 now yields the following obstruction to sliceness.

Theorem 2.6. Let K be a slice knot with Seifert surface F and Seifert pairing θ. Then there exists a direct summand (a so-called *metabolizer*) H of $H_1(F)$ such that

1.) $2\ \mathrm{rank}(H) = \mathrm{rank}(H_1(F))$;

2.) $\theta(H \times H) = 0$.

Note that the intersection pairing on $H_1(F)$ is related to the Seifert pairing by the formula $\theta - \theta^T$ (cf. [45]). Theorem 2.6 is one of the main characterization of slice knots and motivates the following definitions.

Definition 2.7 (Algebraically Slice Seifert Form). A $(2n \times 2n)$-dimensional Seifert form is called *algebraically slice* (or *metabolic*) if there is an n-dimensional summand of the underlying free $\mathbb{Z}$-module on which the form vanishes. Such a summand is called *metabolizer* of the form. Equivalently, a $(2n \times 2n)$-dimensional Seifert matrix is called *algebraically slice* if it is congruent to a matrix with a half-dimensional block of zeros. That is, there exists a non-singular integral matrix P such that

$$PAP^T = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

where the top-left zero denotes the $(n \times n)$-dimensional zero matrix.

Definition 2.8 (Algebraically Slice Knot). A knot K is called *algebraically slice* if it admits a Seifert form which is algebraically slice.

It can be shown that if some Seifert matrix for K is algebraically slice, then all are, independent of the choice of Seifert surface or basis for the first homology (cf. [36]). The following corollary is a direct consequence of Theorem 2.6.

Corollary 2.9. If a knot K is slice, then it is also algebraically slice.

The converse to Corollary 2.9 is not true: there exist knots which are algebraically slice, but not slice (cf. Section 5).

In the previous section, we have defined three knot invariants in terms of Seifert matrices: the Alexander polynomial, the classical signature and the Tristram-Levine signature. It is not surprising that these invariants have special properties for algebraically slice knots.

Proposition 2.10. Let K be algebraically slice.

1.) The Alexander polynomial of K takes the form

$$\Delta_K(t) = \pm t^k f(t) f(t^{-1})$$

for some polynomial $f \in \mathbb{Z}[t]$ with $f(1) = \pm 1$.

2.) The signature of K vanishes, i.e. $\sigma(K) = 0$.

3.) Given $\omega \in S^1 \subset \mathbb{C}$ such that $\Delta_K(\omega) \neq 0$, then the Tristram-Levine signature vanishes as well, i.e. $\sigma_\omega = 0$.

The notion of algebraic sliceness is a consequence of one of the most elementary obstructions to sliceness of a knot. Of course, this is not the only obstruction knot theory has found so far, and we will encounter some more obstructions in the section about Casson-Gordon invariants which we are going to use for our analysis of the twist knots. But first, we are going to discuss how the notion of sliceness can turn equivalence classes of knots into a group.

2.2 Knot Concordance

We have mentioned earlier that the connected sum turns the set of equivalence classes of oriented knots into a commutative monoid. This monoid is not a group because there are in general no additive inverses. However, the notion of sliceness can be used to define another equivalence relation on the set of knots, which turns them into an abelian group, the *knot concordance group*. Throughout this section, orientation plays an important role, and every object will be understood with orientations in mind, even though we might not always explicitly specify a particular orientation.

Before we can start describing this equivalence relation, we need a result regarding the sliceness of the connected sum of knots. Recall that the inverse $-K^*$ of a knot K is defined as the mirror image of K with its orientation reversed (cf. Section 1.2).

Theorem 2.11. Let $K \subset S^3$ be a knot. Then $K\# - K^*$ is slice.

Theorem 2.11 can be proved directly by constructing a slice disk in terms of so-called *spun knots*, first introduced by Artin [1] (see also [45]), or indirectly by showing that for any knot K, $K\# - K^*$ is ribbon (cf. [25]). Theorem 2.11 already gives a glimpse of what the new equivalence relation will look like. The next definition makes this precise.

Definition 2.12 (Knot Concordance). Two knots $K_1, K_2 \subset S^3$ are called *concordant* if $K_1\# - K_2^*$ is slice. If this is the case, we write $K_1 \sim K_2$.

The following theorem provides a useful characterization of knot concordance.

Theorem 2.13. Two knots $K_1, K_2 \subset S^3$ are concordant if and only if they cobound a smooth 2-manifold C diffeomorphic to $S^1 \times I$ in $S^3 \times I$ such that $C \cap (S^3 \times \{1\}) = K_1$ and $C \cap (S^3 \times \{-1\}) = K_2$. Here, $I = [-1, 1]$.

For a proof, see [36]. The next theorem shows that knot concordance induces a well-defined equivalence relation on the set of knots, turning the set of equivalence classes into an abelian group. A proof can be found in [36].

Theorem 2.14.

1.) Knot concordance is an equivalence relation on the set of oriented knots in S^3. Its equivalence classes are called *(knot) concordance classes.*

2.) Isotopic knots are concordant.

3.) The connected sum of knots induces a well-defined binary operation on the set of concordance classes, and under this operation the set of concordance classes is turned into an abelian group, where the class of the unknot represents the identity element and $-K^*$ represents the inverse of the class of a knot K.

Definition 2.15 (Knot Concordance Group). The (classical) knot concordance group $\mathcal{C}$ is the abelian group of concordance classes under the operation of connected sum. The identity element in $\mathcal{C}$ is given by the class of the unknot, which contains all slice knots.

As mentioned in the introduction, the knot concordance group was first introduced in 1966 by Fox and Milnor [14] and has been subject of extensive study ever since. There are also other notions of concordance, one of them being *algebraic concordance*. Algebraic concordance is the form of concordance corresponding to algebraically slice knots, that is, it is an equivalence relation defined on Seifert matrices which turns them into an abelian group, the *algebraic concordance group*, denoted by $\mathcal{G}^{\mathbb{Z}}$. Unlike the classical concordance group, the algebraic concordance group is well-understood. Knowing that there is only little information about the algebraic structure of the classical concordance group, it is remarkable that J. Levine [29, 30] could prove in the late 1960's that

$$\mathcal{G}^{\mathbb{Z}} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty \oplus (\mathbb{Z}/4\mathbb{Z})^\infty.$$

For a detailed treatment of the algebraic concordance group, see [36]. We continue our discussion by introducing an extension of the classical knot genus, known as the slice or 4-ball genus.

2.3 Slice Genus

Recall that the classical genus $g(K)$ of a knot $K \subset S^3$ is defined as the minimal genus of any Seifert surface for K (cf. Section 1.4). Since we have been studying knots that bound a smoothly and properly embedded disk in the 4-ball, one could ask if a knot that does not bound such a disk might bound a smoothly and properly embedded (orientable) surface in the 4-ball instead, and if it does, one could further ask how the minimal genus among all such surfaces compares to the ordinary genus of the knot. This leads to the following definition.

Definition 2.16 (Slice Genus). Let $K \subset S^3$ be a knot. The *slice* or 4-*ball genus* $g_4(K)$ of K is defined as the minimal genus among all smoothly and properly embedded orientable surfaces in B^4 that bound the knot K.

The slice genus is sometimes also simply called 4-genus. By pushing a Seifert surface of a knot K into the 4-ball B^4 while keeping its boundary fixed in S^3, we see that

$$g_4(K) \leq g(K).$$

Since a slice knot bounds a properly and smoothly embedded disk in B^4, we see that for slice knots $g_4 = 0$. On the other hand, non-trivial slice knots (such as the square knot) provide examples of $0 = g_4 < g$. The trefoil and the figure eight knot are examples of non-slice genus one knots, so for each of these we have $g_4 = g = 1$. In Section 7, we will encounter a technique that can be used to improve the bound $g_4(K) \leq g(K)$ in certain cases. Another convenient bound for the 4-genus is given by the following theorem.

Theorem 2.17. If a knot K can be unknotted by changing n crossings, then

$$g_4(K) \leq n.$$

A proof can be found in [36]. Theorem 2.17 implies in particular that if $u(K)$ is the *unknotting number*, i.e. the minimal number of changings of crossings needed to unknot K, then $g_4(K) \leq u(K)$. Another bound was given by Murasugi [41] in terms of the signature of a knot:

$$g_4(K) \geq \frac{1}{2}|\sigma(K)|.$$

This bound generalizes to $2g_4(K) \geq |\sigma_\omega(K)|$ provided that $\Delta_K(\omega) \neq 0$, where $\sigma_\omega(K)$ denotes the Tristram-Levine signature of K and Δ_K is the Alexander polynomial of K (cf. Sections 1.5 and 1.6). If $J_1 \# J_2$ is the connected sum of two knots J_1 and J_2, then we have the trivial bound

$$g_4(J_1 \# J_2) \leq g_4(J_1) + g_4(J_2).$$

In particular, g_4 defines a sub-additive function on the knot concordance group $\mathcal{C}$.

In general, it is difficult to obtain lower bounds for the slice genus g_4. In Section 3.4, we will encounter a theorem based on Casson-Gordon invariants that can be used to obtain lower bounds in certain cases, see for instance [19]. Other than that, we have Taylor's lower bound [50] and bounds obtained from L^2-signatures, Khovanov-Rozansky homology, knot Floer homology and gauge theory, see for instance [9, 12, 26, 43, 44]. Note that Casson-Gordon invariants, Taylor's method and L^2-signatures are the only tools that also provide bounds in the topological setting.

2.4 Stable 4-Genus

In 2010, Livingston [35] defined a variation of the 4-genus defined as follows.

Definition 2.18 (Stable 4-Genus). Let $K \subset S^3$ be a knot. Then the *stable* 4*-genus* $g_{st}(K)$ of K is defined as

$$g_{st}(K) = \lim_{n\to\infty} \frac{g_4(nK)}{n},$$

where nK denotes the n-fold connected sum $K\#\cdots\#K$.

The existence of the limit follows from the subadditivity of g_4. As an example, the stable 4-genus of the figure-eight is 0 (which is a consequence of the figure-eight being amphicheiral), and the stable 4-genus of the trefoil is 1 (cf. [35]).

It is possible to show that g_{st} induces a semi-norm on the rationalized knot concordance group $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes \mathbb{Q}$. The definition of the stable 4-genus is in analogy with the *stable commutator length* from group theory, which can be used to study genus problems of various kinds in topology and geometry (e.g. find a surface of least genus with a particular property in some topological space). An introduction to the stable commutator length and related topics can be found in [4].

There are many open questions around the stable 4-genus g_{st}. For example, it is known that g_{st} is not always an integer, but it is unknown whether $g_{st}(K) \in \mathbb{Q}$ for all knots K or not. Maybe of more interest is the question whether or not g_{st} is a norm on $\mathcal{C}_{\mathbb{Q}}$. That is, if $g_{st}(K) = 0$, does K represent torsion in $\mathcal{C}$? In the same spirit is the question whether or not there is a knot K such that $0 < g_{st}(K) < \frac{1}{2}$, which is closely related to the question whether there is torsion of order greater than 2 in $\mathcal{C}$ (see [35] for more details).

In Section 7, we will provide an upper bound for the stable 4-genus of the twist knots K_n.

3 Casson-Gordon Invariants

In the late 1970's, Casson and Gordon [5, 6] have defined certain invariants of knots, known as $\sigma(K,\chi)$ and $\tau(K,\chi)$, which led to new obstructions for a knot being slice. In particular, they used these obstructions to show that the only slice knots among the twist knots K_n are the unknot K_0 and the Stevedore knot K_2. Roughly speaking, the invariant $\sigma(K,\chi)$ is the difference between a twisted and untwisted signature of the intersection form of a certain 4-manifold W associated to the n-fold cyclic branched covering L_n and a character $\chi : H_1(L_n) \to \mathbb{Z}_m$, which yields an obstruction for ribbon knots (Theorem 1 in [5]). To extend this obstruction to slice knots (Theorem 2 in [5]), Casson and Gordon introduce another invariant $\tau(K,\chi)$, which is an element in the rationalized Witt group $W(\mathbb{C}(t)) \otimes \mathbb{Q}$ and takes, next to finite cyclic branched covers, also infinite cyclic covers into account. The relation between the invariants $\sigma(K,\chi)$ and $\tau(K,\chi)$ is given in Theorem 3 in [5] (also see Section 3.3 below). Casson and Gordon's result about the sliceness of the twist knots then follows after a careful analysis of the invariant $\sigma(K,\chi)$ for 2-bridge knots (cf. Section 4) and applying the obstruction to sliceness obtained from $\sigma(K,\chi)$ and $\tau(K,\chi)$.

For our analysis of the slice genus of the twist knots, we will only be interested in the invariant $\tau(K,\chi)$, and in particular in Gilmer's description of $\tau(K,\chi)$ in terms of curves lying on a Seifert surface for the knot. However, for the sake of completeness, we will also include the definition of $\sigma(K,\chi)$, and state the main results of [5]. But first, we will give a short overview of the objects that arise in the definition of $\sigma(K,\chi)$ and $\tau(K,\chi)$, respectively.

3.1 Preliminaries

Dehn Surgery on Knots

Let $K \subset S^3$ be a knot and X_K the knot exterior. Further, let $r = \frac{p}{q}$ be a rational number with p and q coprime. Recall that ∂X_K is a torus $S^1 \times S^1$ with longitude-meridian pair (λ, μ). By gluing a solid torus to the boundary of X_K, we obtain a closed 3-manifold (which, of course, depends on the gluing map). This yields the following definition.

Definition 3.1 (Dehn Surgery on Knots). The *Dehn surgery* along K with *framing* $r = \frac{p}{q}$ consists of the closed 3-manifold $S^3_r(K)$ obtained by gluing a solid torus $S^1 \times D^2$ to the boundary of the knot exterior ∂X_K, identifying the meridian of $S^1 \times D^2$ with $p\mu + q\lambda$. In symbols,

$$S^3_r(K) := X_K \cup_r S^1 \times D^2.$$

Since the diffeomorphism type of $S^3_r(K)$ is entirely specified by the image of the meridian of the solid torus $S^1 \times D^2$ (cf. [45]), the result of Dehn surgery is well-defined (up to diffeomorphism). A remarkable result is that every closed, orientable 3-manifold can be obtained by performing surgery on a framed link in S^3, as shown by Lickorish and Wallace in the 1960's [32, 56]. An important class of spaces is formed by performing

$\frac{p}{q}$-surgery on the unknot U. Such a space is called *lens space*, and is denoted by $L(p,q)$. Lens spaces were introduced by Heinrich Tietze in 1908 [52] and are the only known examples of closed 3-manifolds that are not entirely determined by their fundamental group and homology. We will remark in Section 4 that the 2-fold branched cover of the twist knots is a lens space.

For the definition of the invariant τ, we will be interested in the 0-surgery $S^3_{0/1}(K)$, i.e. the closed 3-manifold that is obtained by gluing a solid torus to the knot exterior X_K, where the meridian of the solid torus is identified with the longitude of ∂X_K. To shorten notation, let M denote $S^3_{0/1}(K)$. It can be shown (cf. Lemma 2.18 in [7]) that the first homology group of M is infinite cyclic, that is, $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, generated by a meridian of a tubular neighborhood of K in M. Thus, similar to the construction of the finite cyclic coverings of X_K, we can consider the composition

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \cong \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

in order to obtain an n-fold cyclic covering $M_n \to M$. A key observation that will be needed later is the following.

Lemma 3.2. Let $K \subset S^3$ be a knot with knot exterior X_K, and let $M = S^3_{0/1}(K)$ denote the 0-surgery performed along K. Further, let L_n be the n-fold cyclic branched covering of S^3 and let M_n be the n-fold cylic covering of M. Then:

1.) $H_1(M_n;\mathbb{Z}) \cong H_1(X_K;\mathbb{Z})$;

2.) $H_1(X_K;\mathbb{Z}) \cong H_1(L_n;\mathbb{Z}) \oplus \mathbb{Z}$.

The proof is a standard application of the Mayer-Vietoris exact sequence and can be found for example in [7]. We will use Lemma 3.2 to define a certain character on $H_1(M_n;\mathbb{Z})$.

Twisted Homology and the Intersection Form

The aim of this paragraph is to give the definition of homology with twisted coefficients and see how it compares to the ordinary homology of a space X. Along we will review the intersection form on 4-manifolds and the corresponding twisted version. Our discussion will be based on [7]. A more detailed treatment can be found in [10, 16].

Fix a (not necessarily commutative) ring R with involution ν. Given a right R-module M, let $\overline{M}$ denote the left R-module obtained by turning M into a left R-module via the standard procedure $r \cdot m := m \cdot \nu(r)$.

Let X be a topological space that admits a universal covering $p : \widetilde{X} \to X$. Further, let $Y \subset X$ be a (possibly empty) subspace and set $\widetilde{Y} := p^{-1}(Y)$. Let $C_*(\widetilde{X}, \widetilde{Y})$ denote the singular chain complex of the pair $(\widetilde{X}, \widetilde{Y})$. Then, the usual right action of $\pi_1(X)$ on $\widetilde{X}$ endows $C_*(\widetilde{X}, \widetilde{Y})$ with the structure of a right $\mathbb{Z}[\pi_1(X)]$-module, and the choice of a homomorphism $\varphi : \pi_1(X) \to R$ endows R with a $(\mathbb{Z}[\pi_1(X)], R)$-bimodule structure. Tensoring now $C_*(\widetilde{X}, \widetilde{Y})$ with R over $\mathbb{Z}[\pi_1(X)]$ yields a new chain complex whose homology is the twisted homology of the pair (X, Y).

Definition 3.3 (Homology with Twisted Coefficients). The *twisted homology* of the pair (X, Y) with coefficients in R is defined as

$$H_*(X, Y; R) = H_*(C_*(\widetilde{X}, \widetilde{Y}) \otimes_{\mathbb{Z}[\pi_1(X)]} R).$$

Similarly, the *twisted cohomology* of the pair (X, Y) with coefficients in R is defined as

$$H^*(X, Y; R) = H^*(\mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(\overline{C_*(\widetilde{X}, \widetilde{Y})}, R)).$$

Homology with twisted coefficients is also known as *homology with local coefficients* and was first introduced by Norman E. Steenrod [49] in 1963 as a generalization of ordinary homology theories. In some sense, homology with twisted coefficients captures the homology of all regular coverings of X. Indeed, suppose that $\varphi : \pi_1(X) \to G$ is an epimorphism to some abelian group G, and let $\widehat{X}$ denote the covering of X corresponding to $\ker(\varphi)$. Note that φ equips $\mathbb{Z}[G]$ with a right $\mathbb{Z}[\pi_1(X)]$-module structure. Then it can be shown that the covering map $\widetilde{X} \to \widehat{X}$ induces a chain isomorphism

$$C_*(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[G] \cong C_*(\widehat{X}).$$

Consequently, $H_*(X; \mathbb{Z}[G])$ and $H_*(X; \mathbb{Z})$ are canonically isomorphic. Note that the cases $G = \{1\}$ and $G = \mathbb{Z}[\pi_1(X)]$ yield the usual singular homology of X and $\widetilde{X}$, respectively.

We are going to recall the definition of the intersection form of a compact, orientable 4-manifold W next. Consider the composition

$$\Phi_{\mathbb{Z}} : H_2(W; \mathbb{Z}) \xrightarrow{i_*} H_2(W, \partial W; \mathbb{Z}) \xrightarrow{\mathrm{PD}} H^2(W; \mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_2(W; \mathbb{Z}), \mathbb{Z}),$$

where i_* denotes the map induced by the inclusion $i : (W, \emptyset) \to (W, \partial W)$, PD denotes the Poincaré duality isomorphism and ev denotes the evaluation map. Set $\lambda_{\mathbb{Z}}(x, y) := \Phi_{\mathbb{Z}}(x)(y)$. It turns out that $\lambda_{\mathbb{Z}}$ defines a symmetric bilinear form on W (cf. [24]).

Definition 3.4 (Intersection Form). The symmetric bilinear pairing

$$\lambda_{\mathbb{Z}} : H_2(W; \mathbb{Z}) \times H_2(W; \mathbb{Z}) \to \mathbb{Z}, \quad \lambda_{\mathbb{Z}}(x, y) = \Phi_{\mathbb{Z}}(x)(y)$$

is called *intersection form* of W. The signature of $\lambda_{\mathbb{Z}}$ is called the *signature* of W and is denoted by $\mathrm{sign}_{\mathbb{Z}}(W)$.

Note that the intersection pairing need not be non-degenerate. Indeed, the evaluation map is in general not injective and furthermore, the exact sequence of the pair $(W, \partial W)$ shows that $\lambda_{\mathbb{Z}}$ vanishes on $\mathrm{im}(H_2(\partial W; \mathbb{Z}) \to H_2(W; \mathbb{Z}))$. However, the definition of the intersection form can be adapted to rational coefficients. This yields a pairing $\lambda_{\mathbb{Q}}$ on $H_2(W; \mathbb{Q})$ whose signature coincides with the signature of $\lambda_{\mathbb{Z}}$. This means in particular that if $H_2(\partial W; \mathbb{Q}) = 0$, then $\lambda_{\mathbb{Q}}$ is non-degenerate (and in fact non-singular since $\mathbb{Q}$ is a field). There are also other, more geometric interpretations of the intersection form, see for example [2] or [24].

There is an analogue of the intersection form for homology with twisted coefficients, defined in the exact same way (cf. [7]): consider the composition

$$\Phi_R : H_2(W; R) \xrightarrow{i_*} H_2(W, \partial W; R) \xrightarrow{\mathrm{PD}} H^2(W; R) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_R(\overline{H_2(W; R)}, R),$$

where i_* denotes the map induced by the inclusion inclusion $i : (W, \emptyset) \to (W, \partial W)$, PD denotes the Poincaré duality isomorphism and ev denotes the evaluation map. As above, the map Φ yields a well-defined symmetric bilinear form λ_R on W.

Definition 3.5 (Twisted Intersection Form). The symmetric bilinear pairing

$$\lambda_R : H_2(W; R) \times H_2(W; R) \to R, \quad \lambda_R(x, y) = \Phi_R(x)(y)$$

is called *twisted intersection form* of W. The signature of $\lambda_{\mathbb{R}}$ is called the *twisted signature* of W and is denoted by $\operatorname{sign}_R(W)$.

Note that for $R = \mathbb{Z}$, one recovers the definition of the ordinary intersection form $\lambda_{\mathbb{Z}}$. The twisted and untwisted intersection form and their respective signatures are the powerful tools that lead to the Casson-Gordon invariants.

Linking Forms

There is an analogue to the intersection form of a 4-manifold in the 3-dimensional case. Let M be a compact, orientable, 3-dimensional manifold, and let $\operatorname{Tor} H_1(M; \mathbb{Z})$ be the torsion part of $H_1(M; \mathbb{Z})$, that is,

$$\operatorname{Tor} H_1(M; \mathbb{Z}) := \{x \in H_1(M; \mathbb{Z}) : nx = 0 \text{ for some } n \in \mathbb{N}\}.$$

Then there exists a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form ϕ on $\operatorname{Tor} H_1(M; \mathbb{Z})$, defined as follows. Given $[x], [y] \in \operatorname{Tor} H_1(M; \mathbb{Z})$, let $x, y \in C_1(M)$ be cycles representing $[x]$ and $[y]$, respectively. Further, let $w \in C_2(M)$ be such that $\partial w = ny$ for some $n \in \mathbb{N}$, and suppose that w is transverse to x. Define

$$\phi : \operatorname{Tor} H_1(M; \mathbb{Z}) \times \operatorname{Tor} H_1(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}, \quad \phi([x], [y]) = \frac{\#(x \cap w)}{n},$$

where $\#(x \cap w)$ denotes the number of intersection points counted with signs. It follows from classical results in algebraic topology that ϕ is well-defined and bilinear (see for example [2] or [24]). We have the following definition.

Definition 3.6 (Linking Form). Let M be a compact, orientable, 3-dimensional manifold. The $\mathbb{Q}/\mathbb{Z}$-valued bilinear form ϕ defined above is called *(geometric) linking form* of M. ϕ is called *non-singular* if the correlation map $c : \operatorname{Tor} H_1(M; \mathbb{Z}) \to \operatorname{Hom}(\operatorname{Tor} H_1(M; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ defined by $c_x(\cdot) = \phi(x, \cdot)$ is an isomorphism.

If M is a rational homology 3-sphere, then one can show that L is symmetric and non-singular (cf. [8]). We will be mainly interested in the case where M is the double branched cover L_2 of some knot $K \subset S^3$. Since L_2 is a rational homology 3-sphere (cf. Proposition 1.24), the linking form ϕ of the double branched cover is symmetric and non-singular.

It is also possible to define linking forms in purely algebraic terms (cf. [36]). Let A be a finite abelian group. An *(algebraic) linking form* on A is a bilinear symmetric map $\phi' : A \times A \to \mathbb{Q}/\mathbb{Z}$ that is non-singular. As above, non-singular means that the correlation map $c' : A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ defined by $c'_x(\cdot) = \phi'(x, \cdot)$ is an isomorphism. Note that it is not necessary to require ϕ' to be symmetric and non-singular, but it is customary to do so in the context of knot theory. Also, note that since $H_1(L_2; \mathbb{Z})$ is a finite torsion group (cf. Proposition 4.4), its geometric linking form ϕ is also an algebraic linking form ϕ' as defined above.

In the upcoming sections, we will need some further notions that are associated to linking forms. For convenience, we stay in the algebraic context. If ϕ' is an algebraically defined linking form, then there is a dual form $(\phi')^*$ on $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ defined by the formula $(\phi')^*(c_x, c_y) = \phi(x, y)$. Next, if H is a subgroup of A, let

$$H^\perp := \{x \in A : \phi'(x, y) = 0 \text{ for all } y \in H\}.$$

Similar to the notion of a metabolizer for the Seifert pairing (cf. Definition 2.7), there is a notion of metabolizer for linking forms.

Definition 3.7 (Metabolic Linking Form). Let ϕ' be an algebraically defined linking form. If there is a subgroup H such that $H = H^\perp$, then ϕ' is called *metabolic* and H is called a *metabolizer.*

In contrast to the rank of a metabolizer of the Seifert pairing, which is half the rank of the first homology of a Seifert surface, the order of a metabolizer H of a linking form ψ' satisfies $|H|^2 = |A|$. As we will see, this is a key property of metabolic linking forms that is used in many arguments.

The last notion that we are going to need is that of an even presentation of a linking form (cf. [19]). Let F be a free $\mathbb{Z}$-module of finite rank and $\langle \cdot, \cdot \rangle : F \times F \to \mathbb{Z}$ a non-degenerate symmetric bilinear form. Here, non-degenerate means that the correlation map $F \to F^* = \mathrm{Hom}(L, \mathbb{Z})$ is injective. By tensoring with $\mathbb{Q}$, we can extend $\langle \cdot, \cdot \rangle$ to a form

$$\langle \cdot, \cdot \rangle : F \otimes \mathbb{Q} \times F \otimes \mathbb{Q} \to \mathbb{Q}.$$

Define

$$F^\# := \{x \in F \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in F\}.$$

Identifying F as a subset of $F \otimes \mathbb{Q}$ in the usual way, we have $F \subset F^\#$, and since F is free of finite rank, $F^\#/F$ is a finite abelian group. It is now possible to define a linking form ϕ' on $F^\#/F$ by the formula

$$\phi'(xF, yF) = \langle x, y \rangle \bmod \mathbb{Z}.$$

$\langle \cdot, \cdot \rangle$ is called a *presentation* of ϕ'. If $\langle x, x \rangle$ is even for all $x \in F$, then the presentation is called *even.* It is possible to show that every algebraically defined linking form has an even presentation (cf. [55]).

Witt Groups

We have mentioned earlier that the invariant $\tau(K, \chi)$ is an element of the rationalized Witt group $W(\mathbb{C}(t)) \otimes \mathbb{Q}$, where $\mathbb{C}(t)$ denotes the field of complex rational functions in one variable t. The aim of this paragraph is to briefly recall the definition of Witt groups $W(R)$ and describe, in the case that $R = \mathbb{C}(t)$, a certain averaged signature function $\operatorname{sign}_\omega^{\mathrm{av}} : W(\mathbb{C}(t) \to \mathbb{Q}$. Our discussion follows [7] and [36]. A more detailed account to Witt groups can be found in [38].

As above, let R be a ring with involution ν. To simplify our algebra, we now assume that R is commutative and, furthermore, a principal ideal domain. Given a free R-module M of finite rank, a map $\beta : M \times M \to R$ is called *sesquiliniear* if β is linear in the first and anti-linear in the second variable, that is, $\beta(x, ry) = \nu(r)\beta(x, y)$. Further, β is called *Hermitian* if β is sesquilinear and $\beta(x, y) = \nu(\beta(y, x))$. Finally, β is called *non-degenerate* if the *adjoint map* $M \to \overline{\operatorname{Hom}_R(M, R)}$ given by $x \mapsto \beta(x, \cdot)$ is injective, and *non-singular* if the adjoint map is an (anti-linear) isomorphism. A *Hermitian form* is a pair (M, β), where M is a free R-module of finite rank and β is a non-singular Hermitian pairing.

Let us turn our attention now to the definition of Witt equivalence. Given two Hermitian forms (M_1, β_1) and (M_2, β_2), their *direct sum* $(M_1, \beta_1) \oplus (M_2, \beta_2) = (M_1 \oplus M_2, \beta_1 \oplus \beta_2)$ is formed in the obvious way. A Hermitian form (M, β) is called *metabolic* if there exists a direct summand submodule $P \subset M$ whose orthogonal complement $P^\perp$ is equal to P, i.e.

$$P^\perp = \{x \in M : \beta(x, y) = 0 \text{ for all } y \in P\} = P.$$

Last but not least, two Hermitian forms (M_1, β_1) and (M_2, β_2) are called *Witt equivalent* if the direct sum $(M_1, \beta_1) \oplus (M_2, -\beta_2)$ is metabolic. We have the following theorem:

Theorem 3.8. Let R be a commutative principal ideal domain with involution $\nu : R \to R$. Then the following statements hold:

1.) Witt equivalence is an equivalence relation on the set of isomorphism classes of free finite rank Hermitian forms over R, and the set of equivalence classes forms an abelian group $W(R)$, the *Witt group* of R.

2.) A Hermitian form represents zero in the Witt group if and only if it is metabolic.

3.) The additive inverse of an element $[(M, \beta)] \in W(R)$ is given by $[(M, -\beta)] \in W(R)$.

A proof can be found in [7]. Of course, the group operation on $W(R)$ is given by direct sum. As an example, given $R = \mathbb{R}$ with the trivial involution and $R = \mathbb{C}$ with the involution given by complex conjugation, one can show that the ordinary signature induces isomorphisms $W(\mathbb{R}) \cong \mathbb{Z}$ and $W(\mathbb{C}) \cong \mathbb{Z}$. As mentioned above, we will be interested in the Witt group $W(\mathbb{C}(t))$ (note that the involution on $\mathbb{C}(t)$ is given by conjugating complex numbers and sending t to t^{-1}). Unfortunately, $W(\mathbb{C}(t))$ is rather difficult to describe; it is isomorphic to an infinite direct sum of copies of $\mathbb{Z}$ (cf. [33]). However, the exact description of $W(\mathbb{C}(t))$ is not needed for the definition of τ, so we do not go into further details at this point.

Let us describe now a signature function $\operatorname{sign}_\omega^{\mathrm{av}} : W(\mathbb{C}(t)) \to \mathbb{Q}$, where $\omega \in S^1$, which will relate the invariants $\sigma(K, \chi)$ and $\tau(K, \chi)$ (cf. Theorem

3.14). Given a Hermitian matrix $A(t)$ over $\mathbb{C}(t)$, note that for some fixed $\omega \in S^1$, the signature $\operatorname{sign}(A(\omega))$ is constant in a neighborhood of ω unless $\det(A(\omega)) = 0$ or some entry of $A(\omega)$ is infinite (compare to the Tristram-Levine signature σ_ω in Section 1.6). Thus, $\omega \mapsto \operatorname{sign}(A(\omega))$ is a piecewise-constant function with finitely many discontinuities. At each discontinuitiy ω, we redefine $\operatorname{sign}(A(\omega))$ to be the *averaged signature*

$$\operatorname{sign}^{\mathrm{av}}(A(\omega)) := \lim_{\omega^+,\omega^- \to \omega} \left(\frac{\operatorname{sign}(A(\omega^+)) + \operatorname{sign}(A(\omega^-))}{2} \right) \in \mathbb{Q},$$

where $\omega^+, \omega^- \in S^1$ are unit complex numbers with arguments above and below that of ω. One can check that $\operatorname{sign}^{\mathrm{av}}(A(\omega))$ is well-defined for all $\omega \in S^1$. This leads to the following definition.

Definition 3.9 (Averaged Signature Function). Let $\omega \in S^1$ be fixed. Then the *averaged signature function* $\operatorname{sign}^{\mathrm{av}}_\omega$ is defined as

$$\operatorname{sign}^{\mathrm{av}}_\omega : W(\mathbb{C}(t)) \to \mathbb{Q}, \quad A(t) \mapsto \operatorname{sign}^{\mathrm{av}}(A(\omega)).$$

It is possible to show that $\operatorname{sign}^{\mathrm{av}}_\omega$ is a well-defined group homomorphism (cf. [5]). Note that every element $[(M, \beta)] \in W(\mathbb{C}(t))$ can be represented by a matrix over $\mathbb{C}(t)$ after choosing a basis for M.

3.2 The Invariant $\sigma(K, \chi)$

In this section we are going to define the Casson-Gordon invariant $\sigma(K, \chi)$ following [5] and [7]. Throughout this section $\mathbb{Z}_n$ will denote the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Let M be a closed 3-manifold and $\chi : H_1(M; \mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism. It follows from bordism theory that there exists a 4-manifold W and a character $\psi : H_1(W; \mathbb{Z}) \to \mathbb{Z}_m$ such that ∂W consists of $r > 0$ copies of M, and ψ agrees with χ on each copy of M. For short, we will write $\partial(W, \psi) = r(M, \chi)$ in this case. Precomposing ψ with the abelianization map yields a surjection $\pi_1(W) \to \mathbb{Z}_m$. Let $W_m \to W$ denote the corresponding m-fold covering.

Since the group of deck transformations of W_m is isomorphic to $\mathbb{Z}_m$, we can regard $H_2(W_m; \mathbb{Z})$ as a $\mathbb{Z}[\mathbb{Z}_m]$-module. By mapping the generator of $\mathbb{Z}_m$ to $\omega = e^{\frac{2\pi i}{m}}$, we obtain a map $\mathbb{Z}[\mathbb{Z}_m] \to \mathbb{Q}(\omega)$ which endows $\mathbb{Q}(\omega)$ with a $(\mathbb{Z}[\mathbb{Z}_m], \mathbb{Q}(\omega))$-bimodule structure. Define

$$H_*(W; \mathbb{Q}(\omega)) := H_*(W_m; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{Q}(\omega).$$

The notation $H_*(W; \mathbb{Q}(\omega))$ is justified since $H_*(W_m; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \mathbb{Q}(\omega)$ is isomorphic to the homology of the twisted chain complex $C_*(\widetilde{W}) \otimes_{\mathbb{Z}[\pi_1(W)]} \mathbb{Q}(\omega)$, where $\widetilde{W}$ denotes the universal cover of W (cf. [7]). Moreover, this isomorphism induces a (twisted) intersection form $\lambda_{\mathbb{Q}(\omega)}$ on $H_*(W; \mathbb{Q}(\omega))$ whose signature will be denoted by $\operatorname{sign}^\psi(W)$. Similarly, denote the signature of the standard (untwisted) intersection for $\lambda_{\mathbb{Z}}$ on $H_2(W; \mathbb{Z})$ by $\operatorname{sign}(W)$. The main definition of this section is the following.

Definition 3.10 (Casson-Gordon Invariant $\sigma(K,\chi)$). Let M be a closed 3-manifold and $\chi : H_1(M;\mathbb{Z}) \to \mathbb{Z}_m$, $m \in \mathbb{N}$, an epimorphism. Suppose that that $\partial(W,\psi) = r(M,\chi)$ for some 4-manifold, character pair (W,ψ) and $r \in \mathbb{N}_{>0}$. The *Casson-Gordon invariant* $\sigma(M,\chi)$ is defined as

$$\sigma(M,\chi) := \frac{1}{r}(\mathrm{sign}^{\psi}(W) - \mathrm{sign}(W)) \in \mathbb{Q}.$$

In the special case that M is the double branched covering L_2 of a knot $K \subset S^3$, the invariant $\sigma(K,\chi)$ is defined as

$$\sigma(K,\chi) := \sigma(L_2,\chi).$$

One can check [7] that $\sigma(M,\chi)$ (and consequently $\sigma(K,\chi)$) is well-defined and does not depend on r, W, or ψ. The following theorem is the main result about $\sigma(K,\chi)$ (Theorem 1 in [5]).

Theorem 3.11. If $K \subset S^3$ is a ribbon knot whose double branched covering L_2 is a lens space, then:

1.) $|H_1(L_2;\mathbb{Z})|$ is a square, say k^2;

2.) if $\chi : H_1(L_2;\mathbb{Z}) \to \mathbb{Z}_m$ is a non-constant character whose order m divides k^2, then $|\sigma(K,\chi)| = 1$.

The proof (cf. [5]) reveals that one can take as W the 2-fold cover of B^4 branched along a pushed-in ribbon disk D for K (note that $\partial W = L_2$), and relies further on the fact that $\pi_1(L_2)$ is finite cyclic and that D has no index 2 critical points (cf. Section 2.1 or Lemma 1 in [5]). Thus, the proof of the theorem above can not be directly applied to arbitrary slice knots, which is why Casson and Gordon introduce a second invariant, $\tau(K,\chi)$.

3.3 The Invariant $\tau(K,\chi)$

In this section we are going to define the invariant $\tau(K,\chi)$ following [7] and [5]. As before, throughout this section $\mathbb{Z}_n$ will denote the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Let $K \subset S^3$ be a knot, and let M be the closed 3-manifold obtained by performing 0-framed surgery on the knot exterior X_K, i.e. $M = S^3_{0/1}(K)$. We have remarked in Section 3.1 that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, giving rise to the composition

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \cong \mathbb{Z} \to \mathbb{Z}_n$$

whose kernel corresponds to an n-fold cyclic covering $M_n \to M$. Now, let L_n denote the n-fold cyclic cover of S^3 branched along K, and suppose we are given a character $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{Z}_m$ of prime-power order $m = p^k$, p prime. Our aim is now to define an epimorphism $\varphi : \pi_1(M_n) \to \mathbb{Z}_m \times \mathbb{Z}$. This is done as follows:

- The covering map $p : M_n \to M$ gives rise to the composition

  $$\pi_1(M_n) \stackrel{p_*}{\to} \pi_1(M) \to H_1(M;\mathbb{Z}) \cong \mathbb{Z}$$

  with image isomorphic to $n\mathbb{Z}$. This allows us to define a surjection $\alpha : \pi_1(M_n) \to \mathbb{Z}$.

- Recall from Lemma 3.2 that $H_1(M_n;\mathbb{Z}) \cong H_1(L_n;\mathbb{Z}) \oplus \mathbb{Z}$. Using projection to the first factor and the character $\chi : H_1(L_n) \to \mathbb{Z}_m$, we get again a composition

$$\pi_1(M_n) \to H_1(M_n;\mathbb{Z}) \to H_1(L_n;\mathbb{Z}) \xrightarrow{\chi} \mathbb{Z}_m.$$

  Let $\widetilde{\chi} : \pi_1(M_n) \to \mathbb{Z}_m$ denote this composition.

Using α and $\widetilde{\chi}$, we can now define a homomorphism

$$\varphi : \pi_1(M_n) \to \mathbb{Z}_m \times \mathbb{Z}, \quad x \mapsto (\alpha(x), \widetilde{\chi}(x)),$$

which is surjective [5].

Now, similarly to the definition of $\sigma(K,\chi)$, bordism theory implies that there exists an $r > 0$, a 4-manifold V_n, and a map $\psi : \pi_1(V_n) \to \mathbb{Z} \times \mathbb{Z}_m$ such that $\partial(V_n, \psi) = r(M_n, \varphi)$. Let $\widehat{V}_n \to V_n$ denote the $\mathbb{Z}_m \times \mathbb{Z}$-cover associated to ψ. Next, equip $\mathbb{C}(t)$ with the $(\mathbb{Z}[\mathbb{Z}_m \times \mathbb{Z}], \mathbb{C}(t))$-bimodule structure that arises from the composition

$$\mathbb{Z}[\mathbb{Z}_m \times \mathbb{Z}] \to \mathbb{Q}[\mathbb{Z}_m \times \mathbb{Z}] \to \mathbb{C}(t),$$

where the last map sends the generator of $\mathbb{Z}_m$ to $\omega = e^{\frac{2\pi i}{m}}$ and the generator of $\mathbb{Z}$ to t. Define

$$H_*(V_n;\mathbb{C}(t)) = H_*(\widehat{V}_n;\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}_m \times \mathbb{Z}]} \mathbb{C}(t).$$

As before, the notation $H_*(V_n;\mathbb{C}(t))$ is justified since $H_*(\widehat{V}_n;\mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}_m \times \mathbb{Z}]} \mathbb{C}(t)$ is isomorphic to the homology of the twisted chain complex $C_*(\widetilde{V}_n) \otimes_{\mathbb{Z}[\pi_1(V_n)]} \mathbb{C}(t)$, where $\widetilde{V}_n$ denotes the universal cover of V_n. Moreover, this isomorphism induces a (twisted) intersection form $\lambda_{\mathbb{C}(t)}$ on $H_*(V_n;\mathbb{C}(t))$. Since m is by assumption a primer-power, the form $\lambda_{\mathbb{C}(t)}$ is non-singular and defines therefore an element $[\lambda_{\mathbb{C}(t)}]$ in the Witt group $W(\mathbb{C}(t))$ (cf. the corollary to Lemma 4 in [5]).

We wish now to compare $[\lambda_{\mathbb{C}(t)}]$ with an element in $W(\mathbb{C}(t))$ that arises from the standard intersection form $\lambda_{\mathbb{Z}}$ on $H_2(V_n;\mathbb{Z})$. As mentioned in Section 3.1, the signature of $\lambda_{\mathbb{Z}}$ agrees with the signature of the intersection form $\lambda_{\mathbb{Q}}$ on $H_2(V_n;\mathbb{Q})$. However, in both cases, the forms might be singular. So instead, we consider the non-singular form $\lambda_{\mathbb{Q}}^{\text{nonsing}}$ induced by $\lambda_{\mathbb{Q}}$ on $H_2(V_n;\mathbb{Q})/\text{im}(H_2(\partial V_n;\mathbb{Q}) \to H_2(V_n;\mathbb{Q}))$, which defines an element $[\lambda_{\mathbb{Q}}^{\text{nonsing}}]$ in the Witt group $W(\mathbb{Q})$. Mapping $[\lambda_{\mathbb{Q}}^{\text{nonsing}}]$ to $W(\mathbb{C}(t))$ via the homomorphism $i : W(\mathbb{Q}) \to W(\mathbb{C}(t))$ induced by the canonical inclusion $\mathbb{Q} \to \mathbb{C}(t)$ then defines an element $i([\lambda_{\mathbb{Q}}^{\text{nonsing}}]) \in W(\mathbb{C}(t))$. We have the following definition.

Definition 3.12 (Casson-Gordon Invariant $\tau(K,\chi)$). Let $K \subset S^3$ be a knot, $n \in \mathbb{N}_{>0}$ a positive integer, and let $M_n \to M$ be the n-fold cyclic cover of M, where M is the result of 0-framed surgery along K. Further, let L_n be the n-fold cyclic cover of S^3 branched along K, and let $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{Z}_m$ be a character of primer-power order $m = p^k$. Finally, let $\varphi : \pi_1(M_n) \to \mathbb{Z}_m \times \mathbb{Z}$ be the epimorphism as defined above, and suppose that $\partial(V_n,\psi) = r(M_n,\varphi)$ for some 4-manifold, homomorphism pair (V_n,ψ) and $r \in \mathbb{N}_{>0}$. The *Casson-Gordon invariant* $\tau(K,\chi)$ is defined as the Witt class

$$\tau(K,\chi) := ([\lambda_{\mathbb{C}(t)}] - i([\lambda_{\mathbb{Q}}^{\text{nonsing}}])) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes \mathbb{Q}.$$

As for the invariant $\sigma(K,\chi)$, one can check [7] that $\tau(K,\chi)$ is well-defined and does not depend on r, V_n, or ψ. Note that tensoring with $\mathbb{Q}$ is necessary in order to obtain a well-defined knot invariant (cf. [5]). The following theorem is the main result about $\tau(K,\chi)$, providing the already mentioned slicing obstruction (Theorem 2 in [5]).

Theorem 3.13. Let $K \subset S^3$ be a knot with n-fold cyclic branched covering L_n for some prime-power n. If K is slice, then there exists a subgroup $G \subset H_1(L_n;\mathbb{Z})$ such that

1.) $|G|^2 = |H_1(L_n;\mathbb{Z})|$;

2.) $\phi(G \times G) = 0$, where ϕ is the geometric linking form of L_n;

3.) if $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{Z}_m$ is a character of prime-power order such that $\chi(G) = 0$, then
$$\tau(K,\chi) = 0.$$

The proof can be found in [5]. Note that the first and second point of Theorem 3.13 say that the linking form ϕ of L_n is metabolic with metabolizer G.

In general, the invariant $\tau(K,\chi)$ is difficult to compute. Fortunately, it is possible to compare $\tau(K,\chi)$ to $\sigma(K,\chi)$ using the averaged signature function $\mathrm{sign}^{\mathrm{av}}_\omega$, which turns Theorem 3.11 into an obstruction for slice knots whose double branched cover is a lens space. Recall from Definition 3.9 that the averaged signature function is defined as

$$\mathrm{sign}^{\mathrm{av}}_\omega : W(\mathbb{C}(t)) \to \mathbb{Q}, \quad A(t) \mapsto \mathrm{sign}^{\mathrm{av}}(A(\omega)),$$

where

$$\mathrm{sign}^{\mathrm{av}}(A(\omega)) := \lim_{\omega^+,\omega^- \to \omega} \left(\frac{\mathrm{sign}(A(\omega^+)) + \mathrm{sign}(A(\omega^-))}{2} \right) \in \mathbb{Q}.$$

We have the following theorem by Casson and Gordon (Theorem 3 in [5]).

Theorem 3.14. Let $K \subset S^3$ be a knot with n-fold cyclic branched covering L_n. Further, let $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{Z}_m$ be a character of prime-power order, inducing an m-fold covering $\widetilde{L}_n \to L_n$. If $H_1(\widetilde{L}_n;\mathbb{Q}) = 0$, then

$$|\sigma(K,\chi) - \mathrm{sign}^{\mathrm{av}}_1(\tau(K,\chi))| \leq 1.$$

A proof can be found in [5]. The following corollary now provides the aforementioned slicing obstruction for knots whose double branched cover is a lens space (compare Theorem 3.11)

Corollary 3.15. Let $K \subset S^3$ be a knot whose double branched covering L_2 is a lens space. If K is slice, then

1.) $|H_1(L_2;\mathbb{Z})|$ is a square, say k^2;

2.) if $\chi : H_1(L_2;\mathbb{Z}) \to \mathbb{Z}_m$ is a non-constant character whose prime-power order m divides k^2, then $|\sigma(K,\chi)| \leq 1$.

The first point of the corollary follows from the first point in Theorem 3.13, while the second point follows from the fact that if L_2 is a lens space, then $H_1(L_2;\mathbb{Z}) = \mathbb{Z}_{k^2}$, so there is a unique subgroup $G \subset H_1(L_2;\mathbb{Z})$ of order k on which χ vanishes. Then Theorem 3.13 applies and shows that $\tau(K,\chi) = 0$, so by Theorem 3.14, $|\sigma(K,\chi)| \leq 1$ (for more details, see [7]).

These are the main results obtained by Casson and Gordon in the 1970's. The Casson-Gordon invariants have been ever since important objects in the study of slice knots and have found interpretations in many different contexts. See the appendix of [5] for further information on subsequent developments.

We conclude this section with a few remarks regarding $\tau(K,\chi)$ that will be needed in the next section.

Remark.

1.) In the definition of $\tau(K,\chi)$, the character χ maps into a finite cyclic group $\mathbb{Z}_m$. It is also possible to consider characters $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{C}^*$ or $\chi : H_1(L_n;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of order m instead, so that the corresponding images are isomorphic to a finite cyclic group. In particular, this allows us to consider χ as an element of $H^1(L_2;\mathbb{C}^*)$ or $H^1(L_2;\mathbb{Q}/\mathbb{Z})$, respectively. The latter case will be used in particular in the next section.

2.) Given $\chi \in H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(L_2;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$, one can consider the inverse character $-\chi$ and the corresponding Casson-Gordon invariant $\tau(K,-\chi)$. It is possible to show that $\tau(K,\chi) = \tau(K,-\chi)$, see for instance [18].

3.) Consider again the averaged signature function $\mathrm{sign}_1^{\mathrm{av}}$. Note that there is a natural map $W(\mathbb{R}) \to W(\mathbb{C}(t))$. We have mentioned in Section 3.1 that the ordinary signature gives an isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$. Together, this yields a map

$$\rho : \mathbb{Q} \to W(\mathbb{C}(t)) \otimes \mathbb{Q}.$$

It is not difficult to see that $\mathrm{sign}_1^{\mathrm{av}} \circ \rho$ is the identity, what in particular implies that ρ is injective. This fact will be needed in the next section about Gilmer's description of the invariant $\tau(K,\chi)$.

3.4 Gilmer's Results about $\tau(K,\chi)$

In [18], Patrick Gilmer gives a relation between the first cohomology $H^1(L_2;\mathbb{Q}/\mathbb{Z})$ of the double branched cover L_2 of a knot $K \subset S^3$ with certain curves lying on a Seifert surface for K, leading to an expression of the Casson-Gordon invariant $\tau(K,\chi)$ in terms of these curves. Moreover, in the case that the genus of K is one, he gives an explicit formula for the computation of $\tau(K,\chi)$ based on these curves.

In a second paper [19], Gilmer provides an extension of Casson and Gordon's slicing obstruction that we stated in Theorem 3.13 above. More precisely, Gilmer's result can be used to derive lower bounds for the slice genus g_4. In combination with Gilmer's results about $\tau(K,\chi)$, one gets powerful tools for the analysis of the slice genus g_4.

As we will see in Section 4, all twist knots have genus one. Therefore, we can use Gilmer's formula for $\tau(K,\chi)$ together with his obstruction to provide a simple proof of Casson and Gordon's theorem that the only slice knots among the twist knots are the unknot and the Stevedore's knot (cf. Section 5). For this, we are going to briefly review the most important points of Gilmer's papers [18] and [19] in the following section.

Let K be a knot in S^3 with Seifert surface $F \subset S^3$ and Seifert pairing $\theta : H_1(F) \times H_1(F) \to \mathbb{Z}$. Let L_2 denote the double branched cover of S^3, branched along K, and let $H^1(L_2;\mathbb{Q}/\mathbb{Z})$ be the first cohomology of L_2 with $\mathbb{Q}/\mathbb{Z}$-coefficients, i.e. the set of characters $\chi : H_1(L_2;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$. Define a map $\varepsilon : H_1(F) \to H^1(F)$ by

$$\varepsilon_x(y) = \theta(x,y) + \theta(y,x),$$

and let $N \subset H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$ be the kernel of $\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}$. Further, let $N' \subset N$ be the subset of elements of N with prime-power order. In Section 1 of [18], Gilmer shows that

$$H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong N,$$

and this isomorphism is natural up to sign. For $\chi \in H^1(L_2,\mathbb{Q}/\mathbb{Z})$, let $\tau(K,\chi)$ be the Casson-Gordon invariant as defined in Section 3.3. Because $\tau(K,\chi) = \tau(K,-\chi)$, we can view τ as defined on N.

In order to describe Gilmer's formula for $\tau(K,\chi)$ in the genus one case, assume that $x \in H_1(F)$ is primitive (i.e. not a multiple of any other element), and let C_x denote the collection of knots in S^3 obtained by representing x first by a simple closed curve γ on F, and then viewing γ in S^3. Note that if $g(F) = 1$, then C_x is a singleton whose element is denoted by J_x. Recall that $H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong N \subset H_1(F) \otimes \mathbb{Q}/\mathbb{Z}$. We have the following result [18]:

Theorem 3.16. If a knot K admits a genus one Seifert surface F, then for any $\chi = x \otimes \frac{s}{m} \in N$, where $x \in H_1(F)$ is primitive, m is a prime power, and $0 < s < m$, we have

$$\tau(K,\chi) = \rho\Big(\underbrace{2\sigma_{\frac{s}{m}}(J_x) + \frac{4(m-s)s}{m^2}\theta(x,x) - \sigma_{\frac{1}{2}}(K)}_{=:C(s)}\Big),$$

where $\rho : \mathbb{Q} \to W(\mathbb{C}(t)) \otimes \mathbb{Q}$ is the homomorphism induced by the ordinary signature $\sigma : W(\mathbb{R}) \to \mathbb{Z}$ and the inclusion $W(\mathbb{R}) \to W(\mathbb{C}(t))$.

As mentioned in Section 3.3, the homomorphism ρ is injective, so that $\tau(K,\chi) = 0$ if and only if the argument $C(s)$ of ρ is zero. In general, not every $\chi \in N$ satisfies the requirements of Theorem 3.16. However, it still makes sense to compute the rational number $C(s)$ for any $\chi \in N$. We will make use of this fact in Section 5 in order to show that the only slice knots among the twist knots K_n are the unknot K_0 and the Stevedore's knot K_2.

Note that there is a more general result of Theorem 3.16, also due to Gilmer, which holds for knots with arbitrary genus. The result states that it is possible to bound the difference $|\mathrm{sign}_1^{\mathrm{av}}(\tau(K,\chi)) - C(s)|$ by the nullity of the Tristram-Levine matrix $(1-\omega^s)A - (1-\omega^{-s})A^T$, where $\omega = e^{\frac{2\pi i}{m}}$, and the first Betti number of a Seifert surface for K. We refer the interested reader to [18], Theorem 3.5.

Let us turn our attention now to Gilmer's extension of Casson and Gordon's slicing obstruction given in [19]. Let L_2 denote again the double branched cover of a knot $K \subset S^3$, and let ϕ denote the geometric linking form of L_2. Further, let β denote the form $-\phi^*$ defined on $H^1(L_2;\mathbb{Q}/\mathbb{Z})$. We have the following theorem [19]:

Theorem 3.17. If $g_4(K) = g$, then $(H^1(L_2;\mathbb{Q}/\mathbb{Z}), \beta)$ can be written as a direct sum $\beta_1 \oplus \beta_2$ such that

1.) β_1 has an even presentation with rank $2g$ and signature $\sigma(K)$;

2.) β_2 has a metabolizer H such that if $\chi \in H$ has prime-power order, then

$$|\mathrm{sign}_\omega^{\mathrm{av}}(\tau(K,\chi)) + \sigma(K)| \leq 4g$$

for all $\omega \in S^1$.

Clearly, if $g = 0$, we recover Casson and Gordon's slicing obstruction described in Theorem 3.13 above. Note that if K is slice (i.e. $g = 0$), then $\beta_1 = 0$ and $\beta_2 = \beta$, so $H \subset H^1(L_2;\mathbb{Q}/\mathbb{Z})$ is a metabolizer for β.

Let us now see what happens in the case that K is a genus one knot. Suppose that $\omega = 1$. Since $\mathrm{sign}_1^{\mathrm{av}} \circ \rho = \mathrm{id}_{\mathbb{Q}}$ as described in Section 3.3, the inequality in the second point of Theorem 3.17 becomes

$$|C(s) + \sigma(K))| \leq 4g.$$

If we further assume that K is slice and that $\sigma(K) = 0$, then the inequality further simplifies to

$$|C(s)| \leq 0.$$

Thus, Theorem 3.17 shows that if K is a genus one knot with signature zero such that $C(s) \neq 0$ for all prime-power order $\chi \in H^1(L_2;\mathbb{Q}/\mathbb{Z})$, then K can not be slice. On the other hand, if there are some χ such that $C(s) = 0$, then these have to be contained in a metabolizer for β, which is a priori not always guaranteed.

As we will see in Section 4, all twist knots K_n are of genus one and have signature zero. We will prove in Section 5 that $C(s) \neq 0$ except when $n = 0, 2$, which shows by the previous discussion that the only possible slice knots are K_0 and K_2.

We have remarked earlier that Theorem 3.17 can be used to obtain lower bounds for the slice genus g_4. This is usually done by assuming a certain value for $g_4(K) = g$, and then show that β does not split according to Theorem 3.17. Although this is in general rather difficult, it is actually possible to obtain exact values for the slice genus from this method in some cases. Sample computations can be found in Gilmer's original paper [19].

The last result that we are going to need from Gilmer is concerned with the behavior of $\tau(K,\chi)$ under connected sum of knots. For this, let $J_1, J_2 \subset S^3$ be two knots with double branched covering L_2^1 and L_2^2, respectively, and set $N_i = H^1(L_2^i; \mathbb{Q}/\mathbb{Z})$ for $i = 1, 2$. Further, let $J = J_1 \# J_2$ be the connected sum of J_1 and J_2, with double branched covering L_2 and $N = H^1(L_2; \mathbb{Q}/\mathbb{Z})$. Then $L_2 = L_2^1 \# L_2^2$ and $N = N_1 \oplus N_2$. The following proposition is due to Gilmer [18].

Proposition 3.18. If $\chi_1 \in N_1$ and $\chi_2 \in N_2$, then

$$\tau(J_1 \# J_2, \chi_1 \oplus \chi_2) = \tau(J_1, \chi_1) + \tau(J_2, \chi_2).$$

Thus, Proposition 3.18 shows that the Casson-Gordon invariant $\tau(K, \chi)$ is additive under connected sum, which will be of great use to us in Section 6.

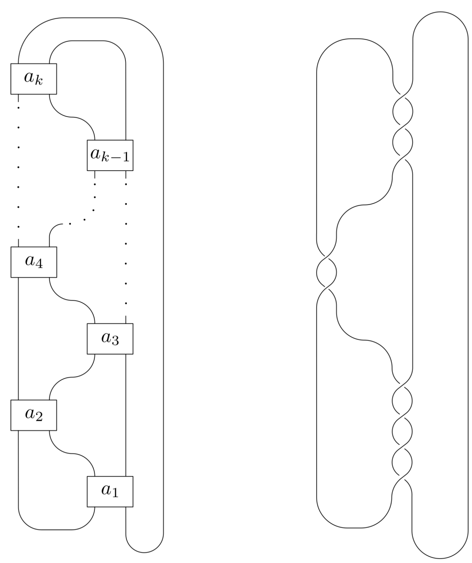
Part II
The Slice Genus of Twist Knots

4 The Twist Knots K_n – A First Pass

In this section, we finally turn our interest to the main actors of this text: the twist knots K_n. We start with definitions and general properties, describe a Seifert surface and Seifert matrix for the twist knots, and study their double branched covering. We further show that the signature of all twist knots vanishes, and show that K_n is algebraically slice if and only if $4n + 1$ is a square.

4.1 Definition and General Properties

Roughly speaking, a twist knot is obtained by applying a certain number of full 2π-right hand twists to an unknot and then linking its end together. More formally, twist knots arise as a special case of so-called 2-*bridge knots* (respectively 2-*bridge links*) $C(a_1, \ldots, a_k)$, where $a_i \in \mathbb{Z}$, depicted below:



$C(a_1, \ldots, a_k)$ $\qquad$ $C(4, -2, 3)$

Figure 4.1: General 2-bridge knot $C(a_1, \ldots, a_k)$ and the 2-bridge knot $C(4, -2, 3)$

Here, $a_i \in \mathbb{Z}$ denotes the number of half-twists in each corresponding box, with the convention that for even (respectively odd) $i \in \mathbb{N}$ and positive a_i, left-hand (respectively right-hand) half-twists are applied (and vice-versa if a_i is negative). Depending on k and the coefficients a_i, $C(a_1, \ldots, a_k)$ is either a knot or a link with two components. On the right hand side of Figure 4.1 is the 2-bridge knot $C(4, -2, 3)$. Note that depending on the parity of k, there are two different ways to connect the top and bottom of $C(a_1, \ldots, a_k)$. The left hand side of Figure 4.1 shows the connection for even k, while the example on the right hand side shows how the strands are connected for odd k.

By setting $k = 2$, $a_1 = 2n$ for $n \in \mathbb{N}$, and $a_2 = 2$, we obtain the twist knot K_n.

Definition 4.1 (Twist Knot). For $n \in \mathbb{N}$, the 2-bridge knot $C(2n, 2)$ is called *twist knot with* n *full* 2π*-twists* and is denoted by K_n.

For short, we simply call K_n a twist knot. Note that the twists in K_n are right-handed (also called positive). Figure 4.2 below shows the general form of a twist knot.

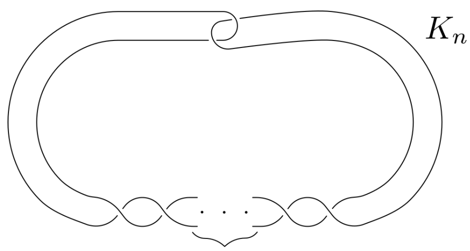


Figure 4.2: The twist knots K_n

The most famous knots among the twist knots are the unknot K_0, the figure-eight $K_1 = 4_1$, and the Stevedore's knot $K_2 = 6_1$ shown below.

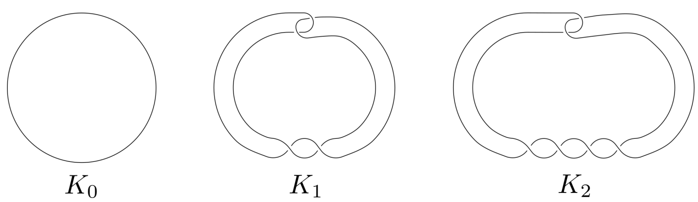


Figure 4.3: The unknot, figure-eight, and Stevedore's knot

All twist knots are reversible (i.e. equivalent to itself with reversed orientation), but only the figure-eight K_1 is amphicheiral (i.e. equivalent to its mirror image with reversed orientation). A possible Seifert surface for K_n is already visible from Figure 4.2. However, we will work with a more convenient representation of that Seifert surface in the form of a disk with a number of curled and linked bands attached, as seen in Figure 4.4 below.

The surface in Figure 4.4 will be denoted by F_n, and when speaking of a Seifert surface for the twist knots K_n, we will always mean the surface F_n. It is not difficult to see that F_n actually bounds the twist knot K_n; just note that the boundary of a curl corresponds to a full 2π-twist as shown in Figure 4.5 below.

The advantage of the Seifert surface F_n is that there is a basis for $H_1(F_n)$ in terms of simple closed curves. The basis elements we are going to work with are represented by the curves labelled a and b in Figure 4.4,

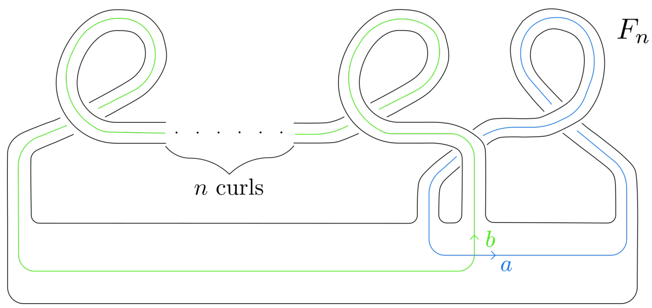


Figure 4.4: Seifert surface for K_n

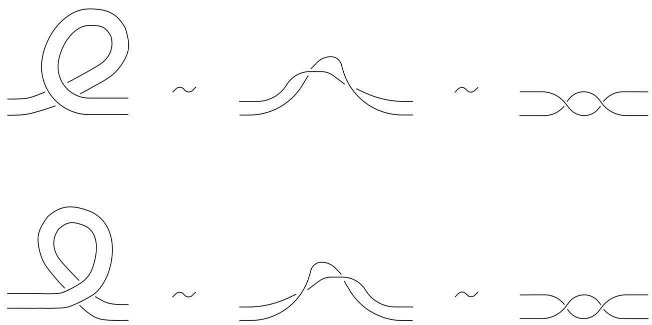


Figure 4.5: Curls in bands correspond to twists

equipped with the indicated orientation. In this basis, a Seifert matrix for K_n is given as

$$A_n := \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}.$$

When speaking of a Seifert matrix for K_n, we will always mean the matrix A_n defined above. From this Seifert matrix, the Alexander polynomial $\Delta_{K_n}(t)$ of the twist knots is readily computed:

$$\Delta_{K_n}(t) = A_n - tA_n^T = \begin{pmatrix} t-1 & 1 \\ -t & n(1-t) \end{pmatrix} \doteq -nt + (2n+1) - nt^{-1}.$$

Another characteristic of the twist knots is that they all have vanishing signature. Recall from Section 1.6 that the Tristram-Levine signature of a knot K is defined as $\sigma_\omega(K) = \operatorname{sign}((1-\omega)A + (1-\overline{\omega})A^T)$, where $\omega \in S^1 \setminus \{1\}$ and A is a Seifert matrix for K. Note that $\sigma_{-1}(K)$ is the ordinary signature of K.

Proposition 4.2. $\sigma_\omega(K_n) = 0$ for all $n \in \mathbb{N}$ and $\omega \in S^1 \setminus \{1\}$.

Proof. Let $\omega \in S^1$, $\omega \neq 1$, and $n \in \mathbb{N}$ be arbitrary, and let

$$A_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

denote the usual matrix of the Seifert form of K_n as defined above. Then the signature $\sigma_\omega(K_n)$ is given as the signature of the matrix

$$A_\omega = (1-\omega)A + (1-\overline{\omega})A^T = \begin{pmatrix} \omega+\overline{\omega}-2 & 1-\omega \\ 1-\overline{\omega} & n(2-\omega-\overline{\omega}) \end{pmatrix}.$$

Our aim is to find a diagonal matrix D_ω congruent to the Hermitian matrix A_ω. First note that $\omega+\overline{\omega}-2=0$ implies that $\omega+\overline{\omega}=2$, so $\operatorname{Re}(\omega)=1$ and hence $\omega=1$, the case we excluded. So we can assume $\omega+\overline{\omega}-2\neq 0$ and perform the following row and column operations to A_ω: multiply the first row with $-\frac{1-\overline{\omega}}{\omega+\overline{\omega}-2}$ and add it to the second row, and then multiply the first column with $-\frac{1-\omega}{\omega+\overline{\omega}-2}$ and add it to the second column. This yields the following matrix:

$$D_\omega = \begin{pmatrix} \omega+\overline{\omega}-2 & 0 \\ 0 & n(2-\omega-\overline{\omega}) - \frac{(1-\omega)(1-\overline{\omega})}{(\omega+\overline{\omega}-2)} \end{pmatrix}.$$

Note that

$$\begin{aligned} n(2-\omega-\overline{\omega}) - \frac{(1-\omega)(1-\overline{\omega})}{(\omega+\overline{\omega}-2)} &= n(2-\omega-\overline{\omega}) - \frac{1-\omega-\overline{\omega}+1}{(\omega+\overline{\omega}-2)} \\ &= n(2-\omega-\overline{\omega})+1, \end{aligned}$$

so

$$D_\omega = \begin{pmatrix} \omega+\overline{\omega}-2 & 0 \\ 0 & n(2-\omega-\overline{\omega})+1 \end{pmatrix}.$$

Since $\omega\neq 1$, we know that $\omega+\overline{\omega}-2<0$, so

$$\sigma_\omega(K_n) = -1 + \operatorname{sign}(n(2-\omega-\overline{\omega})+1).$$

Thus we are done if we can show that $n(2-\omega-\overline{\omega})+1>0$ for all $\omega\in S^1$ unequal to 1. But this is clear, since $\omega\neq 1$ implies that

$$2-\omega-\overline{\omega} = 2-2\operatorname{Re}(\omega) > 0.$$

□

Note that it is also possible to consider twist knots with left-hand instead of right-hand twists. However, one can show that twist knots with left-hand twists all have signature zero (cf. [7]), so by Proposition 2.10, none of them are slice. Therefore, twist knots with right-hand twists are the only ones that are interesting to us.

4.2 Genus and Algebraic Sliceness

The classical genus $g_3(K_n)$ of the twist knots can directly be computed from the Seifert surface F_n. Since F_n is a disk with two bands attached, we have that $g(F_n)=1$. By looking at the Alexander polynomial $\Delta_{K_n}(t)$, we can see that all twist knots are different from the unknot (except the unknot itself), so we conclude that for $n\in\mathbb{N}\setminus\{0\}$,

$$g_3(K_n)=1.$$

When it comes to the slice genus $g_4(K_n)$, we can see from the trivial bound $g_4 \leq g_3$ that

$$g_4(K_n) \leq g_3(K_n) = 1.$$

As we will see in Section 5, $g_4(K_n) = 0$ if and only if $n = 0$ or $n = 2$, so

$$g_4(K_n) = 1$$

for all $n \in \mathbb{N} \setminus \{0, 2\}$. Although only K_0 and K_2 are slice, it would be interesting to know which twist knots are algebraically slice (cf. Section 2.1). It turns out that there is a simple condition on the number of twists that ensures algebraic sliceness.

Proposition 4.3. K_n is algebraically slice if and only if $4n+1$ is a square.

Proof. Let K_n, $n \in \mathbb{N}$, be arbitrary with the usual Seifert matrix A_n as described in Section 4. Our aim is to find conditions that guarantee the existence of a metabolizer H of $H_1(F_n)$ satisfying 1. and 2. in Theorem 2.6. We will do so by considering what the possible generators of such a metabolizer are. Since $\operatorname{rank}(H_1(F_n)) = 2$, we know that $\operatorname{rank}(H) = 1$, so a possible metabolizer is generated by a single element of $H_1(F)$. Now, if $(x \quad y)$ generates H, then it has to fulfill the equation

$$(x \quad y) \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^2 + ny^2 + yx = 0. \tag{4.1}$$

This equation is homogeneous in the sense that any multiple of a solution is a solution again. Now, if $y = 0$ then $x = 0$, leading to the trivial solution, so without loss of generality assume that $y \neq 0$. By scaling over $\mathbb{R}$ we might further assume that $y = 1$, so the equation reads as

$$-x^2 + x + n = 0,$$

which has two solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4n+1}}{-2} \in \mathbb{R}.$$

We would like to know if there exists some $n \in \mathbb{N}$ such that $x_1, x_2 \in \mathbb{Z}$. The first condition for x_1 and x_2 to be integers is certainly that $4n+1$ is a square ℓ^2. If this is the case, then ℓ is odd since the square root of an odd number is odd, so $-1 \pm \ell$ is even. But if this is the case, then the nominator is a multiple of 2, and hence $x_1, x_2 \in \mathbb{Z}$. Thus, if $4n+1$ is a square ℓ^2, then the two elements

$$\alpha = \begin{pmatrix} (1-\ell)/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} (1+\ell)/2 \\ 1 \end{pmatrix}$$

are contained in $H_1(F_n)$, satisfy equation (4.1), and are therefore possible generators of a metabolizer H, proving that such a metabolizer exists if $4n+1$ is a square. $\square$

Since slice knots are algebraically slice, Proposition 4.3 will be very convenient in order to rule out many possible candidates for slice knots in the proof that only K_0 and K_2 are slice in Section 5.

4.3 The Double Branched Cover of Twist Knots

For our further study, we will need to work with the double branched cover L_2 of the twist knots K_n. In the case of twist knots, L_2 admits a nice description: it is the lens space $L(4n+1,2)$. More generally, if $C = C(a_1, \ldots, a_k)$ is a 2-bridge knot, then the double branched cover of C is the lens space $L(p,q)$, where p and q are defined via the continuous fraction

$$\frac{q}{p} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}}.$$

For a proof, see [7] or [45]. For our purposes, the first homology of the double branched cover will be the most important. Recall from Proposition 1.25 that for a knot K with double branched cover L_2, $H_1(L_2;\mathbb{Z})$ is isomorphic to the cokernel of $A + A^T$, where A is a Seifert matrix for K. The following proposition shows that in the case of the twist knots, $H_1(L_2;\mathbb{Z})$ is finite cyclic.

Proposition 4.4. Let A_n be the standard Seifert matrix for K_n as described above. Then

$$\operatorname{coker}(A_n + A_n^T) \cong \mathbb{Z}_{4n+1}.$$

Proof. Recall that in our situation, $\operatorname{coker}(A_n + A_n^T)$ is defined as

$$\operatorname{coker}(A_n + A_n^T) = \mathbb{Z}^2 / \operatorname{im}(A_n + A_n^T).$$

In order to identify the quotient on the right-hand side of the equation above. For this, we are going to compute the Smith normal form of $A_n + A_n^T$. We have:

$$\begin{aligned} A_n + A_n^T &= \begin{pmatrix} -2 & 1 \\ 1 & 2n \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -2 \\ 2n & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 \\ 2n & 4n+1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 \\ 0 & 4n+1 \end{pmatrix}. \end{aligned}$$

Thus, the elementary divisors are 1 and $4n+1$, so we have

$$\operatorname{coker}(A_n + A_n^T) \cong \mathbb{Z}_{4n+1}.$$

□

Corollary 4.5. Let L_2 be the double branched cover of S^3 branched along K_n. Then

$$H_1(L_2;\mathbb{Z}) \cong \mathbb{Z}_{4n+1}.$$

In particular, $H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{4n+1}$.

Proof. It follows from Proposition 1.25 and 4.4 that

$$H_1(L_2;\mathbb{Z}) \cong \operatorname{coker}(A_n + A_n^T) \cong \mathbb{Z}_{4n+1}.$$

The second statements follows from the isomorphism $H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong H_1(L_2;\mathbb{Z})$ (cf. Corollary 1.26). $\square$

The fact that $H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{4n+1}$ will be used in the upcoming section. Note that the above also follows from the description of L_2 as the lens space $L(4n+1,2)$.

5 Sliceness of K_n

5.1 Main Result

In this section, we are going to provide another proof of the following well-known result by Casson and Gordon [5, 6]:

Theorem 5.1. K_n is slice if and only if $n = 0$ (the unknot) or $n = 2$ (the Stevedore knot).

While the original proof in [5] respectively [6] is rather long and based on extensive computations, further developments in the theory allow one to prove Theorem 5.1 in a much simpler way. More precisely, we are going to use Gilmer's results about the Casson-Gordon invariant $\tau(K, \chi)$ described in Section 3.4 in order to show that there is no metabolizer for the dual linking form $-\phi^*$ as described in Theorem 3.17, except when $n = 0, 2$. As we will see, the proof of Theorem 5.1 essentially boils down to the computation of signatures of torus knots.

5.2 Preliminary Lemmas

In order to prove Theorem 5.1, we are going to need a few preliminary lemmas. As we have mentioned above, we are going to use Gilmer's results about the Casson-Gordon invariant $\tau(K, \chi)$ described in Section 3.4. For this, we are going to quickly recall the most important points from this section. Let K be a knot with Seifert surface F, Seifert pairing θ, and double branched covering L_2. We have the isomorphism $H^1(L_2; \mathbb{Q}/\mathbb{Z}) \cong N$, where N denotes the kernel of the map $\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}$, and where $\varepsilon : H^1(F) \to H_1(F)$ is defined as $\varepsilon_x(y) = \theta(x, y) + \theta(y, x)$. Assuming that $g(F) = 1$ and $\chi = x \otimes \frac{s}{m} \in N$ with $0 < s < m$ and m a prime-power, we know from Theorem 3.16 that

$$\tau(K, \chi) = \rho\Big(\underbrace{2\sigma_{\frac{s}{m}}(J_x) + \frac{4(m-s)s}{m^2}\theta(x, x) - \sigma_{\frac{1}{2}}(K_n)}_{=:C(s)}\Big), \tag{5.1}$$

where $\rho : \mathbb{Q} \to W(\mathbb{C}(t)) \otimes \mathbb{Q}$ is the homomorphism described in Section 3.3, and where J_x denotes the curve that represents x on F, but viewed as a knot in S^3.

The main task in the proof of Theorem 5.1 will be to compute the rational number $C(s)$ in the case of the twist knots K_n. The following results serve as a preparation for this. As a start, we are going to compute the kernel N for the twist knots K_n.

Lemma 5.2. For the twist knots K_n, the kernel N is generated by the element

$$\widetilde{\chi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \frac{1}{4n+1} \in H_1(F_n) \otimes \mathbb{Q}/\mathbb{Z}$$

for any $n \in \mathbb{N}$.

Proof. Let

$$A_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

be the usual Seifert matrix for K_n and write $x = (1 \quad 2)$. For $y = (u \quad v) \in H_1(F)$, we have:

$$\begin{aligned}
\varepsilon_x(y) &= \theta(x,y) + \theta(y,x) \\
&= (1 \quad 2)\begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + (1 \quad 2)\begin{pmatrix} -1 & 0 \\ 1 & n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\
&= (1 \quad 2)\begin{pmatrix} -2 & 1 \\ 1 & 2n \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\
&= (0 \quad 4n+1)\begin{pmatrix} u \\ v \end{pmatrix} \\
&= (4n+1)\pi_2(y),
\end{aligned}$$

where π_2 is the projection to the second component. Thus,

$$\begin{aligned}
\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}\left(x \otimes \frac{1}{4n+1}\right) &= \varepsilon_x(\cdot) \otimes \frac{1}{4n+1} \\
&= (4n+1)\pi_2(\cdot) \otimes \frac{1}{4n+1} \\
&= \pi_2(\cdot) \otimes 1 \\
&= 0.
\end{aligned}$$

Hence $\widetilde{\chi} \in \ker \varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}} = N$. Now, let L be the double branched cover of K_n. Since $H^1(L;\mathbb{Q}/\mathbb{Z})$ is isomorphic to the cokernel of $A + A^T$, we know that $N \cong \mathbb{Z}_{4n+1}$ (cf. Section 3.4 and 4). But $\widetilde{\chi}$ is an element of order $4n+1$, thus a generator of N. $\square$

Lemma 5.2 shows that N is finite cyclic in the case of twist knots. As a consequence, there is only one curve J_x that has to be taken under consideration, namely the one that represents $(1 \quad 2) \in H_1(F_n)$. Fortunately, this curve can be easily described.

Lemma 5.3. The element $x = (1 \quad 2) \in H_1(F_n)$ is represented by a $(2, 2n+1)$-torus knot. That is, $J_x = T(2, 2n+1)$.

Proof. Consider the following illustration.

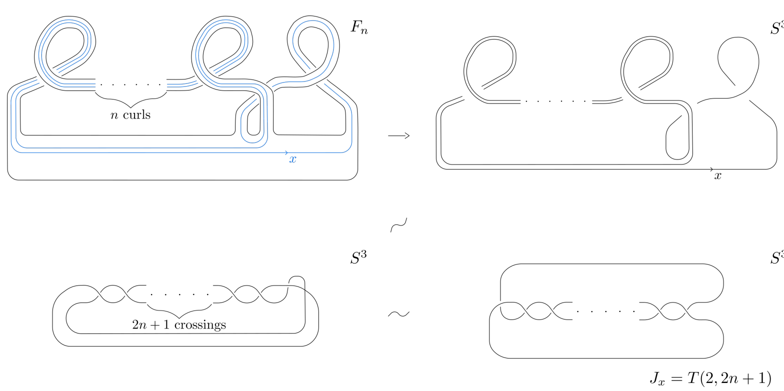


Figure 5.1: The curve J_x is a $(2, 2n+1)$-torus knot

In the top-left corner of Figure 5.1, we can see the Seifert surface F_n together with $x \in H_1(F_n)$ represented as a simple closed curve, indicated in blue. If we now consider this curve as contained in S^3, we arrive at the knot labelled J_x shown at the top-right. The transformations on the lower half of Figure 5.1 show that J_x is a $(2, 2n+1)$-torus knot. □

Looking back at Equation 5.1, Lemma 5.2 and 5.3 show that we are interested in the signatures $\sigma_{\frac{s}{4n+1}}(J_x)$, where J_x is the $(2, 2n+1)$-torus knot and $0 < s < 4n+1$. Recall that the Tristram-Levine signature σ_ω is a piecewise constant function in ω with jumps only occurring at zeros of the Alexander polynomial of the knot. Away from the discontinuities, the function is even-valued (see Section 1.6). The Alexander polynomial of the torus knot $T(2, 2n+1)$ is given as (cf. [40])

$$\Delta_{T(2,2n+1)}(t) = \frac{(t^{2(2n+1)} - 1)(t-1)}{(t^2-1)(t^{2n+1}-1)} = \frac{t^{2n+1}+1}{t+1},$$

thus the zeros are exactly the $(2n+1)$st roots of -1, that is, $\exp(\frac{\pi i(2t-1)}{2n+1})$ for $t = 1, 2, \ldots, 2n+1$. In particular, all zeros are of multiplicity one, so we know that the signature function of J_x can only jump by ± 2 at a discontinuity. Thus, in order to derive a formula for $\sigma_{\frac{s}{4n+1}}(J_x)$, we certainly need to know what values $\exp(\frac{2\pi is}{4n+1})$ lie between which zeros of $\Delta_{T(2,2n+1)}(t)$ on S^1.

Lemma 5.4. Let $\omega_s = \frac{2\pi is}{4n+1}$ and $\xi_{2t-1} = \frac{\pi i(2t-1)}{2n+1}$ be the arguments of the $(4n+1)$st roots of unity and the arguments of the $(2n+1)$st roots of minus one, respectively. Then, for $t = 1, \ldots, 2n$,

$$\omega_s \in [\xi_{2t-1}, \xi_{2t+1}] \iff s = 2t-1, 2t,$$

and for $t = 2n+1$,

$$\omega_s \in [\xi_{2n+1}, \xi_{2n+3}] \iff s = 4n+1.$$

In other words, on S^1, between each two roots of minus one there are exactly two roots of unity, except between $\exp(\xi_{2n+1})$ and $\exp(\xi_1)$, where there is only one root, namely $\exp(\omega_{4n+1}) = 1$.

Proof. The lemma is basically a consequence of the pigeonhole principle, but it can also be proved by direct computation. Let ω_s and ξ_{2t-1} be as above. Then

$$\omega_s = \frac{2\pi is}{4n+1} = \frac{\pi is(4n+2)}{(4n+1)(2n+1)}, \quad \xi_{2t-1} = \frac{\pi i(2t-1)}{2n+1} = \frac{\pi i(2t-1)(4n+1)}{(2n+1)(4n+1)}.$$

Write $\widetilde{\omega}_s = s(4n+2)$ and $\widetilde{\xi}_{2t-1} = (2t-1)(4n+1)$. Then the statement from the lemma is equivalent to

$$\widetilde{\omega}_s \in [\widetilde{\xi}_{2t-1}, \widetilde{\xi}_{2t+1}] \iff s = 2t-1, 2t$$

for $t = 1, \ldots, 2n$, and

$$\widetilde{\omega}_s \in [\widetilde{\xi}_{2n+1}, \widetilde{\xi}_{2n+2}] \iff s = 4n+1.$$

Fix $t \in \{1, \dots, 2n\}$. Then

$$\begin{aligned}
\widetilde{\omega}_{2t-2} < \widetilde{\xi}_{2t-1} &\iff \\
(2t-2)(4n+2) < (2t-1)(4n+1) &\iff \\
8nt - 8n + 4t - 4 < 8nt - 4n + 2t - 1 &\iff \\
-4n + 2t - 3 < 0,
\end{aligned}$$

which is clearly satisfied. Similarly,

$$\begin{aligned}
\widetilde{\xi}_{2t+1} < \widetilde{\omega}_{2t+1} &\iff \\
(2t+1)(4n+1) < (2t+1)(4n+2) &\iff \\
4n + 1 < 4n + 2,
\end{aligned}$$

which is satisfied as well. So, for $t = 1, \dots, 2n$, the only candidates for being contained in $[\widetilde{\xi}_{2t-1}, \widetilde{\xi}_{2t+1}]$ are $\widetilde{\omega}_{2t-1}$ and $\widetilde{\omega}_{2t}$. We have

$$\widetilde{\xi}_{2t-1} < \widetilde{\omega}_{2t-1} \iff (4n+1)(2t-1) < (4n+2)(2t-1)$$

and

$$\begin{aligned}
\widetilde{\omega}_{2t} < \widetilde{\xi}_{2t+1} &\iff \\
(4n+2)2t < (4n+1)(2t+1) &\iff \\
2t < 4n + 1,
\end{aligned}$$

which is satisfied for $t = 1, \dots, 2n$. This proves the first statement. The case $t = 2n + 1$ is treated similarly. $\square$

Figure 5.2 below shows the distribution of the roots in the case $n = 5$.

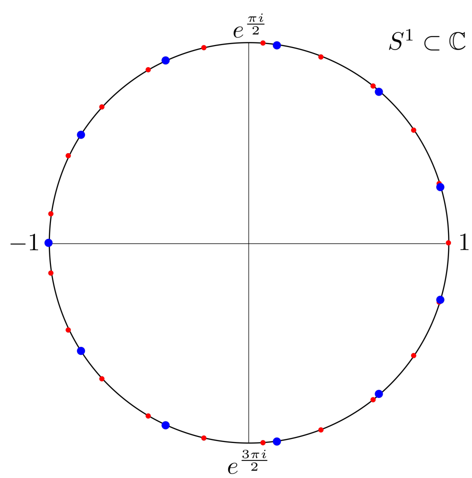


Figure 5.2: Distribution of the roots on S^1 described in Lemma 5.4 in the case $n = 5$. The thin red dots mark the 21st roots of unity, while the thick blue dots mark the 11th roots of minus one.

Knowing that the ordinary signature of a $(2, 2n+1)$-torus knot is $-2n$ and that its ω-signature jumps by -2 on the upper and by $+2$ on the lower hemisphere of S^1 and stays constant at $\omega = -1$ (cf. [40]), a formula for $\sigma_{\frac{s}{4n+1}}(T(2, 2n+1))$ is now readily derived by using Lemma 5.4:

$$\sigma_{\frac{s}{4n+1}}(T(2, 2n+1)) = \begin{cases} -2\left\lceil \frac{s}{2} \right\rceil, & s = 1, \dots, 2n \\ -2\left\lceil \frac{(4n+1)-s}{2} \right\rceil, & s = 2n+1, \dots, 4n. \end{cases}$$

Note that $\sigma_{\frac{s}{4n+1}}(T(2, 2n+1))$ is symmetric about $2n$. Figure 5.3 below shows the values of the formula above in the case $n = 5$.

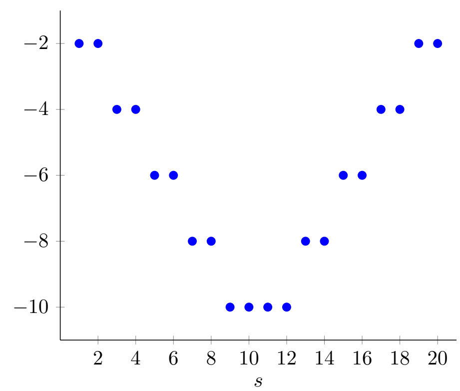


Figure 5.3: The values of $\sigma_{\frac{s}{21}}(T(2, 11))$ for $s = 1, \dots, 20$

The last lemma that we are going to need will be essential in order to derive a contradiction to the sliceness obstruction given in Theorem 3.17. Recall the statement from Theorem 3.17: If a knot K is slice, then there exists a metabolizer $H \subset H^1(L_2; \mathbb{Q}/\mathbb{Z})$ of the dual linking form $-\phi^*$ such that $|\mathrm{sign}^{\mathrm{av}}_\omega(\tau(K, \chi)) + \sigma(K)| = 0$ for all $\chi \in H$ of prime-power order and $\omega \in S^1$.

Lemma 5.5. Let K_n be slice with metabolizer H as in Theorem 3.17. Then there is at least one non-trivial element $\chi \in H$ of prime-power order.

Proof. Let K_n be slice. Since a slice knot is always algebraically slice, we may assume that $4n + 1$ is a square ℓ^2. Let $\chi \in H$ be arbitrary and non-trivial. Since $H^1(L_2; \mathbb{Q}/\mathbb{Z}) \cong N$ and N is generated by $x \otimes \frac{1}{\ell^2}$, where $x = (1 \quad 2) \in H_1(F_n; \mathbb{Z})$, we may assume that $\chi = x \otimes \frac{s}{\ell^2}$ for some $0 < s < \ell^2$. If χ is already of prime-power order, there is nothing to show. Otherwise, suppose that $\ell^2 = p_1^{r_1} \cdots p_k^{r_k}$ is the prime decomposition of ℓ^2. Since a metabolizer is a subgroup, we have

$$(p_2^{r_1} \cdots p_k^{r_k})(x \otimes \frac{s}{\ell^2}) = x \otimes \frac{1}{p_1^{r_1}} \in H.$$

Thus, $x \otimes \frac{1}{p_1^{r_1}}$ is an element of prime-power order that is contained in the metabolizer H. $\square$

5.3 Proof of Theorem 5.1

We are now finally ready to prove Theorem 5.1: K_n is slice if and only if $n = 0$ or $n = 2$. The main part of the proof will reveal that the only possible candidates for slice knots are K_0 and K_2. While the unknot is certainly slice, it is not obvious that the same is true for the Stevedore knot K_2. Since the proof that K_2 is slice is in a slightly different spirit than the rest of the proof of Theorem 5.1, we are first going that show that K_2 is indeed slice.

Proposition 5.6. The Stevedore knot K_2 is slice.

Proof. Consider the following transformations of the Stevedore knot K_2:

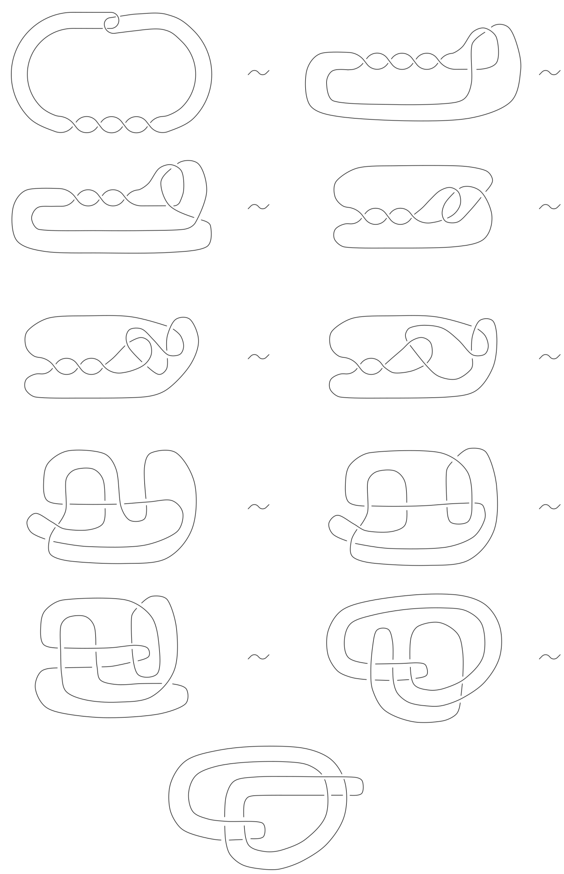


Figure 5.4: Ribbon disk for the Stevedore's knot K_2

We can clearly see that the end product of this transformation is a ribbon disk according to Definition 2.2 that bounds K_2. Since ribbon knots are slice by Theorem 2.3, we conclude that the Stevedore's knot K_2 is indeed slice. □

Note that we have actually encountered the ribbon disk for the Stevedore's knot already in Figure 2.2. We are now ready to prove Theorem 5.1.

Proof (Theorem 5.1). Let K_n, $n \in \mathbb{N}$, be an arbitrary twist knot. Since a slice knot is always algebraically slice, we can assume that $4n+1=\ell^2$ is a square (cf. Proposition 4.3). Recall once more the formula for τ from Theorem 3.16: If $\chi = x \otimes \frac{s}{m} \in H^1(L_2;\mathbb{Q}/\mathbb{Z}) \cong N = \ker\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}$ with m a prime-power and $0<s<m$, then

$$\tau(K_n,\chi) = \rho\underbrace{\Big(2\sigma_{\frac{s}{m}}(J_x) + \frac{4(m-s)s}{m^2}\theta(x,x) - \sigma_{\frac{1}{2}}(K_n)\Big)}_{=:C(s)}. \tag{5.2}$$

Now, assume for a contradiction that K_n is slice. Then the dual linking form $-\phi^*$ admits a metabolizer H as in Theorem 3.17. We have shown in Lemma 5.5 above that there is at least one non-trivial element χ of prime-power order contained in H that satisfies

$$\mathrm{sign}^{\mathrm{av}}_\omega(\tau(K_n,\chi)) = \mathrm{sign}^{\mathrm{av}}_\omega(\rho(C(s))) = 0.$$

If we set $\omega=1$, then $\mathrm{sign}^{\mathrm{av}}_1 \circ \rho = \mathrm{id}_{\mathbb{Q}}$, so if K_n is slice, then there is at least one non-trivial element $\chi \in H$ of prime-power order such that

$$C(s)=0.$$

Our goal is now to show the contrary: $C(s)\neq 0$ for any $\chi \in H^1(L_2;\mathbb{Q}/\mathbb{Z})$ except when $n=0,2$.

Recall from Lemma 5.2 that $H^1(L_2;\mathbb{Q}/\mathbb{Z})$ is generated by an element of order $4n+1$, so we can set $m=4n+1=\ell^2$. Let $\chi = (1 \quad 2)\otimes \frac{s}{\ell^2} \in N$ be arbitrary. Since $\theta(x,x)=\ell^2$ for $x=(1 \quad 2)$ and $\sigma_{\frac{1}{2}}(K_n)=0$ (cf. Proposition 4.2), we have

$$\begin{aligned} C(s) &= 2\sigma_{\frac{s}{\ell^2}}(T(2,2n+1)) + \frac{4(\ell^2-s)s}{\ell^2} \\ &= \begin{cases} -4\left\lceil \frac{s}{2}\right\rceil + \frac{4(\ell^2-s)s}{\ell^2}, & s=1,\dots,2n \\ -4\left\lceil \frac{(4n+1)-s}{2}\right\rceil + \frac{4(\ell^2-s)s}{\ell^2}, & s=2n+1,\dots,4n. \end{cases} \end{aligned}$$

By the symmetry of $C(s)$ about $2n$, is is sufficient to consider the case $s\in\{1,\dots,2n\}$. If s is not a multiple of ℓ, then $\frac{4(\ell^2-s)s}{\ell^2}\in\mathbb{Q}\setminus\mathbb{Z}$ while $-4\left\lceil\frac{s}{2}\right\rceil\in\mathbb{Z}$, so $C(s)\neq 0$. If s is a (non-zero) multiple of ℓ, i.e. $s=r\ell$ for some $r\in\mathbb{N}_{>0}$, then

$$C(r\ell) = -4\left\lceil\frac{r\ell}{2}\right\rceil + \frac{4(\ell^2-r\ell)r\ell}{\ell^2} = -4\left\lceil\frac{r\ell}{2}\right\rceil + 4r(\ell-r).$$

Now, if r is even, then $-4\left\lceil\frac{r\ell}{2}\right\rceil = -2r\ell$, so

$$\begin{aligned} C(r\ell) &= -2r\ell + 4r(l-r) \\ &= 2r\ell - 4r^2 \\ &= 2r(\ell-2r), \end{aligned}$$

which is zero if and only if $\ell = 2r$, a contradiction because ℓ is odd. On the other hand, if r is odd, then $-4\left\lceil \frac{r\ell}{2} \right\rceil = -2(r\ell + 1)$, yielding

$$\begin{aligned} C(r\ell) &= -2(r\ell + 1) + 4r(l - r) \\ &= 2r\ell - 4r^2 - 2 \\ &= 2(r\ell - 2r^2 - 1), \end{aligned}$$

which is zero if and only if

$$r(\ell - 2r) = 1 \Leftrightarrow \ell - 2r = \frac{1}{r}.$$

However, the only possible integer solution to this equation is given by $\ell = 3$ and $r = 1$, which can only appear if $n = 2$.

The computations above show that $C(s) \neq 0$ for all $\chi \in H^1(L_2; \mathbb{Q}/\mathbb{Z})$ except when $n = 0, 2$, which is in particular true for all χ of prime-power order (so that the formula for $\tau(K_n, \chi)$ in Equation 5.2 holds). Thus, if $n \neq 0, 2$, then there is no metabolizer for the dual linking form $-\phi^*$ as in Theorem 3.17. Therefore, the only possible slice knots are the unknot K_0 and the Stevedore knot K_2. The unknot is clearly slice, and the Stevedore knot is slice by Proposition 5.6. This concludes the proof that K_n is not slice except when $n = 0$ or $n = 2$. $\square$

6 Sliceness of the Connected Sum rK_n

We have seen in the previous section that the only slice knots among the twist knots are the unknot K_0 and the Stevedore knot K_2. An interesting question would now be to ask whether the same is true for a connected sum of some twist knot K_n. That is, if $K_n \# \ldots \# K_n$ is a finite connected sum of the twist knot K_n, is such a sum slice if and only if $n = 0$ or $n = 2$? Or is there some other $n \in \mathbb{N}$ and a certain number of summands such that the connected sum becomes slice? In fact, what we are asking for is the order of K_n in the knot concordance group $\mathcal{C}$ (cf. Section 2.2). In this section, we provide a way based on our previous computations that could possibly lead to an answer to the above questions.

Before we start, let us fix some notation and conventions. Given $n, r \in \mathbb{N}$, let

$$rK_n := \underbrace{K_n \# \cdots \# K_n}_{r \text{ times}}$$

denote the r-fold connected sum of the twist knot K_n. A Seifert surface for rK_n is given as the r-fold connected sum of the Seifert surface F_n, shown in the figure below.

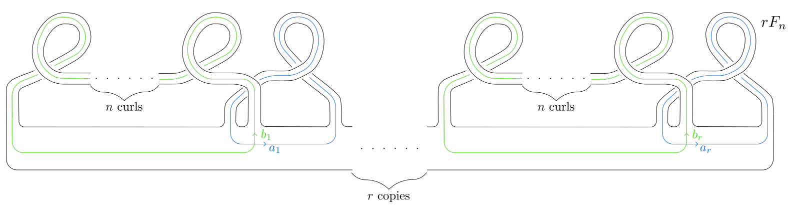


Figure 6.1: A Seifert surface for rK_n

We will denote this Seifert surface by rF_n. A basis for $H_1(rF_n; \mathbb{Z})$ is given by the curves $a_1, b_1, \ldots, a_r, b_r \in H_1(rF_n; \mathbb{Z})$, with orientation as indicated in Figure 7.1. In this basis, a Seifert matrix for rK_n is given as

$$\mathcal{A}_n := \bigoplus_1^m A_n = \begin{pmatrix} -1 & 1 & & & \\ 0 & n & & & \\ & & \ddots & & \\ & & & -1 & 1 \\ & & & 0 & n \end{pmatrix}.$$

Let us turn our attention now to the question if there are some $n, r \in \mathbb{N}$, $n \neq 0, 2$, such that rK_n is slice. If $n = 1$, then K_1 is the figure-eight which is amphicheiral (i.e. $K_1 = -K_1^*$), so the connected sum $K_1 \# K_1$ is slice, and the order of the figure-eight in the knot concordance group $\mathcal{C}$ is 2 (cf. [36]). So we may assume that $n > 2$.

Since a slice knot is always algebraically slice, the first question to ask is what knots rK_n are algebraically slice. In contrast to the case $r = 1$, one can show that if a knot has Tristram-Levine signature zero, then it is of finite order in the algebraic concordance group $\mathcal{G}^{\mathbb{Z}}$ (cf. [36]). Since the signature is additive under connected sum and all twist knots have signature zero, we have by Levine's results that K_n is of order at most 4 in $\mathcal{G}^{\mathbb{Z}}$ (cf. Section 2.2). Therefore, it makes sense to take all twist knots K_n, $n > 2$, into account for our further considerations.

In the following, fix some $n > 2$ and the corresponding twist knot K_n. Let L^i, $i = 1, \dots, r$, be r copies of the double branched cover of K_n, and set $N_i = H^1(L^i; \mathbb{Q}/\mathbb{Z})$. If L denotes the double branched cover of rK_n and $N = H^1(L; \mathbb{Q}/\mathbb{Z})$, then $L = L^1 \# \cdots \# L^r$ and $N = N_1 \oplus \cdots \oplus N_r$, as remarked in Section 3.4. Since $N_i = \mathbb{Z}_{4n+1}$ for all $i = 1, \dots, r$, we have that

$$N = H^1(L; \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{4n+1}.$$

Now, given some $\chi = \chi_1 \oplus \cdots \oplus \chi_r \in N$, we have by Gilmer's additivity result (cf. Proposition 3.18)

$$\tau(rK_n, \chi) = \sum_{i=1}^{r} \tau(K_n, \chi_i).$$

Next, recall once more the formula for $\tau(K_n, \chi_i)$ from Theorem 3.16: If $\chi_i = x \otimes \frac{s}{m} \in N_i$, with $0 < s < m$ and m a prime-power, then

$$\tau(K_n, \chi_i) = \rho\Big(\underbrace{2\sigma_{\frac{s}{m}}(J_x) + \frac{4(m-s)s}{m^2}\theta(x,x) - \sigma_{\frac{1}{2}}(K_n)}_{=:C(s)}\Big),$$

where $\rho : \mathbb{Q} \to W(\mathbb{C}(t)) \otimes \mathbb{Q}$ is the injective homomorphism described in Section 3.3. Supposing that $\chi = \chi_1 \oplus \cdots \oplus \chi_r \in N = N_1 \oplus \cdots \oplus N_r$ is a character such that each χ_i is of prime-power order, then

$$\tau(rK_n, \chi) = \sum_{i=1}^{r} \tau(K_n, \chi_i) = \sum_{i=1}^{r} \rho(C(s_i))$$

for some suitable $0 < s_i < m$, $i = 1, \dots, r$.

Assume that rK_n is slice. Then by Theorem 3.17, there exists a metabolizer $H \subset H^1(L; \mathbb{Q}/\mathbb{Z}) = N$ for the dual linking form $-\phi^*$ such that

$$\operatorname{sign}_{\omega}^{\mathrm{av}}(\tau(rK_n, \chi)) = \operatorname{sign}_{\omega}^{\mathrm{av}}(\rho(C(s))) = 0$$

for all $\chi \in H$ of prime-power order. Similarly as before, set $\omega = 1$ and suppose that $\chi = \chi_1 \oplus \cdots \oplus \chi_r \in H$ is of prime-power order (note that H does not necessarily split as a direct sum). Then

$$\begin{aligned}
\operatorname{sign}_{1}^{\mathrm{av}}(\tau(rK_n, \chi)) &= \operatorname{sign}_{1}^{\mathrm{av}}(\sum_{i=1}^{r} \tau(K_n, \chi_i)) \\
&= \operatorname{sign}_{1}^{\mathrm{av}}(\sum_{i=1}^{r} \rho(C(s_i)) \\
&= \sum_{i=1}^{r} \underbrace{\operatorname{sign}_{1}^{\mathrm{av}} \circ \rho}_{=\mathrm{id}_{\mathbb{Q}}}(C(s_i)) \\
&= \sum_{i=1}^{r} C(s_i) \\
&= 0,
\end{aligned}$$

for suitable $0 < s_i < m$, $i = 1, \ldots, r$. Thus, in order to know if rK_n is slice, the above suggests that we try to find solutions to the equation

$$C(s_1) + C(s_2) + \cdots + C(s_r) = 0 \tag{6.1}$$

and then check if the corresponding characters are contained in a metabolizer for $-\phi^*$. Note that since a metabolizer satisfies $|H|^2 = |H^1(L; \mathbb{Q}/\mathbb{Z})|$, r has either to be even or $4n+1$ has to be a square.

The first step in finding solutions is to check if a solution is possible at all. For this, recall that N_i is generated by the element $x \otimes \frac{1}{4n+1}$, where $x = (1 \quad 2) \in H_1(F_n; \mathbb{Z})$ (cf. Lemma 5.2). Each character $\chi_i = x \otimes \frac{s}{4n+1} \in N_i$ defines a rational number $C(s)$. Although χ_i might not be of prime-power order, we can still check if there is some $0 < s < 4n+1$ and a corresponding character such that $C(s) < 0$, since this is clearly needed in order to make the sum in Equation 6.1 vanish (note that $C(s) = 0$ is not possible as seen in the proof of Theorem 5.1). From Section 5, we know that

$$\begin{aligned} C(s) &= 2\sigma_{\frac{s}{4n+1}}(T(2, 2n+1)) + \frac{4((4n+1)-s)s}{4n+1} \\ &= \begin{cases} -4\left\lceil \frac{s}{2} \right\rceil + \frac{4((4n+1)-s)s}{4n+1}, & s = 1, \ldots, 2n \\ -4\left\lceil \frac{(4n+1)-s}{2} \right\rceil + \frac{4((4n+1)-s)s}{4n+1}, & s = 2n+1, \ldots, 4n. \end{cases} \end{aligned}$$

We claim the following:

Lemma 6.1. $C(s) < 0$ if and only if $s = 1$ or $s = 4n$.

Proof. By the symmetry of $C(s)$ about $2n$, it is sufficient to consider the cases $s = 1, \ldots, 2n$ and show that $C(s) < 0$ if $s = 1$ and $C(s) > 0$ otherwise. Consider first the case $s = 1$. Then

$$C(1) = -4 + \frac{16n}{4n+1} = \frac{-4}{4n+1} < 0,$$

so $C(1) < 0$. In order to shorten notation for the second case, set $m = 4n+1$. Suppose that $s \geq 2$. Then

$$C(s) = -4\left\lceil \frac{s}{2} \right\rceil + \frac{4(m-s)s}{m} > 0 \iff 4(m(s - \left\lceil \frac{s}{2} \right\rceil) - s^2) > 0.$$

Looking at the right-hand side of the equivalence, if s is even, then

$$m(s - \left\lceil \frac{s}{2} \right\rceil) - s^2 = \frac{ms}{2} - s^2 = s(\frac{m}{2} - s) > s(2n - s) \geq 0$$

since $s \leq 2n$. On the other hand, if s is odd, then

$$m(s - \left\lceil \frac{s}{2} \right\rceil) - s^2 = \frac{m(s-1)}{2} - s^2 = \frac{ms - m - 2s^2}{2}.$$

Considering $-2s^2 + ms - m$ as a polynomial in s, its zeros are

$$x_1 = \frac{-m + \sqrt{m^2 - 8m}}{-4}, \quad x_2 = \frac{-m - \sqrt{m^2 - 8m}}{-4}.$$

We have

$$\begin{aligned} x_1 < 3 &\iff \sqrt{m^2 - 8m} > m - 12 \\ &\iff m^2 - 8m > m^2 - 24m + 144 \\ &\iff 16m > 144 \\ &\iff 64n > 128, \end{aligned}$$

which is satisfied since we assume that $n > 2$. Similarly, we have

$$\begin{aligned} 2n - 1 < x_2 &\iff 8n - m - 4 < \sqrt{m^2 - 8m} \\ &\iff 64n^2 - 16nm - 64n + 16m + 16 < 0 \\ &\iff -16n + 32 < 0 \\ &\iff 32 < 16n, \end{aligned}$$

which is again satisfied because $n > 2$ by assumption. Since the leading coefficient of $-2s^2 + ms - m$ is negative, we see that

$$-2s^2 + ms - m > 0$$

for all odd $s \in \{1, \ldots, 2n\}$. Thus if $s \geq 2$, then $C(s) > 0$. All together, we have shown that $C(s) < 0$ if and only if $s = 1$ or $s = 4n$. □

Lemma 6.1 implies that if $C(s_1) + \cdots + C(s_r) = 0$, then $s_i = 1$ or $s_i = 4n$ for at least one $i \in \{1, \ldots, r\}$. In particular, every corresponding character has order $4n + 1$. Also, we see that solutions to Equation 6.1 are indeed possible.

The next step is to try to find actual solutions to the Equation 6.1. The best way in doing so is probably to solve the equation

$$\alpha_1 C(1) + \alpha_2 C(2) + \cdots + \alpha_{4n} C(4n) = 0,$$

where $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$, and then build solutions to Equation 6.1 for some fixed $r > 0$. To demonstrate the procedure, we do the computation in the case $n = 3$, that is, for the twist knot K_3.

Example 6.2. Consider the twist knot K_3. In this case, $4n + 1 = 13$, so we have to consider the values $C(1), \ldots, C(12)$. By the symmetry of $C(s)$ about $2n$, we can restrict our considerations to the values $C(1), \ldots, C(6)$ and then count solutions with multiplicity. The values $C(1), \ldots, C(6)$ are the following:

$$\begin{aligned} C(1) &= -4 + \frac{48}{13} = -\frac{4}{13} \qquad & C(4) &= -8 + \frac{144}{13} = \frac{40}{13} \\ C(2) &= -4 + \frac{88}{13} = \frac{36}{13} \qquad & C(5) &= -12 + \frac{160}{13} = \frac{4}{13} \\ C(3) &= -8 + \frac{120}{13} = \frac{16}{13} \qquad & C(6) &= -12 + \frac{168}{13} = \frac{12}{13} \end{aligned}$$

Thus, we would like to solve the equation

$$\alpha_1 \cdot (-\frac{4}{13}) + \alpha_2 \cdot \frac{36}{13} + \alpha_3 \cdot \frac{16}{13} + \alpha_4 \cdot \frac{40}{13} + \alpha_5 \cdot \frac{4}{13} + \alpha_6 \cdot \frac{12}{13} = 0,$$

where $\alpha_i \in \mathbb{N}$, which is equivalent to

$$-4\alpha_1 + 36\alpha_2 + 16\alpha_3 + 40\alpha_4 + 4\alpha_5 + 12\alpha_6 = 0.$$

Now, if we set

$$\alpha_1 = 9\alpha_2 + 4\alpha_3 + 10\alpha_4 + \alpha_5 + 3\alpha_6,$$

then we can construct for any given natural numbers $\alpha_2, \dots, \alpha_6 \in \mathbb{N}$ a sum $C(s_1) + \cdots + C(s_r)$, $0 < s_i \leq 6$, that vanishes. For example, if we set $\alpha_5 = 2$ and $\alpha_i = 0$ for $i \neq 1, 5$, then

$$C(1) + C(1) + C(5) + C(5) = 0.$$

Here $r = 4$, so the sum $C(1) + C(1) + C(5) + C(5)$ corresponds to the character

$$\chi = (x \otimes \frac{1}{13}) \oplus (x \otimes \frac{1}{13}) \oplus (x \otimes \frac{5}{13}) \oplus (x \otimes \frac{5}{13}) \in H^1(L; \mathbb{Q}/\mathbb{Z}),$$

where L is the double branched cover of $4K_3$, and we see that we have found a character χ such that

$$\mathrm{sign}_1^{\mathrm{av}}(\tau(4K_3, \chi)) = 0.$$

The above procedure can be used to find all characters χ such that $\mathrm{sign}_1^{\mathrm{av}}(\tau(rK_3, \chi)) = 0$ for any given (even) $r > 0$. For instance, if $r = 2$, we have the characters

$$\begin{aligned}
\chi_1 &= (x \otimes \frac{1}{13}) \oplus (x \otimes \frac{5}{13}) \\
\chi_2 &= (x \otimes \frac{5}{13}) \oplus (x \otimes \frac{1}{13}).
\end{aligned}$$

Taking the symmetry of $C(s)$ into account, we get further the characters

$$\begin{aligned}
\chi_3 &= (x \otimes \frac{1}{13}) \oplus (x \otimes \frac{8}{13}) \\
\chi_4 &= (x \otimes \frac{8}{13}) \oplus (x \otimes \frac{1}{13}) \\
\chi_5 &= (x \otimes \frac{12}{13}) \oplus (x \otimes \frac{5}{13}) \\
\chi_6 &= (x \otimes \frac{5}{13}) \oplus (x \otimes \frac{12}{13}) \\
\chi_7 &= (x \otimes \frac{12}{13}) \oplus (x \otimes \frac{8}{13}) \\
\chi_8 &= (x \otimes \frac{8}{13}) \oplus (x \otimes \frac{12}{13}).
\end{aligned}$$

Thus, we end up with a total of 8 characters such that

$$\mathrm{sign}_1^{\mathrm{av}}(\tau(2K_3, \chi)) = 0.$$

Note that this actually implies that $2K_3$ is not slice since a metabolizer $H \subset H^1(L; \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{13} \oplus \mathbb{Z}_{13}$ would need to have exactly 13 elements.

The computations in Example 6.2 can be done for any $n \in \mathbb{N}$. The reason for this is that $C(1) = -\frac{4}{4n+1}$ (resp. $C(4n) = -\frac{4}{4n+1}$) for any $n \in \mathbb{N}$, and that both summands in the formula for $C(s)$ are a multiple of 4 (see above). Therefore, we conclude that there are indeed many possible solutions to the equation $C(s_1) + \cdots + C(s_r) = 0$ for any sum rK_n, and it is not possible to say something about the sliceness of rK_n yet.

The next step would be to take the metabolizer into account and check if the characters that satisfy $\operatorname{sign}_1^{\mathrm{av}}(\tau(rK_n, \chi)) = 0$ are contained in some metabolizer H. However, this is where the difficulties start. First of all, given a sum rK_n, it is difficult to compute possible generators of a metabolizer $H \subset H^1(L; \mathbb{Q}/\mathbb{Z})$ for the dual form $-\phi^*$. Second, the computations done in Example 6.2 get more and more involved as r and n grow, which makes it difficult to determine all suitable characters χ.

Another option arises if we assume that $4n+1$ is a prime-power. For if this is the case, then every character $\chi \in H^1(L; \mathbb{Q}/\mathbb{Z})$ is of prime-power order, so the characters that satisfy $\operatorname{sign}_1^{\mathrm{av}}(\tau(rK_n, \chi)) = 0$ have to form a metabolizer themselves. Consider for example the sum $4K_6$. Then $4n+1 = 25$, and

$$H^1(L; \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{25}.$$

In this case, a metabolizer H would need to have 25^2 elements, and all of them need to satisfy $\operatorname{sign}_1^{\mathrm{av}}(\tau(4K_6, \chi)) = 0$. It is not obvious if there are enough such elements at all (compare with Example 6.2 above). However, as already remarked earlier, it is difficult to explicitly compute all suitable characters χ in the general case. A possible workaround could be to consider the asymptotic behavior of the order of H and an estimate of the number of characters χ that satisfy $\operatorname{sign}_1^{\mathrm{av}}(\tau(rK_n, \chi)) = 0$. For this, further investigations are needed.

7 An Upper Bound for $g_{st}(K_n)$

In this section, we provide an upper bound for the stable 4-genus of the twist knots $g_{st}(K_n)$. Recall the definition of the stable 4-genus of knots from Section 2.4: if $K \subset S^3$ is a knot, then the stable 4-genus of K is defined as

$$g_{st}(K) = \lim_{n\to\infty} \frac{g_4(nK)}{n},$$

where nK denotes the n-fold connected sum $K\#\cdots\#K$. The basic approach to compute $g_{st}(K)$ is to find upper and lower bounds for the 4-genus $g_4(nK)$ of connected sums nK. While it is in general difficult to find suitable lower bounds, there is a convenient method for bounding the 4-genus from above by finding so-called *surgery curves* on a Seifert surface for K. We will describe this method in Section 7.1 and then use it in Section 7.2 to construct the upper bound for $g_{st}(K_n)$. In Section 7.3, we discuss to what extent the upper bound can be improved. Throughout the following sections, we frequently abuse notation and denote curves on surfaces and their homology class with the same symbols.

7.1 A Method for Bounding the Slice Genus

The method goes as follows (cf. [25]):

Let $K \subset S^3$ be a knot with Seifert surface F and Seifert pairing θ. Suppose that there exists an embedded curve $\alpha \subset F$ such that

1.) $[\alpha] \neq 0 \in H_1(F)$;

2.) α has framing zero, i.e. $\theta([\alpha],[\alpha]) = 0$;

3.) α traces out a slice knot in S^3.

A curve α satisfying 1.) - 3.) is called a *surgery curve*. Now, given a surgery curve α, we can perform surgery on F along α as follows. Since α has framing zero, we can embed a cylinder $\alpha \times I$ in F, and since α is slice, there is an embedding of $D \times S^0$ into B^4, where $D \subset B^4$ is a slice disk for α. Now perform surgery by cutting out $\alpha \times I \subset F$ and gluing back in the embedded $D \times S^0 \subset B^4$. This creates a new surface F' that is, after smoothing corners, properly and smoothly embedded in B^4. The surgery process is best visualized by taking two copies of α on F, cutting out the annulus between them (possible since $\theta(x,x) = 0$), and then adding two slice disks in B^4 at the resulting boundary curves. Note that it is important to be in dimension 4; for example, if α is a surgery curve that is an unknot in S^3, then adding a disk along α in S^3 could possibly produce ribbon or clasp singularities (this can be seen by considering F as a disk with a number of twisted, knotted, and linked bands attached).

The surgery leaves the boundary untouched, so the resulting surface F' has the same boundary as F, i.e. $\partial F' = \partial F = K$. Furthermore, the process of adding slice disks in B^4 reduces the genus of the surface by one, so we have

$$g(F') = g(F) - 1.$$

In particular, this yields the upper bound for the slice genus

$$g_4(K) \le g(F) - 1.$$

If there are multiple disjoint surgery curves available, say $\alpha_1, \dots, \alpha_n$, then it is possible to perform surgery along all of them in order to produce a surface F' with $\partial F' = \partial F = K$ and $g(F') = g(F) - n$, provided the curves satisfy the following additional properties:

1.)$'$ $[\alpha_1], \dots, [\alpha_n]$ need to be linearly independent in $H_1(F)$;

2.)$'$ $\theta([\alpha_i], [\alpha_j]) = 0$ for all $i, j = 1, \dots, n$;

3.)$'$ $a_1 \cup \dots \cup \alpha_n$ is a *slice link* in S^3.

Here, slice link means that the link bounds n disjoint properly and smoothly embedded disks D in the 4-ball B^4.

7.2 Constructing the Upper Bound for $g_{st}(K_n)$

Having the technique from the previous section in mind, our aim is now to find a surgery curve α for a connected sum of twist knots in order to bound the slice genus from above and derive an upper bound for the stable 4-genus of K_n. As in Section 6, let rK_n denote the m-fold connected sum of the twist knot K_n, where $m, n \in \mathbb{N}$, together with the Seifert surface rF_n shown below.

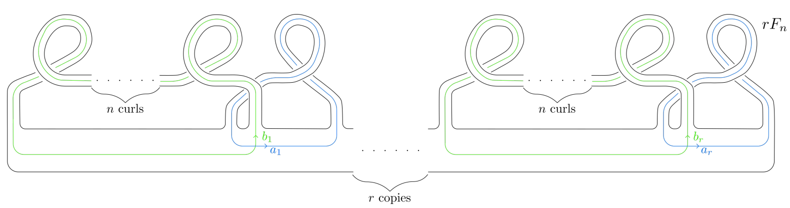


Figure 7.1: A Seifert surface for rK_n

In the basis $a_1, b_1, \dots, a_r, b_r \in H_1(rF_n)$ shown in Figure 7.1, the corresponding Seifert matrix is given as

$$\mathcal{A}_n := \bigoplus_1^m A_n = \begin{pmatrix} -1 & 1 & & & & \\ 0 & n & & & & \\ & & \ddots & & & \\ & & & & -1 & 1 \\ & & & & 0 & n \end{pmatrix}.$$

We wish to find a curve α on rF_n (for some suitable $r \in \mathbb{N}$) such that $[\alpha] \neq 0 \in H_1(mF_n)$, $\theta([\alpha], [\alpha]) = 0$, where θ is the Seifert pairing for mK_n, and such that α is a slice knot in S^3. Consider first the cases $n = 0$ and $n = 1$. If $n = 0$, then $mK_0 = K_0$ is the unknot for all $m \in \mathbb{N}$, and since the unknot is slice, there is no need to find a surgery curve. If $n = 1$, then K_1 is the figure-eight, which is amphicheiral (i.e. $K_1 = -K_1^*$). Since $K_1 \# - K_1^*$ is slice, it follows that $g_4(2K_1) = 0$, so $g_4(2mK_1) = 0$ and

$g_4((2m+1)K_1) = 1$ for all $m \in \mathbb{N}$. Thus, there is again no need to find a surgery curve. Note that $g_4(2K_1) = 0$ also implies that $g_{st}(K_1) = 0$.

Now, suppose that $n \geq 2$ and set $r = n - 1$. Consider the curve α on the Seifert surface $(n-1)F_n$ shown in Figure 7.2 below.

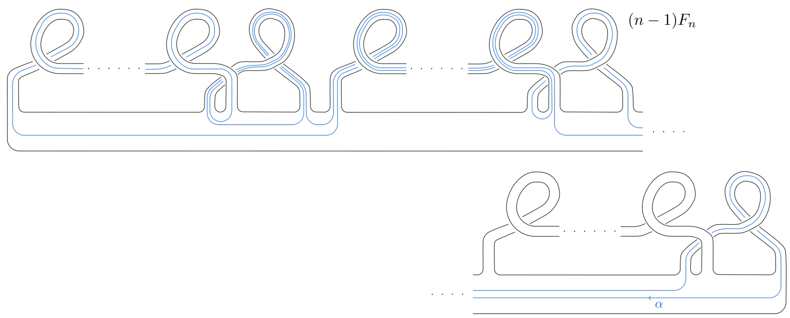


Figure 7.2: A surgery curve for $(n-1)F_n$

The following proposition shows that α is a surgery curve for $(n-1)F_n$.

Proposition 7.1. Suppose that $n \geq 2$. Then the curve α shown in Figure 7.2 forms a surgery curve for $(n-1)F_n$. In particular,

$$g_4((n-1)K_n) \leq n-2.$$

Proof. In order to show that α is a surgery curve, we have to check three things.

1.) In the basis $a_1, b_1, \ldots, a_{n-1}, b_{n-1} \in H_1((n-1)F_n)$ shown in Figure 7.1, α represents the element

$$[\alpha] = (2 \quad 1 \quad -1 \quad 0 \quad \cdots \quad -1 \quad 0) \in H_1((n-1)F_n)$$

which is certainly not zero in $H_1((n-1)F_n)$. Note that in order to draw the curve α as in Figure 7.2, it is necessary to pass through the band with n-curls on the second copy of F_n once in each direction. However, this does not affect the homology class of α.

2.) Let ξ denote the basis representation of $[\alpha]$ from above. Then

$$\mathcal{A}_{n-1}\xi = \begin{pmatrix} -1 & 1 & & & & & \\ 0 & n & & & & & \\ & & -1 & 1 & & & \\ & & 0 & n & & & \\ & & & & \ddots & & \\ & & & & & -1 & 1 \\ & & & & & 0 & n \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \\ \vdots \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ n \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix},$$

so

$$\theta([\alpha],[\alpha]) = \xi^T \mathcal{A}_n \xi = 0,$$

showing that α has framing zero.

3.) Consider the curve α in S^3 as shown below.

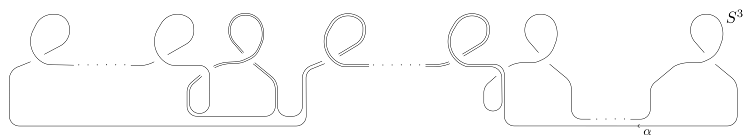


Figure 7.3: The curve α represents the unknot in S^3

Clearly, the curve α can be untangled to the unknot in S^3. Since the unknot is slice, α represents a slice knot in S^3.

The points above show that α is a surgery curve for $(n-1)F_n$. Thus, we can perform surgery on $(n-1)F_n$ along α to obtain a new surface $((n-1)F_n)'$ that is properly and smoothly embedded in B^4, bounds $(n-1)K_n$ in S^3, and satisfies $g(((n-1)F_n)') = g((n-1)F_n) - 1 = n-2$. Therefore

$$g_4((n-1)K_n) \leq n-2.$$

□

As a consequence, we can derive an upper bound for the stable 4-genus of the twist knots K_n.

Corollary 7.2. The stable 4-genus of the twist knots $g_{st}(K_n)$ satisfies

$$g_{st}(K_n) \leq \frac{n-2}{n-1},$$

provided $n \neq 0, 1$.

Proof. We know from Proposition 7.1 that

$$g_4((n-1)K_n) \leq n-2,$$

which implies that

$$\frac{g_4((n-1)K_n)}{n-1} \leq \frac{n-2}{n-1}.$$

Thus, by extracting a subsequence and using the subadditivity of g_4, we get

$$\begin{aligned} g_{st}(K_n) &= \lim_{m\to\infty} \frac{g_4(mK_n)}{m} \\ &= \lim_{m\to\infty} \frac{g_4(m(n-1)K_n)}{m(n-1)} \\ &\leq \lim_{m\to\infty} \frac{g_4((n-1)K_n)}{(n-1)} \\ &\leq \frac{n-2}{n-1}. \end{aligned}$$

□

Although the bound in Corollary 7.2 tends to 1 as n goes to infinity, it shows that for each individual $n \in \mathbb{N} \setminus \{0, 1\}$, the stable 4-genus $g_{st}(K_n)$ is not an integer (except possibly 0) and smaller than 1.

7.3 Perspectives on Improving the Bound

The question that now arises is if there are any further surgery curves to improve the upper bound in Propsition 7.1 (respectively Corollary 7.2). However, finding another surgery curve seems difficult because of the additional requirements that arise when performing surgery along multiple curves (see 2.)′ and 3.)′ in Section 7.1 above). Let us elaborate.

First of all, there are not too many curves available on F_n that are slice knots in S^3. Recall that the basis for $H_1(F_n)$ is given by the simple closed curves a and b shown below.

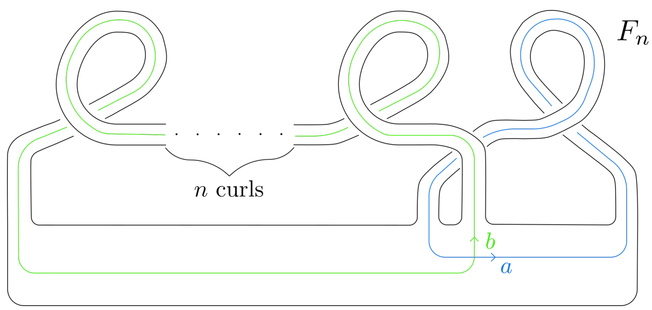


Figure 7.4: Seifert surface for K_n

One can show that any element of the form $x \cdot a + b$ is a non-slice knot for $x \notin \{-1, 0, 1, 2\}$. To convince the reader, consider the curves shown below.

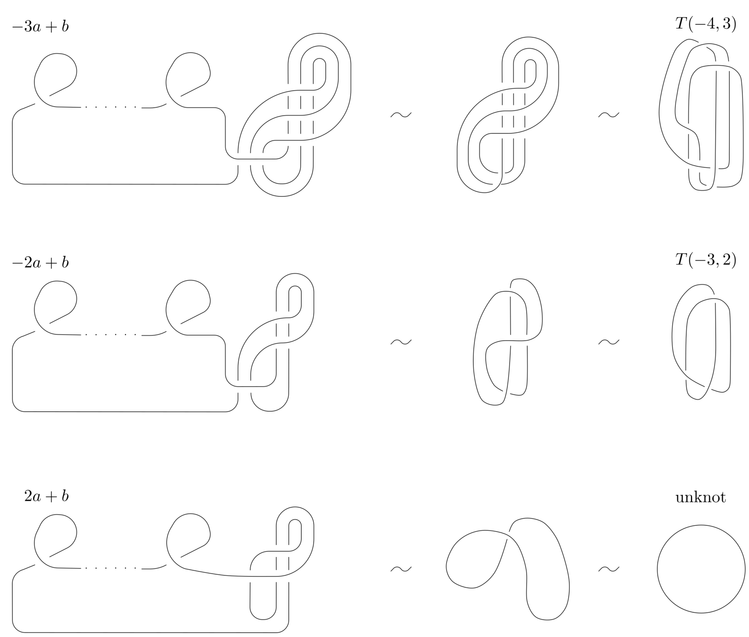


Figure 7.5: The curves $-3a + b$, $-2a + b$ and $2a + b$

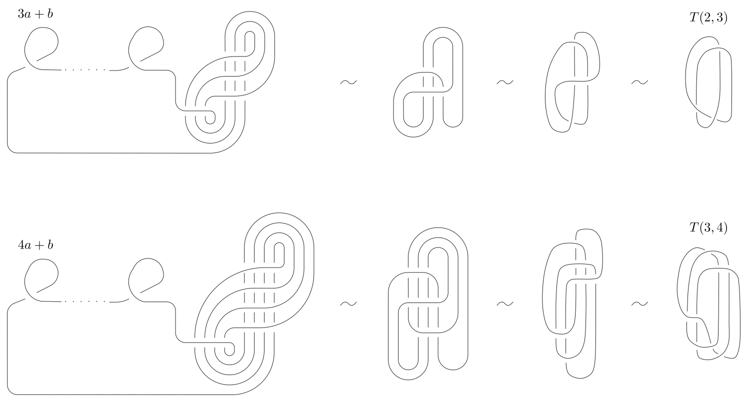


Figure 7.6: The curves $3a + b$ and $4a + b$

The curves shown in Figure 7.5 and 7.6 are the torus knots $T(-4, 3)$, $T(-3, 2)$, $T(1, 2)$, $T(2, 3)$ and $T(3, 4)$, respectively, which are all not slice except $T(1, 1)$ (note that the curve $2a + b$ is an unknot as shown in the proof of Proposition 7.1). In general, the curve $xa + b$ represents the torus knot $T(x - 1, x)$, which is not slice for $x \notin \{-1, 0, 1, 2\}$ (cf. [25]). Therefore, it is likely that most curves $xa + yb$ do not represent a slice knot in S^3.

Second, a surgery curve also needs to have framing zero. Consider for example some twist knot K_n, $n \geq 2$, and an arbitrary element $v = (x \quad y) \in H_1(F_n)$. Then

$$\theta(v, v) = v^T A_n v = (x \quad y) \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^2 + ny^2 + xy \in \mathbb{Z}.$$

Let $S(x, y) := -x^2 + ny^2 + xy$. Then, for $r \in \mathbb{N}$ and any simple closed curve β on rF_n, its framing is given as

$$\theta(\beta, \beta) = S(x_1, y_1) + S(x_2, y_2) + \cdots + S(x_r, y_r),$$

where $x_1, y_1, \ldots, x_r, y_r \in \mathbb{Z}$ are the coefficients of $\beta \in H_1(rF_n)$ in the basis shown in Figure 7.1 above (here, we are slightly abusing notation by denoting the Seifert pairing of F_n and of rF_n with θ). If β was a surgery curve, then the sum on the right-hand side would have to vanish. Since β also has to be a slice knot in S^3, the possibilites for β remain manageable. Consider for example the first few values for $S(x, y)$ in the case $n = 6$:

$x \mid y$	0	1	2	3	4	5	6	7	8
-1	-1	4	21	50	91	144	209	286	375
0	0	6	24	54	96	150	216	294	384
1	-1	6	25	56	99	154	221	300	391
2	-4	4	24	56	100	156	224	304	396

Table 7.1: The first few values for $S(x, y)$ in the case $n = 6$

Since β has to be a slice knot in S^3, we only need to consider the values $x = -1, 0, 1, 2$ by the discussion above. Also, note that we are omitting negative values for y in Table 7.1 because of the symmetries $S(-x, -y) = S(x, y)$ and $S(-x, y) = S(x, -y)$. From Table 7.1, we can construct various surgery curves β, for instance,

$$\beta = (-1 \quad 1 \quad -1 \quad 0 \quad -1 \quad 0 \quad -1 \quad 0 \quad -1 \quad 0) \in H_1(5F_6),$$

However, if we now wish to perform surgery along the curve α from the previous section and β, then we would additionally need that $\theta(\alpha, \beta) = 0$ and that $\alpha \cup \beta$ is a slice link in S^3, and this is where most surgery curves β get excluded. For example, a short computation shows that for the curve β above,

$$\theta(\alpha, \beta) = 6, \quad \theta(\beta, \alpha) = 3,$$

so β can not be used to perform surgery along α and β. A similar behavior can be observed for other values of $n \in \mathbb{N}$ and surgery curves β.

Although this discussion does certainly not prove that there are no other curves available to improve the bound from Corollary 7.2, it shows that most likely other methods are needed in order to gain further informations about the stable 4-genus of the twist knots $g_{st}(K_n)$.

8 References

[1] E. Artin, *Zur Isotopie Zweidimensionaler Flächen im R_4*, Abhandlungen Hamburg, Vol. 4, No. 1, 1925, pp. 174-177.

[2] G. E. Bredon, *Geometry and Topology*, corrected third printing of the 1993 original, Graduate Texts in Mathematics, Vol. 139, Springer, New York, 1997.

[3] G. Burde, H. Zieschang, M. Heusener, *Knots*, 3rd, fully revised and extended edition, de Gruyter Studies in Mathematics, Vol. 5, Walter de Gruyter, Berlin, 2014.

[4] D. Calegari, *scl*, MSJ Memoirs, Vol. 20, Mathematical Society of Japan, Tokyo, 2009.

[5] A. J. Casson, C. McA. Gordon, *Cobordism of Classical Knots*, À la Recherche de la Topologie Perdue, Progress in Mathematics, Vol. 62, Birkhäuser Boston, 1986, pp. 181-199. With an appendix by P. M. Gilmer.

[6] A. J. Casson, C. McA. Gordon, *On Slice Knots in Dimension Three*, Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, 1976), Part 2, Proc. Sympos. Pure Math., Vol. 32, Am. Math. Soc., Providence, Rhode Island, 1978, pp. 39-53.

[7] A. Conway, *Algebraic Concordance and Casson-Gordon Invariants*, notes of a reading group held at the university of Geneva, Geneva, 2017, available online at http://www.unige.ch/math/folks/conway/Notes/AlgebraicConcordanceCassonGordon.pdf

[8] A. Conway, S. Friedl, G. Herrmann, *Linkg Forms Revisited*, Pure Appl. Math. Q., Vol. 12, No. 4, 2016, pp. 493-515.

[9] T. D. Cochran, K. E. Orr, P. Teichner, *Knot Concordance, Whitney Towers, and L^2-Signatures*, Ann. of Math. (2), Vol. 157, No. 2, 2003, pp. 433-519.

[10] J. F. Davis, P. Kirk, *Lecture Notes in Algebraic Topology*, Graduate Studies in Mathematics, Vol. 35, American Mathematical Society, Oxford University Press, 2001.

[11] M. Dehn, *Die beiden Kleeblattschlingen*, Math. Ann., Vol. 75, No. 3, 1914, pp. 402-413.

[12] S. K. Donaldson, *An Application of Gauge Theory to Four-Dimensional Topology*, J. Differential Geom., Vol. 18, No. 2, 1983, pp. 279-315.

[13] R. H. Fox, *A Quick Trip through Knot Theory*, Topology of 3-Manifolds and Related Topics (Proc. of the Univ. of Georgia Institute, 1961), Prentice Hall, Englewood Cliffs, New Jersey, 1962, pp. 120-167.

[14] R. H. Fox, J. Milnor, *Singularities of 2-Spheres in 4-Space and Cobordism of Knots*, Osaka J. Math., Vol. 3, No. 2, 1966, pp. 257-267.

[15] F. Frankl, L. Pontrjagin, *Ein Knotensatz mit Anwendungen auf die Dimensionstheorie*, Math. Ann., Vol. 102, No. 1, 1930, pp. 785-789.

[16] S. Friedl, T. Kim, *The Thurston Norm, Fibered Manifolds and Twisted Alexander Polynomials*, Topology, Vol. 45, No. 6, 2006, pp. 929-953.

[17] S. Friedl, S. Vidussi, *The Thurston Norm and Twisted Alexander Polynomials*, J. Reine Angew. Math., Vol. 707, 2015, pp. 87-102.

[18] P. M. Gilmer, *Slice Knots in* S^3, Quart. J. Math. Oxford (2), Vol. 34, No. 135, 1983, pp. 305-322.

[19] P. M. Gilmer, *On the Slice Genus of Knots*, Invent. Math., Vol. 66, No. 2, 1982, pp. 191-197.

[20] R. E. Gompf, *Smooth Concordance of Topologically Slice Knots*, Topology, Vol. 25, No. 3, 1986, pp. 353-373.

[21] C. McA. Gordon, *Some Aspects of Classical Knot Theory*, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), Lecture Notes in Math., Vol. 685, Springer, Berlin, 1978, pp. 1-60.

[22] C. McA. Gordon, J. Luecke, *Knots are Determined by their Complements*, Bull. Am. Math. Soc. 2, Vol. 20, No. 1, 1989, pp. 83-87.

[23] J. E. Grigsby, D. Ruberman, S. Strle, *Knot Concordance and Heegaard Floer Invariants in Branched Covers*, Geom. Topol., Vol. 12, No. 4, 2008, pp. 2249-2275.

[24] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[25] L. H. Kauffman, *On Knots*, Annals of Mathematics Studies, Vol. 115, Princeton University Press, Princeton, 1987.

[26] P. B. Kronheimer, T. S. Mrowka, *Gauge Theory for Embedded Surfaces I*, Topology, Vol. 32, No. 4, 1993, pp. 773-826.

[27] J. M. Lee, *Introduction to Topological Manifolds*, Second Edition, Graduate Texts in Mathematics, Vol. 202, Springer, New York, 2011.

[28] J. M. Lee, *Introduction to Smooth Manifolds*, Second Edition, Graduate Texts in Mathematics, Vol. 218, Springer, New York, 2013.

[29] J. Levine, *Invariants of Knot Cobordism*, Invent. Math., 8, 1969, pp. 98-110; addendum ibid. 8 1969 355, pp. 98-110.

[30] J. Levine, *Knot Cobordism Groups in Codimension Two*, Comment. Math. Hel., Vol. 44, 1969, pp. 229-244.

[31] W. B. R. Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, Vol. 175, Springer, New York, 1997.

[32] W. B. R. Lickorish, *A Representation of Orientable Combinatorial 3-Manifolds*, Ann. of Math. (2), Vol. 76, No. 3, 1962, pp. 531-540

[33] R. A. Litherland, *Cobordism of Satellite Knots*, Contemp. Math., Vol. 35, Am. Math. Soc., 1984, pp. 327-362.

[34] C. Livingston, *The Stable 4-Genus of Knots*, Algebr. Geom. Topol., Vol. 10, No. 4, 2010, pp. 2191-2202.

[35] C. Livingston, *A Survey of Classical Knot Cobordance*, Handbook of Knot Theory, Chapter 7, Elsevier B. V., Amsterdam, 2005, pp. 319-347.

[36] C. Livingston, S. Naik, *Introduction to Knot Concordance*, work in progress, http://pages.iu.edu/∼livingst/Book.pdf, draft: 26 Aug. 2016.

[37] J. W. Milnor, *Most Knots are Wild*, Fund. Math., Vol. 54, 1964, pp. 335-338.

[38] J. W. Milnor, D. Husemoller, *Symmetric Bilinear Forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer, New York-Heidelberg, 1973.

[39] J. R. Munkres, *Topology*, Second Edition, Prentice Hall, Inc., Upper Saddle River, New Jersey, 2000.

[40] K. Murasugi, *Knot Theory and its Applications*, translated from the 1993 Japanese original by Bohdan Kurpita, reprint of the 1996 translation, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, 2008.

[41] K. Murasugi, *On a certain Numerical Invariant of Link Types*, Trans. Amer. Math. Soc., Vol. 117, 1965, pp. 387-422.

[42] K. Murasugi, *On the Signature of Links*, Topology, Vol. 9, 1970, pp.283-298.

[43] P. Ozsváth, Z. Szabó, *Knot Floer Homology and the Four-Ball Genus*, Geom. Topol., Vol. 7, 2003, pp. 615-639.

[44] J. Rasmussen, *Khovanov Homology and the Slice Genus*, Invent. Math., Vol. 182, No. 2, pp. 419-447.

[45] D. Rolfsen, *Knots and Links*, AMS Chelsea Edition, AMS Chelsea Publishing, Providence, Rhode Island, 2003.

[46] L. Rudolph, *The Slice Genus and the Thurston-Bennequin Invariant of a Knot*, Proc. Amer. Math. Soc., Vol. 125, No. 10, 1997, pp. 3049-3050.

[47] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann., Vol. 110, No.1, 1934, pp. 571-592.

[48] H. Seifert, *Verschlingungsinvarianten*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. 1933, No. 26-29, 1933, pp. 811-828.

[49] N. E. Steenrod, *Homology with Local Coefficients*, Ann. Math. (2), Vol. 44, No. 4, 1963, pp. 610-627.

[50] L. R. Taylor, *On the Genera of Knots*, Topology of Low-Dimensional Manifolds (Proc. sec. Sussex Conf., Chelwood Gate, 1977), Lecture Notes in Math., Vol. 722, Springer, Berlin, 1979, pp. 144-154.

[51] P. Teichner, *Slice Knots: Knot Theory in 4th Dimension*, lecture course given by Peter Teichner in San Diego in 2001, notes by Justin Roberts, typed from the original version by Julia Collins and Mark Powell, 22 June 2010.

[52] H. Tietze, *Ueber die Topologischen Invarianten Mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys., Vol. 19, No. 1, 1908, pp. 1-118.

[53] M. B. Thistlewaite, J. Hoste, J. Weeks, *The First 1'701'936 Knots*, Math. Intelligencer, Vol. 20, No. 4, 1998, pp. 33-48.

[54] A. G. Tristram, *Some Cobordism Invariants for Links*, Proc. Cambridge Philos. Soc., Vol. 66, 1969, pp. 251-264.

[55] C. T. C. Wall, *Quadratic Forms on Finite Groups, and Related Topics*, Topology, Vol. 2, 1964, pp. 281-298.

[56] A. H. Wallace, *Modifications and Cobounding Manifolds*, Can. J. Math., Vol. 12, 1960, pp. 503-528.

[57] W. Whitten, *Knot Complements and Groups*, Topology, Vol. 26, No. 1, 1987, pp. 41-44.