Université Paris VII - Denis Diderot

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

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Homologies de Khovanov-Rozansky, toiles nouées pondérées et genre lisse

Khovanov-Rozansky homologies, knotted weighted webs and the slice genus

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dirigée par Christian Blanchet

Soutenue le 28 juin 2013 devant le jury composé de :

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Remerciements

Je tiens d'abord à remercier chaleureusement Christian Blanchet, mon directeur de thèse, qui m'a initié au monde de la recherche. Il a toujours été disponible pour partager son grand savoir mathématique et me donner des conseils profonds et pertinents. Je lui suis très reconnaissant pour m'avoir guidé pendant ces quatre années.

Continuing in English, I express my profound gratitude to Francesco Costantino and Andrew Lobb for being the *rapporteurs* of this thesis, and for their valuable comments on the manuscript. Also thanks to Andrew Lobb for welcoming me at Durham University.

I am happy that Jacob Rasmussen, on whose work much of this thesis builds, and Sebastian Baader, who initiated me to Khovanov homology during my master's thesis, have agreed to participate in the jury. It is an honour that Pierre Vogel and Michael Eisermann have accepted to be jury members.

In the course of my doctorate, I had the opportunity to attend a number of conferences, which were great sources of inspiration and perspective; many thanks in particular to the organisers of the three Swiss Knots conferences, to Thomas Fiedler (Université Toulouse III) for the winter schools of La Llagonne, and to the organisers of the workshop "Geometry and topology of smooth 4-manifolds" in Bonn.

I wish to thank Pedro Vaz (Université catholique de Louvain), Alexander Shumakovitch (George Washington University), Paul Turner (Université de Genève) and Anna Beliakova (Universität Zürich) for their encouragement, and Victoria Lebed (Université Paris 7), Arnaud Mortier (Université Toulouse III), Ana G. Lecuona (Université d'Aix-Marseille), Peter Feller (Universität Bern) and my two *frères de thèse* Louis-Hadrien Robert and Hoel Queffelec for the inspiring exchanges of ideas. Je repasse en français pour remercier Louis-Hadrien pour les discussions presque quotidiennes le long de nos années communes de thèses. J'ai hautement apprécié son enthousiasme et sa rigueur.

Je remercie les gestionnaires de l'Institut de Mathématiques de Jussieu et de l'Université Paris-Diderot pour m'avoir aidé à traverser la jungle administrative : merci en particulier à Michèle Wasse, Christine Le Sueur, Regine Guittard et Pascal Chiettini.

Merci aux étudiants de MI2, MI2 et MM2 pour qui j'ai eu la chance d'enseigner : enseigner, c'est aussi apprendre.

L'ambiance à Chevaleret et Sophie-Germain est extraordinaire et pour cela je remercie, dans un ordre chaotique, mes co-bureaux Elodie, Robert, Martin et Sadek ; mes co-BDDs Ismaël, Baptiste et Louis-Hadrien ; pour la musique, Daniel et Pierre-Guy ; pour l'appréciation du cactus, Paloma ; merci à Sarah, Fathi, Mathieu, David, Delphine, Hendrik, Christophe, Victoria, Giovanni, Eirini, Claudia, Elena, Alex, Alfredo, Hoel, Johan, Yohan, Dragos, Arnaud, Isidor, Julie, à ceux que j'ai oublié (pardon !), et à tous – pour partager le petit café d'après-midi (ouuu...), les pauses de thé, pour le foot, les films, les concerts, les jeux. Merci à mes colocataires Martin et Irina, et *special thanks* à Genéviève ; merci à Alexis pour la musique, et merci à Narimane, Jeff et Monica. Merci de tout mon cœur pour le bon temps passé ensemble.

Ich danke meinen Freunden aus Deutschland und der Schweiz, die mich im Lauf der Jahre in Paris besucht haben: Stefan, Charles (trotz des unbequemen Sofas), Anne, Kristian (unabsichtlich), Max, Helen, Johannes (immer mit coq au vin), Philipp, Sandra, Andrin (der eine Wette verliert), Martin und Andreas; und Dank an Andrée (für vielerlei Narretei), Gidon, Kerstin, Anne, Manuel, Stefan, Teresa und Saskia (die Silvestergemeinschaft). Tausend Dank für die großartigen Stunden und Tage – und kommt doch mal nach England.

Dank an Katrins Familie, an Christine, Karl-Heinz, Nadine und Patrick, bei denen ich mich immer sehr wohl gefühlt habe.

Ich danke meiner Familie, Conny, Matthias, Mirjam, Ida, Nadine, Stephan und Yannik, und vor allem meinen Eltern, Sybille und Siegfried, die mich immer uneingeschränkt ermutigt und unterstützt haben. Ohne sie hätten weder ich noch diese Arbeit je das Licht der Welt erblickt.

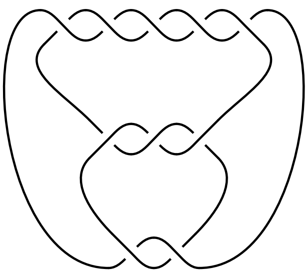
Mein größter Dank gilt Katrin, die mit mir die geographische Entfernung überwindet und ihr Leben teilt. Diese Arbeit ist ihr gewidmet.

Contents

Introduction

This PhD-thesis is composed of three largely independent projects: an examination of the interrelations of the different Khovanov-Rozansky homologies (Chapter I), an implementation of an algorithm to compute $\mathfrak{sl}_3$-foam homology (Chapter III), and an extension of $\mathfrak{sl}_3$-homology to a class of knotted graphs (Chapter IV). Chapter II contains a review of $\mathfrak{sl}_3$-homology, on which Chapter III and IV depend. The guiding idea uniting all three projects is to deepen our understanding of the Khovanov-Rozansky homologies and consequently gain geometrical information.

Before stating the results obtained in this thesis let us outline its mathematical context. Classical knot theory is the study of embeddings of a circle into the 3-space, up to continuous deformation. A question which presents itself immediately is: how to tell if a given knot is trivial? Or more generally, given two knots, are they equal? Another, more specific problem is to find the slice genus of a knot: the minimal genus of a surface smoothly embedded into the 4-space that bounds the knot.



A diagram of the $(5, -3, 2)$-pretzel knot.

Apart from methods stemming from algebraic topology, it is possible to analyse a knot via one of its diagrams. In this way, in 1984 Jones [Jon85] found a new polynomial knot invariant which triggered what is frequently called a revolution in knot theory. The Jones polynomial often (but not always) distinguishes different knots; whether it detects the unknot is still an open problem. It is also linked to the representation theory of quantum $\mathfrak{sl}_2$. Reshetikhin and Turaev generalised this approach to all quantum versions of classical lie algebras, defining the $\mathfrak{sl}_N$-polynomials on the way.

Categorification has become what one might call a second revolution in knot theory. To *categorify* means to step up one rung on the abstraction ladder. For example, finite dimensional vector spaces over a fixed field are a categorification of the natural numbers: a number n is replaced by spaces of dimension n. The categorified structure is richer, because there are (non-trivial) homomorphisms between spaces, something which has no counterpart in the uncategorified world.

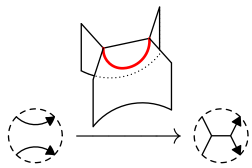
Categorification in knot theory began in 1999 with Khovanov's categorification of the Jones polynomial [Kho00]. On the one hand, Khovanov homology is simply a stronger invariant than the Jones polynomial, distinguishing knots the latter cannot. It has even been proven to detect the unknot [KM11] (cf. also [GW10, Hed09]). On the other hand, its structure is richer, because it comprises cobordisms between knots. So, unlike the Jones polynomial, Khovanov homology contains geometrical information: Rasmussen [Ras10] showed how to extract a lower bound to the slice genus from it, which is strong enough to solve problems that could before only be tackled by gauge theory, as the Milnor conjecture.

In the last decade categorification has flowered. Ozsváth and Szabó [OS04], and independently Rasmussen [Ras03] categorified the Alexander polynomial; this *knot Floer homology* proved to be a powerful invariant, detecting e.g. the 3-genus of knots. The $\mathfrak{sl}_N$-polynomials and the HOMFLYPT-polynomial were categorified by Khovanov and Rozansky, cf. [KR08a] and [KR08b], respectively (see also [DGR06]). These *Khovanov-Rozansky* homologies (by which term we mean both the $\mathfrak{sl}_N$-homologies and the HOMFLYPT-homology) form the ecosystem in which this thesis lives. Let us have a closer look at how they arise and in what different flavours they come.

Categorification is not an automatic process. There is a magic ingredient one has to add, from which the to-be-defined invariant will draw its power: this ingredient is the category over which one works, in particular its morphisms. For Khovanov homology, the morphisms are cobordisms between 1-manifolds. By means of a TQFT, these subsequently yield graded abelian groups and graded homomorphisms between them.

The category used for Khovanov-Rozansky homologies is of a different nature. Its objects are certain plane trivalent bipartite graphs with thick edges, called *webs*. The calculus of these webs has been developed in [MOY98]. By contrast with Khovanov homology, Khovanov and Rozansky give no geometrical morphisms between such webs; instead, they first render the objects algebraic, associating *matrix factorisations* to webs, and then use homomorphisms of matrix factorisations as morphisms. So a priori, Khovanov-Rozansky homology is of a less geometric character than Khovanov homology.

This is different in the special case $N = 3$. For representation theoretic reasons, one may work with simpler webs over $\mathfrak{sl}_3$: thick edges are unnecessary, and a web becomes just a plane trivalent oriented graph, whose every vertex is a source or a sink, and that may have additional edges that are vertex-less circles. Morphisms may then be defined geometrically, as cobordisms between webs; just as webs are 1-manifolds except in finitely many points, where they resemble the letter Y, *foams* are mostly 2-manifolds, except close to finitely many circles or intervals, where they resemble the letter Y times a circle or an interval. This description of $\mathfrak{sl}_3$-homology [Kho04] predates the general Khovanov-Rozansky homologies.



A foam, cobordism of webs.

In fact, a slightly more complicated construction allows also the morphisms of the Khovanov-Rozansky homologies to be understood as foams, see [MSV09]. Let us summarise: for all $N \geq 2$, there are $\mathfrak{sl}_N$-homology theories based on matrix factorisations, and geometric $\mathfrak{sl}_N$-homology theories, which are particularly simple for $N = 2$ and $N = 3$. These different constructions are isomorphic for the same N [KR08a, MV08a, MSV09], and so it makes sense to speak of *the* $\mathfrak{sl}_N$-homology. HOMFLYPT-homology may equivalently be constructed using Soergel-bimodules [Kho07].

Let us now present the most important results of chapters I, III and IV.

Khovanov-Rozansky homologies and the $\mathfrak{sl}_N$-concordance invariants

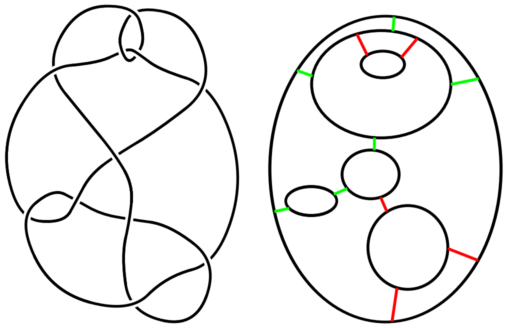
As a matter of fact, the Khovanov-Rozansky construction yields a homology theory for every complex polynomial $p(X)$, the $\mathfrak{sl}_N$-homology corresponding to $p(X) = X^N$. Taking for p a polynomial with N distinct roots, like $p(X) = X^N - 1$, yields a homology theory which is no longer graded, but merely filtered [Gor04]. Let us call it *filtered* $\mathfrak{sl}_N$*-homology*. For a knot K, it has a particularly simple form, being entirely described by an even integer s'_N. The normalisation $s_N = s'_N/(-N-1)$ is a concordance invariant [Lob12], which we will call the $\mathfrak{sl}_N$-concordance invariant (see definition I.3.4). This was first discovered by Rasmussen for the $N = 2$ case [Ras10], building on the work of Lee [Lee05]. The relationship of the different s_N-invariants is an open problem. Lobb conjectures the following, stating that "We hope that this is not true, and do not know whether to expect it to be true":

Conjecture ([Lob12, Conjecture 1.5 and 1.6])**.**

(i) *For any knot K and $N, N' \geq 2$, we have $s_N(K) = s_{N'}(K)$.*

(ii) *Or, failing that, $s_N(K) \in 2\mathbb{Z}$ for all N.*

The $\mathfrak{sl}_N$-concordance invariants belong to the class of *slice-torus knot concordance invariants* (Livingston [Liv04] was the first to consider this class, but did not baptise it): i.e., they give a lower bound to the slice genus [Lob09, Wu09], which is sharp for positive torus knots. Analysing this class of invariants, we prove that switching the sign of a crossing from negative to positive, the value of a slice-torus invariant y does not decrease, and may increase by at most 2. We show that y can easily be determined from a quasi-alternating diagram. Furthermore we generalise to links and find similar statements to hold for *slice-torus link invariants*. Finally, we prove the following inequality:



$K = 11n_{53}$ and its Seifert graph (positive edges are green, negative red). $w = 1, n = 6, O^{\pm} = 3 \Rightarrow 0 \leq y(K) \leq 2$.

Definition I.1.11. *The* Seifert graph $\Gamma(D)$ *of a link diagram D is a plane bipartite graph whose edges carry a sign ($+$ or $-$). It is constructed as follows:*

- *The vertices of $\Gamma(D)$ correspond to the circles of the Seifert resolution of D.*
- *A fixed crossing of D is adjacent to two different Seifert circles, which correspond to two vertices in $\Gamma(D)$. For any crossing, let $\Gamma(D)$ have an edge between these two vertices. The edge's sign indicates if the crossing is positive or negative.*

Let $\Gamma^+(D)$ ($\Gamma^-(D)$) be the subgraph of $\Gamma(D)$ that contains only the positive (negative) edges. Let $O^{\pm}(D)$ be the number of connected components of $\Gamma^{\pm}(D)$.

Theorem I.1.12. *Let D be a diagram of a knot K, with writhe w and n Seifert circles. Then*

$$-1 + w - n + 2O^+ \leq y(L) \leq 1 + w + n - 2O^-.$$

The upper and a lower bound agree for homogeneous diagrams. These results on slice-torus invariants are straight-forward generalisations of the work undertaken in [Liv04, Shu07, Kaw, Lob11, Wu07, Abe11].

Every $\mathfrak{sl}_N$-homology has as well a *reduced* version. Reduced Khovanov homology was defined in [Kho03], and reduced $\mathfrak{sl}_N$-homology, along with the unreduced version, in [KR08a]. While unreduced $\mathfrak{sl}_N$-homology categorifies the $\mathfrak{sl}_N$-polynomial normalised to give the value $[N]_q$ for the unknot, reduced $\mathfrak{sl}_N$-homology categorifies the version of the polynomial giving 1 for the unknot. But whereas the polynomials just differ by a constant factor, the relationship of unreduced and reduced homology is more complicated, and of yet not well understood. HOMFLYPT-homology, on the other hand, has several versions of varying reducedness, but they just differ by a constant factor.

We prove the existence of a spectral sequence which links reduced ($\overline{[\![\cdot]\!]}_N$) to unreduced $\mathfrak{sl}_N$-homology ($[\![\cdot]\!]_N$):

Theorem I.4.3. *There is a spectral sequence with first page $[N]_{qr} \cdot \overline{[\![L]\!]}_N$, where r is an additional degree. Ignoring this degree, the sequence's limit is isomorphic to $[\![L]\!]_N$. The spectral sequence respects the q-degree and converges on the N-th page. Its higher pages are invariants of links with a marked component. The k-th differential is homogeneous of degree $tr^{2k}q^0$.*

We have pointed out a variety of different Khovanov-Rozansky homology theories: one for each $N \geq 2$, and one for the HOMFLYPT-polynomial, each with an unreduced, filtered and reduced version. We define a reduced version of filtered $\mathfrak{sl}_N$-homology, thus completing the picture of the different versions of Khovanov-Rozansky homologies and their interrelations.

These different homology theories are related to each other by means of spectral sequences: there are Gornik's spectral sequences from $\mathfrak{sl}_N$-homology to filtered $\mathfrak{sl}_N$-homology [Gor04]. For the $N = 2$ case this was found by [Lee05]. The existence of this spectral sequence allows often (but not always) to extract the value of s_N from unreduced or reduced $\mathfrak{sl}_N$-homology.

Then, there are Rasmussen's spectral sequences from reduced HOMFLYPT-homology to reduced $\mathfrak{sl}_N$-homology [Ras06]. For sufficiently large N, they converge on the first page; in other words, HOMFLYPT-homology is the stabilisation of the $\mathfrak{sl}_N$-homologies as $N \to \infty$. It is not yet clear what this implies concerning the behaviour of the s_N-concordance invariants as $N \to \infty$.

Rasmussen's spectral sequences endow the $\mathfrak{sl}_N$-homology with an extra grading, which we exploit to define a new link invariant:

Definition I.4.6. *Let L be a link. Proposition I.4.1 gives a spectral sequence from $\overline{[\![L]\!]}_\infty$ converging to a regraded version of $\overline{[\![L]\!]}_1$. But since* $\operatorname{xdim} \overline{[\![L]\!]}_1 = 1$, *the limit of this spectral sequence has graded dimension $q^{s_\infty(L)}a^{-s_\infty(L)}$ for some $s_\infty(L) \in 2\mathbb{Z}$. The spectral sequence is a link invariant, and therefore $s_\infty(L)$ is as well.*

Conjecture I.4.8. *The link invariant s_∞ is a slice-torus link invariant.*

Combining Gornik's, Rasmussen's and our spectral sequence, we deduce a relationship between reduced HOMFLYPT-homology and the $\mathfrak{sl}_N$-concordance invariants. This implies in particular a lower bound to the slice genus from HOMFLYPT-homology.

Corollary I.5.3. *Let K be a knot and $N \geq 2$. Then there are integers α, β, α', β', such that the* HOMFLYPT*-homology of K contains generators of degrees $q^\alpha a^\beta$ and $q^{\alpha'}a^{\beta'}$ and*

$$s'_N(K) - 2N + 2 \leq \alpha + N\beta \leq s'_N(K) \leq \alpha' + N\beta' \leq s'_N(K) + 2N - 2.$$

For the normalised s_N, these inequalities read

$$s_N(K) - 2 \leq -(\alpha' + \beta')/(N-1) - \beta' \leq s_N(K) \leq -(\alpha + \beta)/(N-1) - \beta \leq s_N(K) + 2.$$

Combining the sharper slice-Bennequin-inequality with the bounds given by HOMFLYPT-homology, we prove exemplarily that the $\mathfrak{sl}_N$-concordance invariants differ:

Theorem I.6.1. *Let $\ell > m \geq 3$, $n \geq 2$, and $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$. Then*

$$
\begin{aligned}
s_\infty(P(\ell,-m,n)) &= \ell - m - 2 && \text{(assuming } s_\infty \text{ is slice-torus)}\\
s_2(P(\ell,-m,n)) &= \ell - m && \text{for } m > n,\\
s_2(P(\ell,-m,n)) &= \ell - m - 2 && \text{for } m < n,\\
s_N(P(\ell,-m,n)) &= \ell - m - 2 && \text{for } n > 2, N > 2,\\
s_N(P(\ell,-m,2)) &\in \left\{\ell - m - 2, \ell - m - 2 + \frac{2}{(N-1)}\right\} && \text{for } N > 2.
\end{aligned}
$$

Note that the infinite family of pretzel knots $P(\ell, m, n)$ with $\ell > m > n \geq 2$, $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$ on which s_2 disagrees with all s_N for $N \geq 3$ is composed of quasi-alternating knots. For these pretzel knots, the Rasmussen invariant gives the best slice genus bound among the $\mathfrak{sl}_N$-concordance invariants. However, we also give an example of a knot ($12n_{340}$) for which s_N with $N \geq 3$ gives a better bound than s_2. Altogether, we answer (i) of Lobb's conjecture in the negative.

While chapter I sheds some light on the relationship between the different Khovanov-Rozansky homologies, lots of questions remain open. Our results prompt in particular the following conjecture:

Conjecture. *Let K be a knot.*

(i) *For all $N \geq 2$ and $x \in \frac{2}{N-1}\mathbb{Z}$, there is a knot K with $s_N(K) = x$.*

(ii) *For all $N > N' \geq 2$, there is a knot K with $s_N(K) \neq s_{N'}(K)$.*

(iii) *The value of the $\mathfrak{sl}_N$-concordance invariant of K converges for $N \to \infty$.*

(iv) *It even stabilises.*

(v) *The limit is $s_\infty(K)$.*

Let us mention two possible applications: in the spirit of [Liv08], one could use the s_N-invariants to prove the existence of a larger free summand in the subgroup of the smooth knot concordance group composed of topologically slice knots; if part (ii) of the above conjecture proves to be correct, one might even detect a countably infinitely generated free summand.

Secondly, an s_N-invariant with $N \geq 3$ could be used to find an exotic smooth structure on S^4, following the approach of [FGMW10].

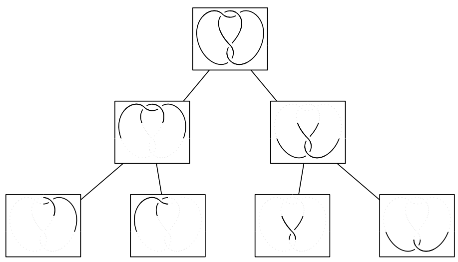
Automated $\mathfrak{sl}_3$-foam homology calculations

All of the homologies in consideration are completely combinatorial in nature – unlike e.g. the original knot Floer homology – meaning that their definition is in itself a description how to compute them. But by hand, this direct way of computation is hardly practicable for any but the smallest knots. So for manual calculation of HOMFLYPT-homology and graded $\mathfrak{sl}_N$-homology, one is better off to use the following three tools: firstly, the homology of two-bridge knots is *thin*, which means it can be read directly form its Euler characteristic. Secondly, the homologies of the two knots and the link which locally differ as vs. vs. are related by a long exact sequence in homology. Thirdly one may use the various spectral sequences which relate the different homologies to each other. Mackaay and Vaz demonstrate [MV08b] how the combination of these techniques allows to calculate the HOMFLYPT-homology of quite complicated knots such as the Conway knot and the Kinoshita-Terasaka knot.

The filtered $\mathfrak{sl}_N$-homology is the hardest to calculate, yet one of the most interesting due to its geometrical applications. Being filtered is much weaker a property than being graded, and neither the first nor the second tool of calculation mentioned in the last paragraph is available to calculate filtered homology. However, the third tool – a spectral sequence converging to filtered $\mathfrak{sl}_N$-homology – readily offers itself to application.

The situation is different if one attempts to do computer-aided calculations. Even the straight-forward method, as implemented with some tweaks in the programme `KhoHo` [Shu03] by Shumakovitch, can already compute the Khovanov homology of knots with up to ca. 20 crossings. Bar-Natan's extension of $\mathfrak{sl}_2$–homology from link diagrams to tangles [BN05] (see also [Kho02]) led subsequently to a divide-and-conquer algorithm to compute $\mathfrak{sl}_2$–homology [BN07]. The speed of this algorithm depends primarily on the *girth* of the link diagram: this is the maximal number of intersection points of a horizontal line with the diagram (see e.g. [Fre09], and cf. section III.1 for details). An implementation by Green and Morrison called `JavaKh` [GM05] is able to compute the Khovanov homology of knots of girth up to 14, e.g. the $(8, 7)$-torus knot. Mackaay and Vaz [MV07] and Morrison and Nieh [MN08] then extended $\mathfrak{sl}_3$–homology to tangles, and the latter describe in detail the ensuing algorithm. Before this thesis, this algorithm had not been implemented. Then, there is Carqueville and Murfet's programme [CM11], which calculates $\mathfrak{sl}_N$-homology for general N, but is only fast enough to calculate knots with up to six crossings; and Webster's programme [Web05], which calculates HOMFLYPT-homology, but is mainly restricted to three-stranded braids. Hence, as far as computer calculations are concerned, there are programmes to efficiently calculate the Khovanov homology of a given knot or link, but there are no such programmes for any other Khovanov-Rozansky homology.

In chapter III, we present an implementation of Morrison and Nieh's algorithm [MN08] as a C++-programme called `FoamHo`, which is able to compute the unreduced and reduced integral $\mathfrak{sl}_3$-homology of knots and links.



A sub-tangle tree of the figure-eight-knot with girth 4.

We show how Morrison and Nieh's algorithm to compute $\mathfrak{sl}_3$-homology can be improved by gluing sub-tangles in a more flexible way, along a *sub-tangle tree* instead of one after the other. This leads to the notion of the *recursive girth* of a link, which replaces the girth as main factor limiting calculation speed, and is in general smaller than the girth.

Definition III.1.1. *Let D be a non-split link diagram with crossings enumerated from* 1 *to* n. *Decompose D into small tangle diagrams D_i that contain the i–th crossing, respectively. A* sub-tangle tree *of D is a full binary tree with a tangle diagram at each node, such that*

- *The leaves are decorated by the D_i.*
- *The root is decorated by D.*
- *Every node which is not a leaf has two children decorated with adjacent tangle diagrams, and is decorated itself with the union of those two tangle diagrams.*

Let the girth of a sub-tangle tree be the maximum number of boundary points of the tangle *diagrams at all nodes. The* recursive girth *of D is the minimum of the girths of all its sub-tangle trees.*

Even though this improvement is not implemented in `FoamHo` yet, the programme is still fast enough to calculate the $\mathfrak{sl}_3$-homology of links with girth up to 12, such as the $(7,6)$–torus knot. Gornik's spectral sequences allows (in most cases) to extract the $\mathfrak{sl}_3$–concordance invariant s_3 from the value of $\mathfrak{sl}_3$-homology.

Calculation results are in agreement with all known results and conjectures,except for part (ii) of Lobb's conjecture: the most striking result obtained with `FoamHo` are the first known examples of knots for which s_3 differs from the Rasmussen invariant s_2. In particular, we find that s_3 may be odd. Calculations for small ℓ, m, n suggest the following conjecture, which agrees with theorem I.6.1:

Conjecture III.5.1. *If $\ell > m \geq 3$, $n \geq 2$, and $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$, then the $(\ell, -m, n)$–pretzel knot $P(\ell, -m, n)$ has s_3–invariant*

$$s_3(P(\ell, -m, n)) = \ell - m + \delta_n,$$

where $\delta_2 = -1$ and $\delta_n = -2$ for $n > 2$.

$\mathfrak{sl}_3$-foam homology and knotted weighted webs

In chapter IV, we analyse knotted webs, i.e. smooth unframed embeddings of a web into 3-space. We show that graded and filtered $\mathfrak{sl}_3$-homology may, up to a shift in both degrees, be readily generalised to knotted webs. Next, let us consider weighted webs:

Definition IV.1.9. *A* weighting *of an abstract web W is a function $\{\text{edges of } W\} \to \{0, 1, 2\}$, such that any two intersecting edges have a different weight. A crossing is called* equiponderate *if its two strands have the same weight, and* antiponderate *otherwise. The* weighted writhe *w_o of the diagram D of a knotted weighted web is defined as*

$$w_o(D) = \#\{\text{positive antiponderate crossings}\} - \#\{\text{negative antiponderate crossings}\}.$$

Introducing shifts which depend on the weighting, $\mathfrak{sl}_3$-homology becomes an invariant of knotted weighted webs:

Definition IV.1.12. *Let D be the diagram of a knotted weighted web. Let the* weighted $\mathfrak{sl}_3$-homology *$[\![\,\cdot\,]\!]_o$ of D be defined as*

$$[\![D]\!]_o = (t^{-1}q^4)^{-w_o(D)} \cdot [\![D]\!].$$

Theorem IV.1.13. *The weighted $\mathfrak{sl}_3$-homology is an invariant of knotted weighted webs. In particular, it is an invariant of knotted (unweighted) theta-graphs. It agrees with $\mathfrak{sl}_3$-link homology, if one weighs all components of a link with one fixed weight.*

In analogy to the case of knots, filtered homology has a particularly simple form: its total dimension equals the number of proper weightings of the web. This allows us to define an analogue of the Rasmussen invariant for knotted weighted webs.

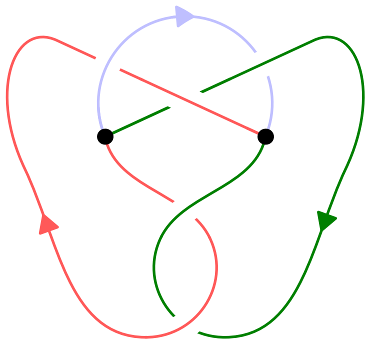
Foams are the natural class of cobordisms for knotted weighted webs. The *slice degree* $\chi_4(W)$ of a web W is the minimal degree of a foam smoothly embedded into the 4-space that bounds the web and satisfies certain admissibility conditions (see definition IV.4.11).

The main result is that the s_3-invariant of a knotted weighted web does indeed induce a lower bound on its slice degree:

Theorem IV.4.14. *Let W be a knotted weighted web. Then*

$$\chi_4(W) \geq 2s_3(W) - 2.$$

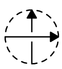
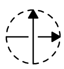
It should be possible to generalise this concept to all $N \geq 4$, using Mackaay and Vaz' foams [MSV09].



A knotted weighted web of slice degree -1.

Notations and conventions

$[N]_q$	the N-th *quantum integer* $\dfrac{q^N - q^{-N}}{q - q^{-1}} = q^{-N+1} + q^{-N+3} + \ldots + q^{N-1}$.
$\mathbb{N}$	the natural numbers $\{0, 1, 2, \ldots\}$.
$\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots$	categories (as in [Bor94]).
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	planar algebras.

Orientations All knots and links are oriented, unless stated otherwise. A crossing is positive if the upper strand points east, and the lower strand north, and negative otherwise:  is positive, and  is negative. The standard generators of the n-stranded braid group are denoted by $\sigma_1, \ldots \sigma_n$. In the closure $\mathrm{tr}(B)$ of a braid B, σ_i is a positive crossing, and σ_i^{-1} a negative crossing. The classical knot signature σ is normalised such that the right-handed trefoil knot, which has positive crossings, has positive signature. For the $(a_1, \ldots a_n)$-pretzel link, which is also denoted by $P(a_1, \ldots a_n)$, the sign of a_i signifies whether the corresponding crossings are right-handed $(-)$ or left-handed $(+)$. See fig. I.1.4 and fig. I.6.1 for examples.

Enumeration of knots For knots with crossing number up to 10, we use Rolfsen's enumeration [Rol76], and for knots with crossing number between 11 and 16 the enumeration by Hoste-Thistlethwaite-Weeks [HTW98, HT99]. Cf. also the highly useful knot databases Knot Atlas [BNM] and KnotInfo [CL].

Chapter I

The Khovanov-Rozansky concordance invariants

The goal of this chapter is to fathom the relationship of the different Khovanov-Rozansky homologies: the $\mathfrak{sl}_N$-homologies for different N, the HOMFLYPT-homology, graded or filtered, reduced or unreduced. We are interested in particular in the $\mathfrak{sl}_N$-concordance invariants, which we show to be not all equal.

The chapter is organised as follows: in section 1, a certain class of concordance invariants is analysed, called in this text *slice-torus concordance invariants*. This class encompasses the $\mathfrak{sl}_N$-concordance invariants. We determine the value of slice-torus invariants of quasi-positive and homogeneous knots. We consider as well the natural generalisation to links. While the results are for the first time stated in this generality, they will not surprise the expert.

Section 2 reviews the aspects of Khovanov-Rozansky homology that are essential for the purpose of the chapter. All our results may be deduced from *formal* properties of the cochain complexes, so we do not give the actual definitions of the various homology theories, and matrix factorisations are not discussed.

In the following section 3, we sketch the definition of the Khovanov-Rozansky concordance invariants and introduce a reduced version of filtered homology: this variation of Khovanov-Rozansky homology was till now missing in the picture.

In section 4 we explain how the different versions of homology are linked by various spectral sequences. We use the fact that the higher pages of the considered spectral sequences are link invariants to introduce a new link invariant $s_\infty \in 2\mathbb{Z}$. We also introduce a new spectral sequence which relates reduced and unreduced homology.

In section 5 we show how stringing together this new spectral sequence with two others links the $\mathfrak{sl}_N$-concordance invariants to reduced HOMFLYPT-homology (corollary 5.3).

This tool, combined with an inequality for slice-torus invariants from section 1 enables us in section 6 to calculate bounds for the value of the $\mathfrak{sl}_N$-concordance invariants for a certain class of pretzel knots. We thereby show that s_∞ and s_2 (the Rasmussen invariant) are distinct, and different from all the other s_N for $N \geq 3$. In chapter III, we use computer calculations to advance one step and show that s_3, too, is different from s_∞ and all s_N with $N \neq 3$.

1 The slice-torus concordance invariants

For a link L, let $-L$ be the orientation-reversed mirror image of L. Smooth concordance classes of knots form an abelian group, the *smooth knot concordance group*, with the connected sum as addition, the unknot $\bigcirc$ as neutral element and $-K$ as the inverse of K. A *real concordance homomorphism* is a homomorphism from the smooth knot concordance group to the reals.

Definition 1.1. *A* slice-torus knot invariant *is a real concordance homomorphism y with the following properties*[*]*:*

The invariant y is a lower bound to twice the slice genus, i.e. for all knots K: $y(K) \leq 2g_4(K)$. (1.1)

For positive torus knots, this bound is sharp, i.e. $\forall p, q \in \mathbb{Z}^+ \colon \ (p,q) = 1 \implies y(T(p,q)) = 2g_4(T(p,q)) = (p-1)(q-1)$. (1.2)

Generalising to oriented links, a slice-torus link invariant *is a real-valued link invariant y' satisfying (1.2) and for all links L, L_1, L_2:*

$$y'(L_1 \sqcup L_2) = y'(L_1) + y'(L_2) - 1. \tag{1.3}$$

$$y'(L) + y'(-L) \leq 0. \tag{1.4}$$

If there is a cobordism of Euler characteristic χ between L_1 and L_2, whose every component intersects L_1 (such a cobordism is called weakly connected, *and* biweakly connected *if every component intersects L_2 as well), then $y'(L_1) - y'(L_2) \leq -\chi$.* (1.5)

The conditions (1.1), (1.2) in this form come from Livingston [Liv04], who notes that the Ozsváth-Szabó concordance invariant τ satisfies them, and that many properties of τ may be deduced formally just from those conditions.

Slice-torus knot invariants form a convex subset of the space of all real concordance homomorphisms. Let us list some known slice-torus knot invariants, each normalised as to satisfy the condition (1.2):

- the concordance invariant τ stemming from knot Floer homology [OS03, Ras03],
- the Rasmussen invariant s_2 [Ras10],
- more generally, the $\mathfrak{sl}_N$-concordance invariants s_N [Wu09, Lob09, Lob12], see definition 3.4.
- the generalised Rasmussen invariants $s_2^{\mathbb{F}}$, defined for all fields $\mathbb{F}$ [BN05, Tur06] (see also [MTV07]), and

The $\mathfrak{sl}_N$-concordance invariants are the only known slice-torus knot invariants which may take other values than even integers. The classical knot signature σ, on the other hand, is not a slice-torus invariant, since it does not satisfy (1.2), e.g. $\sigma(T(5,4)) = 8$; nor are the Lipshitz-Sarkar invariants [LS12], which are no concordance homomorphisms.

The conditions of definition 1.1 are quite restrictive, to such an extent that few linearly independent slice-torus knot invariants are known – the above list is exhaustive to the author's knowledge. It is an on-going challenge to show that the slice-torus knot invariants in this list are indeed distinct: see [HO08] for a proof of $\tau \neq s_2$. At the moment, no example of $s_2 \neq s_2^{\mathbb{F}}$ is known. In section 6 and chapter III, we show that s_2 and s_3 are different from each other and from all the s_N with $N \geq 4$.

Slice torus knot invariants may be able to detect knots that are topologically but not smoothly slice. Restricting oneself to the subgroup of topologically slice knots, the listed

[*] In [LN06], Livingston and Naik use ν instead of y.

slice torus knot invariants may still linearly independent (see e.g. [Liv08]), which can be used to show that this subgroup contains direct summands isomorphic to $\mathbb{Z}$.

In the remainder of this section, we deduce some properties of slice-torus invariants. Throughout, let y be a slice-torus knot invariant and y' a slice-torus link invariant. The main results are that the value of both y and y' is determined, and can easily be computed, for quasi-positive (theorem 1.9) and for homogeneous (theorem 1.12) knots and links. Although these results are stated for the first time in this generality, they will not surprise the expert: the proves of those results are straight-forward adaptions of [Shu07, Pla06] and [Kaw, Lob11, Abe11], respectively.

Proposition 1.2. *If there is a connected smooth cobordism of Euler characteristic χ between two knots K_0 and K_1, then*

$$|y(K_0) - y(K_1)| \leq -\chi.$$

Proof. Such a cobordism gives rise to a surface F' embedded in $[0,\infty) \times \mathbb{R}^3$ with $\partial F' = F' \cap \{0\} \times \mathbb{R}^3 = K_0 \#(-K_1)$ and Euler characteristic $\chi+1$. Therefore, $g_4(K_0\#-K_1) \leq -\chi/2$. By (1.1), $y(K_0\# - K_1) \leq 2g_4(K_0\# - K_1) = -\chi$, but $y(K_0\# - K_1) = y(K_0) - y(K_1)$ since y is a concordance homomorphism. So $y(K_0) - y(K_1) \leq -\chi$. Reflecting F along a hyperplane, thereby exchanging the role of K_0 and K_1, yields $y(K_0) - y(K_1) \leq -\chi$. Hence $|y(K_0) - y(K_1)| \leq -\chi$. □

Proposition 1.3.

(i) *Slice-torus link invariants are invariant under link concordance.*

(ii) *Let L_3 be any connected sum of L_1 and L_2. Then*

$$y'(L_1) + y'(L_2) - 2 \leq y'(L_3) \leq y'(L_1) + y'(L_2).$$

If L_1 is a knot, then $y'(L_3) = y'(L_1) + y'(L_2)$.

(iii) *For every link $L : 2 - 2|L| \leq y'(-L) + y'(L)$.*

(iv) *Let $\chi_4(L)$ be the maximal Euler characteristic of an oriented surface without closed components embedded smoothly into D^4 whose boundary is L. Then*

$$y'(L) \leq 1 - \chi_4(L).$$

(v) *The restriction of y' to knots is a slice-torus knot invariant.*

Proof. **(i)** This is a direct consequence of (1.5).
(ii) A saddle move gives a cobordism between L_3 and $L_1 \sqcup L_2$, which is weakly connected in both directions. So

$$|y'(L_3) - y'(L_1 \sqcup L_2)| \leq 1 \implies y'(L_1) + y'(L_2) - 2 \leq y'(L_3) \leq y'(L_1) + y'(L_2).$$

If L_1 is a knot, then there is a in biweakly connected cobordism of Euler characteristic 0 from $L_3\# - L_1$ to L_2. Therefore

$$y'(L_2) = y'(L_3\# - L_1) \leq y'(L_3) + y'(-L_1) \leq y'(L_3) - y'(L_1),$$

and this implies the statement.

(iii) There is a surface S in D^4 bounding $-L \sqcup L$ that consists of $|L|$ cylinders. Cutting out discs from all the connected components of S yields a weakly connected cobordism from the $|L|$-component unlink to $-L \sqcup L$ of Euler characteristic $-|L|$. Therefore

$$\begin{aligned} & y'(\sqcup^{|L|}\bigcirc) - y'(-L \sqcup L) \leq |L| \\ \implies \quad & 1 - |L| - y'(-L) - y'(L) + 1 \leq |L| \\ \implies \quad & y'(-L) + y'(L) \geq 2 - 2|L|. \end{aligned}$$

(iv) By removing a disc from it, such an oriented surface can be made into a weakly connected cobordism from L to the unknot. So the statement is a direct consequence of (1.5).

(v) This follows from (ii) and (iii). □

Remark 1.4. It is an open problem whether all slice-torus link invariants satisfy $y'(L_3) = y'(L_1) + y'(L_2)$. Remark 3.8 gives a hint how this could be proved for the $\mathfrak{sl}_N$-slice torus link invariants.

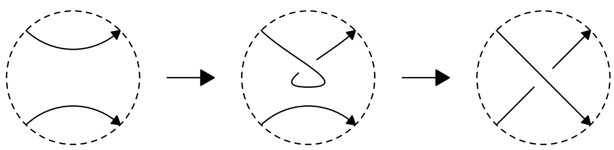


Figure 1.1: A cobordism of Euler characteristic -1 inserting a positive crossing. The cobordism consists of a Reidemeister I move and a saddle move.

Lemma 1.5. *Let B be a positive braid, i.e. a braid whose word contains only the σ_i, not the σ_i^{-1}. Suppose B has n strands and k crossings. Then $y'(\mathrm{tr}(B)) = 1 - \chi_4(\mathrm{tr}(B)) = 1 + k - n$. If the closure of B is a knot, $y(\mathrm{tr}(B)) = 2g_4(\mathrm{tr}(B)) = 2g_3(\mathrm{tr}(B)) = 1 + k - n$.*

Proof (following [Liv04]). Let $\ell \geq k$ so that $(n, \ell) = 1$. The (n, ℓ)-torus knot is the closure of the braid $(\sigma_1 \cdots \sigma_{n-1})^\ell$. This braid may be obtained from B by inserting $k(n-2) + (\ell - k)(n-1) = \ell n - k - l$ additional crossings: replacing each σ_i by $\sigma_1 \cdots \sigma_{n-1}$ and appending $(\sigma_1 \cdots \sigma_{n-1})^{\ell - k}$ to the end of the braid. Geometrically, this translates to a biweakly connected cobordism between $\mathrm{tr}(B)$ and the (n, ℓ)-torus knot of Euler characteristic $k + \ell - \ell n$, which is the composition of copies of the basic cobordism shown in fig. 1.1. By (1.5) and (1.2), we have $y'(\mathrm{tr}(B)) \geq y'(T(n, \ell)) + k + \ell - \ell n = 1 + k - n$. If $\mathrm{tr}(B)$ is a knot, the same inequality holds for y using proposition 1.2.

On the other hand, the canonical Seifert surface of $\mathrm{tr}(B)$ has n discs and k twisted bands, and thus a Euler characteristic of $n - k$. Hence $y'(\mathrm{tr}(B)) \leq 1 + k - n$, and the same inequality holds for y if $\mathrm{tr}(B)$ is a knot. □

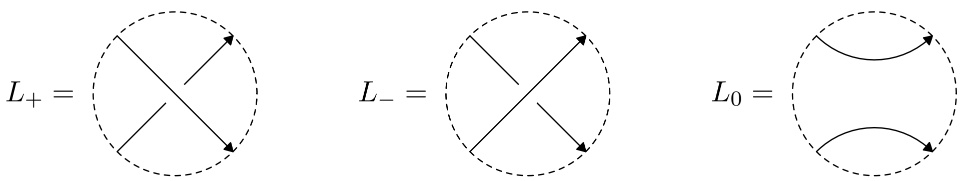


Figure 1.2: The links $L_\pm$ and L_0.

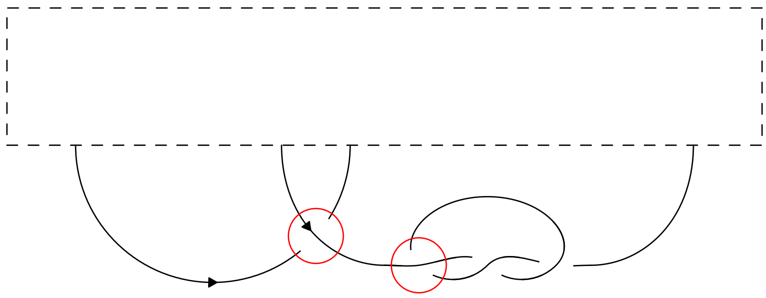


Figure 1.3: The sign of the two crossings encircled in red can be switched simultaneously by a cobordism of Euler characteristic -2 (saddle — Reidemeister move 2 — saddle)

Proposition 1.6. *If L_+ and L_- are links which have diagrams that are identical but for the sign of one crossing, which is given in the subscript (see fig. 1.2), then*

$$0 \leq y'(L_+) - y'(L_-) \leq 2.$$

If $L_\pm$ are knots, then

$$0 \leq y(L_+) - y(L_-) \leq 2.$$

Proof (following [Liv04]). There is a cobordism composed of two saddles between L_+ and L_-, built of the cobordism of fig. 1.1 and its mirror image. So (1.5) implies $|y'(L_+) - y'(L_-)| \leq 2$, and proposition 1.2 implies the corresponding statement for knots.

Next, consider $L_+\#T(-3,2)$. There is a weakly connected cobordism (depicted in fig. 1.3) from L_- to this link. Thus $y'(L_-)-y'(L_+\#T(-3,2)) \leq 2 \implies y'(L_-)-y'(L_+) \leq 0$. In the case of knots, the slice genus of $L_+\#(-L_-)\#T(-3,2)$ is at most 1, and the analogous statement follows. □

Lemma 1.7. *Let B be a braid with n strands, k_+ positive and k_- negative crossings, and let $w = k_+ - k_-$ be its writhe. Then $y'(\mathrm{tr}(B)) \geq 1 + w - n$, and if the braid's closure is a knot, then $y(\mathrm{tr}(B)) \geq 1 + w - n$.*

Proof. Let B' be the braid obtained from B by switching the sign of all negative crossings, i.e. replacing σ_i^{-1} by σ_i. By the previous proposition, $y'(\mathrm{tr}(B)) \geq y'(\mathrm{tr}(B')) - 2k_-$. Furthermore, using lemma 1.5, $y'(\mathrm{tr}(B')) = 1+k_++k_--n \implies y'(\mathrm{tr}(B)) \geq 1+k_+-k_--n$. The proof for y and knots is similar. □

Definition 1.8. *A braid B is said to be* quasi-positive *if it is the product of braid-words that are conjugate to one of the σ_i; i.e. $B = \prod_j w_j\sigma_{i_j}w_j^{-1}$, where w is any braid-word. If additionally, each w_j is of the form $\sigma_{k_j}\sigma_{k_j+1}\cdots\sigma_{i_j-1}$, then B is said to be* strongly quasi-positive*.*

Quasi-positivity has been introduced and studied by Rudolph, see e.g. [Rud83, Rud05]. The following theorem has been proven for the Rasmussen invariant by Shumakovitch [Shu07]; for τ it is an immediate consequence of the results of Plamenevskaya [Pla06]. The relationship between the τ-invariant, quasi-positivity and fibredness were studied by Hedden [Hed10].

Theorem 1.9. *Let B be a quasi-positive braid with writhe w and n strands. Then $y'(\mathrm{tr}(B)) = 1 - \chi_4(\mathrm{tr}(B)) = 1 + w - n$. Moreover, if $\mathrm{tr}(B)$ is a knot, then $y(\mathrm{tr}(B)) = 2g_4(\mathrm{tr}(B)) = 1 + w - n$.*

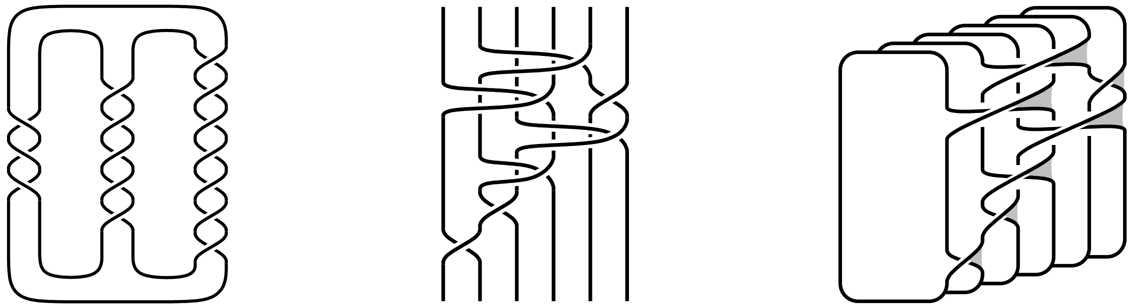


Figure 1.4: The $(-3, 5, 7)$-pretzel knot, its representation as strongly quasi-positive braid, and the corresponding Seifert surface. Figure taken from Shumakovitch [Shu07].

Proof. The previous lemma implies $2g_4(\mathrm{tr}(B)) \geq y(\mathrm{tr}(B)) \geq 1 + w - n$. On the other hand, there is a surface embedded in $\mathbb{R}^4$ bounding $\mathrm{tr}(B)$ with genus $(1 + w - n)/2$, as drawn exemplarily in fig. 1.4. The proof for y' and links is similar. □

Lemma 1.10. *Let D be a positive link diagram of a link L, i.e. a diagram with only positive crossings. Let k be the number of crossings, and n the number of Seifert circles of the Seifert resolution of D. Then $y'(L) = 1 - \chi_4(L) = 1 + k - n$. If L is a knot, $y(L) = 2g_4(L) = 1 + k - n$.*

Proof. Since positive links are quasi-positive [Rud99], the previous theorem implies $y'(K) = 1 - \chi_4(K)$. Note that $1 - \chi_4(L) = 1 + k - n$ remains to be shown, because D is not necessarily a braid diagram. The Seifert surface has Euler characteristic $n - k$, so $1 - \chi_4(L) \leq 1 + k - n$. To show $1 - \chi_4(L) \geq 1 + k - n$, we use the extension of the Rasmussen invariant s_2 to links [Weh08]. This is a slice-torus link invariant and it satisfies the statement of this corollary. So $s_2(L) = 1 + k - n \leq 1 - \chi_4$. The proof for y and knots is similar. □

One of the strongest restrictions that can be deduced from the slice-torus conditions is an inequality à la Bennequin. Its first version stated by Rasmussen [Ras10], Shumakovitch [Shu07] and Plamenevskaya [Pla06] for the Rasmussen invariant. It was subsequently sharpened by Kawamura [Kaw07], and honed yet more independently by Lobb [Lob11] and Kawamura [Kaw].* Given a diagram D of a knot K, the sharper slice-Bennequin inequality gives an upper and lower bound for $y(K)$. Those bounds are easily computable from D, depending only the *Seifert graph* $\Gamma(D)$. The first sharpening of the inequality was generalised by Wu to the s_N-invariants [Wu07]. In this text, we generalise the second sharpening to all slice-torus link invariants.

Definition 1.11. *The* Seifert graph $\Gamma(D)$ *of a link diagram D is a plane bipartite graph whose edges carry a sign (+ or −). It is constructed as follows:*

- *The vertices of $\Gamma(D)$ correspond to the circles of the Seifert resolution of D.*
- *A fixed crossing of D is adjacent to two different Seifert circles, which correspond to two vertices in $\Gamma(D)$. For any crossing, let $\Gamma(D)$ have an edge between these two vertices. The edge's sign indicates if the crossing is positive or negative.*

Let $\Gamma^+(D)$ ($\Gamma^-(D)$) be the subgraph of $\Gamma(D)$ that contains only the positive (negative) edges. Let $O^\pm(D)$ be the number of connected components of $\Gamma^\pm(D)$.

*That the second sharpening is indeed strictly stronger than the first can be seen e.g. considering the standard diagram of $11n_{53}$, see fig. 1.5.

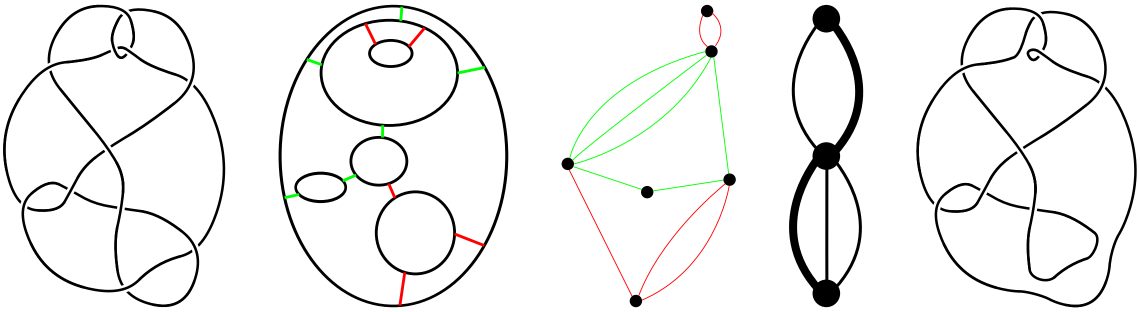


Figure 1.5: From left to right (see the prove of theorem 1.12 for details): a diagram D of the knot $11n_{53}$ (drawn with knotscape [HT99]); its Seifert resolution; its Seifert graph $\Gamma(D)$ (positive edges are green, negative red); the graph $G(D)$, the tree T drawn with thick lines; and the diagram D'.

Theorem 1.12 (The sharper slice-Bennequin inequality)**.** *Let D be a diagram of a link L, with writhe w, n Seifert circles and c split components. Then*

$$-c + w - n + 2O^+ \leq y'(L) \leq c + w + n - 2O^-.$$

In particular, if L is a knot,

$$-1 + w - n + 2O^+ \leq y(L) \leq 1 + w + n - 2O^-.$$

Proof (following [Abe11]). Let us first prove the lower bound of the inequality for y'. The upper bound then follows from $y'(-L) \leq -y'(L)$, and the inequality for y can be proven similarly, as will be discussed at the end. Let us also suppose without loss of generality that $c = 1$. For split diagrams, the inequality can then be deduced from $y'(L_1 \sqcup L_2) = y'(L_1) + y'(L_2) - 1$.

For an example of the following constructions, see fig. 1.5. Let $G(D)$ be the graph that has as vertices the components of $\Gamma^+(D)$, and has for each negative edge in $\Gamma(D)$ an edge between the corresponding components of $\Gamma^+(D)$. Then $G(D)$ is a connected graph with O^+ many vertices and k^- many edges. Pick $O^+ - 1$ edges that form a tree T. Gluing together $k^- - O^+ - 1$ many copies of the mirror image of the cobordism drawn in fig. 1.1 gives a weakly connected cobordism with Euler characteristic $-k^- + O^+ - 1$ from a link L' to L. The link L' has a diagram D' such that $G(D')$ is the tree T. So by (1.5),

$$y'(L') - y(L) \leq k^- - O^+ + 1 \qquad \Longrightarrow \qquad y'(L') - k^- + O^+ - 1 \leq y(L). \tag{1.6}$$

The $O^+ - 1$ many negative crossings of D' are nugatory and may be removed by twists; each twist diminishes the number of Seifert circles by one, so the ensuing diagram D'' is positive, with k^+ many crossings and $n - O^+ + 1$ many Seifert circles. Thus $y'(L') = k^+ - n + O^+$. Putting this together with (1.6) concludes the proof.

To prove the inequality for y and knots, one small modification has to be made to avoid that L' has multiple components: should that be the case, connect the components by inserting positive crossings, using the cobordism of fig. 1.1. Each such insertion increases $y(L')$ by one, but also decreases the degree of the applied cobordism by one; this cancels out, and the same inequality is obtained in the end. □

Let us restate the implied inequality for braid diagrams:

Corollary 1.13. *Let B be a braid with n strands and writhe w. Let $\Omega^\pm \subset \{1, \ldots n-1\}$ such that $i \in \Omega^\pm$ if and only if $\sigma_i^{\pm 1}$ occurs in the word B. Let $b^\pm = |\Omega^\pm|$ and $c = |\Omega^+ \cup \Omega^-|$. Then*

$$c + w - 2b^+ \leq y'(\mathrm{cl}(B)) \leq -c + w + 2b^-.$$

If $\mathrm{cl}(B)$ *is a knot,*

$$n + w - 2b^+ \leq y(\mathrm{cl}(B)) \leq -n + w + 2b^-.$$

Corollary 1.14. *For an alternating knot K, the value of a slice-torus knot invariant equals the value of the classical knot signature: $y(K) = \sigma(K)$. For an alternating link L with c split components, $y'(L) = \sigma(L) + 1 - c$.*

Proof. Use [Lee02, proposition 3.3], which is proven using Göritz matrices [GL78] (see e.g. [Lew09, section I.6]). □

If D is a link diagram with $O^+ + O^- = n + 1$, then the lower bound of theorem 1.12 equals the upper bound, and thus y' is determined by the inequalities. Such diagrams are called *homogeneous*, and, consequently, a link is called homogeneous if it has a homogeneous diagram. This notion was introduced by Cromwell [Cro89] and its relationship with the Rasmussen invariant was studied by Abe [Abe11]. Consequently, if two slice-torus invariants assume different values for a knot, e.g. if some slice-torus invariant is not an even integer for a knot, then this knot is not homogeneous.

Corollary 1.15. *A homogeneous link L with c split components satisfies $y'(L) + y'(-L) = 1 - c$.*

Remark 1.16. Alternating and positive knots are homogeneous, but quasi-alternating [MO08] or quasi-positive knots are generally not: e.g., the $(\ell, -m, n)$-pretzel knots, with ℓ and m odd, n even and $\ell > m > n \geq 2$ are quasi-alternating ([CK09, Gre10]), but not homogeneous, as will be shown in section 6. On the other hand, the $(\ell, m, -n)$-pretzel knots with ℓ, m, n odd and $\ell \geq m > n$ are quasi-positive [Rud93], but some of them, such as the $(-3, 5, 7)$-pretzel knot (see fig. 1.4), have trivial Alexander polynomial, and are hence not homogeneous [Cro04, Theorem 7.6.2].

Further bounds which can be generalised from the Ozsváth-Szabó and Rasmussen invariant to all slice-torus invariants include Livingston and Naik's bounds on twisted doubles of a knot [LN06].

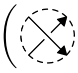
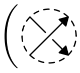
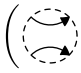
2 A brief formal overview over the Khovanov-Rozansky homologies

In this short section, we intend to give a survey of the different Khovanov-Rozansky homologies, and present the main tools which allow to calculate HOMFLYPT-homology. Throughout, all cochain complexes and homologies* are rational, with the exception of Gornik's filtered homology, which is complex.

Let D be a diagram of a link L. For all integers $N \geq 1$, Khovanov and Rozansky [KR08a] define a cochain complex $C_N(D)$ of graded rational vector spaces whose homology

*Here, we follow Vaz' terminology [Vaz08]: since the differentials decrease the homological degree, the complex is called a *cochain* complex, but because taking its homology is a covariant functor, we speak of *homology*, and not of cohomology.

$[\![L]\!]_N$, the $\mathfrak{sl}_N$-homology, is a link invariant. We consider $[\![L]\!]_N$ as a doubly graded vector space, with a homological (t), and a quantum (q) degree. We write $[\![L]\!]^i_N$ for the subspace of homological degree i. The $\mathfrak{sl}_N$-homology categorifies the $\mathfrak{sl}_N$-polynomial P_N of Reshetikhin and Turaev [RT90], i.e. $\operatorname{xdim}([\![L]\!]_N)(-1,q) = P_N(L)$. The $\mathfrak{sl}_N$-polynomial is given by its value of $[N]_q = (q^{-N+1} + q^{-N+3} + \ldots + q^{N-1})$ on the unknot and the following skein relation:

$q^N \cdot P_N\Big($  $\Big) - q^{-N} \cdot P_N\Big($  $\Big) = (q - q^{-1}) \cdot P_N\Big($  $\Big)$ (2.1)

Khovanov and Rozansky also provide the following definition of the reduced version of this homology, which categorifies the reduced $\mathfrak{sl}_N$-polynomials $\overline{P_N} = P_N/[N]_q$.

Definition 2.1. *Let D be a diagram of the link L with a marked component. Let $A = \mathbb{Q}[X]/(X^N)$, a graded algebra with grading $\deg X^i = 2i$ for $i \in \{0, \ldots N-1\}$. Then $C_N(D)$ has the structure of a free graded A-module in a way defined in [KR08a]. This structure is respected by the differential of $C_N(D)$, and it may depend on the choice of the marked component. Let $\overline{Q}$ be the graded A-module $A/(X)$ with a shift of $N-1$ in the q-grading. This is a one-dimensional rational vector space. Let $\overline{C_N(D)} = C_N(D) \otimes_A \overline{Q}$. This is a graded rational cochain complex, whose homology $\overline{[\![L]\!]}_N$ is an invariant of links with a marked component.*

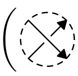
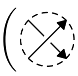
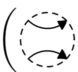
Note that the relationship between unreduced and reduced $\mathfrak{sl}_N$-homology is more intricate than between unreduced and reduced $\mathfrak{sl}_N$-polynomial; in particular, the value of neither determines the other. This issue will be further pursued in section 4.

For small $N \in \{1, 2, 3\}$ the $\mathfrak{sl}_N$-homology offers nothing new:

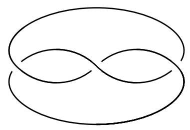
- For $N = 1$, every link L has the same homology, $\operatorname{xdim} [\![L]\!]_1 = \operatorname{xdim} \overline{[\![L]\!]}_1 = 1$.
- For $N = 2$, one recovers the dual of rational Khovanov homology [Kho00].
- For $N = 3$, the issuing homology is isomorphic to the rational foam homology [Kho04], as conjectured by Khovanov and Rozansky, and later proven by Mackaay and Vaz [MV08a].

For $N \geq 2$, Gornik defines a filtered version of the homology [Gor04] (generalising Lee's work for $N = 2$ [Lee05]; see also [BNM06, Weh04]): a (ascendingly) filtered rational cochain complex $C^f_N(D)$, whose associated graded is $C_N(D)$. The homology $[\![L]\!]^f_N$ of this filtered cochain complex is a link invariant. It takes a particularly simple form, which allows the extraction of the slice-torus link invariants s_N for $N \geq 2$ (see definition 3.4). For the filtered homology it is possible as well to define a reduced version, as we will see in the next section.

Furthermore, Khovanov and Rozansky [KR08b] introduce a cochain complex $C_\infty(D)$ of doubly graded rational vector spaces, which is only defined for a braid diagram D. Its homology is a link invariant called the Homflypt-homology, which categorifies the Homflypt-polynomial. The Homflypt-polynomial is determined by its value of 1 on the unknot, and the following skein relation:

$a \cdot P_\infty\Big($  $\Big) - a^{-1} \cdot P_\infty\Big($  $\Big) = (q - q^{-1}) \cdot P_\infty\Big($  $\Big)$ (2.2)

There are several versions of this homology. Rasmussen [Ras06] e.g. presents a reduced and an unreduced version, and an interpolation of the two; but all these versions carry the same information (as has been remarked before, this is not the case for the reduced and unreduced version of $\mathfrak{sl}_N$-homology). In this text, we stick to the reduced version, denoted by $\overline{[\![\cdot]\!]}_\infty$. For a knot this is, unlike the unreduced version, a finite dimensional space. We follow similar grading conventions as Mackaay and Vaz [MV08b], but exchanging t and t^{-1}, i.e.

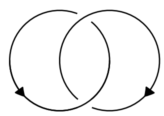
xdim $\overline{[\![}$  positive trefoil $\overline{]\!]}_\infty$ $= a^{-2}q^2 + t^2a^{-2}q^{-2} + t^3a^{-4}.$ (2.3)

In [Ras06], Rasmussen follows still another grading convention; the monomial $q^ia^jt^k$ in that convention corresponds to the monomial $q^ia^jt^{(k-j)/2}$ in ours.

HOMFLYPT-homology is well-behaved under taking the connected sum:

Proposition 2.2 (see [Ras06, Lemma 7.8])**.** *Let L_1 and L_2 be links, and L_3 any connected sum of L_1 and L_2. Then $\overline{[\![L_3]\!]}_\infty = \overline{[\![L_1]\!]}_\infty \cdot \overline{[\![L_2]\!]}_\infty$.*

There is yet another version of HOMFLYPT- and $\mathfrak{sl}_N$-homology, only defined for two-component links: *totally reduced homology*, denoted by $\overline{\overline{[\![\cdot]\!]}}_\infty$ and $\overline{\overline{[\![\cdot]\!]}}_N$, respectively. We will not give a definition, because we only need the following value, which is calculated in [MV08b]:

xdim $\overline{\overline{[\![}}$  positive Hopf link $\overline{\overline{]\!]}}_\infty$ $= (a^{-1}q^2 + ta^{-1} + t^2a^{-1}q^{-2} + t^3a^{-3}) \cdot t^{-1/2}.$ (2.4)

Like reduced homology, totally reduced homology stabilises, i.e. for large enough N,

$$\operatorname{xdim}(\overline{\overline{[\![\cdot]\!]}}_\infty)(q, q^N, t) = \operatorname{xdim}(\overline{\overline{[\![\cdot]\!]}}_N)(q, t).$$

Let us now introduce the two main tools to calculate HOMFLYPT-homology: the notion of thinness, and the skein long exact sequence.

Definition 2.3. *Let the δ-grading on $\overline{[\![\cdot]\!]}_\infty$ be defined by $\delta(q^ia^jt^k) = i + 2j + 2k$. A knot K is* KR-thin *if its* HOMFLYPT*-homology is supported in a single δ-degree that is equal to minus its signature.*

Proposition 2.4. *The* HOMFLYPT*-homology of a KR-thin knot K is determined by its* HOMFLYPT*-polynomial $P_\infty(K)$ and its signature $\sigma(K)$:*

$$\operatorname{xdim} \overline{[\![L]\!]}_\infty = (-t)^{-\sigma(K)/2} \cdot P_\infty(qt^{-1/2}, at^{-1}).$$

Proof. This immediately follows from the fact that $\operatorname{xdim} \overline{[\![L]\!]}_\infty(q, a, -1) = P_\infty(q, a)$. □

Proposition 2.5 ([Ras06, Corollary 1])**.** *Two-bridge knots are KR-thin.*

Proposition 2.6 (The skein long exact sequences, [Ras06, Lemma 7.6])**.** *Let K_+, K_- and L_0 be two knots and one two-component link which look the same everywhere except near one crossing, where they differ as shown in fig. 2.1. Then for all $N \geq 2$, there is a long exact sequence*

$$\cdots \longrightarrow \overline{[\![K_-]\!]}_N \xrightarrow{(-N,\frac{1}{2})} \overline{\overline{[\![L_0]\!]}}_N \xrightarrow{(-N,\frac{1}{2})} \overline{[\![K_+]\!]}_N \xrightarrow{(2N,-2)} \overline{[\![K_-]\!]}_N \longrightarrow \cdots$$

The differentials' (t, q)-degree is indicated above the arrows.

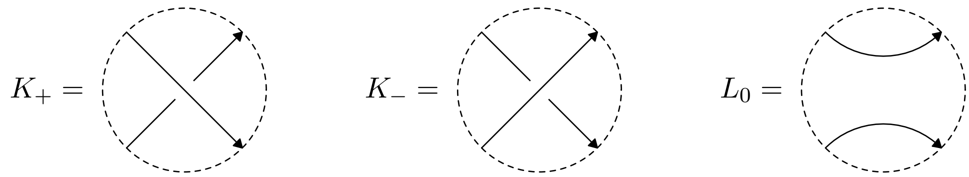


Figure 2.1: The knots $K_\pm$ and the link L_0.

This proposition talks about $\mathfrak{sl}_N$-homology; to make a statement about HOMFLYPT-homology, we need the following technical lemma:

Lemma 2.7. *Let $A, B \in \mathbb{N}[q^{\pm 1}, a^{\pm 1}]$. Suppose that for infinitely many N, $A(q, q^N) \leq B(q, q^N)$ ("$\leq$" is understood in the sense of appendix A). Then $A(q, a) \leq B(q, a)$.*

Proof. Let $i_{\max}$ and $i_{\min}$ be the maximal and minimal exponent of q occurring in A and B. Choose N such that $A(q, q^N) \leq B(q, q^N)$ and $|N| > i_{\max} - i_{\min}$. Then different monomials in $A(q, a)$ and $B(q, a)$ yield different polynomials in $A(q, q^N)$ and $B(q, q^N)$. To show this, consider two monomials $c \cdot q^i a^j$ and $c' \cdot q^{i'} a^{j'}$ in $A(q, a)$ (with $c, c' \neq 0$). Then $cq^{i+Nj} = c'q^{i'+Nj'}$ implies $c = c'$ and $i + Nj = i' + Nj' \implies i - i' = N(j' - j) \implies |N| \cdot |j' - j| \leq i_{\max} - i_{\min} \implies j' = j \implies i = i'$.

So for fixed i and j, let c and c' be the coefficients of the monomial $q^i a^j$ in $A(q, a)$ and $B(q, a)$, respectively. Then c and c' are also the respective coefficients of the monomial q^{i+Nj} in $A(q, q^N)$ and $B(q, q^N)$, and thus $c \leq c'$. □

Corollary 2.8. *Suppose $K_\pm$ and L_0 are given as in fig. 2.1, then*

$$\begin{aligned} \operatorname{xdim} \overline{[\![K_+]\!]}_\infty &\leq t^2 \cdot a^{-2} \cdot \operatorname{xdim} \overline{[\![K_-]\!]}_\infty + t^{1/2} \cdot a^{-1} \cdot \operatorname{xdim} \overline{\overline{[\![L_0]\!]}}_\infty, \\ \operatorname{xdim} \overline{[\![K_-]\!]}_\infty &\leq t^{-2} \cdot a^{2} \cdot \operatorname{xdim} \overline{[\![K_+]\!]}_\infty + t^{-1/2} \cdot a \cdot \operatorname{xdim} \overline{\overline{[\![L_0]\!]}}_\infty. \end{aligned}$$

Proof. We will just prove the first equation, the second one follows similarly. The long exact sequence can be broken up into short ones; i.e., for some quotient space A of $\overline{\overline{[\![L_0]\!]}}_N$ and subspace B of $\overline{[\![K_-]\!]}_N$ there is a short exact sequence

$$0 \longrightarrow A \xrightarrow{(-N, \frac{1}{2})} \overline{[\![K_+]\!]}_N \xrightarrow{(-2N, 2)} B \longrightarrow 0.$$

This is equivalent to $\overline{[\![K_+]\!]}_N \cong (q^{-N} t^{1/2} \cdot A) \oplus (q^{-2N} t^2 \cdot B)$. In terms of graded dimensions, this implies

$$\begin{aligned} &\operatorname{xdim} \overline{[\![K_+]\!]}_N = q^N \cdot t^{-1/2} \cdot \operatorname{xdim} A + q^{2N} \cdot t^{-2} \cdot \operatorname{xdim} B \\ \implies &\operatorname{xdim} \overline{[\![K_+]\!]}_N \leq q^N \cdot t^{-1/2} \cdot \operatorname{xdim} \overline{\overline{[\![L_0]\!]}}_N + q^{2N} \cdot t^{-2} \cdot \operatorname{xdim} \overline{[\![K_-]\!]}_N. \end{aligned}$$

For large enough N, the three polynomials in this inequality stabilise, i.e.

$$\begin{aligned} (\operatorname{xdim} \overline{[\![K_+]\!]}_\infty)(q, q^N) &= (\operatorname{xdim} \overline{[\![K_+]\!]}_N)(q), \\ (\operatorname{xdim} \overline{\overline{[\![L_0]\!]}}_\infty)(q, q^N) &= (\operatorname{xdim} \overline{\overline{[\![L_0]\!]}}_N)(q), \\ (\operatorname{xdim} \overline{[\![K_-]\!]}_\infty)(q, q^N) &= (\operatorname{xdim} \overline{[\![K_-]\!]}_N)(q). \end{aligned}$$

So using lemma 2.7, the statement follows. □

3 Reduced filtered $\mathfrak{sl}_N$-homology

The aim of this section is to construct a reduced filtered $\mathfrak{sl}_N$-homology theory. Not wanting to involve ourselves too deeply in matrix factorisations, we will nevertheless be able to give the definition 3.5, but not to derive the homology theory's essential properties (see conjecture 3.7). Let us first review the relevant parts of Gornik's paper [Gor04], giving the definition of the $\mathfrak{sl}_N$-concordance invariants along the way.

Let D be a diagram of the link L and X the set of its crossings. Let $P : X \to \{0, 1\}$ assign 0 to positive and 1 to negative crossings. A *resolution* Γ is a function $X \to \{0, 1\}$. Let $h(\Gamma) = \sum_{c \in X} \Gamma(c)$. If $\Gamma(c) = 0$, denote by Γ^c the resolution which agrees with Γ on $X \setminus \{c\}$, and sends c to 1.

Definition 3.1. *Fix some $N \geq 2$. The* filtered $\mathfrak{sl}_N$-cochain (C, ∂) complex *is defined as follows: To every resolution Γ, a certain filtered complex vector space C_Γ is associated. If $\Gamma(c) = 0$, a filtered homomorphism $f_\Gamma^c : C_\Gamma \to C_{\Gamma^c}$ is defined, in such a way that the following homomorphisms $C_\Gamma \to C_{\Gamma^{c,d}}$ agree:*

$$f_{\Gamma^c}^d \circ f_\Gamma^c = f_{\Gamma^d}^c \circ f_\Gamma^d.$$

Then define (C, ∂) by

$$C^i = \bigoplus_{h(\Gamma)=i} C_\Gamma, \qquad \partial^i = \sum_{\substack{h(\Gamma)=i,\\ \Gamma(c)=0}} (-1)^* f_\Gamma^c,$$

where $*$ denotes a clever sprinkling of signs which makes all squares of maps anticommutative, and thus ∂ a differential. The homology of C is denoted by $[\![D]\!]_N^f$.

The associated graded cochain complex gives Khovanov-Rozansky's graded unreduced $\mathfrak{sl}_N$-homology $[\![L]\!]_N$.

Proposition 3.2. *Forgetting the filtration, the total dimension of $[\![D]\!]_N^f$ is $N^{|L|}$.*

Sketch of the proof. Consider the connected components of $D \setminus X$. Let an *arc* be the closure of such a component in D. Let A be the set of arcs of D. A *state* is a function $S : A \to \{1, \ldots N\}$. Fix a crossing c, and let $a_1, \ldots a_4$ be the four arcs adjacent to c, in counterclockwise order, such that a_1 and a_2 are oriented towards c. The state S is said to be 0-compatible if $S(a_1) = S(a_4)$ and $S(a_2) = S(a_3)$, and 1-compatible if $S(a_1) \neq S(a_2)$ and $\{S(a_1), S(a_2)\} = \{S(a_3), S(a_4)\}$. The state is *compatible with the resolution* Γ if is 0-compatible for all c with $\Gamma(c) = P(c)$, and 1-compatible for all other c.

Denote by C_Γ^u the vector space C_Γ, forgetting the filtration. Then C_Γ^u decomposes as a direct sum

$$C_\Gamma^u = \bigoplus_{S \text{ state}} C_{\Gamma,S}^u.$$

The dimension of $C_{\Gamma,S}^u$ is one if S is compatible with Γ, and zero otherwise. Moreover, the differential behaves nicely with respect to these decompositions. For every resolution Γ and crossing c with $\Gamma(c) = 0$, we have

$$f_\Gamma^c(C_{\Gamma,S}^u) = C_{\Gamma^c,S}^u.$$

Let C^u be the cochain complex C forgetting the filtration. So C^u is the direct sum of one cochain complex $C^u(S)$ per state. If a state S is both 0- and 1-compatible at any crossing c, then $C^u(S)$ is acyclic. If S is, on the other hand, only either 0- or 1-compatible at every crossing c, then S is compatible to a single resolution Γ, and $C^u(S)$ consists only of $C^u_{\Gamma,S}$, supported in homological degree $h(\Gamma)$.

What does it mean for a state to be either 0- or 1-compatible at a crossing? 0-, but not 1-compatible implies $S(a_1) = S(a_2) = S(a_3) = S(a_4)$; 1-, but not 0-compatible means $S(a_1) = S(a_3) \neq S(a_2) = S(a_4)$. So this condition is just equivalent to $S(a_1) = S(a_3)$ and $S(a_2) = S(a_4)$; in other words, at c arcs which belong to the same connected component of the link have the same image under Γ. So a state S is either 0- or 1-compatible at every crossing if and only if it is induced by a function $\pi_0(L) \to \{1, \dots N\}$. The statement follows. $\square$

Wu [Wu09] and Lobb [Lob12] then independently proved the following:

Proposition 3.3. ***(i)*** *For a knot K with a diagram D, the graded dimension of $[\![D]\!]^f_N$ is $q^s \cdot [N]_q \cdot t^0$ for some even integer s.*

(ii) *This even integer is a knot invariant.*

(iii) *Its normalisation $\dfrac{s}{1-N}$ is a slice-torus invariant.*

Definition 3.4. *The normalisation of s is called the $\mathfrak{sl}_N$-concordance invariant of K and is denoted by $s_N(K)$; by $s'_N(K) = (1-N) \cdot s_N(K)$ we denote the unnormalised $\mathfrak{sl}_N$-concordance invariant.*

Let us now define the *reduced* filtered Khovanov-Rozansky homology:

Definition 3.5. *Let a_0 be a fixed arc. For all $i \in \{1, \dots N\}$, let $\overline{C}_i \subset C$ be the sum of all $C^u(S)$ with $S(a_0) = i$. The $\overline{C}_i$ are subcomplexes, on which the filtration of C induces a filtration. Let the* reduced filtered $\mathfrak{sl}_N$-homology $\overline{[\![L]\!]}^f_N$ *be the filtered homology of $\overline{C}_1$, with a filtration-shift of N.*

Proposition 3.6. *Ignoring the filtration, we have*

$$\dim [\![L]\!]^f_N = N \cdot \dim \overline{[\![L]\!]}^f_N.$$

Proof. This is clear from the construction. $\square$

We conjecture that a closer look at the structure of the filtration of this homology will reveal the following:

Conjecture 3.7.

(i) *All the $\overline{C}_i$ are isomorphic as filtered cochain complexes.*

(ii) *The homology of the associated graded cochain complex of $\overline{C}_1$ is the reduced $\mathfrak{sl}_N$-homology.*

(iii) *The unreduced filtered $\mathfrak{sl}_N$-complex is, as filtered cochain complex, isomorphic to $[N]_q \cdot \overline{C}_1$, where the multiplication by q^i denotes a filtration-shift of i. In particular, the reduced filtered homology does not depend on the choice of the arc a_0.*

Remark 3.8. Presumably, Beliakova and Wehrli's construction of a Rasmussen invariant of links [BW05, Weh08] generalises naturally to $\mathfrak{sl}_N$-homology; thus one obtains the $\mathfrak{sl}_N$-slice-torus link invariants s_N. One application of the reduced filtered homology is that it allows to show that s_N is well-behaved under the sum of links, i.e. $s_N(L \# L') = s_N(L) + s_N(L')$. So far, this has not even been proven in the case $N = 2$.

4 A swarm of spectral sequences

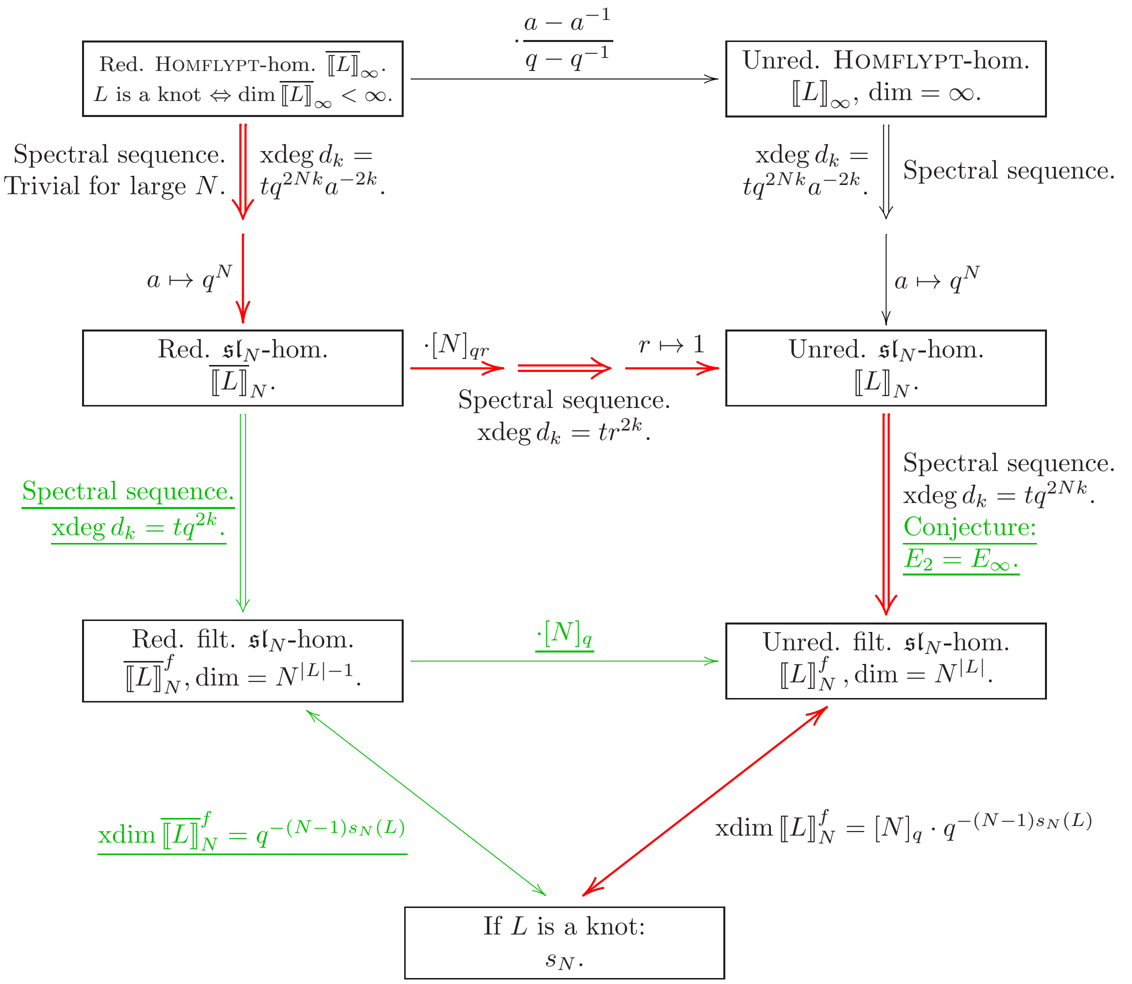


Figure 4.1: Relationship of the different Khovanov-Rozansky homologies. Parts of the diagram relying on conjectures are underlined and coloured green. Theorem 5.1 follows the thick red path.

Let us now introduce several spectral sequences, which link the different Khovanov-Rozansky homologies. The situation is summarised schematically in fig. 4.1.

Proposition 4.1 ([Ras06])**.** *Let L be a link with a marked component. For every $N \geq 1$, there is a spectral sequence with first page $\overline{[\![L]\!]}_\infty$, whose limit is a regraded version of the reduced $\mathfrak{sl}_N$-homology of L. Explicitly, the regrading of the (t, q, a)-degree is $(i, j, \ell) \mapsto (i, j + N\ell)$. The k-th differential has degree $tq^{2Nk}a^{-2k}$. For $k \geq 1$, the k-th page of this sequence is a link invariant. If L is a knot, then for sufficiently large N, this sequence converges on the first page.*

Remark 4.2 ([Ras06, Theorem 6.1]). For the sake of completeness, let us mention another spectral sequence, which we will not use in our calculations. Let K be a knot. Then there is a spectral sequence from $\overline{[\![K]\!]}_\infty$ whose limit has total dimension 1. Its differential d_k is homogeneous of degree $t^{2k-1}q^{-2k+2}a^{-2k+2}$.

Theorem 4.3. *In addition to the homological (t) and the quantum (q) grading, let $\overline{[\![L]\!]}_N$ carry a* reduced grading (r)*, which is* 0 *for all elements. Then there is a spectral sequence* *with first page* $[N]_{qr} \cdot \overline{[\![L]\!]}_N$ *that respects the q-degree and converges on the N-th page. Forgetting its r-degree, the limit equals* $[\![L]\!]_N$. *The k-th differential of the spectral sequence is homogeneous of degree* $tr^{2k}q^0$.

Proof. Recall from definition 2.1 that $C_N(D)$ is a module over $A = \mathbb{Q}[X]/(X^N)$. Denote the scalar multiplication by $*$. Let us introduce a descending filtration $\mathcal{F}$ on $C_N(D)$, given by $\mathcal{F}_{2i-N+1}C_N(D) = X^i * C_N(D)$. Indeed, we have

$$C_N(D) = \mathcal{F}_{-N+1}C_N(D) \supset \mathcal{F}_{-N+3}C_N(D) \supset \ldots \supset \mathcal{F}_{N-1}C_N(D) = 0.$$

Since the differential of $C_N(D)$ commutes with the scalar multiplication, it also respects this filtration. So there is a spectral sequence converging to $[\![L]\!]_N$.

Let $C_{N,j}(D)$ be the subcomplex of $C_N(D)$ of q-degree j. Note that the filtration $\mathcal{F}$ and the q-grading are compatible in the sense of proposition B.7. Hence the spectral sequence induced on $C_N(D)$ by the filtration respects the q-degree, and its differentials are homogeneous of degree $tr^{2k}q^0$.

It remains to analyse the 0-th page of that spectral sequence, i.e. the associated graded cochain complex. We have $\overline{C_N(D)} = C_N(D) \otimes_A \overline{Q}$, and the latter cochain complex is isomorphic to $\mathcal{F}_{-N+1}C_N(D)/\mathcal{F}_{-N+3}C_N(D)$ with a shift of $N-1$ in the q-grading.

Since $C_N(D)$ is a *free* A-module, for all $i \in \{1, \ldots N-1\}$ the multiplication by X^i is a isomorphism of rational vector spaces from $\mathcal{F}_{-N+1}C_N(D)/\mathcal{F}_{-N+3}C_N(D)$ to $\mathcal{F}_{2i-N+1}C_N(D)/\mathcal{F}_{2i-N+3}C_N(D)$. Because the scalar multiplication commutes with the differential, this map is in fact an isomorphism of cochain complexes. It shifts the q-grading and the r-grading by $2i$. The 0-th page of the spectral sequence is the sum of the $\mathcal{F}_{2i-N+1}C_N(D)/\mathcal{F}_{2i-N+3}C_N(D)$, and as such isomorphic to

$$\bigoplus_{i \in \{-N+1, -N+3, \ldots, N-1\}} (qr)^i \cdot \overline{C}_N.$$

Taking homology yields the claimed result for the first page. □

Of the following theorem, we only include sketch the proof; note that other results of this text do not rely on it.

Theorem 4.4. *The spectral sequence of the previous theorem has higher pages that invariants of links with a marked component.*

Sketch of a proof. First note that the reduced cochain complex is defined for a link diagram with a base-point. Moving the base-point along its strand results in a quasi-isomorphic cochain complex. Therefore, it is sufficient to prove invariance under Reidemeister moves which do not involve the strand which carries the base-point. Let us write D as the gluing of a small tangle diagram D_1, in which the Reidemeister move shall happen, and D_2, the complement of D_1. Let D_1' be the small tangle diagram which differs from D_1 by a Reidemeister move, and D' the gluing of D_1' and D_2. Khovanov and Rozansky prove the existence of a quasi-isomorphism between the cochain complexes of D_1 and D_1'. Tensoring with the identity of the cochain complex of D_2, this yields a quasi-isomorphisms between $C_N(D)$ and $C_N(D')$. Note that this map respects the r-grading, since both factors do – one being the identity, the other not involving the base-pointed strand. Using lemma B.5, it is now sufficient to prove that this quasi-isomorphism respects the r-grading. □

Remark 4.5. The limit of Rasmussen's spectral sequence from the reduced HOMFLYPT-homology is the reduced $\mathfrak{sl}_N$-homology with an *additional* $\mathbb{Z}$-grading. Since the higher pages of the spectral sequence are link invariants, so is this grading, and thus one obtains a refinement of $\mathfrak{sl}_N$-homology. Something similar is true for the spectral sequence from N copies of the reduced $\mathfrak{sl}_N$-homology to unreduced $\mathfrak{sl}_N$-homology: it endows the unreduced $\mathfrak{sl}_N$-homology with an additional $\mathbb{Z}/N\mathbb{Z}$-grading, and this grading is a link invariant, too.

One can exploit these refinements of the homologies, e.g. as follows:

Definition 4.6. *Let L be a link. Proposition 4.1 gives a spectral sequence from $\overline{[\![L]\!]}_\infty$ converging to a regraded version of $\overline{[\![L]\!]}_1$. But since* $\operatorname{xdim} \overline{[\![L]\!]}_1 = 1$*, the limit of this spectral sequence has graded dimension $q^{s_\infty(L)}a^{-s_\infty(L)}$ for some $s_\infty(L) \in 2\mathbb{Z}$. The spectral sequence is a link invariant, and therefore $s_\infty(L)$ is as well.*

Remark 4.7. Notice that if the reduced Homflypt-homology of a link has only one generator whose degree is of the form $q^s a^{-s}$, then $s_\infty = s$. So in most cases, s_∞ can be easily read from the value of the reduced Homflypt-homology.

Conjecture 4.8. *The link invariant s_∞ is a slice-torus link invariant.*

This conjecture seems plausible in light of the fact that on KR-thin knots, in particular on two-bridge knots, s_∞ agrees with all the other $\mathfrak{sl}_N$-slice torus concordance invariants (this is a consequence of corollary 5.3). The invariance of the following spectral sequence under Reidemeister moves is a result of Wu, who, however, does not mention part (ii) of the following proposition:

Proposition 4.9. ***(i)*** [Wu09, Theorem 1.2] *There is a spectral sequence starting at unreduced $\mathfrak{sl}_N$-homology and converging to filtered $\mathfrak{sl}_N$-homology. Its higher pages are link invariants.*

(ii) *The homogeneous k-th differential has degree tq^{2Nk}.*

Proof of (ii). Note that the differentials of Gornik's cochain complex preserve the q-degree mod $2N$ (see [Gor04]). So the hypothesis of proposition B.6 are satisfied. □

This spectral sequence is conjectured to converge on the second page, although this may well be false; e.g. Bar-Natan and Shumakovitch [BN07] demonstrate that certain torus knots give counterexamples if one works over finite fields.

In the reduced case, the differential does *not* preserve the degree mod N, so all we get is the following:

Proposition 4.10. *There is a spectral sequence starting at reduced $\mathfrak{sl}_N$-homology and converging to reduced filtered $\mathfrak{sl}_N$-homology. The homogeneous k-th differential has degree tq^{2k}.*

Remark 4.11. Of course, we expect the higher pages of this spectral sequence to be link invariants*; to avoid delving into matrix factorisations, a proof is not included.

This spectral sequence does not always converge on the second page, e.g. it does not for 10_{125} (see fig. 4.2).

*In fact, the contrary would be even more interesting; in any case, we do not use the invariance in this text.

$\overline{[\![\mathbf{10_{125}}]\!]}_\infty$

$q \backslash t$	−3	−2	−1	0	1	2	3	4	5
6		●							
4	●				●				
2				●●					
0			●	○			●		
−2						●●			
−4					●				●
−6								●	

$\delta(\bullet) = 2, \delta(\circ) = 0.$

$\overline{[\![\mathbf{10_{125}}]\!]}_\mathbf{3}$

$q \backslash t$	−3	−2	−1	0	1	2	3	4	5
10									1
8									
6							1	1	
4									
2					1	2			
0				1					
−2				2	1				
−4									
−6		1	1						
−8									
−10	1								

$[\![\mathbf{10_{125}}]\!]_\mathbf{3}$

$q \backslash t$	−3	−2	−1	0	1	2	3	4	5
12									1
10									1
8							1		
6							1	1	
4					1	1		1	
2						2			
0				2		1			
−2				3	1				
−4		1		1	1				
−6		1	1						
−8			1						
−10	1								
−12	1								

Figure 4.2: The first arrow (from the left) depicts a part of the first-page-differential of the spectral sequence from $\overline{[\![10_{125}]\!]}_\infty$ to $\overline{[\![10_{125}]\!]}_1$. The only survivor is the hollow dot, i.e. the generator of degree 1. So $\widetilde{s_1}(10_{125}) = 0$. The two other arrows come from the first- and second-page differential of the spectral sequence from $\overline{[\![10_{125}]\!]}_3$ to $\overline{[\![10_{125}]\!]}_3^f$. The only survivor has degree q^{-2}, but this cannot be deduced from considering only $\overline{[\![10_{125}]\!]}_3$.

5 From HOMFLYPT-homology to the $\mathfrak{sl}_N$-concordance invariants

The following theorem combines this information of the spectral sequence of proposition 4.1, theorem 4.3 and proposition 4.9.

Theorem 5.1. *Let K be a knot, and let $N \geq 2$. There are polynomials $\overline{P}'_N \in \mathbb{N}[t^\pm, q^\pm, a^\pm]$, $P'_N \in \mathbb{N}[t^\pm, q^\pm, r^\pm]$ and for all $k \geq 1$ polynomials $f_N^k \in \mathbb{N}[t^\pm, q^\pm, a^\pm]$, $g_N^k \in \mathbb{N}[t^\pm, q^\pm, r^\pm]$, $h_N^k \in \mathbb{N}[q^\pm, t^\pm]$, such that for large enough $N : \forall k : f_N^k = 0$, and such that the following decompositions hold:*

$$
\begin{aligned}
\operatorname{xdim} \overline{[\![K]\!]}_\infty &= \overline{P}'_N(t,q,a) + \sum_{k=1}^{\infty} (1 + tq^{-2Nk}a^{2k}) f_N^k(t,q,a), \\
\operatorname{xdim} \overline{[\![K]\!]}_N &= \overline{P}'_N(t,q,q^N), \\
\operatorname{xdim} \overline{[\![K]\!]}_N \cdot [N]_{qr} &= P'_N(t,q,r) + \sum_{k=1}^{N-1} (1 + tr^{2k}) g_N^k(t,q,r), \\
\operatorname{xdim} [\![K]\!]_N &= P'_N(t,q,1), \\
\operatorname{xdim} [\![K]\!]_N &= q^{s'_N(K)} \cdot [N]_q + \sum_{k=1}^{\infty} (1 + tq^{2Nk}) h_N^k(q,t).
\end{aligned}
$$

See e.g. fig. 4.2 for the reduced HOMFLYPT-homology, reduced $\mathfrak{sl}_3$-homology and unreduced $\mathfrak{sl}_3$-homology of a knot that shows interesting behaviour.

Given $\operatorname{xdim} \overline{[\![K]\!]}_\infty$, there are only finitely many choices for the other polynomials and for s'_N. So this gives restrictions on s'_N, which may be practically computed from the value of HOMFLYPT-homology. This theorem may appear unwieldy; and in fact, we will only use the following corollary which skips all intermediary steps between the HOMFLYPT-homology and s'_N.

Remark 5.2. The mixture of reduced and unreduced homologies in the theorem may seem roundabout: why not simply just use either reduced or unreduced homologies? On the one hand, the spectral sequence from graded to filtered homology is weaker in the reduced case, since its differentials have degree tq^{2k}, and not tq^{2Nk}. On the other hand, unreduced HOMFLYPT-homology is infinite dimensional, so the spectral sequence from it to unreduced $\mathfrak{sl}_N$-homology is more difficult to handle, and in particular it is not trivial for large enough N.

Among all paths from the top to the bottom level of the diagram in fig. 4.1, the one taken in the theorem should lead to the maximum of information.

Corollary 5.3. *Let K be a knot and $N \geq 2$. Then there are integers α, β, α', β', such that the* HOMFLYPT*-homology of K contains generators of degrees $q^\alpha a^\beta$ and $q^{\alpha'} a^{\beta'}$ and*

$$s'_N(K) - 2N + 2 \leq \alpha + N\beta \leq s'_N(K) \leq \alpha' + N\beta' \leq s'_N(K) + 2N - 2.$$

For the normalised s_N, these inequalities read

$$s_N(K) - 2 \leq -(\alpha' + \beta')/(N-1) - \beta' \leq s_N(K) \leq -(\alpha + \beta)/(N-1) - \beta \leq s_N(K) + 2.$$

Proof. Let us use the equations of theorem 5.1, climbing from the bottom up:

$$\begin{aligned}
& q^{s'_N(K)-N+1} \leq \operatorname{xdim} [\![K]\!]_N \\
\implies \quad & q^{s'_N(K)-N+1} r^i \leq P'_N \text{ for some } i \\
\implies \quad & q^{s'_N(K)-N+1} r^i \leq \operatorname{xdim} \overline{[\![K]\!]}_N \cdot [N]_{qr} \\
\implies \quad & q^{s'_N(K)-N+1-j} \leq \operatorname{xdim} \overline{[\![K]\!]}_N \text{ for some } j \in \{1-N, 3-N, \dots, N-1\} \\
\implies \quad & q^{\alpha} a^{\beta} \leq \overline{P}'_N \text{ for some } \alpha, \beta \text{ with } s'_N(K) - 2N + 2 \leq \alpha + N\beta \leq s'_N(K) \\
\implies \quad & q^{\alpha} a^{\beta} \leq \operatorname{xdim} \overline{[\![K]\!]}_{\infty}.
\end{aligned}$$

A similar reasoning yields $q^{s'_N(K)+N-1} \leq \operatorname{xdim} [\![K]\!]_N \implies q^{\alpha'} a^{\beta'} \leq \overline{[\![K]\!]}_{\infty}$ for some α', β' with $s'_N(K) \leq \alpha' + N\beta' \leq s'_N(K) + 2N - 2$. □

Note also that the power of theorem 5.1 is limited:

Proposition 5.4. *Let K be a knot, and let $N \geq 2$. Suppose there are polynomials $i_N^0, \dots i_N^{N-1} \in \mathbb{N}[t^{\pm 1}, a^{\pm 1}, q^{\pm 1}]$ and some α, β such that the following decomposition holds:*

$$\operatorname{xdim} \overline{[\![K]\!]}_{\infty} = q^{\alpha} a^{\beta} + \sum_{k=0}^{N-1} (1 + t a^{-2} q^{2k}) \cdot i_N^k.$$

Then there is also a decomposition as in theorem 5.1 with $\alpha + N\beta$ at the place of $s'_N(K)$, i.e. the theorem cannot be used to show $s'_N(K) \neq \alpha + N\beta$.

Proof. Let N be fixed. Setting $f_N^k = 0$ for all k, $h_N^k = 0$ for $k \geq 2$ and

$$h_N^1 = \sum_{k=0}^{N-1} [N-k]_q \cdot q^k \cdot i_N^k,$$

$$g_N^k = [N-k]_{qr} \cdot (qr)^{-k} \cdot i_N^{N-k}$$

gives the desired decomposition. □

Remark 5.5. If the spectral sequence from $\overline{[\![K]\!]}_{\infty}$ to the regraded version of $\overline{[\![K]\!]}_1$ converges on the second page, it gives a decomposition as in the above proposition, with $i_N^k = 0$ for $k \neq 1$ and $\alpha = -\beta = s_{\infty}$. So, if the spectral sequence does indeed always converge on the second page, theorem 5.1 cannot be used to show that $s_N(K) \neq s_{\infty}(K)$ (see definition 4.6), and in particular, that theorem alone is not enough to prove that the $\mathfrak{sl}_N$-concordance invariants are distinct.

6 The $\mathfrak{sl}_N$-concordance invariants are not all equal

Pretzel knots are a practical family of candidates to disprove the conjecture that all the $\mathfrak{sl}_N$-concordance invariants are equal: they show sufficiently complex behaviour, yet their diagrams allow easy calculations, because they invite an inductive approach.

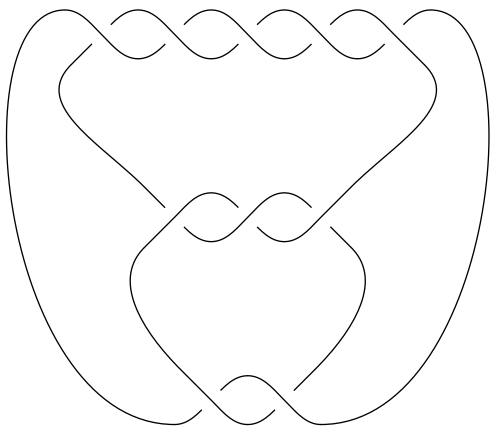


Figure 6.1: The $(5, -3, 2)$-pretzel knot K with $s_\infty(K) = 0$, $s_2(K) = 2$, $s_3(K) = 1$ and $s_N(K) \in \{0, 2/(N-1)\}$ for all $N \geq 4$.

Theorem 6.1. *Let $\ell > m \geq 3$, $n \geq 2$, and $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$. Then*

$$
\begin{aligned}
s_\infty(P(\ell, -m, n)) &= \ell - m - 2 && \textit{(assuming conjecture 4.8)} && \text{(i)}\\
s_2(P(\ell, -m, n)) &= \ell - m && \textit{for } m > n, && \text{(ii)}\\
s_2(P(\ell, -m, n)) &= \ell - m - 2 && \textit{for } m < n, && \text{(iii)}\\
s_N(P(\ell, -m, n)) &= \ell - m - 2 && \textit{for } n > 2, N > 2, && \text{(iv)}\\
s_N(P(\ell, -m, 2)) &\in \left\{\ell - m - 2, \ell - m - 2 + \frac{2}{(N-1)}\right\} && \textit{for } N > 2. && \text{(v)}
\end{aligned}
$$

Remark 6.2. Note that, under the assumption that s_∞ is a slice torus knot invariant,

$$\lim_{N\to\infty} s_N(P(\ell, -m, n)) = s_\infty(P(\ell, -m, n)).$$

Remark 6.3. In particular, the pretzel knots $P(\ell, -m, n)$ with $m > n$ are examples for which s_2 differs from all s_N with $N \geq 3$ and thus present an infinite family of counter-examples for part (i) of Lobb's conjecture (see introduction). Computer calculations give $s_3(P(\ell, -m, 2)) = \ell - m - 1$ for small cases (see section III.5). So the pretzel knots $P(\ell, -m, 2)$ are examples of knots for which s_2 and s_3 are different from each other and from all s_N with $N \geq 4$.

Of course, this prompts the conjecture *that all the s_N are all mutually distinct.* To prove this, however, new methods will be necessary: while the s_2-invariant equals the signature for quasi-alternating knots (see below), and both the s_2- and the s_3-invariant can be efficiently calculated by a computer [GM05, Lew12a], for $N \geq 4$ there are few tools available which could distinguish the s_N-invariants.

In the proof of the theorem, we use only corollary 5.3 and not theorem 5.1 itself; but even theorem 5.1 would not be strong enough to completely determine the value of $s_N(P(\ell, -m, 2))$. For example, using e.g. Webster's programme [Web05] or the skein long exact sequence, one finds that

$$
\begin{aligned}
\overline{[\![P(5, -3, 2)]\!]_\infty} = t^{-3}a^2q^4 + t^{-2}q^6 + t^{-1}a^2 + (2q^2 + 1) +& \\
t(a^{-2}q^4 + a^2q^{-4}) + 2t^2q^{-2} + t^3a^{-2} + t^4q^{-6} + t^5a^{-2}q^{-4}.& \quad (6.1)
\end{aligned}
$$

But this polynomial has several different decompositions as in proposition 5.4, among them one with $\alpha = \beta = 0$, and one with $\alpha = 2, \beta = 0$.

The remainder of this section is devoted to the proof of theorem 6.1. Part (ii) and (iii) are well-known facts: the $(\ell,-m,n)$-pretzel knot is quasi-alternating for $m > n$ (see Champanerkar and Kofman [CK09] and Greene [Gre10]), and hence its s_2–invariant equals its classical knot signature (see Manolescu and Ozsváth [MO08]):*

$$s_2(P(\ell,-m,n)) = \sigma(P(\ell,-m,n)) = \ell - m.$$

This value of the signature can be easily computed using Göritz matrices and the formula of Gordon and Litherland [GL78]. This covers (ii); for (iii), see e.g. [Man11, Man13].

Lemma 6.4. *For odd $\ell \geq 3$, we have*

$$\overline{[\![T(\ell,2)]\!]}_\infty = a^{1-\ell}q^{\ell-1} \cdot \left(1 + (t^2q^{-4} + t^3a^{-2}q^{-2}) \cdot \frac{t^{\ell-1}q^{2-2\ell}-1}{t^2q^{-4}-1}\right).$$

Proof. First, one may inductively calculate the HOMFLYPT-polynomial of $T(\ell,2)$, using the skein relation shown in (2.2). Then, since the $(\ell,2)$-torus knot is two-bridge, proposition 2.5 gives the HOMFLYPT-homology. □

Lemma 6.5. *For all $N \geq 2$ and odd $\ell \geq 5$,*

$$s_\infty(P(\ell,2-\ell,2)) = 0 \tag{6.2}$$

$$s_N(P(\ell,2-\ell,2)) \in \left[0, \frac{2}{N-1}\right] \tag{6.3}$$

$$s_N(P(\ell,2-\ell,4)) \in \begin{cases} [0,2] & N = 2, \\ [\frac{4}{N-1}-2, 0] & N \geq 3. \end{cases} \tag{6.4}$$

Proof. Let $K_- = P(\ell,2-\ell,2)$. Switching one of the two negative crossings of the last strand, one obtains the sum of two torus knots: $K_+ = T(\ell,2)\#T(2-\ell,2)$. Resolving that crossing, one obtains the positive Hopf link (to get its standard diagram, apply $(\ell-2)$ Reidemeister II moves). Lemma 6.4 and proposition 2.2 give xdim $\overline{[\![K_+]\!]}_\infty$, and see eq. 2.4 for xdim $\overline{\overline{[\![L_0]\!]}}_\infty$. So using corollary 2.8, one finds that

$$\operatorname{xdim} \overline{[\![P(\ell,2-\ell,2)]\!]}^0_\infty \leq (\ell-2)q^{-2} + 1.$$

This proves eq. 6.2. By corollary 5.3, concludes the proof of eq. 6.3, too. Notice also that

$$\operatorname{xdim} \overline{[\![P(\ell,2-\ell,2)]\!]}^2_\infty \leq (\ell-4)q^{-6} + a^{-2}.$$

Now let $K_- = P(\ell,2-\ell,4)$, and fix once again one of the negative crossings of the last strand. Then $K_+ = P(\ell,2-\ell,2)$, and once again L_0 is the positive Hopf link. So

$$\operatorname{xdim} \overline{[\![P(\ell,2-\ell,4)]\!]}^0_\infty \leq (\ell-4)a^2q^{-6} + 2.$$

Applying corollary 5.3 finishes the proof. □

*Only the Rasmussen invariant of a quasi-alternating knots equals its signature, not, in general, the other $\mathfrak{sl}_N$-concordance invariants. This may heuristically be explained as follows: whether a knot is quasi-alternating is determined by an *unoriented* skein relation, and s_2 is the sole invariant among the $\mathfrak{sl}_N$-concordance invariants which may not just be defined on the basis of an oriented skein relation, but as well using an unoriented one.

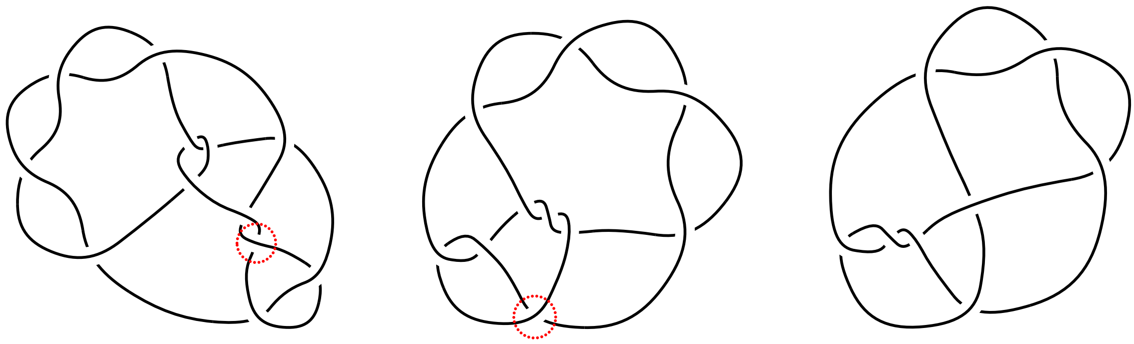


Figure 6.2: From left to right: $12n_{340}$, 10_{141} and 8_9 (drawn with Knotscape [HT99]).

Lemma 6.6. *Let $y = s_\infty$ or $y = s_N$ for some $N \geq 2$, and let ℓ, m, n as in theorem 6.1. Then*

$$y(P(\ell, -m, n)) \in [\ell - m - 2, \ell - m].$$

Proof. The standard diagram of the $(\ell, -m, n)$-pretzel knot, as shown exemplary in fig. 6.1, has writhe $(\ell - m - n)$, has $(n + 1)$ many Seifert circles, and $O^+ = n$ and $O^- = 1$. So the statement follows from the sharper slice Bennequin inequality (theorem 1.12). □

Let us now assemble the proof of theorem 6.1:

Proof. To show (iv), let ℓ, m and n be given, and let $N \geq 3$. By lemma 6.5, we have

$$s_N(P(m + 2, -m, 4)) \in \left[\frac{4}{N-1} - 2, 0\right]. \tag{6.5}$$

It takes $\frac{n-4}{2}$ many crossing switches from positive to negative, and $\frac{\ell-m-2}{2}$ many crossing switches from negative to positive to go from $P(m+2, -m, 4)$ to $P(\ell, -m, n)$. Thus eq. 6.5 implies (using proposition 1.6)

$$s_N(P(\ell, -m, n)) \in \left[\frac{4}{N-1} + 2 - n, \ell - m - 2\right]. \tag{6.6}$$

But by lemma 6.6, $s_N(P(\ell, -m, n)) \in [\ell - m - 2, \ell - m]$. This leaves $s_N(P(\ell, -m, n)) = \ell - m - 2$ as only value in the intersection of the two intervals.

Assuming that s_∞ is a slice torus knot invariant, part (i) may be shown similarly.

To show (v), by the same method one finds that $s_N(P(\ell, -m, 2)) \in [\ell - m - 2, \ell - m - 2 + 2/(N-1)]$. The only two elements of $\frac{2}{N-1}\mathbb{Z}$ in this interval are $\ell - m - 2$ and $\ell - m - 2 + 2/(N-1)$. □

Let us compute another example, to illustrate that the Rasmussen invariant does not necessarily give the best slice genus bound among all the $\mathfrak{sl}_N$-concordance invariants.

Example 6.7. *Let $K = 12n_{340}$, then $s_\infty(K) = 2, s_2(K) = 0, s_3(K) = 1$ and for $N \geq 4$: $s_N(K) \in \{2 - 2/(N-1), 2\}$.*

Thus yet once more we have $\lim_{N\to\infty} s_N(K) = s_\infty(K)$.

Proof. The value of s_2 and s_3 may be computed using javakh [GM05] and foamho [Lew12a], respectively; the other values can be read from $\llbracket K \rrbracket_\infty$, which we are going to compute using the skein long exact sequence. Notice that the calculation is rather quick, and that we do not need to determine the HOMFLYPT-homology of K completely (this would be possible using Rasmussen's spectral sequences, see proposition 4.1 and remark 4.2).

Resolving the crossing indicated in fig. 6.2 gives K as K_+, 10_{141} as K_- and the positive Hopf link as L_0. Resolving once more the indicated crossing of 10_{141} gives 10_{141} as K_+, 8_9 as K_- and the positive Hopf link as L_0. The knot 8_9 is two-bridge, so its reduced HOMFLYPT-homology is determined by its HOMFLYPT-polynomial and its signature. One finds

$$\operatorname{xdim} \overline{[\![8_9]\!]}_\infty^{-4} = q^4a^2$$

Applying corollary 2.8 two times gives

$$\operatorname{xdim} \overline{[\![K]\!]}_\infty \leq t^4a^{-4} \operatorname{xdim} \overline{[\![8_9]\!]}_\infty + (t^{1/2}a^{-1} + t^{5/2}a^{-3}) \operatorname{xdim} \overline{\overline{[\![\text{pos. Hopf links}]\!]}}_\infty,$$

and therefore

$$\operatorname{xdim} \overline{[\![K]\!]}_\infty^0 \leq q^4a^{-2} + q^2a^{-2}.$$

This immediately gives $s_\infty(K) = 2$ (see remark 4.7), and by corollary 5.3, one finds that

$$\forall N \geq 2 : s_N(K) \in \{2 - 2/(N-1), 2\}.$$

□

Chapter II

$\mathfrak{sl}_3$-foam homology of links*

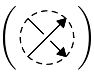
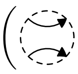
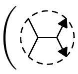
This chapter gives a definition of the $\mathfrak{sl}_3$–polynomial and its categorification, the $\mathfrak{sl}_3$–foam homology. Except for the definition of reduced homology using foams (section 5), this chapter contains nothing essentially new. We just review the parts of [Kho04, MN08] which are relevant to the purpose of this chapter – which is to provide a self-contained definition of $\mathfrak{sl}_3$–homology with the objective of calculation in mind. Thus we not dwell on the representation theoretic origins of webs and the $\mathfrak{sl}_N$–polynomials [RT90, Kup96].

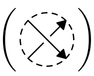
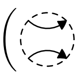
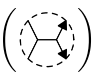
Instead of choking tori we use dots, like Khovanov [Kho04] and Mackaay and Vaz [MV07].

1 The $\mathfrak{sl}_3$–polynomial, naively

The $\mathfrak{sl}_3$–polynomial can be defined by a single skein relation involving only link diagrams. We will instead use the two skein relations (Sk^+) and (Sk^-), see below, because this allows a categorification using foams. These skein relations involve webs: a *closed web* is a plane oriented trivalent graph whose every vertex is either a source or a sink, and that may have vertex-less circles as additional edges. Note that the orientation of a single edge of a connected closed web already determines the orientations of all its edges. Thus, in drawings of webs, only the orientations of one edge per component will be specified.

A *tangle diagram* is the generic intersection of a link diagram with a disc; generic means that the disc's boundary intersects the diagram's strands transversely, and does not pass through a crossing. Let us define a map V from the set of smooth isotopy classes of link diagrams to the free $\mathbb{Z}[q^{\pm 1}]$–module on the set of smooth isotopy classes of closed webs. The map V is uniquely determined by the following two local relations, which are interpreted naively for now (i.e. apply these relations to all crossings of the link at once, then expand):

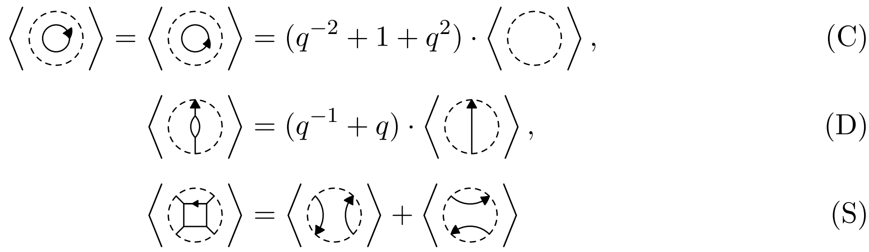
$V\Bigg($  $\Bigg) = q^2 \cdot V\Bigg($  $\Bigg) - q^3 \cdot V\Bigg($  $\Bigg)$ and (Sk^-)

$V\Bigg($  $\Bigg) = q^{-2} \cdot V\Bigg($  $\Bigg) - q^{-3} \cdot V\Bigg($  $\Bigg)$. (Sk^+)

Next, we define an evaluation $\langle \cdot \rangle$ of closed webs, called the *Kuperberg bracket* [Kup96, Jae92], which associates to a closed web a Laurent polynomial in q. This evaluation is

*Chapter II and III form the preprint [Lew12b].

given by the three relations



The $\mathfrak{sl}_3$–polynomial, which associates a Laurent polynomial in q to a link diagram, can now be obtained by composing V with $\langle \cdot \rangle$, and identifying the empty web with 1. Categorifying this construction is going to yield the $\mathfrak{sl}_3$–homology. However, it is advantageous to formalise the *localness* of the relations (Sk$^{\pm}$, C, D, S) before proceeding.

2 Planar algebras

While 2–categories do give a framework for webs and foams, they make sense only if one aims at interpreting webs as maps between oriented 0–manifolds; this aspect is not essential to the calculation of $\mathfrak{sl}_3$–homology, and so we use planar algebras instead. Planar algebras were introduced by Jones [Jon98] to identify subfactors. They were subsequently used to describe locally defined knot invariants such as the Jones polynomial. Bar-Natan [BN05] introduced a categorified version of planar algebras called *canopolis* to describe Khovanov homology, a method adaptable to $\mathfrak{sl}_3$–homology [MN08]. We will use a slightly generalised version of planar algebras, working over arbitrary monoidal categories instead of over the category of vector spaces over a fixed field. In this way, a canopolis is a planar algebra as well.

Let $B_0 \subset \mathbb{R}^2$ be a closed disc, and $B_1, \dots B_n \subset B_0^\circ$ be n pairwisely disjoint smaller closed discs. Punching out these discs yields a disc with holes, $H = B_0 \setminus \bigcup_{i=1}^n B_i^\circ$. Let M be a compact oriented one-dimensional smooth submanifold of H with $M \cap \partial H = \partial M$; in other words, M is a collection of circles and of intervals whose endpoints lie on the boundary of the big discs or one of the smaller discs. An *input diagram* consists of M, H, and on each boundary component of H a base point which is not in ∂M. We consider input diagrams up to smooth isotopy, in the course of which boundary points of M may not cross the base points.

For every $i \in \{0, \dots n\}$, the intersection $M \cap B_i$ is a finite set; at each of its points, the corresponding interval of M is either oriented towards the boundary ($+$ for $i > 0$, $-$ for $i = 0$), or away from it ($-$ for $i > 0$, $+$ for $i = 0$). Moreover, these points have a canonical order, given by starting from the base point and walking once around the circle in the counterclockwise direction. Thus the isotopy type of $M \cap B_i$ may be written as a sign-word ε_i, i.e. a word over the alphabet $\{+, -\}$. The *boundary* of H is the tuple $(\varepsilon_0, \dots \varepsilon_n)$.

Now suppose (M, H) and (M', H') are two input diagrams, such that $\varepsilon'_0 = \varepsilon_k$ for a fixed $k \in \{1, \dots n\}$. Then (M', H') may be shrunk and glued into B_k, base point on base point and boundary points on boundary points, resulting in a new input diagram with $n + n' - 1$ holes.

Let I be a subset of the set of all sign-words. Let C be a monoidal category, in the easiest case just the category Set of sets, and in the classical case the category of vector spaces over a fixed field. Then a *planar algebra* $\mathcal{P}$ over I and C consists of the following data:

- For each $\varepsilon \in I$, an object $\mathcal{P}_\varepsilon \in \mathsf{C}$.

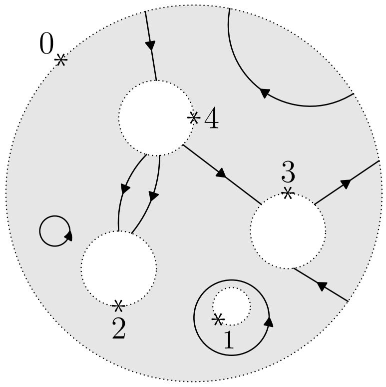


Figure 2.1: Example of an input diagram (also called spaghetti-and-meatballs diagram) with boundary $(+-+-+, \varnothing, ++, ++-, +---)$.

- For each input diagram H with boundary $(\varepsilon_0, \ldots \varepsilon_n)$ such that $\forall i : \varepsilon_i \in I$, a C–morphism
$$\mathcal{P}_H : \bigotimes_{i=1}^{n} \mathcal{P}_{\varepsilon_i} \to \mathcal{P}_{\varepsilon_0}.$$

This data is required to satisfy the following axioms:

- Suppose H is an input diagram with $n = 1$ and $\varepsilon_0 = \varepsilon_1$ that consists only of appropriately oriented radial strands. Then $\mathcal{P}_H : \mathcal{P}_{\varepsilon_0} \to \mathcal{P}_{\varepsilon_1}$ is the identity morphism.
- Let H and H' be two input diagrams with boundary $(\varepsilon_0, \ldots \varepsilon_n)$ and $(\varepsilon'_0, \ldots \varepsilon'_n)$, respectively. Suppose that for a fixed $k \in \{1, \ldots n\}$, $\varepsilon'_0 = \varepsilon_k$. Let H'' be the input diagram obtained from gluing H' into the k–th hole of H. Then the morphism $\mathcal{P}_{H''}$ is equal to the composition of the morphisms $\mathcal{P}_H$ and $\mathcal{P}_{H'}$, i.e.
$$\mathcal{P}_{H''} = \mathcal{P}_H \circ \big(\mathrm{id}_{\bigotimes_{i=1}^{k-1} \mathcal{P}_{\varepsilon_i}} \otimes \mathcal{P}_{H'} \otimes \mathrm{id}_{\bigotimes_{i=k+1}^{n} \mathcal{P}_{\varepsilon_i}}\big).$$

If $F : \mathsf{C} \to \mathsf{C}'$ is a monoidal functor, one may define the planar algebra $F(\mathcal{P})$ over I and C' by $F(\mathcal{P})_\varepsilon = F(\mathcal{P}_\varepsilon)$ and
$$F(\mathcal{P})_H : \bigotimes_{i=1}^{n} F(\mathcal{P})_{\varepsilon_i} \to F(\mathcal{P})_{\varepsilon_0}$$
to be the composition $F(\mathcal{P}_H) \circ \gamma$, where γ is the natural transformation
$$\bigotimes_{i=1}^{n} F(\mathcal{P}_{\varepsilon_i}) \to F\Big(\bigotimes_{i=1}^{n} \mathcal{P}_{\varepsilon_i}\Big)$$
which comes with the functor F because it is monoidal. Examples of this construction include, for a planar algebra $\mathcal{P}$ over Set, replacing for all $\varepsilon \in I$ the set $\mathcal{P}_\varepsilon$ by the free R–modules for some ring R by means of applying the left-adjoint of the forgetful functor from the category of R–modules to Set; or the quotient $\mathcal{P}$ by an equivalence relation on $\mathcal{P}$, by which we mean a collection of equivalence relations on all the $\mathcal{P}_\varepsilon$ which respect the planar algebra structure.

Suppose $\mathcal{P}$ and $\mathcal{P}'$ are planar algebras over I, I' and C, C', respectively, such that $I \subset I'$. A *planar algebra morphism* from $\mathcal{P}$ to $\mathcal{P}'$ consists of a functor $F : \mathsf{C}' \to \mathsf{C}$ and an I–indexed collection of C–morphisms $\mathcal{P}_\varepsilon \to F(\mathcal{P}'_\varepsilon)$ which respect the planar algebra structure, i.e. commute with the maps $\mathcal{P}_H$ and $\mathcal{P}'_H$. The functor F will typically be a forgetful functor.

Tangle diagrams with a base point on the boundary, considered – as input diagrams – up to smooth isotopy, form a planar algebra $\mathcal{T}$ over Set and the set I_0 of sign-words $\varepsilon_1 \cdots \varepsilon_m$ with $\sum_{j=1}^m \varepsilon_j = 0$. Let us elaborate this example: the planar algebra $\mathcal{T}$ associates to a sign-word ε with an equal number of both signs the (countably infinite) set $\mathcal{T}_\varepsilon$ of all tangles diagrams with boundary ε, modulo smooth isotopy; and to an input diagram H with n holes (see e.g. fig. 2.1) a function which maps a tuple of n tangle diagrams with appropriate boundaries to a new, bigger tangle diagram, by gluing each of the n tangle diagrams into the corresponding hole of H. One easily verifies that the planar algebra axioms are satisfied.

3 The $\mathfrak{sl}_3$–polynomial in the context of planar algebras

Suppose the unit circle intersects a closed web generically; as in the definition of tangle diagrams, this means that the circle intersects the edges of the closed web transversely, and does not pass through a vertex. Then the intersection of the closed web with the unit disc is called a *web*. As for input diagrams, we fix a base point on the boundary of a web, and encode the isotopy type of the boundary by a sign-word, in which $+$ stands for a strand oriented away from the boundary, $-$ for a strand oriented towards it. Note that a sign-word $\varepsilon_1 \cdots \varepsilon_m$ is the boundary of some web if and only if $\sum_{j=1}^m \varepsilon_j \equiv 0 \pmod 3$. Denote by I_3 the set of all such sign-words. Webs, up to smooth isotopy, form a planar algebra $\mathcal{W}$ over I_3 and Set.

Let $\mathcal{W}^q_\varepsilon$ be the free $\mathbb{Z}[q^{\pm 1}]$–module on $\mathcal{W}_\varepsilon$. Then $\mathcal{W}^q$ forms a planar algebra over I_3 and the category of $\mathbb{Z}[q^{\pm 1}]$–modules. In $\mathcal{W}^q$, we may interpret the relations (C), (D) and (S) as relations on $\mathcal{W}^q_\varnothing$, $\mathcal{W}^q_{-+}$ or $\mathcal{W}^q_{+-}$ and $\mathcal{W}^q_{-+-+}$ or $\mathcal{W}^q_{+-+-}$, respectively. Denote by $\mathcal{W}^{qr}$ the quotient by the generated equivalence relation.

Let W, W' be two webs and $\varphi : W \to W'$ a diffeomorphism – just of the webs themselves, not taking into account the ambient discs. We call φ a *web diffeomorphism* if it preserves the order of the boundary points, and the cyclic ordering of edges around vertices. Note that in the quotient $\mathcal{W}^{qr}$, the equivalence class of a web is already determined by its *web diffeomorphism* type. In $\mathcal{W}$, this distinction is slightly coarser than the isotopy type, since e.g. web diffeomorphisms do not take the orientation and relative position of closed components into account.

The two skein relations (Sk$^\pm$) determine a unique morphism $V : \mathcal{T} \to \mathcal{W}^q$ of planar algebras, $\mathcal{T}$ being the planar algebra of tangles. A link diagram L may be seen as element of $\mathcal{T}_\varnothing$. The equivalence class $[V(L)] \in \mathcal{W}^{qr}$ has a unique member that is a $\mathbb{Z}[q^{\pm 1}]$–multiple of the empty web. The coefficient equals the $\mathfrak{sl}_3$–polynomial of the link diagram. Reidemeister invariance may be shown by proving that the tangle diagrams with two, four and six boundary points corresponding to the Reidemeister moves I, II and III have in each case the same image under V.

4 The $\mathfrak{sl}_3$–homology in the context of canopolis

To categorify the $\mathfrak{sl}_3$–polynomial, one needs to understand foams, the cobordisms of webs. Suppose that for all $i \in \{1, 2, 3\}$, Σ^i are compact oriented smooth (generally not

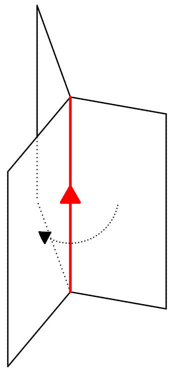


Figure 4.1: Cyclic ordering of facets around a singular circle of a closed foam. Singular circles are drawn as a thick red line.

connected) 2–manifolds with m boundary components $S_1^i, \ldots, S_m^i$ each. Let $\varphi_j : S_j^1 \to S_j^2$ and $\psi_j : S_j^1 \to S_j^3$ be orientation preserving diffeomorphisms. Consider the quotient of $\Sigma^1 \sqcup \Sigma^2 \sqcup \Sigma^3$ by the equivalence relation generated by $[x] = [\varphi_j(x)] = [\psi_j(x)]$ for all j. The images of the S_j^1 in the quotient are called *singular circles*, and the images of connected components of the Σ^i are called *facets*. There are three facets adjacent to each singular circle. Associate a non-negative integer to each facet. Such an integer d will graphically be represented by drawing d dots on the facet, which may roam the facet freely, but may not cross a singular circle. Such a quotient, together with the dots and with a choice of cyclic ordering of the three facets around each singular circle is called a *prefoam*.

Now consider a smooth embedding of a prefoam into $\mathbb{R}^3$, i.e. an embedding that is smooth on the facets and on the singular circles. Such an embedding induces cyclic orderings of the facets around each singular circle by the left-hand rule (see figure 4.1). If these cyclic orderings agree with those given by the prefoam, the image of the embedding is called a *closed foam*. Under the following conditions, the intersection of a closed foam with the cylinder $B \times [0,1] = \{(x,y,z) \mid x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}$ is called a *foam*:

- The boundary of the cylinder intersects the facets and singular circles of the closed foam transversely.
- The side $(\partial B) \times [0,1]$ of the cylinder intersects the closed foam in finitely many vertical lines, and is disjoint from all singular circles.
- The intersections with the top and bottom of the cylinder are webs.
- The base point of the top and the base point of the bottom web have the same x– and y–coordinates.

See figure 4.2 for an example.

We consider foams up to isotopies which, on the side of the cylinder, do not depend on the z–coordinate. A connected component of the intersection of a singular circle with the cylinder is called a *singular edge*. The web on the bottom of the cylinder is called *domain* of the foam, and the *codomain* of the foam is defined as the web on the top of the cylinder, with the orientation of each edge reversed. As usual, let χ denote the Euler characteristic. Then the *degree* of a foam f is defined by

$$\deg f = \chi(\text{domain of } f) + \chi(\text{codomain of } f) + 2(\text{total number of dots on } f) - 2\chi(f).$$

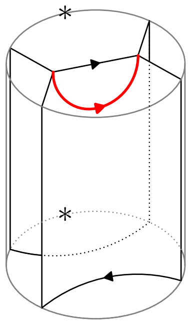
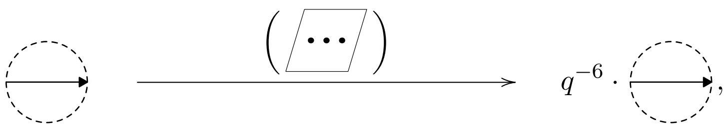


Figure 4.2: A singular saddle is the intersection of a closed theta foam (not drawn) with a can (drawn grey). It has three facets, one singular edge and degree 1. Orientations and base-points of the domain and codomain are specified.

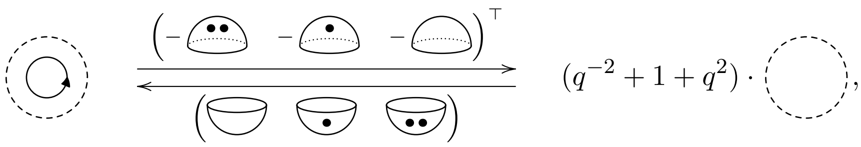
Foams can be glued in two ways: if the domain of one foam agrees with the codomain of another, by stacking them on top of each other. Or, by gluing them into the cylindrical holes of a thickened input diagram. The degree is additive with respect to both of these operations.

Webs with a fixed boundary and the foams between them thus constitute a *graded category*, i.e. a category whose morphisms have an integral rank which is additive under composition. Let us define a planar algebra $\mathcal{W}^c$ over I_3 and the category GCat of small graded categories:* to $\varepsilon \in I_3$, associate the category whose set of objects is $\mathcal{W}_\varepsilon$, and whose morphisms $W \to W'$ between two webs $W, W' \in \mathcal{W}_\varepsilon$ are the foams with domain W and codomain W'. If H is a planar input diagram, then $\mathcal{W}^c_H$ is the functor that acts as $\mathcal{W}_H$ on the objects, and glues foams into a thickened version of H.

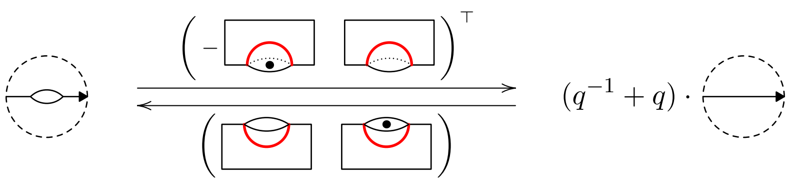
Next, $\mathcal{W}^{cq}$ may be constructed by applying a functor from GCat to ACat, the category of small additive categories: replace webs by $\mathbb{Z}[q^{\pm 1}]$–linear combinations of webs, and foams by matrices of $\mathbb{Z}$–linear combination of foams, where morphisms from $q^\alpha \cdot W$ to $q^\beta \cdot W'$ are the foams with degree $\alpha - \beta$. So the categories $\mathcal{W}^{cq}_\varepsilon$ are not graded, but have instead a shift operator for their objects. In this planar algebra, consider the following morphisms:



(T^c)

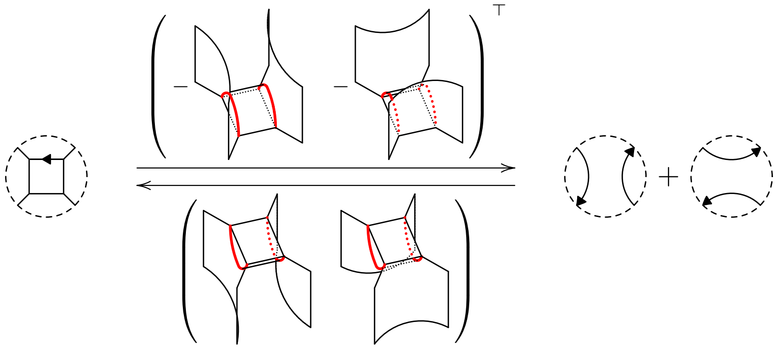


(C^c)



and (D^c)

*The superscript "c" stands for categorification.



(S^c)

Demanding that the first morphism be zero, and the following three pairs of morphisms be mutually inverse, for any placement of the base point, generates an equivalence relation on the planar algebra $\mathcal{W}^{cq}$. Let $\mathcal{W}^{cqr}$ be the quotient of $\mathcal{W}^{cq}$ by this equivalence relation.

Notice that the relations (T^c), (C^c), (D^c), (S^c) are not those of Khovanov's original definition [Kho04]; rather, Khovanov builds on the TQFT given by the Frobenius algebra $\mathbb{Z}[X]/(X^3)$, and defines the evaluation of foams by fixing the evaluation of the closed theta foam, and then using the universal BHMV-construction [BHMV95]. From this, the above relations follow. Since this text's aim is to give a self-contained definition of $\mathfrak{sl}_3$-homology focusing on calculations, these relations are used as definition, not as a corollary of the definition. See [MN08, Lemma 3.5] for a proof that they actually are sufficient.

Let us consider some examples of $\mathcal{W}^{cqr}_\varepsilon$. For $\varepsilon = \varnothing$, the isomorphism classes of objects of this category are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the empty web $\varnothing$. Non-zero morphisms $q^\alpha \cdot \varnothing \to q^\beta \cdot \varnothing$ exist only if $\alpha = \beta$, and in this case the morphism $\mathbb{Z}$–module is just $\mathbb{Z}$. The $\mathbb{Z}[q^{\pm 1}]$–module Khovanov [Kho04] associates to a closed web W can be recovered as

$$\bigoplus_{\alpha \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{W}^{cqr}_\varnothing}(q^0 \cdot \varnothing, q^\alpha \cdot W).$$

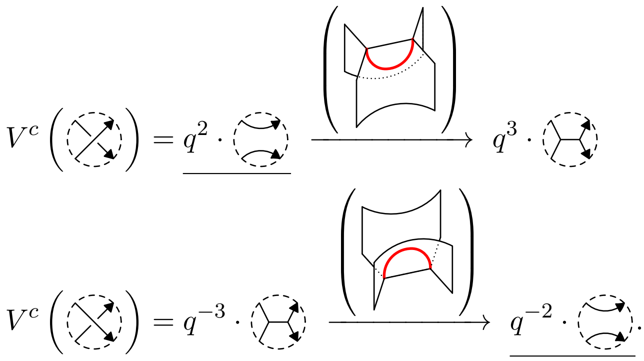
Similarly, the isomorphism classes of objects of $\mathcal{W}^{cqr}_{+-}$ are in correspondence with the elements of the free $\mathbb{Z}[q^{\pm 1}]$–module generated by the web, which is just an interval. The morphisms are more complicated, however, since there are non-zero foams of three different degrees: rectangles with none, one, or two dots. So the morphism $\mathbb{Z}$–module from $q^\alpha \cdot$ $\longrightarrow q^\beta \cdot$ is $\mathbb{Z}$ if $\alpha - \beta \in \{0, 2, 4\}$, and trivial otherwise.

Let a diffeomorphism between two foams be called a *foam diffeomorphism* if its restriction to the top (bottom) of the cylinder constitutes a web diffeomorphisms between the domains (codomains) of the two foams. In $\mathcal{W}^{cqr}$, the equivalence class of a foam is already determined by its foam diffeomorphism type; this is not the case in $\mathcal{W}^{cq}$, where e.g. a cylindrical foam tied into a knot is not the identity of the circular web.

Finally, let $\mathcal{W}^{cqrt}$ be the planar algebra obtained from $\mathcal{W}^{cqr}$ by setting $\mathcal{W}^{cqrt}_\varepsilon$ to be the category of bounded cochain complexes (up to chain homotopy equivalence) over $\mathcal{W}^{cqr}_\varepsilon$. More formally, this means applying a monoidal functor $K : \mathsf{ACat} \to \mathsf{ACat}$. The natural transformation $K(C_1) \otimes K(C_2) \to K(C_1 \otimes C_2)$ is given by

$$(P_i, g_i)_i \otimes (Q_j, h_j)_j \mapsto \Big(\bigoplus_{i+j=k} P_i \otimes Q_j, \sum_{i+j=k} g_i \otimes \mathrm{id}_{Q_j} + (-1)^i \, \mathrm{id}_{P_i} \otimes h_j \Big)_k. \tag{*}$$

In the notation of cochain complexes, the module at t–degree 0 is underlined. We may now define the $\mathfrak{sl}_3$–cochain complex as the planar algebra morphism $V^c : \mathcal{T} \to \mathcal{W}^{cqrt}$ uniquely determined by the skein relations



and (Sk^{-c})

(Sk^{+c})

Note that these relations each just give the cochain complex of one tangle diagram, and no cone or flattening is involved. The minus-sign which must intervene to arrange anti-commutativity of differentials along squares is hidden in (*). Although the placement of this sign is somewhat arbitrary, V^c is still unique, since cochain complexes are considered only up to homotopy equivalence.

Reidemeister invariance may, just as for the $\mathfrak{sl}_3$–polynomial, be shown by inspecting the small tangle diagrams corresponding to Reidemeister moves [MN08, section 4.2]. So the $\mathfrak{sl}_3$–cochain complex is, up to chain homotopy equivalence, an invariant of tangles.

A link diagram lives in $\mathcal{T}_\varnothing$, and is mapped to a cochain complex in the category of $\mathbb{Z}[q^{\pm 1}]$–linear combination of closed webs and matrices of $\mathbb{Z}$–linear combination of foams between them. Due to the relations, there is a homotopy equivalence to a cochain complex which contains only empty webs and closed foams. The latter are in turn just integral multiples of the empty foam. Setting the empty foam to 1, a cochain complex in the category of free graded abelian groups is obtained. Its homology is the $\mathfrak{sl}_3$*–homology* of the link.

5 Reduced $\mathfrak{sl}_3$–homology

A *reduced* version of $\mathfrak{sl}_N$–homology has been introduced by Khovanov and Rozansky [KR08a] in the context of matrix factorisations. This section contains a definition in the context of foams.

Let D be a diagram of a link L with a base-point. Cutting D open at the base-point, one obtains a tangle diagram D' with boundary $+-$. Let us consider how D' changes if the base-point is slid along the component of the link on which it lies. As long as the base-point does not pass a crossing, the isotopy type of D' clearly does not change. On passing a crossing, D' changes in the way depicted in figure 5.1: the strand that does not contain the base-point is pulled from one side of D' to the other, over the rest of the link. This pulling-over of the strand can of course be realised as a sequence of Reidemeister moves. Therefore, we may equivalently consider links with a marked component instead of a base-point. Two tangle diagrams with boundary $+-$ represent the same link with a marked component if and only if they are connected by a finite sequence of Reidemeister moves: because, if a Reidemeister move of a link diagram happens away from the base-point, it directly translates to a Reidemeister move of the corresponding $+-$-tangle. Otherwise, the base-point may be slid away prior to the Reidemeister move.

So, the homotopy type of the $\mathfrak{sl}_3$–chain complex C of D' is an invariant of links with a marked component; in particular, it is a knot invariant. As noted in section 4, if C is fully simplified, it contains only one kind of web – an interval –, and three kinds of foams:

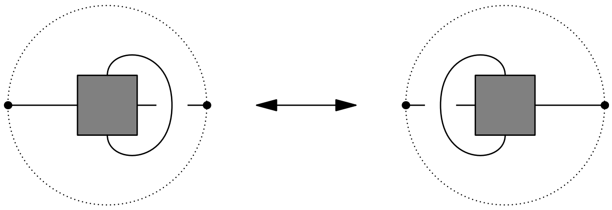


Figure 5.1: The effect on the corresponding $+-$-tangle diagram of sliding the base-point through a crossing.

the identity foam of the interval with none, one or two dots. Now map the identity foam to one, and foams of higher degree to zero. Thus one obtains a cochain complex of free graded abelian groups. Its homology is the reduced $\mathfrak{sl}_3$–homology of L, an invariant of links with a marked component. As simple examples show, its value may indeed depend on the choice of marked component, and it does neither determine the unreduced version, nor is it determined by it (see section III.5).

Chapter III

Automated $\mathfrak{sl}_3$-foam homology calculations*

In section 1, we review the algorithm and discuss sub-tangle trees and the recursive girth of a link. The implementation is fairly straight-forward, some details are discussed in section 3. Section 4 presents the usage and characteristics of the programme itself. In section 2 we show how the $\mathfrak{sl}_3$–concordance invariant s_3, as defined by Lobb [Lob12] (see also Wu [Wu07] and Lobb [Lob09]), may (in most cases) be extracted from the $\mathfrak{sl}_3$–homology by means of the spectral sequence converging to the filtered version of homology. This method was used for $\mathfrak{sl}_2$–homology by Freedman et al. [FGMW10]. It does not depend on the conjectured convergence of the spectral sequence on the second page.

Calculatory results are discussed in section 5. The most striking result obtained with `FoamHo` concerns the s_3–invariant: we find the first known examples of knots for which s_3 is odd and differs from the Rasmussen invariant s_2 (giving a counter-example for part (ii) of Lobb's conjecture, see introduction).

Conjecture 5.1. *If $\ell > m \geq 3$, $n \geq 2$, and $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$, then the $(\ell, -m, n)$–pretzel knot $P(\ell, -m, n)$ has s_3–invariant*

$$s_3(P(\ell, -m, n)) = \ell - m + \delta_n,$$

where $\delta_2 = -1$ and $\delta_n = -2$ for $n > 2$.

This statement is called a "conjecture" since it is established by `FoamHo`-calculations only for small values of ℓ, m and n. In section I.6, the conjecture is proven for the case $n > 2$.

1 The algorithm

The above definition of $\mathfrak{sl}_3$–homology gives a straightforward way of practical calculation: for an n–crossing diagram, take n copies of the cochain complex of $\mathrm{Sk}^{\pm c}$, and take the tensor product (*) of these cochain complexes as determined by the diagram. This is the same as forming the *cube of resolutions*. Transform this cochain complex of closed webs and foams between them into a homotopy equivalent cochain complex of empty webs and closed foams using the three relations (C^c), (D^c), (S^c); finally, evaluate the closed foams, and calculate the homology of the emerging integral cochain complex.

*Chapter II and III form the preprint [Lew12b].

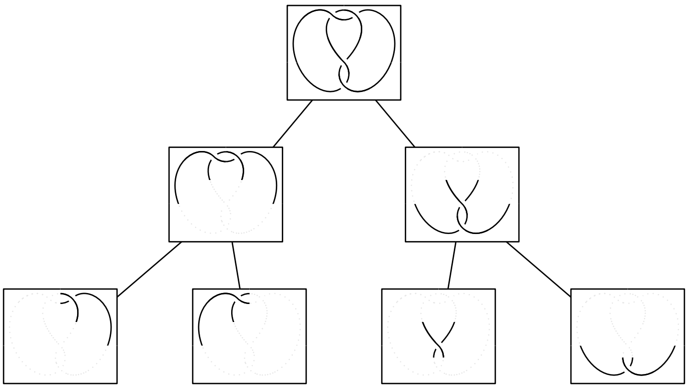


Figure 1.1: Example of a sub-tangle tree of the figure-eight-knot with girth 4.

In the algorithm described by Morrison and Nieh [MN08], it is just the order in which these steps are taken which is changed: we do not apply ($\mathrm{Sk}^{\pm c}$) to all crossings at once, but to one after the other, and try to simplify the cochain complex at each step as much as possible. Examples for the manual application of this algorithm can be found in [MN08]. In section IV.1, we will follow this algorithm to analyse the behaviour of $\mathfrak{sl}_3$-homology under Reidemeister-like moves.

At each step, one manipulates a cochain complex of tangles; if these tangles have less boundary points, then there are fewer different tangles and cobordisms between them and thus the calculations demand less memory and will, heuristically, go faster. Therefore, one glues the crossings in such an order that the cardinal of the boundary of the intermediate tangles is minimised (or at least, as low as one sees possible). This minimum is precisely the *girth* of the link diagram, which is thus the main factor limiting the speed of the algorithm.

In this text, we propose a variation of this algorithm based on sub-tangle trees, which is potentially faster, because intermediate tangles will have smaller boundary.

Definition 1.1. *Let D be a non-split link diagram with crossings enumerated from 1 to n. Decompose D into small tangle diagrams D_i that contain the i–th crossing, respectively. A* sub-tangle tree *of D is a full binary tree with a tangle diagram at each node, such that*

- *The leaves are decorated by the D_i.*
- *The root is decorated by D.*
- *Every node which is not a leaf has two children decorated with adjacent tangle diagrams, and is decorated itself with the union of those two tangle diagrams.*

Let the girth of a sub-tangle tree be the maximum number of boundary points of the tangle diagrams at all nodes. The recursive girth *of D is the minimum of the girths of all its sub-tangle trees.*

In general, the recursive girth of a diagram is smaller than its girth. For example, pretzel links have recursive girth 4, and girth 6; and the 222–crossing link considered in [FGMW10] has girth ≈ 24, and recursive girth at most 16.

Let us give a description of the algorithm. Given a link diagram D, find a sub-tangle tree of D with small girth. The algorithm consists in simplifying the $\mathfrak{sl}_3$–cochain complex of the tangle diagram at each node, one after the other. In the final step, the cochain complex of the diagram at the root is handled, which is D itself. After simplifications, this will be a cochain complex of vectors of q–shifted empty webs and matrices whose entries are integer multiples of the empty foam; identifying the empty foam with 1, this becomes a cochain complex of free graded abelian groups, and its homology may be calculated – separately for each q–degree – using the Smith normal form.

In the beginning, only the cochain complexes at the leaves are known, which are given by the relations ($\mathrm{Sk}^{\pm c}$). During each step, fix a node whose two children have cochain complexes C_1 and C_2 that are already known and simplified. Now, decorate this node with the tensor product $C_1 \otimes C_2$ (see (*)) and simplify this cochain complex as described below. After as many steps as D has crossings, the process reaches the root and is finished.

There are two ways to simplify a cochain complex: firstly, apply the circle, digon or square relation wherever possible. Secondly, apply Gaußian elimination to all matrix entries which are plus or minus an invertible foam:

Lemma 1.2 (Gaußian elimination, [BN07])**.** *Over an additive category with an isomorphism h, the cochain complex*

$$\dots \longrightarrow P \xrightarrow{\begin{pmatrix} * \\ g \end{pmatrix}} Q \oplus R \xrightarrow{\begin{pmatrix} h & i \\ j & k \end{pmatrix}} S \oplus T \xrightarrow{\begin{pmatrix} * & \ell \end{pmatrix}} U \longrightarrow \dots$$

is homotopy equivalent to

$$\dots \longrightarrow P \xrightarrow{g} R \xrightarrow{k - j \circ h^{-1} \circ i} T \xrightarrow{\ell} U \longrightarrow \dots.$$

The additive category in question is $\mathcal{W}_\varepsilon^{cqr}$, whose objects are $R[q^{\pm 1}]$–linear combinations of webs with boundary ε, and whose morphisms are matrices of R–linear combinations of foams. The base ring R is usually $\mathbb{Z}$; but one may also just calculate homology over some field. Gaußian elimination may then be applied more often, namely to all non-zero multiples of invertible foams instead of just plus or minus an invertible foam. This may increase the algorithm's speed.

As a rule, use the circle/digon/square-relations only when it is not possible to apply Gaußian elimination. This is to prevent those relations from altering a foam which is plus or minus the identity and could thus be removed.

2 Extracting the $\mathfrak{sl}_3$–concordance invariant from homology

There is a spectral sequence $E_\bullet$ from the graded $\mathfrak{sl}_N$–homology converging to a the filtered version. For $N = 2$, Freedman et al. [FGMW10, section 5.2] show how this allows to extract the Rasmussen invariant from Khovanov homology. We strengthen their method and adapt it to $N = 3$. For details, see section I.4.

The filtered $\mathfrak{sl}_3$–homology has Poincaré polynomial $q^{-2s_3}(q^{-2}+1+q^2)$. The differential on the k–th page of the spectral sequence has (t,q)–degree $(1,-6k)$: this is because the differential of the filtered $\mathfrak{sl}_3$-cochain complex preserves the degree modulo 6. Let KR_3 be the Poincaré polynomial of the graded $\mathfrak{sl}_3$–homology of a fixed knot. Then $E_\bullet$ gives rise to a decomposition of the form

$$KR_3(t,q) = q^{-2s_3}(q^{-2}+1+q^2) + \sum_{k=1}^{\infty} \zeta_k(t,q) \cdot (1 + tq^{6k}),$$

for some Laurent polynomials $\zeta_k(t,q)$ with non-negative coefficients. Reversely, given the value of KR_3, one may easily determine all possible such decompositions. The conjecture that $E_\bullet$ converges on the second page translates as $\forall k \geq 2 : \zeta_k(t,q) = 0$. Assuming the conjecture is true, the above decomposition is unique; otherwise, however, it is generally not. But even if there are are several possible decompositions, it may happen they all share the same value for s_3. This need not be the case, either, as the example $KR_3(t,q) = 1+q^2+q^4+q^6+tq^{12}$ demonstrates: either $s_3 = -1, \zeta_1(t,q) = q^6, \zeta_{\neq 1}(t,q) = 0$, or $s_3 = -2, \zeta_2(t,q) = 1, \zeta_{\neq 2}(t,q) = 0$. The first knots for which s_3 cannot be uniquely determined from $\mathfrak{sl}_3$–homology are $12n_{118}$, $12n_{210}$, $12n_{214}$ and $12n_{318}$ (see section 5 for further examples). At any rate, one obtains a list of possible values for s_3, and for most small knots, only one value is possible.

3 Implementation issues

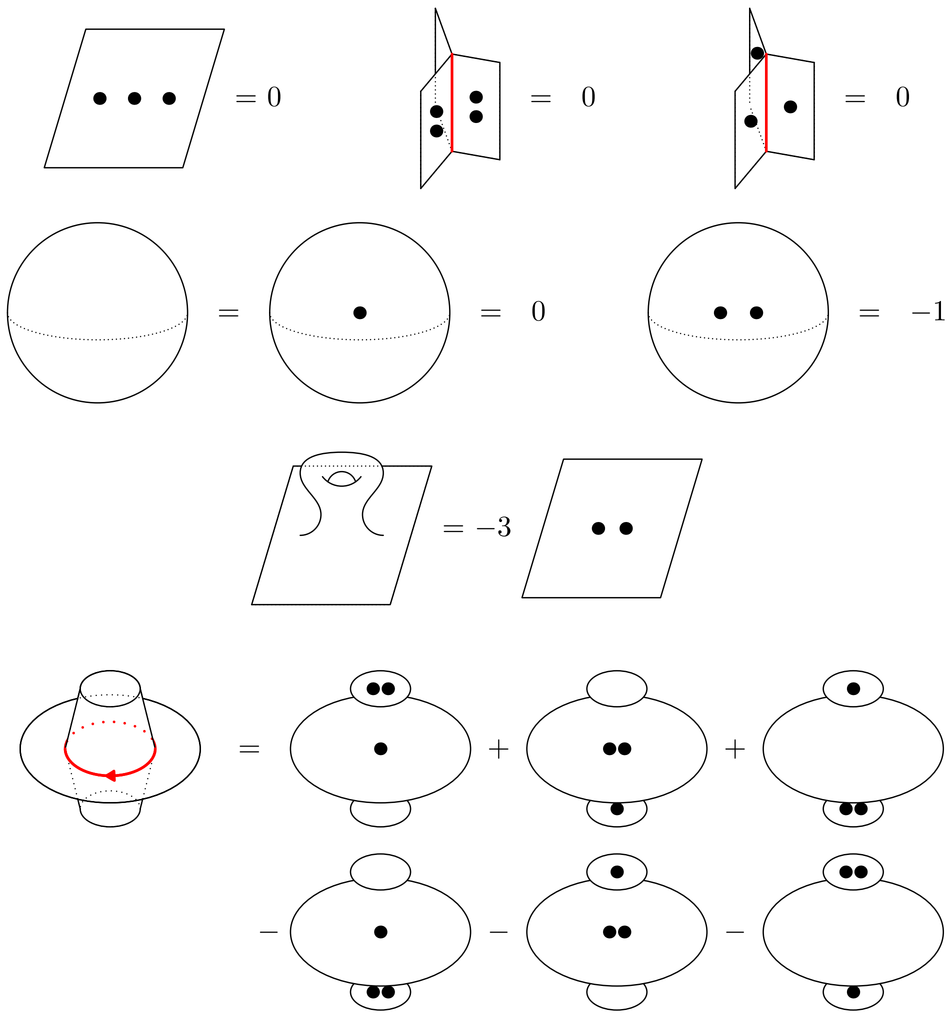


Figure 3.1: Foam relations.

An implementation of the algorithm of section 1 is possible because webs and foams need only be considered up to web and foam diffeomorphisms (see sections 3 and 4), and their diffeomorphism type contains only finitely much information. A web is determined by the following data:

- Its boundary, given as a sign-word.
- The number of edges that are circles.
- For each vertex, the triple of the vertices or boundary points it is connected to, listed in counterclockwise order.

Likewise, a foam is completely encoded by the following information:

- Its domain and codomain web.
- The start and end point of every singular edge that is an interval.
- For each singular edge, the three facets adjacent to it, in the order specified by the left-hand rule (see fig. 4.1).
- The genus of and number of dots on each facet.
- Each boundary component of each facet, given as an ordered tuple of edges; in this tuple, edges of the domain and codomain web, and singular edges alternate.

In order to reduce the complexity of foams, `FoamHo` applies the well-known relations shown in fig. 3.1 whenever possible. In particular, these relations are sufficient to evaluate closed foams.

Gaußian elimination is in fact only applied to plus or minus an identity foam; there are other isomorphisms than those, but they are difficult to detect for a computer programme, and so rare that looking for them does not seem worthwhile.

4 `FoamHo`, an $\mathfrak{sl}_3$–calculator

The algorithm described in the previous sections was implemented by the author as a C++-programme*. It computes the integral or rational (homology over finite fields is not yet implemented), reduced or unreduced homology of knot or link diagrams given in braid, planar diagram or Dowker-Thistlethwaite notation; it also attempts to find the value of the s_3–concordance invariant, using the spectral sequence from $\mathfrak{sl}_3$–homology to filtered $\mathfrak{sl}_3$–homology. If it is impossible to extract the s_3-invariant, a list of possible values is printed, and the value which corresponds to the convergence of the spectral sequence on the second page is highlighted.

The current version does not make use of the sub-tangle trees yet, and glues instead one crossing after the other. Apart from that, there is surely some room for rendering the programme faster and less memory hungry by optimising the code, without changing the algorithm; e.g., webs and foams are encoded in a redundant way, and tightening this would reduce the memory consumption.

The programme was baptised `FoamHo` in recognition of Shumakovitch's `KhoHo` [Shu03]. It has been released under the GPL†, and its source code as well as compiled versions for Linux and Windows may be downloaded from [Lew12a]. There are also tables of the homology and the s_3–concordance invariant of small knots and links available.

*Using the PARI/GP-library [PAR12] to calculate the Smith normal form, and the MPIR library [MPI12] for arbitrary precision integers and rationals.

†See `http://www.gnu.org/licenses/gpl.html`.

Let us give a rough idea of the capabilities of the programme for large knots. On an AMD Opteron (2.2 GHz), the (6,5)–torus knot's homology can be calculated in six minutes, using 80 MB of RAM; the (8,5)–torus knot takes 50 minutes and 275 MB of RAM; but the (7,6)–torus knot is out of the reach of 5 GB of RAM. In other words, the homology of links of girth 10 can be calculated until ca. 40 crossings, while links of girth 12 would demand a well-equipped computer.

Detailed usage instructions for `FoamHo` can be found in the README-file which is distributed along with the programme's source code and binaries. The following appetising example session demonstrates the computation of the $(4,3)$–torus knot's homology. User input begins with a dollar sign and is printed bold:

```
$ ./foamho -h

foamho, a sl3-homology calculator, version 1.1.

Usage: foamho [OPTIONS] braid | pd | dt NOTATION

For example, the following three commands all compute the figure-8-knot's
integral homology:

                   foamho braid aBaB
                   foamho pd "[[4,2,5,1],[8,6,1,5],[6,3,7,4],[2,7,3,8]]
                   foamho dt "[4,6,8,2]"

Options:
-q                 Compute rational homology instead of integral.
-r                 Compute reduced homology instead of unreduced. You may
                   give a number right after -r to indicate the marked
                   strand (useful for links).

-g                 Do not attempt to optimise the girth.
-v                 Display some progress information.
-vv                Display more detailed progress information.
-t                 Display time and memory consumption.
-h                 Display this help message and exit.

Written in 2012/2013 by Lukas Lewark, lewark@math.jussieu.fr.
All feedback is welcome.

$ ./foamho -t braid abababab

Girth-optimised link diagram (modified pd notation) [[2,4,3,1],[5,7,6,4],
    [6,9,8,3],[7,11,10,9],[10,13,12,8],[11,15,14,13],[14,16,1,12],
    [15,5,2,16]].
Girth: 6.
Calculating...
Done. Result:
Rational homology: (q^-14 + q^-12 + q^-10) + t^2(q^-16 + q^-14) +
    t^3(q^-22 + q^-20) + t^4(q^-20 + 2q^-18 + q^-16) +
```

```
    t^5(q^-26 + 2q^-24 + q^-22) + t^6(q^-22) + t^7(q^-28 + q^-26) +
    t^8(q^-32)
Total rank: 19
Rational homology is not self-dual => the link is chiral.
3-torsion: t^3(q^-18) + t^5(q^-22 + q^-20) + t^7(q^-26 + q^-24) +
    t^8(q^-30 + q^-28)
The 3-torsion part of homology is not self-dual => the link is chiral.
s_3-concordance invariant: -12
Run time in seconds: 2
Memory consumption in megabytes: 9.5
```

5 Calculatory results

Pretzel knots:

Conjecture 5.1. *If $\ell > m \geq 3$, $n \geq 2$, and $\ell + 1 \equiv m + 1 \equiv n \equiv 0 \pmod 2$, then the $(\ell, -m, n)$–pretzel knot $P(\ell, -m, n)$ has s_3–invariant*

$$s_3(P(\ell, -m, n)) = \ell - m + \delta_n,$$

where $\delta_2 = -1$ and $\delta_n = -2$ for $n > 2$.

Since pretzel knots have girth only 6, their homology can be computed particularly quickly. `FoamHo` calculations confirm the conjecture for the following values:

- $m + 2 = \ell \leq 69$ and $n = 4$.
- $m < \ell \leq 55$ and $n = 2$.

See section I.6 for a proof of the conjecture for $n > 2$, and a proof that $\delta_2 \in \{-1, -2\}$,

s_3 and s_2 for small knots: Of the 59 937 prime knots with 14 crossings or less, there are 361 for which the s_3 and s_2 concordance invariants differ, and for 63 the absolute value of s_3 is greater, i.e. the lower bound to the slice genus coming from $\mathfrak{sl}_3$ is better than the one coming from $\mathfrak{sl}_2$. For all these 361 knots, the difference between s_2 and s_3 is ± 1. A 16–crossing prime knot example for a greater difference is given by the conjecture 5.1: the $(7, -5, 4)$–pretzel knot, with $s_3 = 0$ and $s_2 = 2$.

The s_3–concordance invariant cannot be determined for certain knots with 14 crossings or less, so the statistics in the preceding paragraph are based on the assumption that the spectral sequence to the filtered version of homology converges on the second page.

s_3 and the Floer-concordance invariant τ: Let τ be the knot concordance invariant coming from Floer homology, as defined by Ozsváth and Szabó [OS03]. Hedden and Ording [HO08] found examples of knots K for which $2 = s_2(K) \neq 2\tau(K) = 0$. `FoamHo` calculations yield the following results for these knots:

$$s_3(D_+(T_{2,3}, 2)) = 2 \qquad s_3(D_+(T_{2,7}, 8)) \in \{2, 3, 4\}$$

$$s_3(D_+(T_{2,5}, 5)) \in \{2, 3\} \qquad s_3(D_+(T_{2,7}, 7)) \in \{2, 3, 4, 5\}$$

$$s_3(D_+(T_{2,5}, 4)) \in \{2, 3, 4\} \qquad s_3(D_+(T_{2,7}, 6)) \in \{2, 3, 4, 5, 6\}$$

All but the first knot are examples of how the method of section 2 may fail to completely determine the value of s_3.

However, since all of the above knots have slice genus equal to 1, they all have an s_3–invariant equal to 2. So these knots do not give examples for $s_2 \neq s_3$, but they demonstrate that $s_3 \neq 2\tau$.

Thinness: Khovanov [Kho03] called a knot *H-thin* if its unreduced Khovanov homology is supported in only two diagonals, or, equivalently, if its reduced Khovanov homology is supported in only one. Thinness was generalised to $\mathfrak{sl}_N$-homologies by Rasmussen [Ras07]. Alternating knots are H-thin, but not necessarily N*-thin* for $N > 2$. We find the knots $11a_{263}, 12a_{36}, 12a_{694}, 12a_{804}, 12a_{811}, 12a_{817}, 12a_{829}, 12a_{832}$ to be the smallest examples of alternating knots that are not 3–thin. None of these knots is two-bridge, which agrees with Rasmussen's theorem [Ras06] that two-bridge knots are N–thin for all N.

Mutation: Integral reduced $\mathfrak{sl}_3$–homology has been proven to be invariant under mutation of knots by Jaeger [Jae11]; for unreduced homology, invariance appears to be an open question. Our calculations confirm that all mutant families up to 13 crossings have the same reduced and unreduced integral $\mathfrak{sl}_3$–homology. We use the lists provided by Stoimenow [Sto].

Torsion: All prime knots and links for which the homology was computed, including knots with up to 12 and links with up to 11 crossings, have 3–torsion, with the exception of the Hopf links, whose homology is torsion-free. This is reminiscent of the omnipresence of 2–torsion in Khovanov homology remarked by Shumakovitch [Shu04]. Exemplary calculations also show the existence of 2–, 4–, 5– and 8–torsion, which are rather scarce. Small knots have torsion-free reduced homology, but large enough knots like the $(8, 5)$–torus knot have not.

Reduced and unreduced homology: Reduced $\mathfrak{sl}_2$–homology was conjectured by Khovanov [Kho03] to be determined by its unreduced counterpart; more precisely, that the rank of reduced homology were one less than the rank of unreduced homology. This is true for small knots, but Shumakovitch produced 15–crossing counterexamples [Shu12]. For $\mathfrak{sl}_3$–homology even considering only knots with crossing number six is enough to see that there is no linear relationship between the ranks of reduced and unreduced homology.

Previous computations: `FoamHo` calculations are in agreement with the results of Carqueville and Murfet [CM11], who compute reduced and unreduced rational $\mathfrak{sl}_3$–homology of all prime knots and links with up to six crossings.

Chapter IV

$\mathfrak{sl}_3$-foam homology and knotted weighted webs

In section 1, we define knotted webs and their diagrams; discuss the equivalent to the Reidemeister moves; and define knotted weighted webs and extend graded $\mathfrak{sl}_3$-homology to them. The next section 2 shows how to extend the filtered homology as well, and how to describe it using **abc**-weighted foams. This leads in section 3 to the definition of an s-invariant for knotted weighted webs. In section 4, we define the *slice degree* of knotted weighted webs and prove that the s-invariant provides a lower bound for it. Finally, section 5 gives examples of knotted weighted webs and discusses further properties.

1 Extending $\mathfrak{sl}_3$-homology to knotted weighted webs

Pondering the definition of $\mathfrak{sl}_3$-foam homology as given in chapter II, or even just of the $\mathfrak{sl}_3$-polynomial, one may remark that it lends itself quite naturally to an extension: one may associate a polynomial not just to *tangles*, but instead to knotted pieces of *webs*, which we will call *web tangles*. Given the proper definition of web tangles, the extension of the $\mathfrak{sl}_3$-polynomial and its categorification are little more than a formality.

Definition 1.1. *An* abstract web W *is a trivalent, oriented, but not necessarily planar graph, whose every vertex is a source or a sink and which may have vertex-less circles as additional edges. An embedding* $\Gamma : W \to \mathbb{R}^3$ *(or* $\Gamma : W \to S^3$*) is called* smooth spatial embedding *of* W *if the restriction of* Γ *to each edge is a smooth map. The image of* Γ *is called a* knotted web*. If* φ *is a diffeomorphism of* $\mathbb{R}^3$*, then we say that* $\operatorname{im}\Gamma$ *is* of the same type *as* $\operatorname{im}\varphi\circ\Gamma$*.*

Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ *be an orthogonal projection. Let us call a point in* $\mathbb{R}^2$ *with two preimages a* double point*. Suppose the following conditions are satisfied:*

(i) *The restriction of* $p\circ\Gamma$ *to each edge has nowhere-vanishing differential.*

(ii) *At every double point, the two intersecting strands of* W *cross transversely.*

(iii) *No point in* $\mathbb{R}^2$ *has more than two preimages.*

(iv) *The differential of two edges is not collinear at a vertex where they meet.*

(v) *No vertex of* W *is sent to a double point.*

Then the image of $p\circ\Gamma$*, endowed with the additional data of which strand passes over and which strand passes under at each double point is called a* diagram of the knotted web $\operatorname{im}\Gamma$*.*

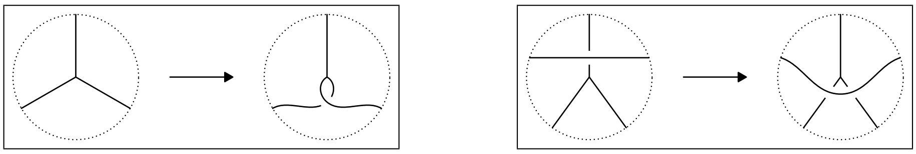


Figure 1.1: The moves W1 and W2.

Remark 1.2. The reason we allow non-planar knotted webs is simply that we can. For the extension of $\mathfrak{sl}_3$-homology to knotted webs it is not necessary that those knotted webs be planar, because the resolutions of a diagram of a non-planar knotted web will still be planar webs.

Remark 1.3. The five conditions in the definition of a web diagram are equivalent to the following single condition: let W' be the disjoint union of the edges of W, and let $P : W' \to \mathbb{R}^2$ be the composition of the projection $W' \to W$ with $p \circ \Gamma$. Let $x_1, \dots x_n \in W'$ all have the same image y under P, and assume that for no vertex v the three preimages of v in W' are all among the x_i. Then the subspace of $T_y\mathbb{R}^2$ generated by the $dP(x_i)$ has dimension n.

Proposition 1.4. *A spatial embedding $\Gamma : W \to \mathbb{R}^3$ and an orthogonal projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ generically yield a diagram of W; that is to say, the set of embeddings Γ and projections p which give a diagram is an open and dense subset of the space of all such embeddings and projections with the C^∞-topology. In particular, every knotted web has a diagram.*

Proof. The proof is similar to the proof of the analogous statement for links and link diagrams. □

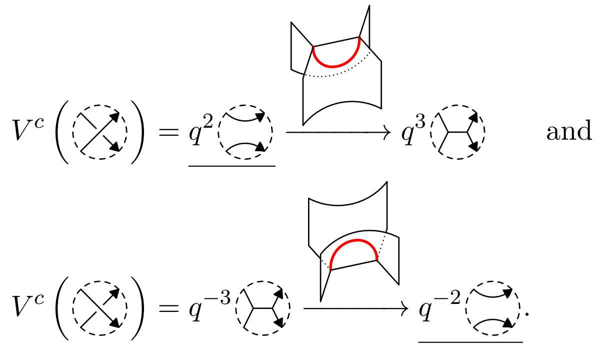
Definition 1.5. *A* web tangle diagram *is a generic intersection of the unit disc with a web diagram.* Generic *means that the unit circle passes neither through a double point, nor through a vertex, and intersects all strands transversely.*

Proposition 1.6. *Two web diagrams represent knotted webs of the same type if and only if they are related by a finite sequence of* web moves*: these moves are the Reidemeister moves R1, R2 and R3, which must happen away from vertices, and the moves W1 and W2, see fig. 1.1. Of course, each of these five moves is considered with its inverse, its mirror image, and any possible orientation of the strands.*

Proof. Cf. [Kau89] for a proof of this statement for PL-graphs. Each of the restrictions (i)–(v) in the definition of a diagram of a knotted web produces a move: (i) $\rightsquigarrow$ R1, (ii) $\rightsquigarrow$ R2, (iii) $\rightsquigarrow$ R3, (iv) $\rightsquigarrow$ W1, (v) $\rightsquigarrow$ W2. □

Remark 1.7. To avoid the moves R1 and W1, one may introduce a *thickening*, i.e. consider a web living inside a surface that contracts to it (cf. [RT90]); alternatively, to allow R1 but still ban W1, one may just put a rigid disc around each vertex (cf. [Kau89]). In this text, we do not pursue any such construction and work with unthickened knotted webs throughout; as will be seen shortly, both $\mathfrak{sl}_3$-polynomial and -homology behave nicely with respect to the W1-move. Our definition may remind of Thurston's knotted trivalent graphs [Thu02]. However, knotted trivalent graphs are thickened graphs as well, and, unlike webs, not necessarily bipartite.

Let us now fit web tangle diagrams in the frame-work of planar algebras. First, recall briefly the terminology of section II.3: the planar algebras of tangles is denoted by $\mathcal{T}$, and



(Sk^{-c})

(Sk^{+c})

Figure 1.2: The two skein relations (see section II.1).

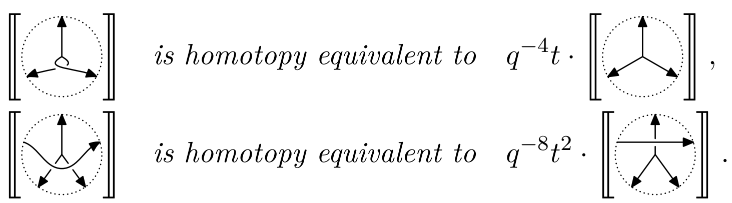
the planar algebra of webs by $\mathcal{W}$. Denoting by $\mathcal{W}^q_\varepsilon$ the free $\mathbb{Z}[q^{\pm 1}]$-module on $\mathcal{W}_\varepsilon$, we obtain a planar algebra $\mathcal{W}^q$. Quotienting $\mathcal{W}^q$ by the equivalence relation generated by the circle, digon and square relation yields the planar algebra $\mathcal{W}^{qr}$. The skein relations (Sk$^\pm$) determine a unique morphism $V : \mathcal{T} \to \mathcal{W}^q$. Let D be a link diagram, i.e. $D \in \mathcal{T}_\varnothing$. Then $[V(D)] \in \mathcal{W}^{qr}$ has a unique representative that is a $\mathbb{Z}[q^{\pm 1}]$-multiple of the empty web. This coefficient is the $\mathfrak{sl}_3$-polynomial of D.

Web tangle diagrams, considered up to smooth isotopy form a planar algebra $\mathcal{U}$ over Set and I_3.* The two skein relations (Sk$^\pm$) allow V to be extended in a unique way to a morphism $V^u : \mathcal{U} \to \mathcal{W}^q$. As for link diagrams, if D is a knotted web diagram, consider $[V^u(D)] \in \mathcal{W}^{qr}$. For some $p \in \mathbb{Z}[q^{\pm 1}]$, $[V^u(D)] = [p \cdot \bigcirc]$. We say that the $\mathfrak{sl}_3$-polynomial of D is p.

The $\mathfrak{sl}_3$-homology may be extended in the same way. Throughout this chapter, we denote the $\mathfrak{sl}_3$-homology by $[\![\cdot]\!]$. We drop the 3 in the subscript, since no other $\mathfrak{sl}_N$-homologies are considered.

The $\mathfrak{sl}_3$-homology of web tangle diagrams is an extension of the homology of tangle diagrams; since Reidemeister moves involve only tangle diagrams, the Reidemeister invariance of the homology of tangle diagrams survives the extension to web tangle diagrams. Let us now investigate how the homology behaves with respect to the web moves. For the W2-move, the following proposition has already been proven in [MN08] for the $\mathfrak{sl}_3$-polynomial, and in [Ros12] for homology.

Proposition 1.8. *The $\mathfrak{sl}_3$ homology of knotted webs behaves as follows under W1 and W2:*



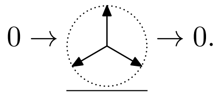
(1.1)

(1.2)

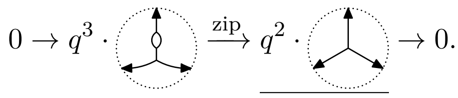
Proof. We will calculate, simplify and compare the cochain complexes of the two diagrams, respectively, using the algorithm described in section III.1. The mirror images and different orientations can be done similarly. To indicate homological degree, the space at degree 0 is underlined. Foams that are just an identity foam with additional dots are depicted by a cross-section with the dots drawn at the appropriate places.

*"U" being the letter between "W" and "T"...

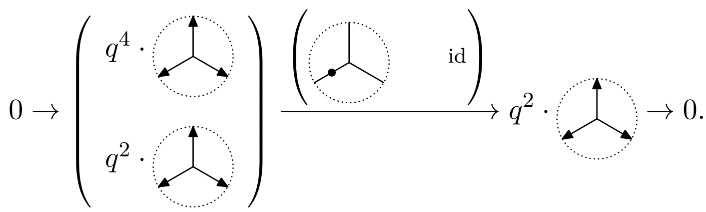
The cochain complex of the diagram on the right-hand side of eq. 1.1 is



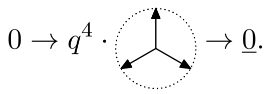
On the left-hand side we have



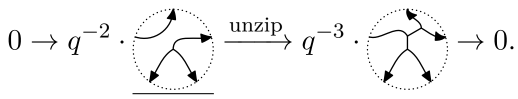
This is isomorphic to the cochain complex



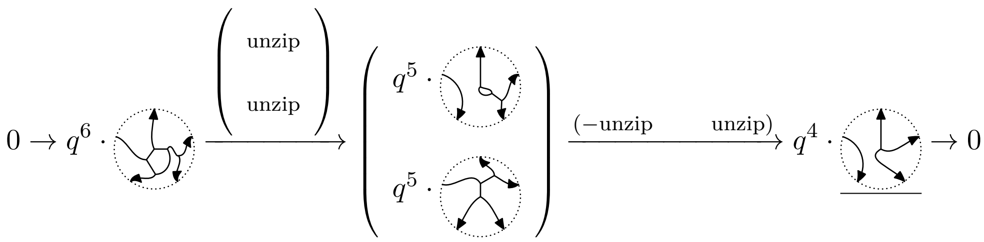
Using Gaußian elimination (lemma III.1.2), this is homotopy equivalent to



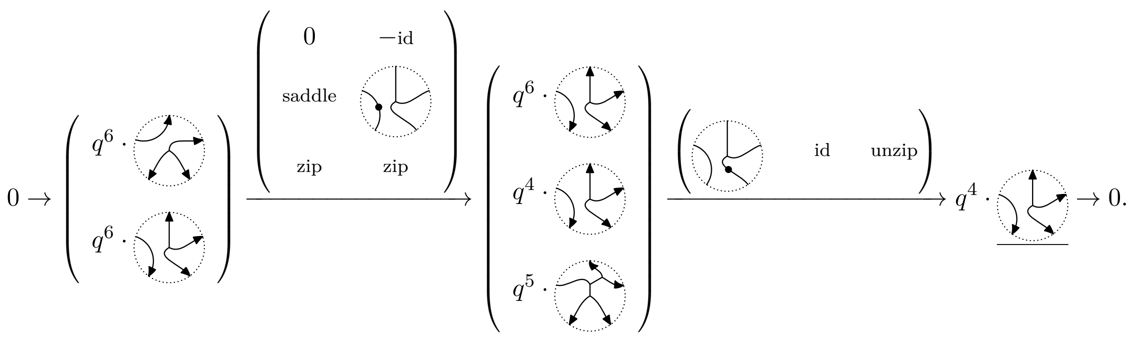
Now, the cochain complex of the diagram on the right-hand side of 1.2 is



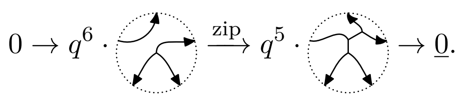
And on the left-hand side:



This is isomorphic to the cochain complex



And, using Gaußian elimination twice, one finds this cochain complex to be in turn homotopy equivalent to



□

This situation is reminiscent of the Kauffman-polynomial, which is invariant under R2 and R3, but suffers a shift under R1. To obtain an invariant, there are two remedies: either consider framed links, or compensate the deviation by introducing a shift which depends on the writhe, which leads to the Jones polynomial. In our case, where we want to extend $\mathfrak{sl}_3$-homology to an invariant of knotted webs, the first way is not possible, because no framing or thickening of the web stops the move W2 from happening. On the other hand, $\mathfrak{sl}_3$-homology does already depend on the writhe, so the second way seems barred as well. It is, however, possible, to define *weightings* of webs which lead to the notion of *weighted writhe*. This will allow a correction term similar to the one in the definition of the Jones polynomial, and hence we can define an $\mathfrak{sl}_3$-homology invariant of *knotted weighted webs*.

Definition 1.9. *A* weighting of an abstract web W *is a function* $\{$*edges of* $W\} \to \{0, 1, 2\}$, *such that any two intersecting edges have a different weight.*[*] *A crossing is called* equiponderate *if its two strands have the same weight, and* antiponderate *otherwise. The* weighted writhe w_o *of the diagram* D *of a knotted weighted web is defined as*

$$w_o(D) = \#\{\textit{positive antiponderate crossings}\} - \#\{\textit{negative antiponderate crossings}\}.$$

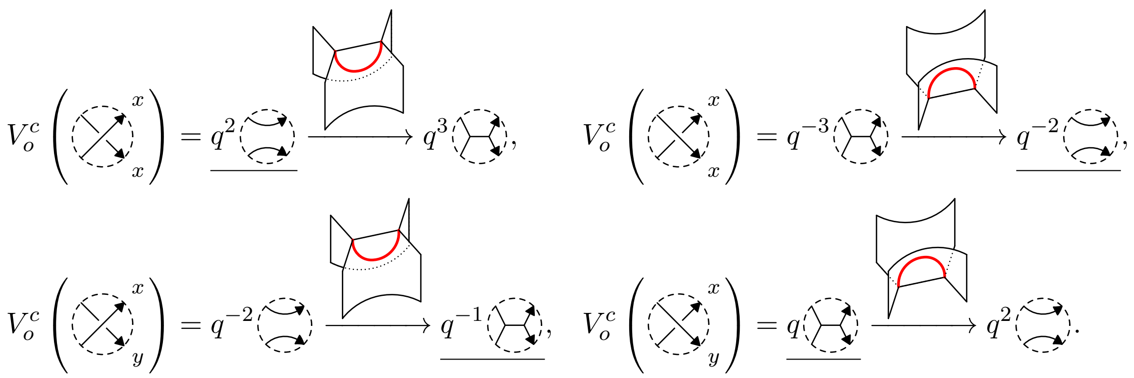
Remark 1.10. Note that permuting the weights, i.e. composing the weighting of a web with one of the six self-bijections of $\{0, 1, 2\}$, does not change the weighted writhe; consequently, the following invariants of knotted weighted webs only depend on the weighting up to permutation.

Remark 1.11. We chose the term “weighting” instead of “colouring” to emphasise the difference between our weightings, which are a priori just combinatorial data, and the colourings used e.g. in [RT90] or [MOY98], which are representations of algebras.

Definition 1.12. *Let* D *be the diagram of a knotted weighted web. Let the* weighted $\mathfrak{sl}_3$-homology $[\![\cdot]\!]_o$ *of* D *be defined as*

$$[\![D]\!]_o = (t^{-1}q^4)^{-w_o(D)} \cdot [\![D]\!].$$

Equivalently, one may define $[\![\cdot]\!]_o$ by the following skein relations, where $x, y \in \{0, 1, 2\}$ and $x \neq y$:



Theorem 1.13. *The weighted* $\mathfrak{sl}_3$*-homology is an invariant of knotted weighted webs. In particular, it is an invariant of knotted (unweighted) theta-graphs. It agrees with* $\mathfrak{sl}_3$*-link homology, if one weighs all components of a link with one fixed weight.*

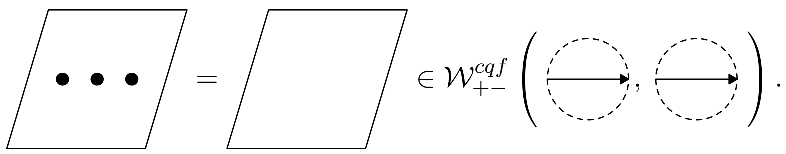
Proof. The Reidemeister moves do not alter the weighted writhe, whereas the W1- and W2-move change it by 1 and 2, respectively. Now the theorem follows quickly from proposition 1.6 and proposition 1.8. Theta webs have, up to permutation, only one weighting, so remark 1.10 implies that their $\mathfrak{sl}_3$-homology does not depend on it. ☐

[*] Such a weighting of a trivalent graph is also called *Tait colouring*.

2 Extending filtered $\mathfrak{sl}_3$-homology

The aim of this section is to construct a filtered $\mathfrak{sl}_3$-theory using foams, not matrix factorisations. This is a special case of the general construction of Mackaay and Vaz [MV07], who also proved in [MV08a] that the resulting homology theory is equivalent to the one constructed by Gornik [Gor04].

Let R be a commutative unital ring. Let $\mathcal{W}^{cqf}$ be the planar algebra of $\mathbb{Z}[q^{\pm 1}]$-linear combinations of webs, and matrices of R-linear combinations of foams between them; where the free abelian group of morphisms from $q^\alpha \cdot W$ to $q^\beta \cdot W'$ is generated by those foams $f : W \to W'$ that satisfy $\deg f - \alpha + \beta \in 6\mathbb{N}$. In this planar algebra as well, demanding that the three pairs of morphisms (C^c), (D^c), (S^c) are mutually inverse (see section II.4) generates an equivalence relation; but we change the relation (T^c) to



(T^{cf})

Quotienting yields $\mathcal{W}^{cqrf}$, and from there, $\mathcal{W}^{cqrtf}$ may be constructed as its filtered counterpart. So to every weighted web tangle we associate a cochain complex of sums of webs with shifted q-degree and matrices of R-linear combinations of foams. In the case of a knotted weighted web the webs are closed. Using the relations (C^c), (D^c) and (S^c), one can find an isomorphic cochain complex in which no closed web but the empty one occurs. Evaluating foams, one arrives at a cochain complex in the category of filtered vector spaces. Its homology is defined to be the *filtered $\mathfrak{sl}_3$-homology of knotted webs.* This is well-defined because the tensor product of cochain complexes commutes with homotopy equivalences.

Remark 2.1. In the graded case, working over a field, repeated application of Gaußian elimination (lemma III.1.2) is sufficient to obtain a cochain complex who is homotopy equivalent to the original ones and all of whose differentials are trivial. In the filtered case, the situation is more complicated, because there are contractible cochain complexes of the form

$$0 \to U \xrightarrow{f} V \to 0,$$

where f is *not* an isomorphism. For example, take $\dim U = \dim V = 1$, and $F_0 U = U, F_1 U = 0, F_1 V = V, F_2 V = 0$, and f an isomorphism of (unfiltered) vector spaces. Then the said cochain complex is contractible, but f is not an isomorphism, since the only filtered map from V to U is the trivial one. In the language of homological algebra, the category of filtered space is pre-abelian, i.e. is an additive category that has all kernels and cokernels, but is *not* abelian: this would mean that additionally, the coimage and image of every map are canonically isomorphic. But the coimage of f is U, and the image is V.

Remark 2.2. The relation (T^{cf}) is induced by the Frobenius algebra $R[X]/(p(X))$, where $p(X) = X^3 - 1$. Taking $R = \mathbb{C}$, all polynomials p with three distinct roots yield isomorphic homology theories [MV07]. However, one may take a different field over which $p(X)$ decomposes as product of linear factors as R, e.g. $p(X) = X^3 - X, R = \mathbb{F}_3$ or $p(X) = X^3 - 1, R = \mathbb{F}_4$. It is an open question whether this results in an isomorphic homology theory as well.

Theorem 2.3. *Let W be a knotted weighted web. There is a spectral sequence from $[\![W]\!]$ to $[\![W]\!]_f$. The degree of the k-th differential is tq^{-6k}, and the higher pages of the sequence are knotted weighted web invariants of W.*

Proof. This is just the spectral sequence associated to the filtered $\mathfrak{sl}_3$-complex. For the invariance of the higher pages, notice that the prove of proposition 1.8 does not involve the equation (T^c). The same is true for the analogous proves of invariance under the Reidemeister moves. Therefore, these proves are also true in the filtered setting, which implies that the filtered cochain complexes of any two diagrams of the same knotted weighted web are homotopy equivalent. By lemma B.5, the invariance of the higher pages of the spectral sequence follows. □

Let us from now on choose as R a field $\Bbbk$ over which the polynomial $X^3 - 1$ decomposes into linear factors, i.e. there is a third root ξ of unity, and $X^3 - 1 = (X-1)(X-\xi)(X-\xi^2)$. Forgetting the filtration, the algebra $\Bbbk[X]/(X^3-1)$ is simply isomorphic to $\Bbbk \oplus \Bbbk \oplus \Bbbk$. In other words, there is a base of three mutually orthogonal idempotents, which are

$$\begin{aligned} \mathbf{a} &= \frac{X^3-1}{3(X-1)} = \frac{1+X+X^2}{3}, \\ \mathbf{b} &= \frac{\xi^2(X^3-1)}{3(X-\xi^2)} = \frac{1+\xi X+\xi^2 X^2}{3}, \quad \text{and} \\ \mathbf{c} &= \frac{\xi(X^3-1)}{3(X-\xi)} = \frac{1+\xi^2 X+\xi X^2}{3}. \end{aligned}$$

There are homogeneous bases of the Hom-sets $\mathcal{W}^{cqr}$ consisting of dotted foams. In what follows, we will instead consider different bases made of **abc**-*weighted foams.* While the basis vectors are inhomogeneous and thus offer no understanding of the filtration, the filtered $\mathfrak{sl}_3$-homology behaves nicely with respect to it. This is an adaption of Lee's $(\mathbf{a},\mathbf{b})$-basis used for filtered Khovanov homology. Mackaay et al. [MPT12] mention this base, but do not discuss it in detail. Let us first generalise our notation:

Definition 2.4. *Let a* polynomially weighted *foam be a foam f with arbitrary polynomials in $\Bbbk[X]$ on its facets, instead of dots. The polynomial X^i is understood to be the same as i dots. Furthermore, polynomially weighted foams shall be distributive, i.e. fix a facet of f and let f_p be f with weight p on that facet, then $f_{p+\lambda q} = f_p + \lambda \cdot f_q$. This way, every polynomially weighted foam is just a linear combination of dotted foams.*

Definition 2.5. *An* **abc**-weighted foam *is a polynomially weighted foam that employs only* **a**, **b** *and* **c** *as weights. It is called* properly **abc**-weighted *if the weights of any two intersecting facets are different.*

Theorem 2.6. *Let W and W' be two webs with boundary ε. Let O be the set of* **abc**-*weightings of W and W' which agree on the boundary. Fixing an arbitrary bijection between $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ and $\{0,1,2\}$, a properly* **abc**-*weighted foam induces weightings on its boundary webs. Then for any $o \in O$, there is an* **abc**-*weighted foam f_o which induces o, and $\{f_o\}_{o\in O}$ is a base of the Hom-space $\mathcal{W}_\varepsilon^{cqrf}(W,W')$.*

Proposition 2.7. *An* **abc**-*weighted closed foam f evaluates to 0 if and only if the weighting is not proper.*

Proof. One easily checks the **abc**-weighted surgery formula and evaluation of **abc**-weighted spheres depicted in fig. 2.1. A theta foam evaluates to zero if two facets bear the same weight; otherwise it gives $\pm\frac{1}{9}(\xi-\xi^2)$, the sign depending on the order of the weights.

As in the case of dotted foams, any **abc**-weighted foam can now be evaluated by the following procedure: at each singular circle c, perform a surgery along three non-singular

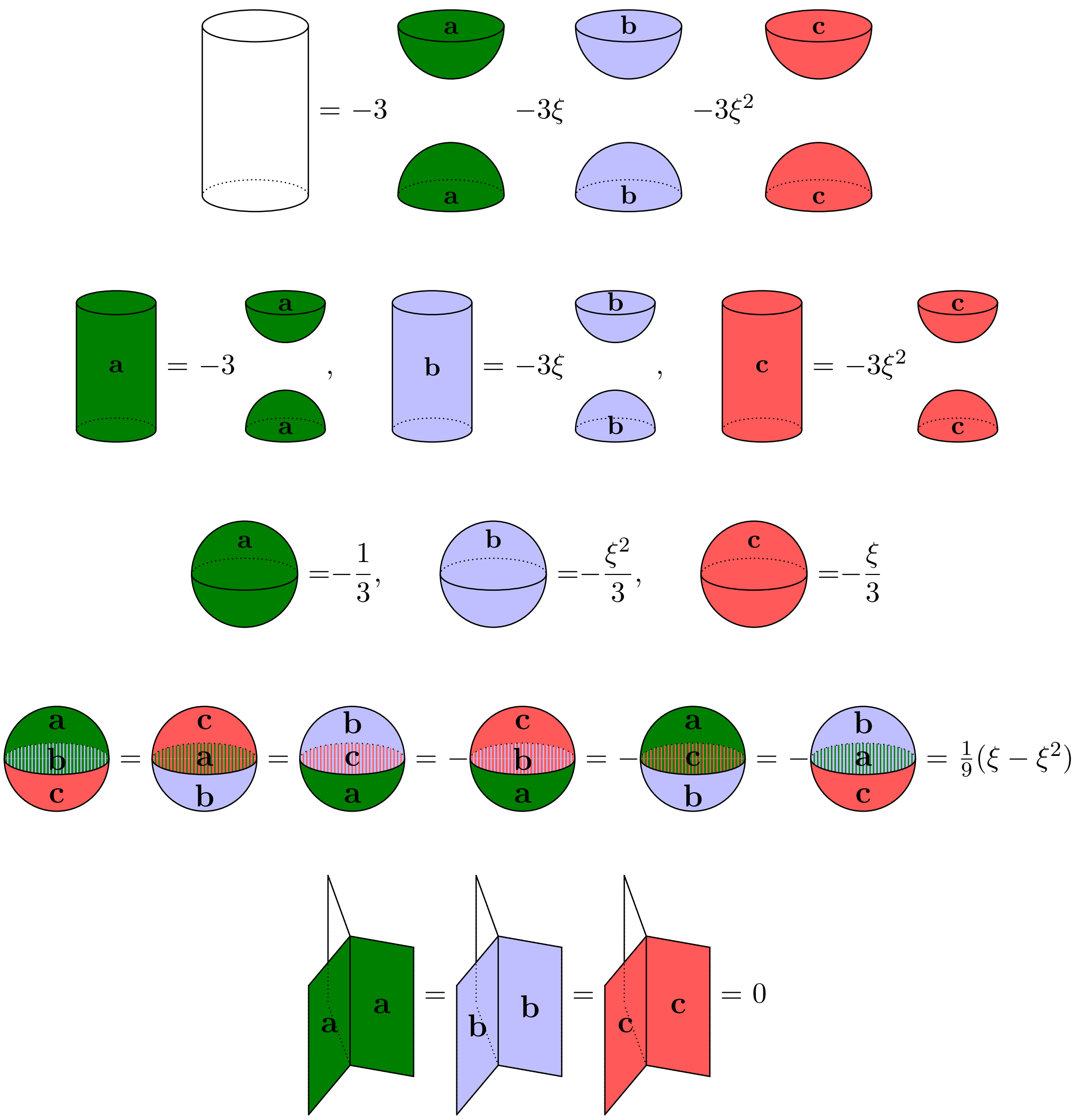


Figure 2.1: **abc**-weighted surgery formulae and evaluation of **abc**-weighted spheres and theta foams.

circles parallel to c in the facets containing c. Then, use surgery to cut off handles until all facets are discs; after this mass-surgery one is left with a sum of foams whose connected components are theta-foams and spheres.

Notice that the outcome of this surgery procedure applied to f is the scalar multiple of a single **abc**-weighted foam f' – not a linear combination of them, as one might expect a priori. Furthermore, during this surgery procedure, there are no changes of weight in the neighbourhood of a singular circle. So if f is not properly **abc**-weighted, then f' has a theta-foam as a connected component that does not have three different weights; thus f evaluates to 0. If, on the other hand, f is properly **abc**-weighted, then f' has only properly **abc**-weighted spheres and theta-foams as connected components, and thus evaluates to a non-zero scalar. □

Furthermore, it is even possible to give an explicit formula for the evaluation of a properly **abc**-weighted foam f, in contrast to the case of dotted foams.

Lemma 2.8. *Let χ_y be the Euler characteristic of the part of the foam weighted by y, let $\theta_\pm$ be the number of singular circles around which the weights are ordered so that the theta foam evaluates to $\pm\frac{1}{9}(\xi - \xi^2)$. Then the evaluation of f only depends on $\chi_a, \chi_b, \chi_c, \theta_+$ and θ_-, namely f evaluates to*

$$3^{-(\theta+\chi_a+\chi_b+\chi_c)/2} \cdot (-1)^{(\chi-3\theta_+-\theta_-)/2} \cdot \xi^{2\chi_c+\chi_b-3\theta} \cdot (\chi - \chi^2)^\theta.$$

Proof. It suffices to verify this formula for a single sphere or theta foam, check its invariance under surgery, and its multiplicativity under the disjoint union of foams. □

Lemma 2.9. *Let W be a web and f and g* **abc***-weighted foams from $\varnothing$ to W. Denote by f^* and g^* the mirror images, i.e. foams from W to $\varnothing$. Then $(f^* \circ f)\cdot(g^* \circ g) = (f^* \circ g)\cdot(g^* \circ f)$.*

Proof. This follows from lemma 2.8. □

Proof of theorem 2.6. The spaces associated to two different weightings have trivial intersection, which proves that $\mathcal{W}^{cqrf}_\varepsilon(W, W')$ decomposes as a direct sum over O. Lemma 2.9 shows that each summand has dimension at most 1. Let us show non-triviality of the summands. In the case $W' = \varnothing$, one may explicitly construct a properly **abc**-weighted foam bounding W' which induces the weighting of W by successive circle-, digon- and square-removal. This is sufficient, since there is an isomorphism $\mathcal{W}^{cf}_\varepsilon(W, W') \to \mathcal{W}^{cf}_\varepsilon(\varnothing, W \otimes W')$, where $W \otimes W'$ is obtained by gluing W and W' along their boundary, reversing the orientation of W. □

3 The s-invariant of knotted weighted webs

Proposition 3.1. *Let $D \subset \mathbb{R}^2$ be a diagram of a knotted web W, and O the set of weightings of W. There is a basis $\{\mathfrak{s}_o\}_{o\in O}$ of the $\mathfrak{sl}_3$-homology $[\![W]\!]$ of W, such that the homological degree of $\mathfrak{s}_o$ is $-w_o(D)$.*

Proof. Let $X \subset D$ be the set of vertices and double-points. An *arc* is the closure in D of a component of $D \setminus X$. Denote the set of arcs by A. An *arc-weighting* of D is a function $\varphi : A \to \{0, 1, 2\}$. It is called *proper* if any two arcs intersecting at a vertex are weighted differently, and in the neighbourhood of each crossing, the arc-weighting resembles one of the local pictures in the top row of fig. 3.1. A weighting of W induces an arc-weighting.

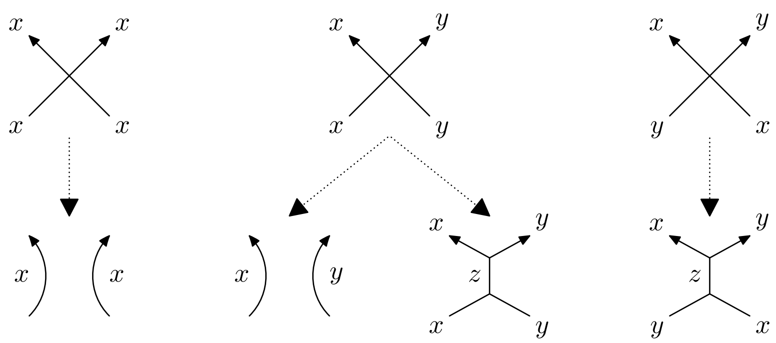


Figure 3.1: Arc-weightings ($x, y \in \{0, 1, 2\}, x \neq y$).

In this way, one obtains all the arc-weightings that look at each crossing like the first or third column of fig. 3.1. Let us associate a subcomplex $[\![D]\!]_\varphi$ of $[\![D]\!]$ to each proper arc-weighting: recall that according to theorem 2.6, the space of foams from $\varnothing$ to a fixed web has a base indexed by colourings of that web. Let $[\![D]\!]_\varphi$ consist of all closed webs which at all crossings look like the web in the bottom row of fig. 3.1 corresponding to φ (or like one of the two webs, in the case of the middle column). Ignoring the filtration, $[\![D]\!]$ decomposes, not just as a vector space, but as a cochain complex, as

$$[\![D]\!] = \bigoplus_{\text{arc-weightings } \varphi} [\![D]\!]_\varphi.$$

Next, the subcomplex associated to a proper arc-weighting is one-dimensional, since at every crossing, only one resolution is compatible with the arc-weighting. All other subcomplexes, on the other hand, are acyclic: fix any crossing where the arc-weighting looks like the web in the middle of the top-row of fig. 3.1. At this crossing, both resolutions are compatible with the arc-weighting, and the two subcomplex of $[\![D]\!]_\varphi$ corresponding to the two resolutions are isomorphic via the differential. This concludes the proof. □

Definition 3.2. *Let W be a knotted weighted web with weighting o. Let s be the q-degree of $[\mathfrak{s}_o]$ in the filtered $\mathfrak{sl}_3$-homology of W. Then the normalised s_3-invariant of W is defined as $-1 - s/2$.*

As an example, consider the following analogue of lemma I.1.10 of chapter I:

Proposition 3.3. *Let D be the diagram of a knotted weighted web W. Suppose the equiponderate crossings of D are negative, and the antiponderate crossings are positive. Let k be half the degree of the Kuperberg bracket of the Seifert resolution of D. Then*

$$s_3(W) = 1 - k + c_+ + 2c_-.$$

Proof. The generator $\mathfrak{s}_0$ is supported in the left-most chain group, and so its quantum degree in homology agrees with its quantum agree in the chain complex. This can be directly calculated to agree with the given formula. □

4 A lower bound to the slice degree of knotted weighted webs

Knotted weighted webs are a generalisation of knots. In this section, we define the objects which take the place of cobordisms.

Definition 4.1. *A* 4-foam *is the intersection of $\mathbb{R}^3 \times [0,1]$ with the image of an embedding of a prefoam f into $\mathbb{R}^4$.*

To understand 4-foams, we visualise them as movies. For the following proposition, see also [Car12].

Proposition 4.2. *Let f be a 4-foam. For $t \in [0,1]$, let $f_t = f \cap \mathbb{R}^3 \times \{t\}$, and D_t be the projection of f_t on $\mathbb{R}^2$. There is a boundary-fixing isotopy, that can be chosen arbitrarily small, which brings f into generic position; that is to say, there is a finite set $S = \{s_1, \ldots s_n\} \subset (0,1)$ of singular values with $\forall i \in \{1, \ldots n-1\} : s_i < s_{i+1}$ such that:*

- ***(i)*** *f_t is a knotted web with diagram D_t if and only if $t \in [0,1] \setminus S$.*
- ***(ii)*** *If t and t' lie in the same connected component of $I \setminus S$, then f induces a isotopy between D_t and $D_{t'}$.*

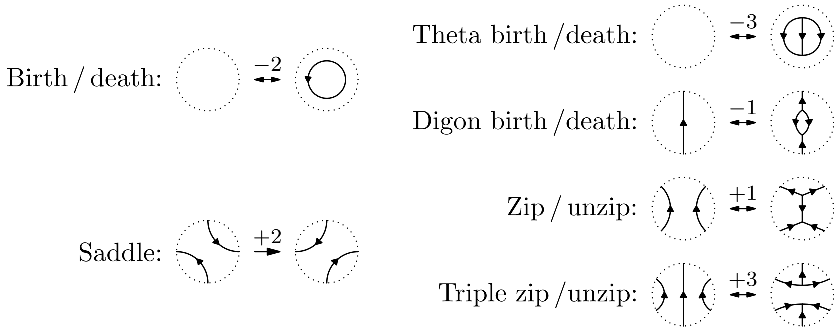


Figure 4.1: The eleven basic foam moves, with indication of their respective degree.

Figure 4.2: Morse moves on the boundary of a surface.

(iii) *For all $i \in \{1, \dots n-1\}$, let us fix some $t_i \in (s_i, s_{i+1})$. Then the discrepancy between D_{t_i} and $D_{t_{i+1}}$ is either a web move (see proposition 1.6) or one of the basic foam moves (see fig. 4.1).*

Proof. There is an arbitrarily small boundary-fixing isotopy of the embedding of f that makes the projection on t a Morse function. A Morse function on a foam is a function that is a regular Morse function on all closed facets. Let S be the set of its singular values. The singular values on facets are responsible for birth, death, and saddles. The singular values on the singular arcs are responsible for the other eight moves: the singular arc may have a minimum or a maximum, and of the three facets attached to it there may be zero, one, two, or three that have a minimum. □

The sequence $D_{t_1}, \dots D_{t_{n-1}}$ of web diagrams is called a *movie* of f. Clearly, a 4-foam f is determined by its movie.

Remark 4.3. In the second column of figure 4.1, one may select one move from left to right, and one move from right to left (e.g. zip and unzip). Then there is an isotopy, albeit not an arbitrarily small one, such that, the only basic foam moves starring in the movie of f are birth, death, the saddle and the two selected moves. This can be proven by deforming the foam in small neighbourhoods of the minima and maxima of its singular circles.

Remark 4.4. However, not every sequence of diagrams linked by web moves and basic foam moves is indeed a movie. For example, while the mock-movies of figure 4.3 do represent geometrical objects, these are not prefoams: because if they were, they would contain self-intersecting facets, which contradicts the fact that by the definition of a prefoam, facets embed into it, and a 4-foam is in turn an embedding of a prefoam. In other words, the composition of prefoams or 4-foams is not well-defined. This shortcoming will be corrected by weights (see proposition 4.6).

Definition 4.5. *A* weighting *of a 4-foam f is a function $\{\text{facets of } f\} \to \{0, 1, 2\}$. The weighting is called proper if intersecting facets carry a different weight. A proper weighting of a 4-foam induces a weighting on its boundary web.*

Proposition 4.6. *Let $D_0, \dots D_n$ be a sequence of diagrams of weighted knotted webs, each element of the sequence linked to its predecessor by a web move or a basic foam move. Then*

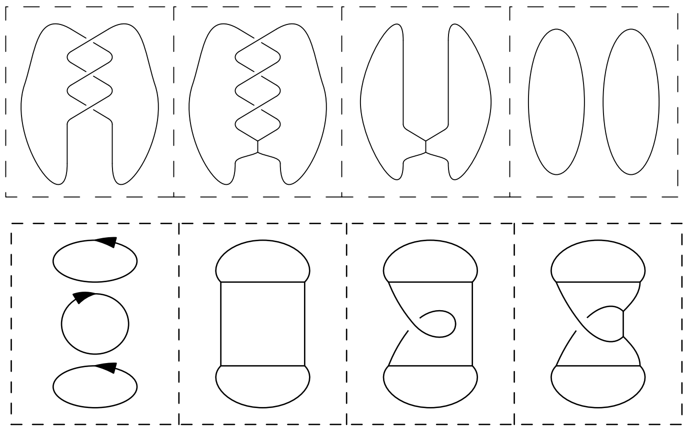


Figure 4.3: "Ceci n'est pas un film": these are not movies because the represented objects are not foams.

$(D_0, \dots D_n)$ *is indeed the movie of some properly weighted 4-foam, that induces the original weighting on* D_0.

Proof. Let us first prove the special case that D_0 and D_n are empty diagrams. For $i \in \{0, 1, 2\}$, and $j \in \{0, \dots n\}$, let D_j^i be the union of those edges of the diagram D_j which have weight i. Since such edges may not intersect, D_j^i is a 1-manifold. Now consider the sequence $D_0^i, \dots D_n^i$ for a fixed i. The only visible moves are Reidemeister moves, birth, death and saddle moves, and the two moves depicted in fig. 4.2. Thus, this is a movie of a surface embedded into $\mathbb{R}^3 \times [0, 1]$, without the restriction that the boundary may only be send to $\mathbb{R}^3 \times \{0, 1\}$. For each $i \in \{0, 1, 2\}$, let Σ^i be this surface. Then the geometrical object described by the movie $D_0, \dots D_n$ can be obtained by gluing together the three surfaces $\Sigma^0, \Sigma^1, \Sigma^2$, and hence it is a prefoam embedded into $\mathbb{R}^4$.

In the general case, one can add a prequel from the empty diagram and a sequel to it to the movie; therefore, the movie represents the intersection of a prefoam embedded into $\mathbb{R}^4$ with $\mathbb{R}^3 \times [0, 1]$, which is by definition a 4-foam. □

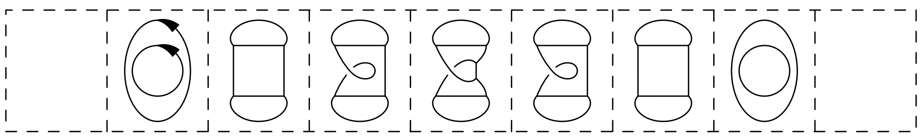


Figure 4.4: The movie of a closed 4-foam without a proper weighting (some obvious frames were skipped).

Remark 4.7. Not every closed 4-foam has a proper weighting, as the movie depicted in figure 4.4 demonstrates.

Definition 4.8. *Let a dotted prefoam* f *be called* admissible *if*

(i) *any facet of* f *that is a disc without dots intersects the boundary, and*

(ii) f *has no closed components.*

Proposition 4.9. *Every knotted weighted web* W *bounds an admissible properly weighted 4-foam* f.

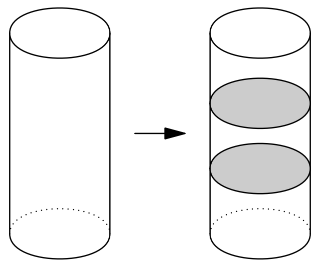


Figure 4.5: Sewing in two discs.

Proof. By the previous proposition, it is sufficient to produce a movie from W to the empty web. Take an arbitrary diagram of W. Firstly, on each antiponderate crossing, use a singular saddle move followed by a W1-move to; and on each equiponderate crossing, a saddle move followed by a R1-move. In this way, one arrives at a planar diagram. Successively removing circles, digons and squares leads to $\varnothing$. Clearly, this foam has no closed components. Putting a dot on each facet that is a disc not intersecting the boundary, one obtains an admissible foam. □

Proposition 4.10. *The degree of admissible dotted properly weighted 4-foams f bounding a fixed knotted weighted web W has a lower bound.*

Proof. Admissibility ensures that each facet has non-negative Euler characteristic. □

The two preceding propositions allow us to define the slice degree of W, generalising the slice genus of knots.

Definition 4.11. *Let W be a knotted weighted web. Let the* slice degree $\chi_4(W)$ *be defined as the minimal degree of an admissible dotted 4-foam f whose boundary is W, and that admits a proper weighting which induces the original weighting on W.*

Remark 4.12. The admissibility conditions are necessary. For assume there is a properly weighted 4-foam f with boundary W. Without (i), fix a part of f that looks like a cylinder. Then, one may sew in a pair of discs (see figure 4.5); that changes the degree by -4. Without (ii), just take the disjoint union of f with an arbitrary number of spheres; each sphere contributes -4 to the degree.

Proposition 4.13. *The slice degree satisfies*

$$\chi_4(W) \equiv \frac{\#\textit{vertices of } w}{2} \pmod 2.$$

Proof. It suffices to verify this for the basic foam moves: each of the eight basic foam moves that changes the number of vertices, changes it by 2 and has odd degree; and each of the other three has even degree. □

Let us now state the main theorem.

Theorem 4.14. *Let W be a knotted weighted web. Then*

$$\chi_4(W) \geq 2s_3(W) - 2.$$

Remark 4.15. This theorem is due to the *functoriality* of $\mathfrak{sl}_3$-homology under 4-foams. However, we do not need or prove full functoriality: this would mean that isotopic 4-foams, who may have different movies, still induce a homotopic map of cochain complexes.

Definition 4.16. *Let W and W' be knotted weighted webs with weightings o and o', respectively. Let the 4-foam $f : W \to W'$ be in general position. Then let $[\![f]\!]$ be the map from $[\![W]\!] \to [\![W']\!]$ defined as follows: If f is a web move, let $[\![f]\!]$ be the associated maps of complexes (see the proof of theorem 2.3). If f is one of the basic foam moves, $[\![f]\!]$ is defined by applying that basic foam move to all webs in the cochain complex $[\![W]\!]$. Any other f can be written as composition of web moves and basic foam moves (see proposition 4.2); so let $[\![f]\!]$ be the composition of the maps associated to those web moves and basic foam moves.*

Proposition 4.17. *The map $[\![f]\!]$ is a filtered map of cochain complexes of degree* $\deg f$*.*

Proof. It is sufficient to prove this for the maps associated to web moves and basic foam moves. In the first case, the maps are known to be quasi-isomorphisms (see the proof of theorem 2.3). In the second case, the filteredness follows immediately from the definition. Commutativity with the differentials is implied by the fact that the basic foam move happens at a horizontal distance from the differentials, and so it does not matter which is applied first. □

Remark 4.18. Of course, one may conjecture that two different movies of the same foam induce the same map; in other words, functoriality of $\mathfrak{sl}_3$-homology of knotted weighted webs with respect to weighted 4-foams. However, this is not necessary for the proof of the main theorem.

Proposition 4.19. *Let W be a knotted weighted web. Let f be a 4-foam with $\partial f = W$. Assume f admits a unique proper weighting that agrees with the weighting o of W. Then the map of the previous proposition sends $\mathfrak{s}_o$ to a non-zero scalar.*

Proof. The generator $\mathfrak{s}_o$ is represented by a single properly **abc**-weighted foam. Composing with f yields a closed foam, which may be written as linear combination with non-zero coefficients of **abc**-weighted foams. Dots do not change the **abc**-weightings, since $X \cdot \mathbf{a} = \mathbf{a}, X \cdot \mathbf{b} = \xi\mathbf{a}$ and $X \cdot \mathbf{c} = \xi^2\mathbf{c}$. However, since f admits only one proper weighting that agrees with the weighting of W, only one of the foams in that linear combination is a properly weighted **abc**-weighted foam. Now the statement follows because a *single* properly **abc**-weighted foam is not zero (proposition 2.7). □

The condition of admissibility is not restrictive enough to proceed directly, since there are admissible 4-foams f who do not satisfy the hypotheses of proposition 4.19. Example! However, for geometric reasons we can further restrict us to admissible irreducible foams, which will be introduced in what follows.

Proposition 4.20. *Let f be an admissible 4-foam. Let o be a (not necessarily proper) weighting of f. Suppose that any three facets intersecting in a singular edge have weights whose sum is divisible by 3, and that any facet touching the boundary has weight 0.*

Then there is an admissible 4-foam g with $\partial g = \partial f$ and $\deg g \leq \deg f$*. Such a 4-foam is called a* reduction of f along o*.*

Proof. Let us remove from f all facets with weight 2, and switch the orientation of all facets with weight 1. Let us analyse what happens around singular circles. They either

- vanish – if $o(f_1) = o(f_2) = o(f_3) = 2$,

- switch their orientation – if $o(f_1) = o(f_2) = o(f_3) = 1$,
- remain untouched – if $o(f_1) = o(f_2) = o(f_3) = 1$,
- or can be smoothed out – if $\{o(f_1), o(f_2), o(f_3)\} = \{0, 1, 2\}$.

So, after smoothing out, we obtain another 4-foam g.

Notice that facets touching the boundary are not modified, since their weight is 0. Therefore, $\partial f = \partial g$.

The foam f being admissible, every facet which does not touch the boundary has zero or negative Euler characteristic. So the removal of facets does not decrease the Euler characteristic, and thus $\deg g \leq \deg f$. □

Let us call an admissible 4-foam *irreducible* if the only weighting satisfying the hypothesis of the proposition is constantly 0.

Lemma 4.21. *If two weightings of an irreducible 4-foam f induce the same weighting on the boundary, then they are identical.*

Proof. Let o and o' be two such weightings. Then we may reduce along $o - o'$, and $o - o'$ sends every exterior facet to 0. Because f is irreducible, $o - o'$ is therefore constantly zero, and thus $o = o'$. □

We are now ready to prove theorem 4.14. Let an admissible foam f be given. Let f' be an irreducible reduction of f. Then f' induces a map of degree $\deg f'$ which sends $\mathfrak{s}_o$ to a non-zero scalar. Since $\deg f' \leq \deg f$, this implies the statement. □

Weighted Links are a special case of knotted weighted webs. Consider links all of whose component have the same weight, such as knots; they have the same $\mathfrak{sl}_3$–homology and s_3–invariant as links as they do as knotted weighted webs. Is it possible that their slice degree as a knotted weighted web is lower than their slice degree as links (properly normalised)? As what follows shows, this is not the case.

Lemma 4.22. *If f is a weighted 4-foam such that ∂f is a collection of circles that are all equally weighted, then there is a 4-foam f' that is in fact a surface weighted with only one weight, and* $\deg f' \leq \deg f$.

Proof. If all facets of f have the same weight, then f is a surface. Otherwise, f is reducible using lemma 4.21, because exchanging the two weights which do not appear on facets intersecting the boundary produces two different weightings of f which agree on the boundary. So let f' be an irreducible reduction of f. □

Corollary 4.23. *Let L be a link, and W this link seen as knotted weighted web, all components with equal weight. Let $\chi_4(L)$ be the maximal Euler characteristic of a smooth surface in D^4 whose boundary lies in S^3 and equals L, and which has no closed components. Then*

$$-2\chi_4(L) = \chi_4(W).$$

In particular, if L is a knot, then

$$4g_4(L) = \chi_4(W) + 2.$$

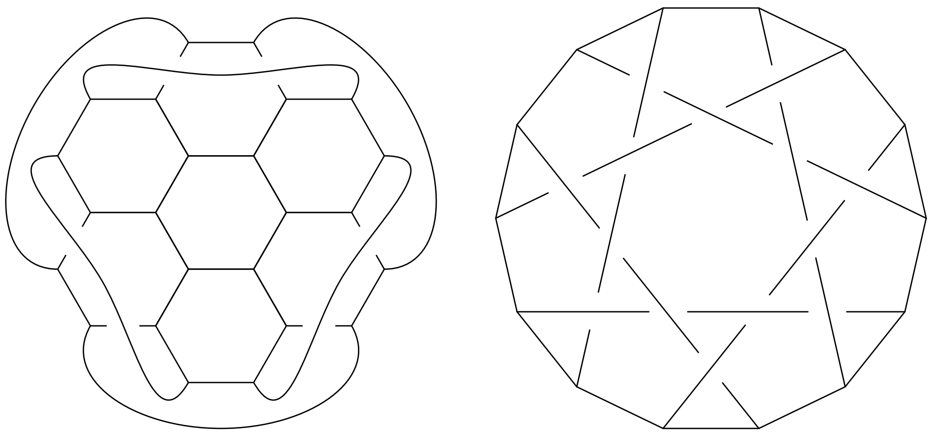


Figure 5.1: Two examples of non-planar knotted webs that contain neither digon nor square. On the right, the Heawood graph.

5 Examples and further properties of knotted weighted webs

Proposition 5.1. *Every abstract web has a weighting.*

Proof. Let W be an abstract web. Since the connected components of W can be weighted independently, assume without loss of generality that W is connected. If W is a circle or a theta, the statement is evident. If W contains a digon, let W' be the web obtained from W by removing this digon. Then any weighting of W' gives rise to a weighting of W. So, without loss of generality, we can assume that W is a connected web that is not a circle, nor a theta, and that contains no digon. Thus W contains no multiple edges; it is a bipartite simple graph. By Kőnig's edge colouring theorem [GY06] the minimum number of weights needed to weigh the edges of such a graph equals the maximum degree of a vertex, which is three. □

The following notion of canonical weighting shows how weightings of a connected web are, in a way, the natural generalisation of the orientation of a connected 1-manifold (i.e. a circle).

Definition 5.2. *Let W be a closed web. Suppose every connected component of $\mathbb{R}^2 \setminus W$ is labelled 0, 1, or 2, such that the unbounded face is labelled 0 and such that the label ℓ on the left of an edge equals 1 plus the label r on the right mod 3. Put on each edge the weight $-\ell - r$ mod 3. Such a weighting is called* canonical.

Proposition 5.3. *Every closed web W has a unique canonical weighting. Furthermore, let W' be an abstract planar connected web that is not a circle. Associating to an embedding $W' \to \mathbb{R}^2$ the ensuing canonical weighting gives a bijection between weightings of W' and up-to-isotopy plane embeddings of W'.*

Proof. Let W be a closed web. Since the label of one connected component of $\mathbb{R}^2 \setminus W$ determines the labels of all adjacent connected components (in the sense of the previous definition), the uniqueness of the canonical weighting follows. To prove existence, one has to verify that the conditions on the labelling of connected components are not contradictory. This is implied by the fact that a closed oriented path which avoids the vertices of a web and intersects edges transversely, passes over as many edges from left to right as from right to left (mod 3).

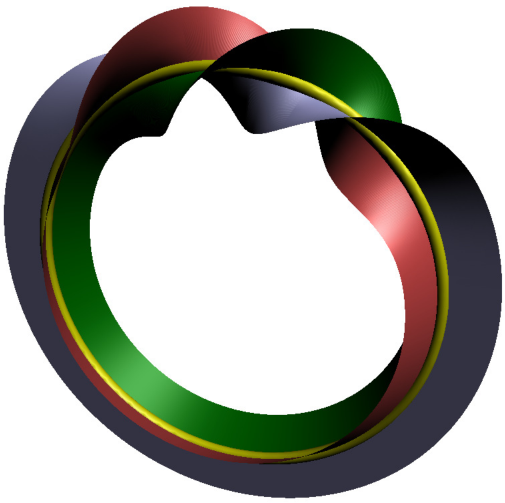


Figure 5.2: A Seifert foam of degree 0 for a weighted $(3,3)$-torus link.

To prove the one-to-one correspondence, it is now sufficient to show that there is a unique way to embed an planar abstract weighted web W such that its weighting is canonical. This follows since the order of edges around each vertex is determined. □

Remark 5.4. One may generalise the notion of Seifert surface to knotted weighted webs. However, not every knotted weighted web has a *Seifert foam*; a simple argument using the linking number shows e.g. that the Hopf links do not admit a Seifert foam, if the two components have different weights. Furthermore, there are non-trivial knotted weighted webs with Seifert foams of degree 0, cf. fig. 5.2.

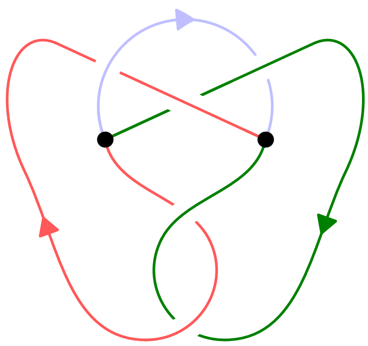


Figure 5.3: Example of a knotted weighted web.

Finally, let us calculate an example, aided by a computer: the $\mathfrak{sl}_3$-homology of the knotted weighted web W shown in fig. 5.3 has graded dimension

$$t^{-4}q^{15} + t^{-3}(q^9 + q^{11}) + t^{-2}q^5 + t^{-1}q^7 + (2q^{-1} + 3q + 2q^3).$$

The spectral sequence of theorem 2.3 can be seen to converge to $2q^{-1} + 2q + 2q^3$, and so we obtain that the s_3-invariant is $1/2$. This implies $\chi_4(W) \geq -1$, and in fact there is an admissible weighted 4-foam of degree -1 bounding W, and thus $\chi_4(W) = -1$. The 4-foam consists of a saddle move and a singular saddle, which lead to a planar web; and a digon and a theta removal, which subsequently lead to $\varnothing$.

Appendix

A Graded and filtered vector spaces

Let G an abelian group. A *G-graded k-vector space* V is a vector space with a decomposition

$$V = \bigoplus_{g \in G} V_g.$$

A *graded homomorphism* $f : V \to W$ is a homomorphism satisfying $\forall g \in G : f(V_g) \subset W_g$. If for some fixed $h \in G$, $\forall g \in G : f(V_g) \subset W_{gh}$, we say that f is *homogeneous of degree h*. Sum, subspace and tensor product of graded spaces are all defined in the natural way. If $G = \mathbb{Z}^n$, we will typically fix a variable x_i for all $i \in \{1, \ldots n\}$, and write

$$\operatorname{xdim} V = \sum_{\alpha \in \mathbb{Z}^n} \dim V_\alpha \cdot \left(\prod_{i=1}^{n} x_i^{\alpha_i} \right) \in \mathbb{N}[x_1^{\pm 1}, \ldots x_n^{\pm 1}].$$

for the *graded dimension* of V. By abuse of notation, we write the shift operator as multiplication by a polynomial, i.e.

$$\prod_{i=1}^{n} x_i^{\alpha_i} \cdot V$$

is the graded vector space V' satisfying $V'_{g+\alpha} = V_g$.

There is a partial order on $\mathbb{N}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]$ given by defining that a polynomial is less than or equal than another if each coefficient of the former is less than or equal than the corresponding coefficient of the latter. Denote this order by $\leq$ and $\geq$. An injective graded map $V \to V'$ exists if and only if $\operatorname{xdim} V \leq \operatorname{xdim} V'$. The degree of a homogeneous homomorphism is in fact a grading on the space of homomorphisms from a fixed space to another. If f is a homogeneous homomorphism of degree α, we write $\operatorname{xdeg} f$ for $\prod_{i=1}^{n} x_i^{\alpha_i}$.

A *(ascendingly) filtered vector space* W is a vector space with a collection $\{F_p W\}_{p \in \mathbb{Z}}$ of subspaces such that $\forall p : F_p W \subset F_{p+1} W$. Descending filtrations are defined equivalently by the condition $\forall p : F_p W \supset F_{p+1} W$. The shift operator may be defined as for graded spaces, and is noted in the same way, as multiplication by a polynomial.

In this text, we will only encounter finite dimensional filtered spaces, and for those, we assume that $F_p W = \{0\}$ for small enough p and $F_p W = W$ for large enough p. A homomorphism $f : W \to W'$ is called *filtered* if $\forall p \in \mathbb{Z} : f(F_p W) \subset F_p W'$, and *filtered of degree d* if $\forall p \in \mathbb{Z} : f(F_p W) \subset F_{p-d} W'$. The *associated graded* of W is the $\mathbb{Z}$-graded space

$$\bigoplus_{p \in \mathbb{Z}} \frac{F_p W}{F_{p-1} W}.$$

The *associated graded* of a filtered map $f : W \to W'$ of degree d is the homogeneous map of degree d induced by f. Sum and tensor product of filtered spaces are defined in the natural way. If $W' \subset W$, then the quotient space W/W' has a filtration defined by setting $F_p(W/W')$ to be the image of $F_p W$ under the projection $W \to W/W'$.

B Spectral sequences

Spectral sequences relate the homology of a filtered cochain complex to the homology of its associated graded cocomplex. In this appendix, we collect the definitions and theorem used throughout the thesis. We follow the standard reference [McC01], but use slightly different grading conventions. Spectral sequences may seem cumbersome for the uninitiated; we recommend the introduction [Cho06]. However, we find ourselves in the easiest case: all spaces are finite-dimensional, and we have no need to understand the differentials on higher pages.

Definition B.1. *A* spectral sequence *is a sequence of $\mathbb{Z}^2$-graded vector spaces, i.e. we have a vector space $E_k^{t,q}$ called the* k-th page of E *for any $(t,q) \in \mathbb{Z}^2$ and $k = 0, 1, \ldots$. Furthermore, for each k there is an endomorphism d_k of $E_k^{*,*}$, which has (t,q)-degree $(1,k)$ and is a differential, i.e. $d_k \circ d_k = 0$. Finally, for $k \geq 0$, the $(k+1)$-th page is the homology of the k-th page, i.e.*

$$E_{k+1}^{t,q} = \frac{\ker d_k|_{E_k^{t,q}} : E_k^{t,q} \to E_k^{t+1,q+k}}{\operatorname{im} d_k|_{E_k^{t-1,q-k}} : E_k^{t-1,q-k} \to E_k^{t,q}}.$$

Let us call all pages but E_0 the higher pages *of E. In the finite-dimensional case, a spectral sequence E is said to* collapse at the K-th page *or* converge on the K-th page *if $\forall k \geq K : d_k = 0$. In this case, we note $E_\infty := E_K$ and say E* converges to E_∞, *or E* is a spectral sequence from E_1 to E_∞, *noted $E_1 \implies E_\infty$.*

In the finite-dimensional case, every spectral sequence converges; this is obvious since the total dimension of every page is less or equal than the total dimension of its predecessor.

Proposition B.2. *[McC01, Theorem 2.6] Let C be a filtered cochain complex. Then there is a spectral sequence with the associated graded cochain complex as* 0*-page that converges to the associated graded of the homology of C.*

Definition B.3. *A* homomorphism of spectral sequences *is a collection of maps $f_k : E_k \to E'_k$ which commute with the differentials, such that f_{k+1} is the map induced by f_k.*

Proposition B.4. *Taking the spectral sequence induced by a filtered cochain complex is functorial, i.e. filtered chain maps induce maps of spectral sequences, and identity and composition law are respected.*

Lemma B.5. *Let $f : C \to C'$ be a map of filtered cochain complexes. Let $E_\bullet$ and $E'_\bullet$ be the respective spectral sequences associated to C and C', and for all $r \geq 0$, let f_r be the induced graded map from E_r to E'_r. If f_R is an isomorphism for some R, then f_r is also an isomorphism for all $\infty \geq r \geq R$.*

Proof. See [McC01, theorem 3.5]; also used by Rasmussen [Ras10, lemma 6.1]. □

Proposition B.6. *Let (C, ∂) be a filtered cochain complex, and suppose the differential of C preserves the degree modulo some $N \geq 2$; i.e. if $v \in F_pC \setminus F_{p+1}C$ and $\partial v \in F_qC \setminus F_{q+1}C$, then $q - p \equiv 0 \pmod N$. Let $(E_\bullet, d_\bullet)$ be the spectral sequence induced by C, then d_k vanishes for all k not divisible by N.*

Proof. The cochain complex C decomposes as a direct sum, and so does the induced spectral sequence. □

Proposition B.7. *Let (C, ∂) be a filtered cochain complex with an additional grading $C = \bigoplus_{i \in \mathbb{Z}} C_i$ that is respected by the differential. The filtration F_p induces a filtration on each C_i by $F_p C_i := C_i \cap F_p C$. If C is as a filtered vector space the sum of the filtered C_i, we say that the filtration is compatible with the grading. In this case, the spectral sequence induced by (C, ∂) respects the grading.*

Proof. Straight-forward. □

In general, the mere existence of a spectral sequence gives combinatorial information. The following lemma is the decategorification of proposition B.2, and contains the information of that spectral sequence one may gain without understanding its differentials.

Lemma B.8. *Let $(E_\bullet, d_\bullet)$ be a spectral sequence of $\mathbb{Z}^n$-graded finite dimensional vector spaces. Then for all $k \geq 1$ there are polynomials $f_k \in \mathbb{N}[x_1^{\pm 1}, \ldots x_n^{\pm 1}]$, such that for all $\ell \geq 1$ there is the following decomposition:*

$$\operatorname{xdim} E_1 = \operatorname{xdim} E_{\ell+1} + \sum_{k=1}^{\ell} (1 + \operatorname{xdeg} d_k) \cdot f_k.$$

In particular,

$$\operatorname{xdim} E_1 = \operatorname{xdim} E_\infty + \sum_{k=1}^{\infty} (1 + \operatorname{xdeg} d_k) \cdot f_k.$$

The spectral sequence converges on the ℓ-th page if and only if $\forall k \geq \ell : f_k = 0$.

Proof. Straight-forward. □

List of Figures

Unless otherwise stated in the figure's caption, all figures were drawn by the author using Asymptote*, MetaPost† or Inkscape‡.

*`http://asymptote.sourceforge.net`
†`http://www.tug.org/metapost.html`
‡`http://inkscape.org`

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Résumé

Cette thèse porte sur les homologies de Khovanov-Rozansky et les invariants de concordance des nœuds qui en proviennent, en prêtant une attention particulière à l'homologie $\mathfrak{sl}_3$ definie par des mousses. Le premier chapitre est consacré aux interdépendances des différentes homologies de Khovanov-Rozansky : les homologies non-réduite et réduite, graduée et filtrée, et les homologies HOMFLYPT et $\mathfrak{sl}_N$ pour différents valeurs de N. Grâce à une composition des suites spectrales connues et nouvelles, on démontre sur des exemples que les invariants de concordance $\mathfrak{sl}_N$ ne sont pas tous égaux ; ce résultat constitue une réponse à un probleme ouvert jusqu'à ici.

Le deuxième et troisième chapitres présentent une implémentation d'un algorithme qui calcule l'homologie $\mathfrak{sl}_3$. Hormis le programme de Bar-Natan, Green et Morrison, donnant l'homologie de Khovanov, il s'agit du seul programme pour calculer une des homologies de Khovanov-Rozansky d'une manière efficace. Les calculs démontrent que l'invariant de concordance $\mathfrak{sl}_3$ peut prendre des valeurs impaires.

Dans le quatrième chapitre, les homologies $\mathfrak{sl}_3$ graduées et filtrées sont étendues à une classe des graphes noués et $\mathbb{F}_3$-pondérés : les toiles nouées pondérées. Les mousses pondérables, qui jouent le rôle des cobordismes orientables pour les toiles pondérées, permettent de définir la notion de degré lisse pour des toiles nouées pondérées. Par analogie avec le travail de Rasmussen, on démontre qu'une borne inférieure au degré lisse des toiles nouées pondérées découle de l'homologie $\mathfrak{sl}_3$ filtrée.

Mots-clefs

théorie des nœuds, homologies de Khovanov-Rozansky, graphes noués, concordance des nœuds, genre lisse, toiles, mousses, suites spectrales

Abstract

This thesis focuses on the Khovanov-Rozansky homologies and the knot concordance invariants issuing from them, paying particular attention to the $\mathfrak{sl}_3$-foam homology. The first chapter treats the interrelation of different Khovanov-Rozansky homologies: unreduced and reduced, graded and filtered, and categorifying the HOMFLYPT-polynomial and the $\mathfrak{sl}_N$-polynomial for varying N. A combination of new and known spectral sequences allows to show exemplarily that the $\mathfrak{sl}_N$-knot concordance invariants may differ, which was unknown until now.

In the second and third chapter, an implementation of an algorithm computing $\mathfrak{sl}_3$-homology is presented. Aside from Bar-Natan, Green and Morrisons' programme calculating Khovanov homology, this is the only existing programme that efficiently computes any Khovanov-Rozansky homology theory. Its calculations show that the $\mathfrak{sl}_3$-knot concordance invariant may be an odd integer.

In the fourth chapter, graded and filtered $\mathfrak{sl}_3$-homology are generalised to a class of knotted $\mathbb{F}_3$-weighted graphs, called *knotted weighted webs*. Weightable foams are defined, which are to knotted weighted webs what orientable cobordisms are to knots, and the slice degree of knotted weighted webs is introduced. In analogy with Rasmussen's result, it is shown that the filtered $\mathfrak{sl}_3$-homology yields a lower bound for the slice degree of knotted weighted webs.

Keywords

knot theory, Khovanov-Rozansky homologies, knotted graphs, knot concordance, four-ball genus, webs, foams, spectral sequences