Exam

1. The intersection of the plane $x + y + 2z = 2$ with the paraboloid $z = x^2 + y^2$ is an ellipse. Find the points on this ellipse with smallest and largest distance to the origin.

2. Consider the surface S given by $z = x(1-x)y(1-y)$ with $0 \le x \le 1$ and $0 \le y \le 1$. Let $K(x, y, z) = (0, 0, x)$. Calculate the integral

$$\int_S K \cdot n \, \mathrm{d}o,$$

where n is the unit normal vector of S, pointing upwards.

3. Determine the area of the domain enclosed by the segment on the x-axis with $-\pi \le x \le 0$ and the curve

$$\sqrt{x^2 + y^2} - \arccos\left(\sqrt{1 - \frac{y^2}{x^2 + y^2}}\right) = 0, \qquad y > 0.$$

4. Using the separation ansatz $u(x, y) = X(x)Y(y)$ determine the eigenvalues λ_{kl} and the eigenfunctions u_{kl} of the eigenvalue problem

$$\begin{aligned} 2u_{xxy} + u_{yy} + \lambda u &= 0 && \text{in } G \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \{0, 2\pi\} \times [0, \pi] \\ u &= 0 && \text{on } [0, 2\pi] \times \{0, \pi\} \end{aligned}$$

on the domain $G = [0, 2\pi] \times [0, \pi]$.

5. Important: In the following multiple choice questions *exactly one* among the given answers is correct. Please mark the correct answers on the exam sheet. Wrong answers result in a deduction of points.

(a) A domain G in the plane has area $|G| = 3$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $T(x, y) = (x - 2y, 3x + 4y)$. What is the area of $T(G)$?

- ○ $|T(G)| = 0.3$.
- ○ $|T(G)| = 1/3$.
- ○ $|T(G)| = 10$.
- ○ $|T(G)| = 27$.
- ○ $|T(G)| = 30$.

(b) Suppose the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ has gradient $\nabla f(x, y, z) = \begin{pmatrix} 3x^2 \\ 4y \\ 8z \end{pmatrix}$. What can one deduce about f?

- ○ $\Delta f(x, y, z) = 6x + 12$.
- ○ $\Delta f(x, y, z) = \begin{pmatrix} 6x \\ 4 \\ 8 \end{pmatrix}$.
- ○ $f(x, y, z) = x^3 + 2y^2 + 4z^2$.
- ○ $\Delta f(x, y, z) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.
- ○ $\Delta f(x, y, z) = 3x^2 + 4y + 8z$.

(c) Let G be the annulus $G = \{(x, y) : 1 < x^2 + y^2 < 4\}$. Suppose the vector field $K(x, y)$ on G satisfies $\operatorname{rot} K = 0$.

○ There always is a function $f\colon G \to \mathbb{R}$ such that $\nabla f = K$.

○ There never is a function $f\colon G \to \mathbb{R}$ such that $\nabla f = K$.

○ There is not enough information to decide whether or not there exists a function $f\colon G \to \mathbb{R}$ such that $\nabla f = K$.

(d) The volume of the solid of revolution obtained by rotating the parabola $y = x^2 + 1$ with $-1 \le x \le 1$ around the x-axis is:

○ $\dfrac{51\pi}{16}$.

○ 3π.

○ $\dfrac{52\pi}{15}$.

○ 4π.

○ $\dfrac{56\pi}{15}$.

(e) What is the most precise description of the surface given by the parameterization

$$\Phi(x, \phi) = (4x^2, 3\sin\phi, 2\cos\phi), \quad 0 \le x \le 1,\ 0 \le \phi \le 2\pi\ ?$$

○ Right circular cylinder.

○ Genuinely oblique circular cylinder.

○ Right elliptic cylinder.

○ Parabolic ellipsoid.

○ Elliptic paraboloid.