From Actuarial to Financial Valuation Principles

Martin Schweizer
Technische Universität Berlin
Fachbereich Mathematik, MA 7–4
Straße des 17. Juni 136
D – 10623 Berlin
Germany

Phone: +49 30 314 24219 / 24602
Fax: +49 30 314 21695
e-mail: mschweiz@math.tu-berlin.de

Abstract: A valuation principle is a mapping that assigns a number (value) to a random variable (payoff). This paper constructs a transformation on valuation principles by embedding them in a financial environment. Given an a priori valuation rule u, we define the associated a posteriori valuation rule h on payoffs by an indifference argument: The u-value of optimally investing in the financial market alone should equal the u-value of first selling the payoff at its h-value and then choosing an investment strategy that is optimal inclusive of the payoff. In an L^2-framework, we explicitly construct in this way the financial transforms of the variance principle and the standard deviation principle. The resulting financial valuation rules involve the expectation under the variance-optimal martingale measure and the intrinsic financial risk of the payoff.

Key words: finance, insurance, variance principle, standard deviation principle, variance-optimal martingale measure

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0. Introduction

"Ask not only what finance can do for insurance. Ask also what insurance can do for finance." This very free adaptation of a famous dictum to the spirit of Hans Bühlmann (1987) could serve as the motto for the present paper. Its starting point is the rather banal observation that the valuation of random amounts is an important topic in both actuarial and financial mathematics and has been studied extensively in both fields. In almost any textbook, one will find a treatment under headings such as premium principles (in insurance) or derivative pricing (in finance). This paper is an attempt to bring these approaches together by embedding an actuarial valuation principle in a financial environment.

The basic idea is quite simple. We begin with an a priori valuation rule which assigns a number ("premium", "price") to any random payoff from a suitable class. Typically, this rule is given or motivated by an *actuarial premium principle*. But the payoffs we consider do not exist in a vacuum. They are surrounded by a financial environment described by the outcomes of trades available to participants in a financial market. Such trades can for instance be used to reduce the risk one has contracted by the sale or purchase of some random amount like an insurance claim or a financial obligation. To value a given payoff in this environment, we compare two alternative strategies. One of them is to ignore the payoff completely and simply trade in the financial market in an optimal way. More precisely, the first strategy tries to obtain via trading from a given initial capital a final outcome with maximal value, where the value is computed according to the given a priori rule. The second strategy starts by selling the payoff under consideration in order to increase the initial capital. Then it looks for a trade whose resulting net final outcome (trading outcome minus payoff) has maximal value. The selling price for the payoff is then defined implicitly by equating these two maximal values; it thus compensates the seller of the payoff for his risk since he becomes indifferent between optimal trading alone and the combination of selling at this price and optimal trading including the payoff. The resulting a posteriori valuation is called the *financial transform* of the a priori valuation rule.

Of course, this abstract program is too general to be useful. We therefore specialize the financial environment to a frictionless market, modelled by a linear subspace $\mathcal{G}$ of L^2 with a riskless asset B. We consider two specific examples of actuarial valuation principles and explicitly find their financial transforms. As a whole, this paper is a joint venture between finance and insurance. Finance makes explicit the transformation mechanism, and insurance provides the input on which the mechanism can operate. In particular, an actuarial justification for the choice of one particular a priori valuation could with this approach lead to a foundation for pricing in an incomplete financial market.

The paper is organized as follows. Section 1 explains the financial background $(\mathcal{G}, B)$, introduces a no-arbitrage type condition and presents some of its implications. In particular, we obtain a decomposition of any random payoff into an attainable part and a non-hedgeable part. The former is riskless, and the latter contains all the financial riskiness of the payoff. Section 2 describes a change of measure required for passing from original to discounted payoffs and defines the important concept of the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$. Section 3 gives a more formal description of the basic idea and determines the financial transform of the actuarial *variance principle*. This turns out to have a very similar structure: It is again a variance principle, but the expectation is taken under $\widetilde{P}$ and the variance component considers only the non-hedgeable part of the payoff. A similar result is obtained in the final section 4 where we find the financial transform of the actuarial *standard deviation principle*.

As it often happens, most of the ideas in this paper have appeared before in some form or other. The use of an indifference argument to implicitly determine a valuation is a standard approach from economics; it has recently been taken up again in a number of papers on option pricing under transaction costs. Pricing and hedging with a mean-variance utility has been studied by Mercurio (1996) in a special case of our framework for $\mathcal{G}$. That paper in fact provided the main single motivation for our work. Replacing the variance in the utility by the standard deviation has been suggested by Aurell/Życzkowski (1996) in the very special case where the financial environment is generated by a trinomial model for stock prices. Both these papers relied on sometimes rather heavy computations to obtain results. The main point of the present paper is that it combines all these isolated ideas into a general and unified framework; the novelty is perhaps that we explicitly point out that this induces a transformation on the set of valuation rules. It would be very interesting to see other examples of financial transforms of particular valuation principles. Explicit examples for the environment $(\mathcal{G}, B)$ that allow for more concrete computations with the resulting financial valuation rule will be discussed elsewhere.

1. Sure profits and projections in L^2

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $L^2 = L^2(\Omega, \mathcal{F}, P)$ the space of all square-integrable real random variables with scalar product $(Y, Z) = E[YZ]$ and norm

$$\|Y\| = \sqrt{(Y,Y)} = \sqrt{E\left[Y^2\right]} \qquad \text{for } Y \in L^2.$$

For any subset $\mathcal{Y}$ of L^2, we denote by

$$\mathcal{Y}^\perp := \left\{ Z \in L^2 \,\middle|\, (Z,Y) = 0 \text{ for all } Y \in \mathcal{Y} \right\}$$

the orthogonal complement of $\mathcal{Y}$ in L^2 and by $\bar{\mathcal{Y}}$ the closure of $\mathcal{Y}$ in L^2.

Now fix $B \in L^2$ with $B > 0$ P-a.s., let $\mathcal{G}$ be a fixed subset of L^2 and set $\mathcal{A} := I\!\!R B + \mathcal{G}$. Clearly, $\mathcal{G}^\perp$ is a closed linear subspace of L^2, and we denote by π the projection in L^2 on $\mathcal{G}^\perp$. The pair $(\mathcal{G}, B)$ will represent the *financial environment* in which the subsequent considerations take place. An element g of $\mathcal{G}$ models the total *gains from trade* resulting from a self-financing trading strategy with initial capital 0. B is interpreted as the final value of some riskless asset with initial value 1; "riskless" here means that there will always be some money left at the end which is captured by the strict positivity of B. An important special case is $B \equiv 1$; this corresponds to working directly with discounted quantities. The set $\mathcal{A}$ is the space of those random payoffs which are *strictly attainable* in the sense that they can be obtained as the final wealth of some trading strategy with some initial capital. Later on, we shall assume that $\mathcal{G}$ is a linear subspace of L^2; this corresponds to a financial market without frictions like transaction costs, constraints or other restrictions on strategies. Square-integrability is imposed to obtain a nice Hilbert space structure and because we want means and variances to exist.

Example. Let $\mathcal{T} \subseteq I\!\!R_+$ be a time index set and $X = (X_t)_{t \in \mathcal{T}}$ an $I\!\!R^d$-valued semimartingale with respect to P and a filtration $I\!\!F = (\mathcal{F}_t)_{t \in \mathcal{T}}$ on $(\Omega, \mathcal{F})$. Let Θ be the space of all $I\!\!R^d$-valued $I\!\!F$-predictable X-integrable processes $\vartheta = (\vartheta_t)_{t \in \mathcal{T}}$ such that the stochastic integral process $G(\vartheta) := \int \vartheta \, dX$ is in the space $\mathcal{S}^2$ of semimartingales. Then we could take $B \equiv 1$ and $\mathcal{G} := G_T(\Theta)$, where $T := \sup \mathcal{T}$ is the time horizon of our economy. In this example, X models the discounted price evolution of d risky assets, and each $\vartheta \in \Theta$ can be interpreted as a self-financing dynamic portfolio strategy so that $G_T(\vartheta)$ describes the total gains that result from trading according to ϑ. For a continuous-time model where $\mathcal{T} = [0, T]$ for some $T \in (0, \infty]$, the space $G_T(\Theta)$ has been studied by Delbaen/Monat/Schachermayer/Schweizer/Stricker (1996) and Gouriéroux/Laurent/Pham (1996), among others. For $d = 1$ and $\mathcal{T} = \{0, 1, \ldots, T\}$ with $T \in I\!\!N$, Schweizer (1995) has studied the projection in L^2 on $G_T(\Theta)$, and Mercurio (1996) has introduced and computed mean-variance utility prices under an additional (restrictive) condition on X. Also in discrete time, Aurell/Życzkowski (1996) have examined mean-standard-deviation utility prices for a very special process X. The results of Mercurio (1996) and Aurell/Życzkowski (1996) have been generalized by Gharagozlou (1997).

Definition. We say that $\mathcal{G}$ *admits no sure profits in* L^2 if $\bar{\mathcal{G}}$ does not contain B.

With the preceding interpretations, this notion is very intuitive: It says that one cannot approximate (in the L^2-sense) the riskless payoff B by a self-financing strategy with initial wealth 0. This is one way to formulate a no-arbitrage condition on the financial environment; loosely speaking, it should be impossible to turn nothing into something without incurring costs. For the case where $B \equiv 1$ and $\mathcal{G}$ consists of elementary stochastic integrals with respect to a given square-integrable stochastic process, a very similar condition has been studied by Stricker (1990).

Lemma 1. *If $\mathcal{G}$ is a linear subspace of L^2, the following assertions are equivalent:*

1) *$\mathcal{G}$ admits no sure profits in L^2.*

2) *$\pi(B)$ is not P-a.s. identically 0.*

3) $E[B\pi(B)] > 0$.

4) $\mathcal{G}^\perp \cap \left(I\!\!RB + \bar{\mathcal{G}}\right) \neq \{0\}$.

5) *There is some Z in $\mathcal{G}^\perp \cap \left(I\!\!RB + \bar{\mathcal{G}}\right)$ with $(B, Z) > 0$.*

Proof. Since $\mathcal{G}$ is a linear subspace of L^2, $\mathcal{G}^{\perp\perp} = \bar{\mathcal{G}}$. Thus $L^2 = \mathcal{G}^\perp \oplus \bar{\mathcal{G}}$, and so the riskless payoff B has the orthogonal decomposition

$$B = \pi(B) + \big(B - \pi(B)\big).$$

In particular, $\pi(B) \in \mathcal{G}^\perp \cap \left(I\!\!RB + \bar{\mathcal{G}}\right)$ and

$$E[B\pi(B)] = E\left[\big(\pi(B)\big)^2\right] \geq 0.$$

This shows that 1) – 4) are all equivalent and imply 5) with $Z = \pi(B)$. As 5) clearly implies 4), the proof is complete.

q.e.d.

Actually, it is easy to show that linearity of $\mathcal{G}$ implies that $\mathcal{G}^\perp \cap \left(I\!\!RB + \bar{\mathcal{G}}\right) = \pi(I\!\!RB)$, but we shall not use this in the sequel. The second part of the next result will not be used either, but may be useful for applications.

Lemma 2. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. Then:*

1) $\mathcal{A}^{\perp\perp} = \bar{\mathcal{A}} = I\!\!RB + \bar{\mathcal{G}} = I\!\!RB + \mathcal{G}^{\perp\perp}$.

2) *$\mathcal{A}$ is closed in L^2 if and only if $\mathcal{G}$ is closed in L^2.*

Proof. 1) Since $\mathcal{A}$ and $\mathcal{G}$ are both linear subspaces, the first and third equalities are clear without further assumptions. Any $g \in \bar{\mathcal{G}}$ is the limit in L^2 of a sequence (g_n) in

$\mathcal{G}$; hence $cB + g_n = a_n$ is a Cauchy sequence in $\mathcal{A}$ and thus converges in L^2 to a limit $a \in \bar{\mathcal{A}}$ so that $cB + g = a \in \bar{\mathcal{A}}$. This gives the inclusion "$\supseteq$" for the middle equality. For the converse, we use the assumption that $\mathcal{G}$ admits no sure profits in L^2. Lemma 1 then gives us a $Z \in \mathcal{G}^\perp \cap \left(I\!RB + \bar{\mathcal{G}}\right)$ with $(B, Z) > 0$. For any $a \in \bar{\mathcal{A}}$, there is a sequence $a_n = c_n B + g_n$ in $\mathcal{A}$ converging to a in L^2. Since $c_n B + g_n \in I\!RB + \mathcal{G}$ for all n, we conclude that

$$(a_n, Z) = (c_n B + g_n, Z) = (c_n B, Z) = c_n (B, Z)$$

converges in $I\!R$ to (a, Z), and since $(B, Z) > 0$, (c_n) converges to $c = \frac{(a,Z)}{(B,Z)}$. Therefore $g_n = a_n - c_n B$ converges in L^2 to $g := a - cB$, and since this limit is in $\bar{\mathcal{G}}$, we have $a = cB + g \in I\!RB + \bar{\mathcal{G}}$ which proves the inclusion "$\subseteq$".

2) The "if" part is immediate from 1). Conversely, let (g_n) be any sequence in $\mathcal{G}$ converging in L^2 to some g_∞ which is in $\bar{\mathcal{A}}$ since $\mathcal{G} \subseteq \mathcal{A}$. If $\mathcal{A}$ is closed in L^2, we thus obtain $g_\infty \in \mathcal{A}$, hence $g_\infty = cB + g$ for some $c \in I\!R$ and $g \in \mathcal{G}$. Since $\mathcal{G}$ admits no sure profits in L^2, we can choose Z as in Lemma 1 to obtain

$$c(B, Z) = (cB + g, Z) = (g_\infty, Z) = \lim_{n\to\infty} (g_n, Z) = 0$$

because of $g_n \in \mathcal{G}$ and $Z \in \mathcal{G}^\perp$. Since $(B, Z) > 0$, we conclude that $c = 0$ so that $g_\infty = g \in \mathcal{G}$, and this proves the "only if" part.

q.e.d.

The next result provides an important decomposition for arbitrary payoffs in L^2. Its interpretation is deferred until the end of section 2.

Corollary 3. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. Then every $H \in L^2$ has a unique decomposition as*

$$H = c^H B + g^H + N^H \tag{1.1}$$

with $c^H \in I\!R$, $g^H \in \bar{\mathcal{G}}$ and $N^H \in \bar{\mathcal{A}}^\perp$. In particular, we have $E\left[BN^H\right] = 0$ and $E\left[N^H g\right] = 0$ for all $g \in \bar{\mathcal{G}}$.

Proof. Writing $L^2 = \bar{\mathcal{A}} \oplus \bar{\mathcal{A}}^\perp$ yields $H = a^H + N^H$ with unique elements $a^H = c^H B + g^H \in \bar{\mathcal{A}}$ and $N^H \in \bar{\mathcal{A}}^\perp$. If $a^H = cB + g$ for some $c \in I\!R$ and $g \in \bar{\mathcal{G}}$ and if Z is as in Lemma 1, we obtain

$$c(B, Z) = \left(a^H, Z\right) = c^H (B, Z)$$

which proves the uniqueness of the decomposition (1.1) because $(B, Z) > 0$.

q.e.d.

Consider now the optimization problem

$$\text{(1.2)} \qquad \text{Minimize } E\left[(H - cB - g)^2\right] \text{ over all } c \in I\!R \text{ and all } g \in \mathcal{G}$$

for a fixed $H \in L^2$. Since $\mathcal{A} = I\!RB + \mathcal{G}$, this can equivalently be rewritten as

$$\text{(1.3)} \qquad \text{Minimize } \|H - a\|^2 \text{ over all } a \in \mathcal{A}$$

and is therefore recognized as the problem of projecting H in L^2 on $\mathcal{A}$. The resulting minimal distance will later turn out to play an important role.

Lemma 4. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. For every fixed $H \in L^2$ with decomposition (1.1), we then have*

$$J_0 := \inf_{(c,g) \in I\!R \times \mathcal{G}} E\left[(H - cB - g)^2\right] = \inf_{a \in \mathcal{A}} \|H - a\|^2 = E\left[\left(N^H\right)^2\right].$$

If g^H is in $\mathcal{G}$, then the solution of (1.2) or (1.3) is given by $\left(c^H, g^H\right) \in I\!R \times \mathcal{G}$. In particular, this is always the case if $\mathcal{G}$ is closed in L^2.

Proof. Clearly, we have $J_0 = \inf_{a \in \mathcal{A}} \|H - a\|^2 = \inf_{a \in \bar{\mathcal{A}}} \|H - a\|^2$. By Corollary 3, the projection of H in L^2 on $\bar{\mathcal{A}}$ is $c^H B + g^H$, and so

$$J_0 = \left\|H - c^H B - g^H\right\|^2 = \left\|N^H\right\|^2 = E\left[\left(N^H\right)^2\right].$$

The remaining assertions are clear.

q.e.d.

2. Changing the measure

In this section, we introduce some additional measures on $(\Omega, \mathcal{F})$. This has two reasons. For one thing, it will allow us to obtain alternative expressions for the quantity J_0 in Lemma 4 and for the constant c^H in the decomposition (1.1) of H. More importantly, however, we shall also need one of them in the formulation of our a priori valuation rules in the next section. We first define a probability measure $P^B \approx P$ by

$$\frac{dP^B}{dP} := \frac{B^2}{E[B^2]}.$$

Note that here and in the sequel, all expectations and variances without sub- or superscripts refer to P. It is clear that a random variable H is in $L^2 = L^2(P)$ if and only if

$\frac{H}{B}$ is in $L^2(P^B)$. In financial terms, dividing by B is called *discounting with respect to* B, since it serves to express everything in units of B. Thus P^B is the natural measure to be used when working with B-discounted quantities. The next result collects some simple properties for later use; superscripts ^{B} in the sequel always refer to P^B.

Lemma 5. *Assume that $H \in L^2$ has a decomposition $H = c^H B + g^H + N^H$ as in (1.1). Then $E^B\left[\frac{N^H}{B}\right] = 0$ and $\mathrm{Cov}^B\left(\frac{g}{B}, \frac{N^H}{B}\right) = 0$ for all $g \in \bar{\mathcal{G}}$ so that under P^B, $\frac{N^H}{B}$ has zero mean and is uncorrelated with $\frac{1}{B}\bar{\mathcal{G}}$.*

Proof. From the definition of P^B, we obtain

$$E^B\left[\frac{N^H}{B}\right] = \frac{1}{E[B^2]} E\left[BN^H\right] = 0$$

by Corollary 3. Hence we have

$$\mathrm{Cov}^B\left(\frac{g}{B}, \frac{N^H}{B}\right) = E^B\left[\frac{g}{B}\frac{N^H}{B}\right] = \frac{1}{E[B^2]} E\left[gN^H\right] = 0$$

for all $g \in \bar{\mathcal{G}}$ again by Corollary 3.

q.e.d.

The next concept has been introduced by Schweizer (1996) for $B \equiv 1$; the extension to positive B follows Gouriéroux/Laurent/Pham (1996).

Definition. A *signed* $(\mathcal{G}, B)$*-martingale measure* is a signed measure Q on $(\Omega, \mathcal{F})$ with $Q[\Omega] = 1$, $Q \ll P$ with $\frac{1}{B}\frac{dQ}{dP} \in L^2$ and

$$E_Q\left[\frac{g}{B}\right] = \left(\frac{1}{B}\frac{dQ}{dP}, g\right) = 0 \qquad \text{for all } g \in \mathcal{G}.$$

A signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$ is called *B-variance-optimal* if

$$\left\|\frac{1}{B}\frac{d\widetilde{P}}{dP}\right\| \leq \left\|\frac{1}{B}\frac{dQ}{dP}\right\| \qquad \text{for all signed } (\mathcal{G}, B)\text{-martingale measures } Q.$$

(To indicate the dependence on B, we could write $\widetilde{P}^B$ instead of $\widetilde{P}$, but as B is fixed throughout, this would only clutter up the notation.)

Remarks. 1) Clearly, $E^B\left[\frac{dQ}{dP^B}\right] = Q[\Omega] = 1$ for every signed $(\mathcal{G}, B)$-martingale measure Q. Since

$$\frac{dQ}{dP} = \frac{dQ}{dP^B}\frac{B^2}{E[B^2]}$$

by the definition of P^B, we obtain

$$\left\| \frac{1}{B} \frac{dQ}{dP} \right\|^2 = \frac{1}{E[B^2]} E^B \left[\left(\frac{dQ}{dP^B} \right)^2 \right] = \frac{1}{E[B^2]} \left(\mathrm{Var}^B \left[\frac{dQ}{dP^B} \right] + 1 \right).$$

This shows that $\widetilde{P}$ is B-variance-optimal if and only if its B-discounted density with respect to P^B has minimal P^B-variance among all signed $(\mathcal{G}, B)$-martingale measures Q; hence the terminology.

2) In the setting of the example in section 1, $\widetilde{P}$ actually turns out to be equivalent to P (and in particular nonnegative) if the asset price process X is continuous. This has been proved by Delbaen/Schachermayer (1996) for $B \equiv 1$ and by Gouriéroux/Laurent/Pham (1996) for B and $\frac{1}{B}$ bounded, and their proofs can be extended to the present setting. But in general, $\widetilde{P}$ will only be a signed measure so that some care is needed in practical applications.

Lemma 6. *The B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$ exists if and only if $\mathcal{G}$ admits no sure profits in L^2. In that case, $\widetilde{P}$ is unique and given by*

$$\frac{d\widetilde{P}}{dP} = \frac{B\pi(B)}{E[B\pi(B)]} \qquad \text{or} \qquad \frac{d\widetilde{P}}{dP^B} = \frac{\frac{\pi(B)}{B}}{E^B\left[\frac{\pi(B)}{B}\right]}. \tag{2.1}$$

Proof. For $B \equiv 1$, this can be found in Delbaen/Schachermayer (1996) or Schweizer (1996). The argument for general $B > 0$ is quite similar, but we include it here for completeness. If Q is a signed $(\mathcal{G}, B)$-martingale measure, then $\frac{1}{B}\frac{dQ}{dP} \in \mathcal{G}^\perp$ by definition, and so

$$1 = Q[\Omega] = \left(\frac{1}{B} \frac{dQ}{dP}, B \right) = \left(\frac{1}{B} \frac{dQ}{dP}, \pi(B) \right).$$

This shows that the set of signed $(\mathcal{G}, B)$-martingale measures is nonempty if and only if $\pi(B) \not\equiv 0$, hence by Lemma 1 if and only if $\mathcal{G}$ admits no sure profits in L^2. Since the set of all B-discounted densities $\frac{1}{B}\frac{dQ}{dP}$ of signed $(\mathcal{G}, B)$-martingale measures Q is convex and closed in L^2, it is clear that the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure then exists and is unique. To prove (2.1), we first note that $\frac{B\pi(B)}{E[B\pi(B)]}$ is well-defined by Lemma 1 since $\mathcal{G}$ admits no sure profits in L^2, and so $\widetilde{P}$ defined by the first equation in (2.1) is a signed $(\mathcal{G}, B)$-martingale measure. Moreover, $\pi(B)$ is in $\mathbb{R}B + \bar{\mathcal{G}}$ by Lemma 1, hence of the form $cB + g$ for some $c \in \mathbb{R}$ and $g \in \bar{\mathcal{G}}$, and so

$$\left(\frac{1}{B} \frac{dQ}{dP}, \frac{1}{B} \frac{d\widetilde{P}}{dP} \right) = \frac{1}{E[B\pi(B)]} \left(\frac{1}{B} \frac{dQ}{dP}, \pi(B) \right) = \frac{c}{E[B\pi(B)]} \tag{2.2}$$

is constant over all signed $(\mathcal{G}, B)$-martingale measures Q. If Q is now an arbitrary signed $(\mathcal{G}, B)$-martingale measure, then so is $R := 2Q - \widetilde{P}$, and $Q = \widetilde{P} + \frac{R - \widetilde{P}}{2}$. Hence we obtain

$$\left\| \frac{1}{B} \frac{dQ}{dP} \right\|^2 = \left\| \frac{1}{B} \frac{d\widetilde{P}}{dP} \right\|^2 + \frac{1}{4} \left\| \frac{1}{B} \frac{d(R - \widetilde{P})}{dP} \right\|^2 \geq \left\| \frac{1}{B} \frac{d\widetilde{P}}{dP} \right\|^2,$$

because the mixed term disappears thanks to (2.2). This proves that $\widetilde{P}$ defined by the first expression in (2.1) is indeed B-variance-optimal. The alternative second expression is obtained by noting that

$$\frac{d\widetilde{P}}{dP^B} = \frac{d\widetilde{P}}{dP} \frac{dP}{dP^B} = \frac{\pi(B) E[B^2]}{B E[B \pi(B)]} = \frac{\frac{\pi(B)}{B}}{E^B \left[\frac{\pi(B)}{B} \right]}$$

by the first half of (2.1) and the definition of P^B.

q.e.d.

For future reference, we shall compute the minimal variance $\mathrm{Var}^B \left[\frac{d\widetilde{P}}{dP^B} \right]$ explicitly in terms of P, B and $\pi(B)$. We first observe that

$$E^B \left[\left(\frac{\pi(B)}{B} \right)^2 \right] = \frac{1}{E[B^2]} E \left[\left(\pi(B) \right)^2 \right] = \frac{1}{E[B^2]} E[B \pi(B)] = E^B \left[\frac{\pi(B)}{B} \right].$$

Thanks to (2.1), we thus obtain

$$\begin{aligned} \mathrm{Var}^B \left[\frac{d\widetilde{P}}{dP^B} \right] &= \frac{1}{E^B \left[\frac{\pi(B)}{B} \right]} - 1 \\ &= \frac{E^B \left[1 - \frac{\pi(B)}{B} \right]}{E^B \left[\frac{\pi(B)}{B} \right]} \\ &= \frac{E \left[B \big(B - \pi(B) \big) \right]}{E[B \pi(B)]} \\ &= \frac{\| B - \pi(B) \|^2}{\| \pi(B) \|^2} \end{aligned} \tag{2.3}$$

by the orthogonality of $\pi(B)$ and $B - \pi(B)$ in L^2.

Corollary 7. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. For $H \in L^2$, consider the representation $H = c^H B + g^H + N^H$ from Corollary 3. Then*

$$c^H = \widetilde{E} \left[\frac{H}{B} \right] = \left(\frac{1}{B} \frac{d\widetilde{P}}{dP}, H \right),$$

where $\widetilde{E}$ denotes expectation with respect to the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$. Moreover,

$$J_0 = \inf_{a \in \mathcal{A}} \|H - a\|^2 = E\left[B^2\right] \mathrm{Var}^B \left[\frac{N^H}{B}\right]. \tag{2.4}$$

Proof. By (2.1) and Lemma 1, $\frac{1}{B}\frac{d\widetilde{P}}{dP}$ is in $\mathcal{G}^\perp \cap \left(I\!\!R B + \bar{\mathcal{G}}\right) = \mathcal{G}^\perp \cap \bar{\mathcal{A}}$ by Lemma 2. Since $g^H \in \bar{\mathcal{G}}$ and $N^H \in \bar{\mathcal{A}}^\perp$, we thus obtain

$$\widetilde{E}\left[\frac{H}{B}\right] = \left(\frac{1}{B}\frac{d\widetilde{P}}{dP}, c^H B + g^H + N^H\right) = c^H \widetilde{P}[\Omega] = c^H.$$

Moreover, Lemma 4 and Lemma 5 imply that

$$J_0 = E\left[\left(N^H\right)^2\right] = E\left[B^2\right] E^B\left[\left(\frac{N^H}{B}\right)^2\right] = E\left[B^2\right] \mathrm{Var}^B \left[\frac{N^H}{B}\right].$$

q.e.d.

In view of Corollary 7, the decomposition

$$H = c^H B + g^H + N^H \tag{1.1}$$

in Corollary 3 has a very intuitive financial interpretation. It tells us that H splits naturally into an *attainable part* $c^H B + g^H \in \bar{\mathcal{A}}$ and a *non-hedgeable part* N^H which is orthogonal to the space $\bar{\mathcal{A}}$ of attainable payoffs. The constant c^H is the initial capital of the attainable part; it can be computed in a simple way as the $\widetilde{P}$-expectation of the B-discounted payoff $\frac{H}{B}$. Moreover, (2.4) tells us that the P^B-variance of the B-discounted non-hedgeable part of H is a measure for the *intrinsic financial risk* of H in terms of approximation in L^2 by attainable, hence riskless payoffs.

3. The financial variance principle

Consider a fixed random variable $H \in L^2$ and think of this as a random payoff. In a financial context, H could represent the net payoff of some derivative product, e.g., of one unit of a European call option. In insurance terms, H should be thought of as the negative of a claim amount to be paid by the insurer. Since financial derivatives are often termed contingent claims, it seems appropriate to call H a general *claim*. How much should we receive or pay if we sell or buy γ units of this claim?

This question has a standard answer from classical option pricing theory, but only in a special case. Assume as before that $\mathcal{G}$ is a linear subspace of L^2 and that H is *strictly attainable* in the sense that $H \in \mathcal{A}$. Thus we can write $H = c^H B + g^H$ with $c^H \in I\!R$ and $g^H \in \mathcal{G}$, and then the price per unit of H must be c^H to avoid the possibility of constructing riskless profit opportunities. To illustrate the underlying idea, suppose for instance that H is nonnegative and offered at a price $x < c^H$. Then one can buy $\frac{c^H}{x}$ units of H and finance this initial investment of c^H with the trading strategy $\left(-c^H, -g^H\right)$. The terminal payoff from this costless deal is then $\frac{c^H}{x}H - c^H B - g^H = \left(\frac{c^H}{x} - 1\right) H \geq 0$; thus one has turned nothing into something, and this is postulated to be impossible. A similar argument applies if one can sell one unit of H for more than c^H. In an *incomplete financial market*, this line of reasoning breaks down: A general contingent claim is typically not attainable, and its price will depend on subjective preferences.

If we think of $-H$ as an insurance risk, there is also a standard approach to its valuation from insurance mathematics: We simply apply a *valuation principle* which seems appropriate for our needs. Thus we choose a mapping u from random variables Y into $I\!R$ and think of $u(Y)$ as the value or the utility associated to the random amount Y. For instance, the classical actuarial variance principle would correspond to

$$u(Y) := E[Y] - A\operatorname{Var}[Y];$$

the slightly unfamiliar choice of signs is due to our convention that $-H$ and not H corresponds to an insurance claim.

Just applying some u to H is of course one possible way to arrive at a valuation. But from a financial perspective, this is too simplistic because it ignores the possibilities of trading represented by $\mathcal{G}$. We therefore use a utility indifference argument to obtain our financial valuation rule. This is a well-known general approach in economics, and we follow here Mercurio (1996) and Aurell/Życzkowski (1996) who combined this with an L^2-framework to determine option prices in some special cases. For a general formulation of the basic idea, we start with an initial capital $c \in I\!R$ and our a priori valuation principle u. In order to decide on the price of γ units of H, we compare the following two alternatives:

(i) Invest optimally into a trading strategy with initial capital c and simply ignore the possibility of selling H. This amounts to maximizing $u(cB + g)$ over all $g \in \mathcal{G}$, and we denote by

$$v(c, 0) := \sup_{g \in \mathcal{G}} u(cB + g)$$

the value of this control problem.

(ii) Sell γ units of H for the amount $h(c, \gamma) \in I\!R$ to increase the initial wealth to $c + h(c, \gamma)$. This can then be invested into a trading strategy and leads to a total final wealth of $\big(c + h(c,\gamma)\big)B + g - \gamma H$, since we have to pay out the claims from H at the end. To obtain an optimal investment, we thus have to maximize $u\Big(\big(c + h(c,\gamma)\big)B + g - \gamma H\Big)$ over all $g \in \mathcal{G}$, and the value of this second control problem is

$$v(c,\gamma) := \sup_{g \in \mathcal{G}} u\Big(\big(c + h(c,\gamma)\big)B + g - \gamma H\Big).$$

Indifference with respect to u prevails if $h(c,\gamma)$ is chosen in such a way that neither of these alternatives is preferred to the other. More formally:

Definition. We say that $h(c,\gamma)$ is a *u-indifference price for γ units of H* if $h(c,\gamma)$ satisfies

$$v(c,\gamma) = v(c,0),$$

i.e.,

$$\sup_{g \in \mathcal{G}} u\Big(\big(c + h(c,\gamma)\big)B + g - \gamma H\Big) = \sup_{g \in \mathcal{G}} u(cB + g). \tag{3.1}$$

Note that for every choice of u, $h(c,0) = 0$ is always a u-indifference price for 0 units of H, although $h(c,\gamma)$ need not be unique. If $H = c^H B + g^H \in \mathcal{A}$ is strictly attainable, it is also easily checked that $h(c,\gamma) = \gamma c^H$ is a u-indifference price whatever u is. In general, the price for γ units of H can depend on the initial capital c and need not be linear in γ; the "price per unit of H", $\frac{h(c,\gamma)}{\gamma}$, is therefore of limited use only. For $\gamma = \pm 1$, we can nevertheless think of $h(c,\pm 1)$ as the ask and bid prices for one unit of H, respectively.

Recall now the probability measure $P^B \approx P$ introduced in section 2. Our goal in this paper is to determine the u-indifference prices for the two valuation principles

$$u_1(Y) := E^B\left[\frac{Y}{B}\right] - A\mathrm{Var}^B\left[\frac{Y}{B}\right] \tag{3.2}$$

and

$$u_2(Y) := E^B\left[\frac{Y}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{Y}{B}\right]},$$

where $A > 0$ is a risk aversion parameter. The corresponding u-indifference prices will be denoted by $h_{1,2}(c,\gamma)$, respectively. Note that both u_1 and u_2 account for the presence of the riskless asset B by first discounting with respect to B and then working with the measure P^B appropriate for B-discounted quantities.

Lemma 8. *For $u \in \{u_1, u_2\}$, the u-indifference price $h(c,\gamma)$ for any $H \in L^2$ does not depend on c and is given by*

$$h(c,\gamma) = h(\gamma) = \sup_{g\in\mathcal{G}} u(g) - \sup_{g\in\mathcal{G}} u(g-\gamma H) = \sup_{g\in\bar{\mathcal{G}}} u(g) - \sup_{g\in\bar{\mathcal{G}}} u(g-\gamma H)$$

for all $\gamma, c \in \mathbb{R}$.

Proof. Since the variance of a random variable does not change if we add a constant, we have

$$\begin{aligned} u_{1,2}\big((c+x)B + g - \gamma H\big) &= E^B\left[c + x + \frac{g-\gamma H}{B}\right] - A\left(\mathrm{Var}^B\left[c+x+\frac{g-\gamma H}{B}\right]\right)^{\beta_{1,2}} \\ &= c + x + u_{1,2}(g-\gamma H) \end{aligned}$$

for any $c, x \in \mathbb{R}$, where $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$. This implies that

$$v(c,\gamma) = \sup_{g\in\mathcal{G}} u\Big(\big(c + h(c,\gamma)\big)B + g - \gamma H\Big) = c + h(c,\gamma) + \sup_{g\in\mathcal{G}} u(g-\gamma H)$$

and

$$v(c,0) = \sup_{g\in\mathcal{G}} u(c+g) = c + \sup_{g\in\mathcal{G}} u(g)$$

for $u \in \{u_1, u_2\}$. Since $h(c,\gamma)$ is defined by (3.1), we obtain the first and second equalities.

For the third equality, it is obviously enough to show that for any $Y \in L^2$, we have

$$L := \sup_{g\in\mathcal{G}} u(g+Y) = \sup_{g\in\bar{\mathcal{G}}} u(g+Y) =: R. \tag{3.3}$$

Clearly, $L \le R$, and so it only remains to show that $L \ge R$. For any $g \in \bar{\mathcal{G}}$, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ converging to g in L^2. This implies that

$$E^B\left[\frac{g_n + Y}{B}\right] = \frac{1}{E[B^2]} E[B(g_n+Y)]$$

converges to $E^B\left[\frac{g+Y}{B}\right]$ and that $(\|g_n + Y\|)_{n\in\mathbb{N}}$ is bounded in n. Since

$$\left|\|g_n+Y\|^2 - \|g+Y\|^2\right| \le \|g_n - g\|\left(\sup_{n\in\mathbb{N}}\|g_n+Y\| + \|g+Y\|\right)$$

and

$$\begin{aligned} &\left|(E[B(g_n+Y)])^2 - (E[B(g+Y)])^2\right| \\ &\le \left|E[B(g_n-g)]\right|\left(\sup_{n\in\mathbb{N}}\left|E[B(g_n+Y)]\right| + \left|E[B(g+Y)]\right|\right) \\ &\le \|B\|^2\|g_n-g\|\left(\sup_{n\in\mathbb{N}}\|g_n+Y\| + \|g+Y\|\right), \end{aligned}$$

we conclude that

$$\left| \mathrm{Var}^B\left[\frac{g_n+Y}{B}\right] - \mathrm{Var}^B\left[\frac{g+Y}{B}\right] \right| \le \frac{1}{E[B^2]} \left| E\left[(g_n+Y)^2\right] - E\left[(g+Y)^2\right] \right| + \frac{1}{(E[B^2])^2} \left| (E[B(g_n+Y)])^2 - (E[B(g+Y)])^2 \right|$$

converges to 0 as $n \to \infty$. Given $\varepsilon > 0$, we thus have $E^B\left[\frac{g_n+Y}{B}\right] \ge E^B\left[\frac{g+Y}{B}\right] - \varepsilon$ and $\mathrm{Var}^B\left[\frac{g_n+Y}{B}\right] \le \mathrm{Var}^B\left[\frac{g+Y}{B}\right] + \varepsilon$ for n sufficiently large. This implies that

$$\begin{aligned} L &\ge u(g_n+Y) \\ &= E^B\left[\frac{g_n+Y}{B}\right] - A\left(\mathrm{Var}^B\left[\frac{g_n+Y}{B}\right]\right)^\beta \\ &\ge E^B\left[\frac{g+Y}{B}\right] - \varepsilon - A\left(\mathrm{Var}^B\left[\frac{g+Y}{B}\right] + \varepsilon\right)^\beta \end{aligned}$$

for n sufficiently large, hence

$$L \ge E^B\left[\frac{g+Y}{B}\right] - A\left(\mathrm{Var}^B\left[\frac{g+Y}{B}\right]\right)^\beta = u(g+Y)$$

by letting ε tend to 0, and since $g \in \bar{\mathcal{G}}$ was arbitrary, we obtain (3.3).

q.e.d.

Theorem 9. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. For any $H \in L^2$ and any $\gamma, c \in \mathbb{R}$, the u_1-indifference price for γ units of H is then*

$$h_1(c,\gamma) = h_1(\gamma) = \gamma c^H + A\gamma^2 \frac{J_0}{E[B^2]} = \gamma \widetilde{E}\left[\frac{H}{B}\right] + A\gamma^2 \mathrm{Var}^B\left[\frac{N^H}{B}\right], \tag{3.4}$$

where $\widetilde{E}$ denotes expectation with respect to the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$.

Proof. By Corollary 3, H can be decomposed as $H = c^H B + g^H + N^H$ as in (1.1) so that

$$u_1(g - \gamma H) = u_1\left(-\gamma c^H B + g - \gamma g^H - \gamma N^H\right) = -\gamma c^H + u_1\left(g - \gamma g^H - \gamma N^H\right).$$

Since $\mathcal{G}$ is a linear subspace of L^2, the mapping $g \mapsto g' := g - \gamma g^H$ is a bijection of $\bar{\mathcal{G}}$ into itself for every fixed $H \in L^2$ and $\gamma \in \mathbb{R}$, and so Lemma 8 implies that

$$h_1(\gamma) = \sup_{g \in \bar{\mathcal{G}}} u_1(g) + \gamma c^H - \sup_{g' \in \bar{\mathcal{G}}} u_1\left(g' - \gamma N^H\right). \tag{3.5}$$

But $E^B\left[\frac{N^H}{B}\right]=0$ and $\mathrm{Cov}^B\left(\frac{g'}{B},\frac{N^H}{B}\right)=0$ for all $g'\in\bar{\mathcal{G}}$ by Lemma 5, and so we obtain

$$\begin{aligned}
u_1\left(g'-\gamma N^H\right) &= E^B\left[\frac{g'-\gamma N^H}{B}\right]-A\mathrm{Var}^B\left[\frac{g'-\gamma N^H}{B}\right]\\
&= E^B\left[\frac{g'}{B}\right]-A\mathrm{Var}^B\left[\frac{g'}{B}\right]-A\gamma^2\mathrm{Var}^B\left[\frac{N^H}{B}\right]\\
&= u_1(g')-A\gamma^2\mathrm{Var}^B\left[\frac{N^H}{B}\right].
\end{aligned}$$

Combining this with (3.5) yields

$$h_1(\gamma)=\sup_{g\in\bar{\mathcal{G}}}u_1(g)+\gamma c^H-\sup_{g'\in\bar{\mathcal{G}}}u_1(g')+A\gamma^2\mathrm{Var}^B\left[\frac{N^H}{B}\right],$$

and this together with Corollary 7 implies the assertion.

q.e.d.

The valuation formula (3.4) in Theorem 9 has a number of very attractive features. To see the most striking of these, let us interpret $h_1(\pm 1)$ as the buying ($\gamma=-1$) and selling ($\gamma=+1$) prices for one unit of H, respectively. Then we see from (3.4) that

$$h_1(\pm 1)=\widetilde{E}\left[\frac{H}{B}\right]\pm A\mathrm{Var}^B\left[\frac{N^H}{B}\right]. \tag{3.6}$$

This looks very similar to the *actuarial valuation principle* (3.2), but differs by two important points. First of all, the expectation of the B-discounted claim $\frac{H}{B}$ is not taken under the original measure P^B, but under the B-variance-optimal signed $(\mathcal{G},B)$-martingale measure $\widetilde{P}$. Secondly, the variance component in (3.6) is not based on H, but only on the non-hedgeable part N^H of H. Thus we have to pass from the real-world measure P^B to the (appropriate) risk-neutral measure $\widetilde{P}$ for computing expectations, and we do not add or subtract a risk-loading (under P^B) for that part of H which can be hedged away by judicious trading. In view of the perfect analogy to (3.2), the prescription (3.6) could therefore be called the *financial variance principle*.

If H is attainable in the sense that $H=c^H+g^H$ is in $\bar{\mathcal{A}}$, then (3.6) reduces to $h_1(\pm 1)=c^H$. This is exactly what we expect from the classical arbitrage arguments from option pricing. For a general claim H, we obtain a *bid-ask spread* of

$$h_1(+1)-h_1(-1)=2A\mathrm{Var}^B\left[\frac{N^H}{B}\right]$$

which is proportional to the risk aversion A and the intrinsic financial risk $\frac{J_0}{E[B^2]}$ of H; see Corollary 7. Finally, note that the valuation in (3.4) does not depend on the initial capital c and that it is not linear in the number γ of claims. It might be interesting to compare this to empirically observed prices.

4. The financial standard deviation principle

In this section, we determine the u_2-indifference prices $h_2(c,\gamma)$ for the *actuarial standard deviation principle*

$$u_2(Y) := E^B\left[\frac{Y}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{Y}{B}\right]}. \tag{4.1}$$

From Lemma 8, we know that

$$h_2(c,\gamma) = h_2(\gamma) = \sup_{g\in\bar{\mathcal{G}}} u_2(g) - \sup_{g\in\bar{\mathcal{G}}} u_2(g-\gamma H)$$

for any $H \in L^2$ and $c,\gamma \in I\!R$. Moreover, Corollary 3 yields the decomposition

$$g - \gamma H = -\gamma c^H B + g - \gamma g^H - \gamma N^H,$$

and since N^H has expectation 0 and is uncorrelated with $\frac{1}{B}\bar{\mathcal{G}}$ under P^B by Lemma 5, we obtain as in the proof of Theorem 9 that

$$u_2(g-\gamma H) = -\gamma c^H + E^B\left[\frac{g-\gamma g^H}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g-\gamma g^H}{B}\right] + \gamma^2\mathrm{Var}^B\left[\frac{N^H}{B}\right]}$$

for any $\gamma \in I\!R$. Since $\mathcal{G}$ is a linear subspace of L^2, the mapping $g \mapsto g' := g - \gamma g^H$ is a bijection of $\bar{\mathcal{G}}$ into itself for every fixed $H \in L^2$ and $\gamma \in I\!R$, and so we get

$$\begin{aligned} h_2(\gamma) = \gamma c^H &+ \sup_{g\in\bar{\mathcal{G}}}\left(E^B\left[\frac{g}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g}{B}\right]}\right) \\ &- \sup_{g'\in\bar{\mathcal{G}}}\left(E^B\left[\frac{g'}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g'}{B}\right] + \gamma^2\mathrm{Var}^B\left[\frac{N^H}{B}\right]}\right). \end{aligned} \tag{4.2}$$

We start by analyzing the last term.

Lemma 10. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. For any $y \in I\!R$, we then have*

$$\begin{aligned} &\sup_{g\in\bar{\mathcal{G}}}\left(E^B\left[\frac{g}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g}{B}\right] + y^2}\right) \\ &= \begin{cases} -A\sqrt{y^2} & \text{for } B \in \mathcal{G}^\perp \\ \sup_{m\in I\!R}\left(m - A\sqrt{\dfrac{m^2}{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]} + y^2}\right) & \text{for } B \notin \mathcal{G}^\perp. \end{cases} \end{aligned} \tag{4.3}$$

Proof. If $B \in \mathcal{G}^\perp$, then $E^B\left[\frac{g}{B}\right] = \frac{1}{E[B^2]}E[Bg] = 0$ for all $g \in \bar{\mathcal{G}}$, and so we can simply minimize the P^B-variance of $\frac{g}{B}$ by choosing $g = 0$. For the case where $B \notin \mathcal{G}^\perp$, the idea is to perform the maximization in two steps: we first restrict attention to those g satisfying the constraint $E^B\left[\frac{g}{B}\right] = m$ and then maximize over $m \in I\!R$. In analogy to Corollary 16 of Schweizer (1996), we therefore define

$$g_m := c_m\big(B - \pi(B)\big) := \frac{mE[B^2]}{E\left[\big(B-\pi(B)\big)^2\right]}\big(B - \pi(B)\big).$$

Since $B \notin \mathcal{G}^\perp$, we have $E\left[B\big(B-\pi(B)\big)\right] = E\left[\big(B-\pi(B)\big)^2\right] > 0$, and so g_m is well-defined and in $\mathcal{G}^{\perp\perp} = \bar{\mathcal{G}}$ since $\mathcal{G}$ is linear. Moreover,

$$E^B\left[\frac{g_m}{B}\right] = \frac{1}{E[B^2]}E[Bg_m] = m.$$

If we take any $g \in \bar{\mathcal{G}}$ with $E^B\left[\frac{g}{B}\right] = m$, then

$$\begin{aligned}\|g - c_mB\|^2 &= \|g - g_m + c_m\pi(B)\|^2 \\ &= \|g - g_m\|^2 + c_m^2\|\pi(B)\|^2 \\ &\geq c_m^2\|\pi(B)\|^2 \\ &= \|g_m - c_mB\|^2,\end{aligned}$$

and so we deduce that

$$\mathrm{Var}^B\left[\frac{g}{B}\right] = \mathrm{Var}^B\left[\frac{g}{B} - c_m\right] = \frac{1}{E[B^2]}\|g - c_mB\|^2 - (m - c_m)^2 \geq \mathrm{Var}^B\left[\frac{g_m}{B}\right].$$

This implies that

$$\begin{aligned}&\sup\left\{E^B\left[\frac{g}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g}{B}\right] + y^2}\ \middle|\ g \in \bar{\mathcal{G}} \text{ with } E^B\left[\frac{g}{B}\right] = m\right\} \\ &= E^B\left[\frac{g_m}{B}\right] - A\sqrt{\mathrm{Var}^B\left[\frac{g_m}{B}\right] + y^2} \\ &= m - A\sqrt{c_m^2\mathrm{Var}^B\left[1 - \frac{\pi(B)}{B}\right] + y^2}.\end{aligned}$$

But

$$\begin{aligned}\mathrm{Var}^B\left[1 - \frac{\pi(B)}{B}\right] &= \frac{1}{E[B^2]}\|B - \pi(B)\|^2 - \left(\frac{1}{E[B^2]}E\left[B\big(B-\pi(B)\big)\right]\right)^2 \\ &= \frac{1}{E[B^2]}\|B - \pi(B)\|^2 - \left(\frac{1}{E[B^2]}E\left[\big(B-\pi(B)\big)^2\right]\right)^2,\end{aligned}$$

and so we get

$$c_m^2 \operatorname{Var}^B\left[1-\frac{\pi(B)}{B}\right] = m^2\left(\frac{E[B^2]}{E\left[\left(B-\pi(B)\right)^2\right]}-1\right) = m^2\frac{\|\pi(B)\|^2}{\|B-\pi(B)\|^2}.$$

Together with (2.3), this proves the assertion.

q.e.d.

The next result is elementary analysis; its proof is only included for completeness.

Lemma 11. *For any $y \in I\!R$, let*

$$s(y) := \sup_{x\in I\!R}\left(x-\sqrt{Cx^2+y^2}\right)$$

for a fixed $C \geq 0$. Then

$$s(0)-s(y) = \begin{cases} \sqrt{y^2}\sqrt{1-\frac{1}{C}} & \text{for } C \geq 1 \\ \text{undefined} & \text{for } C < 1. \end{cases}$$

Proof. Fix $y \in I\!R$ and let $f(x) := x-\sqrt{Cx^2+y^2}$ for $x \in I\!R$. Then

$$\lim_{x\to\pm\infty}\frac{f(x)}{|x|} = \lim_{x\to\pm\infty}\left(\pm 1-\sqrt{C+\frac{y^2}{x^2}}\right) = \pm 1-\sqrt{C}. \tag{4.4}$$

For $C < 1$, this implies that $\lim_{x\to\pm\infty} f(x) = \pm\infty$, and since f is continuous, we conclude that f has no finite maximum so that $s \equiv +\infty$ in this case.

For $C \geq 1$, we write $f(x)$ as

$$f(x) = \frac{x^2-(Cx^2+y^2)}{x+\sqrt{Cx^2+y^2}} = \frac{(1-C)x^2-y^2}{x+\sqrt{Cx^2+y^2}};$$

this shows that $f \leq 0$. If $C = 1$, the numerator does not depend on x and the denominator goes to $+\infty$ for $x \to +\infty$. Hence we conclude that $s \equiv 0$ in this case. If $C > 1$ and $y = 0$, the maximum of $f \leq 0$ is attained in $x = 0$ so that $s(0) = 0$ for $C > 1$. Finally, if $C > 1$ and $y^2 > 0$, f is continuously differentiable with derivative

$$f'(x) = 1-\frac{Cx}{\sqrt{Cx^2+y^2}}.$$

Since $f' > 0$ for $x \leq 0$, it is easily checked that f' vanishes at the unique point $x^* = \sqrt{\frac{y^2}{C(C-1)}}$, and $f(x^*) = -\sqrt{y^2}\sqrt{1-\frac{1}{C}}$ by computation. By (4.4), f must have its maximum at x^*, and so the assertion follows.

q.e.d.

Combining the two previous results, we now obtain

Theorem 12. *Let $\mathcal{G}$ be a linear subspace of L^2 and assume that $\mathcal{G}$ admits no sure profits in L^2. For any $H \in L^2$ and any $\gamma, c \in I\!R$, the u_2-indifference price for γ units of H is then*

(4.5) $\quad h_2(c,\gamma) = h_2(\gamma)$

$$= \begin{cases} \gamma\widetilde{E}\left[\frac{H}{B}\right] + A|\gamma|\sqrt{1 - \frac{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]}{A^2}}\sqrt{\mathrm{Var}^B\left[\frac{N^H}{B}\right]} & \text{for } A^2 \geq \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right] \\ \text{undefined} & \text{for } A^2 < \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right], \end{cases}$$

where $\widetilde{E}$ denotes expectation with respect to the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$.

Proof. We first observe that if $B \in \mathcal{G}^\perp$, then $\pi(B) = B$ and therefore $\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right] = 0$ by (2.1). This shows that for $B \in \mathcal{G}^\perp$, the second case in (4.5) will never occur. If $B \in \mathcal{G}^\perp$, then (4.2) and (4.3) imply that

$$h_2(\gamma) = \gamma c^H + A\sqrt{\gamma^2 \mathrm{Var}^B\left[\frac{N^H}{B}\right]} = \gamma\widetilde{E}\left[\frac{H}{B}\right] + A|\gamma|\sqrt{\mathrm{Var}^B\left[\frac{N^H}{B}\right]}$$

by Corollary 7. If $B \notin \mathcal{G}^\perp$, then (4.2) and (4.3) yield

$$h_2(\gamma) = \gamma c^H + \sup_{m \in I\!R}\left(m - \sqrt{Cm^2}\right) - \sup_{m \in I\!R}\left(m - \sqrt{Cm^2 + A^2\gamma^2 \mathrm{Var}^B\left[\frac{N^H}{B}\right]}\right),$$

where we have set

$$C := \frac{A^2}{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]} \geq 0.$$

From Lemma 11 and Corollary 7, we thus obtain

$$h_2(\gamma) = \gamma\widetilde{E}\left[\frac{H}{B}\right] + A|\gamma|\sqrt{1 - \frac{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]}{A^2}}\sqrt{\mathrm{Var}^B\left[\frac{N^H}{B}\right]}$$

for $C \geq 1$, while $h_2(\gamma)$ is undefined for $C < 1$. This proves the assertion.

q.e.d.

As in the last section, the valuation formula (4.5) in Theorem 12 has a very appealing interpretation. If we write

$$
(4.6)\quad h_2(\pm 1) = \begin{cases} \widetilde{E}\left[\frac{H}{B}\right] \pm A\sqrt{1 - \frac{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]}{A^2}}\sqrt{\mathrm{Var}^B\left[\frac{N^H}{B}\right]} & \text{for } A^2 \geq \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right] \\ \text{undefined} & \text{for } A^2 < \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right], \end{cases}
$$

we see that our approach transforms the actuarial standard deviation principle (4.1) into the *financial standard deviation principle* (4.6). Note that like (3.4), the valuation in (4.5) is based on the expectation under the B-variance-optimal signed $(\mathcal{G}, B)$-martingale measure $\widetilde{P}$ and (the square root of) the intrinsic financial risk of H. The corresponding bid-ask spread is given by

$$
h_2(+1) - h_2(-1) = \begin{cases} 2A\sqrt{1 - \frac{\mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]}{A^2}}\sqrt{\mathrm{Var}^B\left[\frac{N^H}{B}\right]} & \text{for } A^2 \geq \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right] \\ \text{undefined} & \text{for } A^2 < \mathrm{Var}^B\left[\frac{d\widetilde{P}}{dP^B}\right]. \end{cases}
$$

In contrast to (3.4), the valuation (4.5) is piecewise linear in the number γ of claims to be valued; this implies that selling and buying prices for an arbitrary amount of H are proportional to the selling and buying price of 1 unit of H, respectively. A second major difference to the last section is that all these results need a sufficiently high risk aversion for $h_2(c, \gamma)$ to be well-defined. The lower bound on A depends on the P^B-variance of the density of $\widetilde{P}$ with respect to P^B. It is thus determined by the global properties of the financial environment $(\mathcal{G}, B)$ and in particular independent of the individual claim under consideration. In a very special case, a result like Theorem 12 has also been obtained by Aurell/Życzkowski (1996) by means of rather laborious calculations.

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