A stochastic control approach to a robust utility maximization problem

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Abstract: We study a stochastic control problem arising in the context of utility maximization under model uncertainty. The latter is formulated as a sup-inf problem over strategies π and models (measures) Q, and we treat the inner problem of minimizing over Q the sum of a Q-expected utility term and a penalty term based on the relative entropy of Q with respect to a reference measure P. We prove in general that there exists a unique optimal measure Q^* and show that Q^* is equivalent to P. For a continuous filtration, we characterize the dynamic value process of our stochastic control problem as the unique solution of a generalized backward stochastic differential equation with a quadratic driver. Our results extend earlier work by Skiadas (2003) and are based on a different approach.

Key words: robust control, model uncertainty, quadratic BSDE, stochastic control, relative entropy, entropic penalty, martingale optimality principle, utility maximization, multiple priors, robust utility

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** This version corrects a minor glitch in the proof of Proposition 3; see also (2.4) and (2.5).

0. Introduction

This paper takes one step in the problem of *utility maximization* under *model uncertainty*. At a very general level, the latter could be formulated as

$$\text{find } \sup_{\pi} \inf_{Q} \mathbf{U}(\pi, Q), \tag{0.1}$$

where π runs through a set of strategies (portfolios, investment decisions, ...) and $Q \in \mathcal{Q}$ through a set of models. In the simplest case, there is one known model so that $\mathcal{Q} = \{P\}$ for a fixed probability measure P, and $\mathbf{U}(\pi, P)$ has the form of a P-expected utility from terminal wealth and/or consumption, both of which are determined by π. There is a vast literature on this by now classical problem; but there is always the drawback that one assumes the underlying model to be exactly known.

To address this issue, one recent line of research considers a non-singleton set $\mathcal{Q}$ of probability measures while keeping for $\mathbf{U}(\pi, Q)$ a Q-expected utility. Such a setting for $\mathcal{Q}$ is often called a *multiple priors model*, and the corresponding optimization problem (0.1) is known as *robust utility maximization*. Some results in this area have been obtained in Gundel (2005), Quenez (2004) and Schied/Wu (2005), among others, and the overall approach relies a lot on convex duality ideas. The set $\mathcal{Q}$ of models under consideration is assumed to have certain properties, but is otherwise quite abstract and usually not specified in any detail.

Instead of working with a somehow given set of models, an alternative is to allow a priori all possible models Q, but to include in $\mathbf{U}(\pi, Q)$ a *penalization* term; this should only depend on Q (not on π) and quantify the decision maker's attitude towards the model Q. Such an approach has for instance been suggested in Anderson/Hansen/Sargent (2003) and Hansen et al. (2006); they take as $\mathbf{U}(\pi, Q)$ the sum of a Q-expected utility like above plus a penalty term based on the *relative entropy* of Q with respect to a reference model (measure) P. This is also the setting that we use here. For a very recent treatment of a closely related problem via duality methods, see Schied (2007).

The focus of the analysis in Anderson/Hansen/Sargent (2003) and Hansen et al. (2006) is on general insights about the qualitative behaviour of a solution to (0.1) in their framework. This is done via a mostly formal study of the corresponding Hamilton-Jacobi-Bellman (HJB) equations in a Markovian setting. Our goal in contrast is to obtain rigorous mathematical results, and the present paper achieves some progress in that direction for the partial (inner) problem of minimizing $\mathbf{U}(\pi, Q)$ over Q when π is kept fixed. This problem has also been studied by Skiadas (2003) who has obtained very similar results, but with a different approach; see Section 5 for a more detailed comparison.

The paper is structured as follows. Section 1 sets the stage by giving a precise definition of the functional $Q \mapsto \mathbf{U}(\pi, Q)$ with fixed π and of the corresponding optimization problem, and by introducing notations and key assumptions. Section 2 provides a number of auxiliary results for subsequent use. In Section 3, we show with the help of a standard Komlós-type argument that there exists a unique minimizing measure Q^*, and we prove that Q^* is even equivalent to P. This mainly functional analytic approach is complemented by Section 4. There we treat our optimization problem by *stochastic control* methods and show that for a continuous filtration, the corresponding dynamic value process is characterized as the unique solution of a *generalized backward stochastic differential equation (BSDE)* with a quadratic term in its driver. Our BSDE is a slight generalization of an equation studied in detail by Schroder/Skiadas (1999), but our method of attack is rather different. Like in Schroder/Skiadas (1999), however, our BSDE involves unbounded terms in the driver and the terminal value which cannot be handled by existing techniques from the BSDE literature. Hence our approach has to exploit the precise structure of our equation. The final Section 5 contains a brief comparison with some of the most closely related literature.

1. The basic optimization problem

This section gives a precise formulation of our optimization problem and introduces a number of notations for later use.

We start with a filtered probability space $(\Omega, \mathcal{F}, I\!\!F, P)$ over a finite time horizon $T \in (0, \infty)$. The filtration $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ satisfies the usual conditions of right-continuity and P-completeness. For any probability measure $Q \ll P$ on $\mathcal{F}_T$, the density process of Q with respect to P is the RCLL P-martingale $Z^Q = (Z^Q_t)_{0 \le t \le T}$ with

$$Z^Q_t = \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = E_P\left[\frac{dQ}{dP}\,\middle|\,\mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T.$$

Since Z^Q is closed on the right by $Z^Q_T = \frac{dQ}{dP}\big|_{\mathcal{F}_T}$, we can and do identify Z^Q with Q. (More precisely, Z^Q determines the restriction of Q to $\mathcal{F}_T$, but this will be enough for our purposes.)

The basic ingredients for our optimization problem are now

- parameters $\alpha, \alpha' \in [0, \infty)$ and $\beta \in (0, \infty)$;
- progressively measurable processes $\delta = (\delta_t)_{0 \le t \le T}$ and $U = (U_t)_{0 \le t \le T}$;
- an $\mathcal{F}_T$-measurable random variable U'_T.

Interpretations will follow presently. We define the discounting process

$$S^\delta_t := \exp\Big(-\int_0^t \delta_s\,ds\Big) \qquad , \qquad 0 \le t \le T,$$

the auxiliary quantities

$$\mathcal{U}^\delta_{t,T} := \alpha \int_t^T \frac{S^\delta_s}{S^\delta_t} U_s\, ds + \alpha' \frac{S^\delta_T}{S^\delta_t} U'_T \qquad , \qquad 0 \le t \le T,$$

$$\mathcal{R}^\delta_{t,T}(Q) := \int_t^T \delta_s \frac{S^\delta_s}{S^\delta_t} \log \frac{Z^Q_s}{Z^Q_t}\, ds + \frac{S^\delta_T}{S^\delta_t} \log \frac{Z^Q_T}{Z^Q_t} \qquad , \qquad 0 \le t \le T$$

for $Q \ll P$ on $\mathcal{F}_T$ and consider the cost functional

$$c(\omega, Q) := \mathcal{U}^\delta_{0,T}(\omega) + \beta \mathcal{R}^\delta_{0,T}(Q)(\omega).$$

The basic goal is to

(1.1) $\qquad$ minimize the functional $\quad Q \mapsto \Gamma(Q) := E_Q[c(\,\cdot\,, Q)]$

over a suitable class of probability measures $Q \ll P$ on $\mathcal{F}_T$. Note that in the language of the introduction, $\Gamma(Q)$ represents $\mathbf{U}(\pi, Q)$ for fixed π.

A closer look at the cost functional $c(\omega, Q)$ shows that

$$\Gamma(Q) = E_P\left[Z^Q_T\left(\alpha \int_0^T S^\delta_s U_s\, ds + \alpha' S^\delta_T U'_T\right)\right] + \beta E_P\left[\int_0^T \delta_s S^\delta_s Z^Q_s \log Z^Q_s\, ds + S^\delta_T Z^Q_T \log Z^Q_T\right] \tag{1.2}$$

consists of two terms. The first is a Q-expected discounted utility with discount rate δ, utility rate U_s at time s and terminal utility U'_T at time T. Usually, U_s comes from consumption and U'_T from final wealth. As explained above, we consider the strategy decisions π as being frozen for the moment; a maximization over some π determining $U(\pi)$ and $U'_T(\pi)$ will only be done in a second step. The weights α and α' can be used to obtain as special cases the extreme situations of utility rate only or terminal utility only. The second summand is a sort of discounted relative entropy term with both an "entropy rate" as well as a "terminal entropy". The (constant) factor β determines the strength of this penalty term.

Definition. $D^{\exp}_0$ is the space of all progressively measurable processes $y = (y_t)_{0 \le t \le T}$ with

$$E_P\left[\exp\left(\gamma \operatorname*{ess\,sup}_{0 \le t \le T} |y_t|\right)\right] < \infty \qquad \text{for all } \gamma > 0.$$

$D^{\exp}_1$ denotes the space of all progressively measurable processes $y = (y_t)_{0 \le t \le T}$ such that

$$E_P\left[\exp\left(\gamma \int_0^T |y_s|\, ds\right)\right] < \infty \qquad \text{for all } \gamma > 0.$$

Definition. For any probability measure Q on $(\Omega, \mathcal{F})$,

$$H(Q|P) := \begin{cases} E_Q\Big[\log \frac{dQ}{dP}\big|_{\mathcal{F}_T}\Big] & \text{if } Q \ll P \text{ on } \mathcal{F}_T \\ +\infty & \text{otherwise} \end{cases}$$

denotes the relative entropy of Q with respect to P on $\mathcal{F}_T$. We denote by $\mathcal{Q}_f$ the space of all probability measures Q on $(\Omega, \mathcal{F})$ with $Q \ll P$ on $\mathcal{F}_T$, $Q = P$ on $\mathcal{F}_0$ and $H(Q|P) < \infty$. Clearly $P \in \mathcal{Q}_f^e := \{Q \in \mathcal{Q}_f \,|\, Q \approx P \text{ on } \mathcal{F}_T\}$.

For a precise formulation of (1.1), we now assume

(A1) $0 \le \delta \le \|\delta\|_\infty < \infty$ for some constant $\|\delta\|_\infty$;
(A2) the process U is in $D_1^{\exp}$;
(A3) $E_P\left[\exp\left(\gamma|U'_T|\right)\right] < \infty$ for all $\gamma > 0$.

We shall see below that $E_Q[c(\,\cdot\,, Q)]$ is then well-defined and finite for $Q \in \mathcal{Q}_f$. Due to (A1), a simple estimation gives

$$E_P\big[S_T^\delta Z_T^Q \log Z_T^Q\big] \ge -e^{-1} + e^{-\|\delta\|_\infty T} H(Q|P).$$

Hence the second term in $\Gamma(Q)$ explodes unless $H(Q|P) < \infty$. Because we want to minimize $\Gamma(Q)$, this explains why we only consider measures Q in $\mathcal{Q}_f$.

Remark. The special case $\delta \equiv 0$ is much simpler and already gives a flavour of the results we obtain for general δ. In fact, $\delta \equiv 0$ yields $S^\delta \equiv 1$ and allows us to rewrite $\Gamma(Q)$ as

$$\Gamma(Q) = E_Q[\mathcal{U}^0_{0,T}] + \beta H(Q|P) = \beta H(Q|P_{\mathcal{U}}) - \beta \log E_P\big[\exp\big(-\tfrac{1}{\beta}\mathcal{U}^0_{0,T}\big)\big]$$

if we define a new probability measure $P_{\mathcal{U}} \approx P$ by

$$dP_{\mathcal{U}} := \text{const.} \exp\big(-\tfrac{1}{\beta}\mathcal{U}^0_{0,T}\big)\, dP.$$

Hence (1.1) amounts to minimizing the relative entropy of Q with respect to $P_{\mathcal{U}}$, and it is well known from Csiszár (1975) that there exists a unique solution Q^* to this problem and that Q^* is equivalent to $P_{\mathcal{U}}$, hence also to P. In fact, the minimizer obviously is $Q^* = P_{\mathcal{U}}$.

For $\delta \not\equiv 0$, we shall also find that there exists a unique minimizer Q^* of $\Gamma(Q)$ and that $Q^* \approx P$. However, it does not seem possible to reduce the general δ case to $\delta \equiv 0$ in a simple way. We remark that the presence of a discounting term with positive δ is indispensable for an infinite horizon version of (1.1); see Hansen et al. (2006) and forthcoming work by G. Bordigoni for more on this issue. ◇

We later embed the minimization of $\Gamma(Q)$ in a stochastic control problem and to that end now introduce a few more notations. Let $\mathcal{S}$ denote the set of all $I\!\!F$-stopping times τ with values in $[0,T]$ and $\mathcal{D}$ the space of all density processes Z^Q with $Q \in \mathcal{Q}_f$. We define

$$\mathcal{D}(Q,\tau) := \big\{ Z^{Q'} \in \mathcal{D} \,\big|\, Q' = Q \text{ on } \mathcal{F}_\tau \big\},$$
$$\Gamma(\tau,Q) := E_Q[c(\,\cdot\,,Q) \,|\, \mathcal{F}_\tau]$$

and the minimal conditional cost at time τ,

$$J(\tau,Q) := Q\text{ - }\operatorname*{ess\,inf}_{Z^{Q'} \in \mathcal{D}(Q,\tau)} \Gamma(\tau,Q').$$

Then (1.1) can be reformulated to

$$\text{(1.3)} \qquad \text{find} \quad \inf_{Q \in \mathcal{Q}_f} \Gamma(Q) = \inf_{Q \in \mathcal{Q}_f} E_Q[c(\,\cdot\,,Q)] = E_P[J(0,Q)]$$

by using the dynamic programming equation and the fact that $Q = P$ on $\mathcal{F}_0$ for every $Q \in \mathcal{Q}_f$.

2. Auxiliary estimates

In this section, we prove a number of auxiliary estimates that will help us later in establishing our main results. We frequently use the inequalities

$$\text{(2.1)} \qquad x \log x \geq -e^{-1} \qquad \text{for all } x \geq 0,$$
$$|x \log x| \leq x \log x + 2e^{-1} \qquad \text{for all } x \geq 0$$

(where we set $0 \log 0 := 0$) and

$$\text{(2.2)} \qquad xy \leq y \log y - y + e^x \qquad \text{for all } x \in I\!\!R, y \geq 0.$$

The latter is simply the observation that the function $x \mapsto xy - e^x$ on $I\!\!R$ takes its maximum for $y > 0$ in $x = \log y$. *Throughout this section, we assume that (A1) – (A3) hold.*

We first show that $\Gamma(Q)$ can be controlled by $H(Q|P)$.

Lemma 1. *There is a constant $C \in (0,\infty)$ depending only on $\alpha, \alpha', \beta, \delta, T, U, U'_T$ such that*

$$\Gamma(Q) \leq E_Q[|c(\,\cdot\,,Q)|] \leq C\big(1 + H(Q|P)\big) \qquad \textit{for all } Q \in \mathcal{Q}_f.$$

Proof. The first inequality is obvious. For the second, we set $R := \alpha \int_0^T |U_s|\,ds + \alpha' |U'_T|$ and use first the definition of $c(\omega,Q)$, the Bayes formula, (A1) and $0 \leq S^\delta \leq 1$, and then (2.1) and (2.2) to obtain

$$\begin{aligned}E_Q[|c(\,\cdot\,,Q)|] &\le E_P[Z_T^Q R] + \beta E_P\left[\|\delta\|_\infty \int_0^T |Z_s^Q \log Z_s^Q|\,ds + |Z_T^Q \log Z_T^Q|\right]\\ &\le E_P[Z_T^Q \log Z_T^Q - Z_T^Q + e^R]\\ &\quad + 2e^{-1}\beta(\|\delta\|_\infty T + 1) + \beta E_P\left[\|\delta\|_\infty \int_0^T Z_s^Q \log Z_s^Q\,ds + Z_T^Q \log Z_T^Q\right].\end{aligned}$$

By Jensen's inequality and conditioning on $\mathcal{F}_s$, we have

$$E_P[Z_s^Q \log Z_s^Q] \le E_P[Z_T^Q \log Z_T^Q] = H(Q|P)$$

and therefore

$$E_Q[|c(\,\cdot\,,Q)|] \le E_P[e^R] + 2e^{-1}\beta(\|\delta\|_\infty T + 1) + \big(1 + \beta(\|\delta\|_\infty T + 1)\big)H(Q|P).$$

Hence

$$C := \max\big(E_P[e^R] + 2e^{-1}\beta(\|\delta\|_\infty T + 1), 1 + \beta\|\delta\|_\infty T + \beta\big)$$

will do, and $C < \infty$ due to (A1) – (A3) and the definition of R. **q.e.d.**

An immediate but very useful consequence is

Corollary 2. *Assume (A1) – (A3). Then*

$$c(\,\cdot\,,Q) \in L^1(Q) \qquad \text{for every } Q \in \mathcal{Q}_f,$$

and in particular $\Gamma(Q)$ *is well-defined and finite for every* $Q \in \mathcal{Q}_f$.

Our next result now shows that conversely, $H(Q|P)$ can also be controlled by $\Gamma(Q)$. This is a bit more tricky and will be crucial later on. Note how the argument exploits almost the full strength of the integrability assumptions (A2) and (A3).

Proposition 3. *There is a constant* $C \in (0,\infty)$ *depending only on* $\alpha, \alpha', \beta, \delta, T, U, U_T'$ *with*

$$H(Q|P) \le C\big(1 + \Gamma(Q)\big) \qquad \text{for all } Q \in \mathcal{Q}_f. \tag{2.3}$$

In particular, $\inf_{Q \in \mathcal{Q}_f} \Gamma(Q) > -\infty$.

Proof. We first prove for later use an auxiliary inequality in somewhat greater generality. Fix a stopping time $\tau \in \mathcal{S}$. Using the Bayes formula, (A1), $0 \le S^\delta \le 1$ and (2.1) gives

$$\begin{aligned}E_Q\left[\int_0^T \delta_s S_s^\delta \log Z_s^Q\,ds \,\middle|\, \mathcal{F}_\tau\right] &= \int_0^\tau \delta_s S_s^\delta \log Z_s^Q\,ds + \frac{1}{Z_\tau^Q} E_P\left[\int_\tau^T \delta_s S_s^\delta Z_s^Q \log Z_s^Q\,ds \,\middle|\, \mathcal{F}_\tau\right]\\ &\ge \int_0^\tau \delta_s S_s^\delta \log Z_s^Q\,ds - \frac{1}{Z_\tau^Q}\|\delta\|_\infty T e^{-1}.\end{aligned}$$

Similarly, using $1 \geq S_T^\delta \geq e^{-\|\delta\|_\infty T}$ yields

$$\begin{aligned}
E_Q[S_T^\delta \log Z_T^Q \,|\, \mathcal{F}_\tau] &= \tfrac{1}{Z_\tau^Q} E_P[S_T^\delta Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau] \\
&\geq \tfrac{1}{Z_\tau^Q}\big(- e^{-1} + e^{-\|\delta\|_\infty T}(e^{-1} + E_P[Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau])\big) \\
&\geq \tfrac{1}{Z_\tau^Q}\big(- e^{-1} + e^{-\|\delta\|_\infty T} E_P[Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau]\big).
\end{aligned}$$

Moreover, using $0 \leq S^\delta \leq 1$ and again setting $R := \alpha \int_0^T |U_s|\, ds + \alpha' |U'_T|$ gives

$$E_Q[\mathcal{U}_{0,T}^\delta | \mathcal{F}_\tau] \geq -E_Q[R|\mathcal{F}_\tau] = -\tfrac{1}{Z_\tau^Q} E_P[Z_T^Q R \,|\, \mathcal{F}_\tau]$$

so that we get

$$\begin{aligned}
\Gamma(\tau, Q) \geq -\frac{1}{Z_\tau^Q}\Bigg(& E_P\bigg[Z_T^Q \Big(\alpha \int_0^T |U_s|\, ds + \alpha' |U'_T| \Big) \,\bigg|\, \mathcal{F}_\tau \bigg] \\
& + \beta\big(- \|\delta\|_\infty T e^{-1} - e^{-1} + e^{-\|\delta\|_\infty T} E_P[Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau]\big) \Bigg) \\
& + \beta \int_0^\tau \delta_s S_s^\delta \log Z_s^Q \, ds.
\end{aligned} \tag{2.4}$$

To estimate the first term in (2.4), we now use (2.2) with $x = \gamma R$, $y = \frac{1}{\gamma} Z_T^Q$ and $\gamma > 0$ to be chosen later. This yields

$$\begin{aligned}
E_P[Z_T^Q R \,|\, \mathcal{F}_\tau] &\leq E_P\Big[\tfrac{1}{\gamma} Z_T^Q \log Z_T^Q - \tfrac{1}{\gamma} Z_T^Q \log \gamma - \tfrac{1}{\gamma} Z_T^Q \,\Big|\, \mathcal{F}_\tau \Big] + E_P[e^{\gamma R} \,|\, \mathcal{F}_\tau] \\
&= \tfrac{1}{\gamma} E_P[Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau] - \tfrac{1}{\gamma}(\log \gamma + 1) Z_\tau^Q + E_P[e^{\gamma R} \,|\, \mathcal{F}_\tau].
\end{aligned}$$

We plug this into (2.4) to obtain for later use

$$\begin{aligned}
\Gamma(\tau, Q) \geq\ & \frac{1}{\gamma}(\log \gamma + 1) - \frac{1}{Z_\tau^Q} E_P\bigg[\exp\Big(\gamma\alpha \int_0^T |U_s|\, ds + \gamma\alpha' |U'_T| \Big) \,\bigg|\, \mathcal{F}_\tau \bigg] \\
& - \frac{1}{Z_\tau^Q} \beta e^{-1}(\|\delta\|_\infty T + 1) + \frac{1}{Z_\tau^Q} E_P[Z_T^Q \log Z_T^Q \,|\, \mathcal{F}_\tau] \Big(\beta e^{-\|\delta\|_\infty T} - \frac{1}{\gamma} \Big) \\
& + \beta \int_0^\tau \delta_s S_s^\delta \log Z_s^Q \, ds.
\end{aligned} \tag{2.5}$$

If we choose $\tau = 0$ and take expectations, this gives in particular

$$\Gamma(Q) \geq \tfrac{1}{\gamma}(\log \gamma + 1) - E_P[e^{\gamma R}] - \beta e^{-1}(\|\delta\|_\infty T + 1) + H(Q|P)\big(\beta e^{-\|\delta\|_\infty T} - \tfrac{1}{\gamma}\big).$$

For any $\gamma > 0$ such that $\beta e^{-\|\delta\|_\infty T} - \frac{1}{\gamma} \geq \eta > 0$, we thus obtain (2.3) with

$$C := \tfrac{1}{\eta} \max\big(1, E_P[e^{\gamma R}] + \tfrac{1}{\gamma}(|\log \gamma| + 1) + \beta e^{-1}(\|\delta\|_\infty T + 1)\big),$$

and $C < \infty$ due to (A1) – (A3) and the definition of R. The final assertion is clear since $H(Q|P) \geq 0$. **q.e.d.**

A slight modification in the proof of Proposition 3 also yields the following technical estimate.

Lemma 4. *For any $\gamma > 0$ and any set $A \in \mathcal{F}$, we have*

$$(2.6)\ E_Q[|\mathcal{U}^\delta_{0,T}| I_A] \leq \frac{1}{\gamma} H(Q|P) + \frac{1}{\gamma}(e^{-1} + |\log\gamma| + 1) + E_P\left[I_A \exp\left(\gamma\alpha \int_0^T |U_s|\, ds + \gamma\alpha' |U'_T| \right) \right].$$

Proof. We again use (2.2) with $x = \gamma R := \gamma\Big(\alpha \int_0^T S^\delta_s |U_s|\, ds + \alpha' S^\delta_T |U'_T|\Big)$, $y = \frac{1}{\gamma} Z^Q_T$ and then multiply by I_A to obtain

$$Z^Q_T |\mathcal{U}^\delta_{0,T}| I_A \leq Z^Q_T R I_A \leq I_A\big(\tfrac{1}{\gamma} Z^Q_T \log Z^Q_T - \tfrac{1}{\gamma} Z^Q_T (\log\gamma + 1) + e^{\gamma R} \big).$$

Adding e^{-1} and using (2.1) then yields

$$Z^Q_T |\mathcal{U}^\delta_{0,T}| I_A \leq \tfrac{1}{\gamma}(Z^Q_T \log Z^Q_T + e^{-1}) + Z^Q_T \tfrac{1}{\gamma}(|\log\gamma| + 1) + e^{\gamma R} I_A,$$

and (2.6) follows by taking expectations under P. **q.e.d.**

We later want to use the martingale optimality principle from stochastic control theory. Although we know from Corollary 2 that $c(\,\cdot\,, Q)$ is Q-integrable for every $Q \in \mathcal{Q}_f$, this is not enough since we have no uniformity in Q. Therefore we prove here directly that each $J(\tau, Q)$ is Q-integrable.

Lemma 5. *For each $\tau \in \mathcal{S}$ and $Q \in \mathcal{Q}_f$, the random variable $J(\tau, Q)$ is in $L^1(Q)$.*

Proof. By definition,

$$J(\tau, Q) \leq \Gamma(\tau, Q) \leq E_Q\big[|c(\,\cdot\,, Q)| \,\big|\, \mathcal{F}_\tau \big]$$

so that

$$\big(J(\tau, Q)\big)^+ \leq E_Q\big[|c(\,\cdot\,, Q)| \,\big|\, \mathcal{F}_\tau \big]$$

is Q-integrable by Corollary 2. Dealing with the negative part is a bit more delicate. We first fix $Z^{Q'} \in \mathcal{D}(Q, \tau)$ and consider $\Gamma(\tau, Q')$. Our goal is to find a Q-integrable lower bound for $\Gamma(\tau, Q')$ which does not depend on Q' because this will then also work for $J(\tau, Q) = \operatorname*{ess\,inf}_{Z^{Q'} \in \mathcal{D}(Q,\tau)} \Gamma(\tau, Q')$. To that end, we use (2.5) with Q' instead of Q and observe that $Z^{Q'} = Z^Q$

on $[\![0,\tau]\!]$ because $Q' = Q$ on $\mathcal{F}_\tau$. Choosing $\gamma > 0$ to satisfy $\beta e^{-\|\delta\|_\infty T} - \frac{1}{\gamma} = 0$ thus yields

$$\big(\Gamma(\tau,Q')\big)^- \le B := \frac{1}{\gamma}(|\log\gamma| + 1) + \frac{1}{Z^Q_\tau}\Bigg(E_P\bigg[\exp\Big(\gamma\alpha\int_0^T |U_s|\,ds + \gamma\alpha'|U'_T|\Big)\,\bigg|\,\mathcal{F}_\tau\bigg] + \beta e^{-1}(\|\delta\|_\infty T + 1)\Bigg) + \beta\int_0^\tau \delta_s S^\delta_s |\log Z^Q_s|\,ds.$$

But this nonnegative random variable does not depend on Q', and thus we conclude that

$$J(\tau,Q) = \operatorname*{ess\,inf}_{Z^{Q'}\in\mathcal{D}(Q,\tau)} \Gamma(\tau,Q') \ge \operatorname*{ess\,inf}_{Z^{Q'}\in\mathcal{D}(Q,\tau)} -\big(\Gamma(\tau,Q')\big)^- \ge -B$$

so that $\big(J(\tau,Q)\big)^- \le B$. Finally, $B \in L^1(Q)$ because (A1) – (A3) and $Q \in \mathcal{Q}_f$ yield that

$$E_Q[B] \le \frac{1}{\gamma}(|\log\gamma| + 1) + E_P\bigg[\exp\Big(\gamma\alpha\int_0^T |U_s|\,ds + \gamma\alpha'|U'_T|\Big)\bigg] + \beta e^{-1}(\|\delta\|_\infty T + 1) + \beta E_P\bigg[\int_0^T \delta_s S^\delta_s |\log Z^Q_s|\,ds\bigg] < \infty;$$

in fact, the last summand is at most $\beta\|\delta\|_\infty T\big(H(Q|P) + 2e^{-1}\big)$ by the same computation as in the proof of Lemma 1. **q.e.d.**

3. Existence of an optimal measure Q^*

The main result of this section is that the problem (1.1) of minimizing $\Gamma(Q) = E_Q[c(\,\cdot\,,Q)]$ over $Q \in \mathcal{Q}_f$ has a unique solution $Q^* \in \mathcal{Q}_f$, and that Q^* is even equivalent to P. This is proved for a general filtration $I\!\!F$.

Theorem 6. *Assume (A1) – (A3). Then there exists a unique $Q^* \in \mathcal{Q}_f$ which minimizes $Q \mapsto \Gamma(Q)$ over all $Q \in \mathcal{Q}_f$.*

Proof. 1) $x \mapsto x\log x$ is strictly convex and δ and S^δ are nonnegative; hence $Q \mapsto \Gamma(Q)$ is also strictly convex and Q^* must be unique if it exists.

2) Let $(Q^n)_{n\in I\!\!N}$ be a sequence in $\mathcal{Q}_f$ such that

$$\searrow - \lim_{n\to\infty} \Gamma(Q^n) = \inf_{Q\in\mathcal{Q}_f} \Gamma(Q) > -\infty$$

and denote by $Z^n = Z^{Q^n}$ the corresponding density processes. Since each $Z^n_T \ge 0$, it follows from Komlós' theorem that there exists a sequence $(\bar Z^n_T)_{n\in I\!\!N}$ with $\bar Z^n_T \in \operatorname{conv}(Z^n_T, Z^{n+1}_T, \dots)$

for each $n \in I\!N$ and such that $(\bar{Z}^n_T)$ converges P-a.s. to some random variable $\bar{Z}^\infty_T$, which is then also nonnegative but may take the value $+\infty$. Because $\mathcal{Q}_f$ is convex, each $\bar{Z}^n_T$ is again associated to some $\bar{Q}^n \in \mathcal{Q}_f$. We claim that this also holds for $\bar{Z}^\infty_T$, i.e., that $d\bar{Q}^\infty := \bar{Z}^\infty_T \, dP$ defines a probability measure $\bar{Q}^\infty \in \mathcal{Q}_f$. To see this, note first that we have

$$\Gamma(\bar{Q}^n) \le \sup_{m \ge n} \Gamma(Q^m) \le \Gamma(Q^n) \le \Gamma(Q^1) \tag{3.1}$$

because $Q \mapsto \Gamma(Q)$ is convex and $n \mapsto \Gamma(Q^n)$ is decreasing. Hence Proposition 3 yields

$$\sup_{n \in I\!N} E_P[\bar{Z}^n_T \log \bar{Z}^n_T] = \sup_{n \in I\!N} H(\bar{Q}^n|P) \le C\big(1 + \sup_{n \in I\!N} \Gamma(\bar{Q}^n)\big) \le C\big(1 + \Gamma(Q^1)\big) < \infty. \tag{3.2}$$

Thus $(\bar{Z}^n_T)_{n \in I\!N}$ is P-uniformly integrable by de la Vallée-Poussin's criterion and therefore converges in $L^1(P)$ as well. This implies that $E_P[\bar{Z}^\infty_T] = \lim\limits_{n \to \infty} E_P[\bar{Z}^n_T] = 1$ so that $\bar{Q}^\infty$ is indeed a probability measure and $\bar{Q}^\infty \ll P$ on $\mathcal{F}_T$. Because $x \mapsto x \log x$ is bounded below by $-e^{-1}$, Fatou's lemma and (3.2) yield

$$H(\bar{Q}^\infty|P) = E_P[\bar{Z}^\infty_T \log \bar{Z}^\infty_T] \le \liminf_{n \to \infty} E_P[\bar{Z}^n_T \log \bar{Z}^n_T] < \infty. \tag{3.3}$$

Finally, we also have $\bar{Q}^\infty = P$ on $\mathcal{F}_0$; in fact, $(\bar{Z}^n_T)$ converges to $\bar{Z}^\infty_T$ strongly, hence also weakly in $L^1(P)$ and so we have for every $A \in \mathcal{F}_0$

$$\bar{Q}^\infty[A] = E_P[\bar{Z}^\infty_T I_A] = \lim_{n \to \infty} E_P[\bar{Z}^n_T I_A] = \lim_{n \to \infty} \bar{Q}^n[A] = P[A]$$

since all the $\bar{Q}^n$ are in $\mathcal{Q}_f$ and hence agree with P on $\mathcal{F}_0$. This shows that $\bar{Q}^\infty \in \mathcal{Q}_f$.

3) We now want to show that $Q^* := \bar{Q}^\infty$ attains the infimum of $Q \mapsto \Gamma(Q)$ in $\mathcal{Q}_f$ and therefore examine $\Gamma(\bar{Q}^\infty)$ more closely. Let $\bar{Z}^\infty$ be the density process of $\bar{Q}^\infty$ with respect to P. Because we know that $(\bar{Z}^n_T)$ converges to $\bar{Z}^\infty_T$ in $L^1(P)$, Doob's maximal inequality

$$P\bigg[\sup_{0 \le t \le T} |\bar{Z}^\infty_t - \bar{Z}^n_t| \ge \varepsilon\bigg] \le \frac{1}{\varepsilon} E_P\big[|\bar{Z}^\infty_T - \bar{Z}^n_T|\big]$$

implies that $\Big(\sup\limits_{0 \le t \le T} |\bar{Z}^\infty_t - \bar{Z}^n_t|\Big)_{n \in I\!N}$ converges to 0 in P-probability. By passing to a subsequence that we still denote by $(\bar{Z}^n)_{n \in I\!N}$, we may thus assume that $(\bar{Z}^n_\bullet)$ converges to $\bar{Z}^\infty_\bullet$ uniformly in t with P-probability 1. This implies that

$$\bar{Z}^n_T \, c(\,\cdot\,, \bar{Q}^n) \longrightarrow \bar{Z}^\infty_T c(\,\cdot\,, \bar{Q}^\infty) \qquad P\text{-a.s.}$$

and in more detail with

$$\bar{Y}^n_1 := \bar{Z}^n_T \,\mathcal{U}^\delta_{0,T} \qquad , \qquad \bar{Y}^n_2 := \beta\Big(\int_0^T \delta_s S^\delta_s \bar{Z}^n_s \log \bar{Z}^n_s \, ds + S^\delta_T \bar{Z}^n_T \log \bar{Z}^n_T\Big) = \beta \mathcal{R}^\delta_{0,T}(\bar{Q}^n)$$

for $n \in \mathbb{N} \cup \{\infty\}$ that

$$\lim_{n\to\infty} \bar{Y}_i^n = \bar{Y}_i^\infty \qquad P\text{-a.s. for } i = 1, 2.$$

Since $\bar{Y}_2^n$ is by (A1) like $x \log x$ bounded below, uniformly in n and ω, Fatou's lemma yields

$$E_P[\bar{Y}_2^\infty] \le \liminf_{n\to\infty} E_P[\bar{Y}_2^n]. \tag{3.4}$$

We prove below that we also have

$$E_P[\bar{Y}_1^\infty] \le \liminf_{n\to\infty} E_P[\bar{Y}_1^n]. \tag{3.5}$$

Adding (3.5) and (3.4) then yields by (3.1) that

$$\Gamma(\bar{Q}^\infty) = E_P[\bar{Y}_1^\infty + \bar{Y}_2^\infty] \le \liminf_{n\to\infty} \Gamma(\bar{Q}^n) \le \liminf_{n\to\infty} \Gamma(Q^n) \le \inf_{Q\in\mathcal{Q}_f} \Gamma(Q)$$

which proves that $\bar{Q}^\infty$ is indeed optimal.

4) Although $\bar{Y}_1^n$ is linear in $\bar{Z}_T^n$, it is more difficult to handle than $\bar{Y}_2^n$ because the factor $\mathcal{U}_{0,T}^\delta$ is not bounded. However, $\mathcal{U}_{0,T}^\delta$ and $R := \alpha \int_0^T |U_s|\, ds + \alpha' |U_T'|$ are still manageable thanks to the exponential integrability properties from (A2) and (A3); they imply that R is almost bounded in the sense that $e^{\gamma R} \in L^1(P)$ for all $\gamma > 0$. To exploit this, we set

$$\widetilde{R}_m := \mathcal{U}_{0,T}^\delta I_{\{\mathcal{U}_{0,T}^\delta \ge -m\}} \ge -m \qquad \text{for } m \in \mathbb{N}$$

so that we have for each $n \in \mathbb{N} \cup \{\infty\}$

$$\bar{Y}_1^n = \bar{Z}_T^n \mathcal{U}_{0,T}^\delta = \bar{Z}_T^n \widetilde{R}_m + \bar{Z}_T^n \mathcal{U}_{0,T}^\delta I_{\{\mathcal{U}_{0,T}^\delta < -m\}}.$$

Because $\widetilde{R}_m \ge -m$ and each $\bar{Z}_T^n$ has P-expectation 1, Fatou's lemma yields

$$E_P[\bar{Z}_T^\infty \widetilde{R}_m] = -m + E_P[\bar{Z}_T^\infty (\widetilde{R}_m + m)] \le \liminf_{n\to\infty} E_P[\bar{Z}_T^n \widetilde{R}_m]$$

and therefore by adding and subtracting $E_P\big[\bar{Z}_T^n \mathcal{U}_{0,T}^\delta I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big]$

$$\begin{aligned} E_P[\bar{Y}_1^\infty] &\le \liminf_{n\to\infty} E_P[\bar{Z}_T^n \widetilde{R}_m] + E_P\big[\bar{Z}_T^\infty \mathcal{U}_{0,T}^\delta I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big] \\ &\le \liminf_{n\to\infty} E_P[\bar{Y}_1^n] + 2 \sup_{n\in\mathbb{N}\cup\{\infty\}} E_P\big[\bar{Z}_T^n |\mathcal{U}_{0,T}^\delta| I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big]. \end{aligned}$$

Hence (3.5) will follow once we prove that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}\cup\{\infty\}} E_P\big[\bar{Z}_T^n |\mathcal{U}_{0,T}^\delta| I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big] = 0. \tag{3.6}$$

However, Lemma 4 yields for each $n \in \mathbb{N} \cup \{\infty\}$

$$E_P\big[\bar{Z}_T^n |\mathcal{U}_{0,T}^\delta| I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big] = E_{\bar{Q}^n}\big[|\mathcal{U}_{0,T}^\delta| I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big]$$
$$\leq \tfrac{1}{\gamma} H(\bar{Q}^n|P) + \tfrac{1}{\gamma}(e^{-1} + |\log\gamma| + 1) + E_P\big[I_{\{\mathcal{U}_{0,T}^\delta < -m\}} e^{\gamma R}\big]$$

and therefore by using (3.2) and (3.3)

$$\sup_{n \in \mathbb{N} \cup \{\infty\}} E_P\big[\bar{Z}_T^n |\mathcal{U}_{0,T}^\delta| I_{\{\mathcal{U}_{0,T}^\delta < -m\}}\big] \leq \tfrac{1}{\gamma}\big(C(1+\Gamma(Q^1)+e^{-1}+|\log\gamma|+1\big)+E_P\big[I_{\{\mathcal{U}_{0,T}^\delta < -m\}} e^{\gamma R}\big]$$

for each $\gamma > 0$. The first term on the right-hand side becomes arbitrarily small for γ large enough, and the second converges for each fixed γ to 0 as $m \to \infty$ by dominated convergence, due to the exponential integrability of R from (A1) – (A3). This proves (3.6) and completes the proof. **q.e.d.**

Remark. In abstract terms, the proof of Theorem 6 can morally be summarized as follows:

a) Use Komlós' theorem to produce a candidate $\bar{Q}^\infty$ for the optimal measure, where $\bar{Z}_T^\infty$ is a P-almost sure limit of convex combinations $\bar{Z}_T^n$ formed from a minimizing sequence $(Z_T^n)_{n \in \mathbb{N}}$.

b) View $\Gamma(Q)$ like in (1.2) as a function $g(Z^Q)$ defined on density processes Z^Q. Minimality of $\bar{Q}^\infty$ then follows by standard reasoning if g is convex and lower semicontinuous with respect to P-almost sure convergence of Z_T^Q.

While convexity of g is immediate, lower semicontinuity is not obvious at all. For the entropy term $\big($the second summand in (1.2)$\big)$, we can use Fatou's lemma, but we first need the convergence of the entire density process $Z_\bullet^Q$ and not only of its final value Z_T^Q. We have done this above by using $L^1(P)$-convergence of the final values, but this requires of course P-uniform integrability. Due to linearity in Z_T^Q, there is no convergence problem for the integrand of the first summand in (1.2); but we cannot use Fatou's lemma there since we have no uniform lower bound. The arguments in steps 3) and 4) of the above proof show that while g is probably not lower semicontinuous on all of $\mathcal{D}$ with respect to P-almost sure convergence of Z_T^Q, it is so along any sequence $\big(Z_T^{Q^n}\big)_{n \in \mathbb{N}}$ which is bounded in entropy in the sense that $\sup_{n \in \mathbb{N}} H(Q^n|P) < \infty$. Note that we exploit the full strength of the assumptions (A2) and (A3) because we need to let γ tend to ∞.

The above problems disappear if the utility terms U and U_T' are uniformly bounded below or if we have a uniform bound on $H(Q|P)$ for all measures Q we allow in the minimization problem. In Bordigoni (2005), this is for instance achieved by minimizing over a set $\tilde{\mathcal{Q}} \subseteq \mathcal{Q}_f$ which is convex and satisfies $\sup_{Q \in \tilde{\mathcal{Q}}} \big\| \frac{dQ}{dP} \big\|_{L^p(P)} < \infty$ for some $p > 1$. One major achievement of the present work is that it avoids such restrictive assumptions on U, U_T' and Q. ◇

Having established existence and uniqueness of an optimal Q^*, our next goal is to prove that Q^* is equivalent to P. This uses an adaptation of an argument by Frittelli (2000), and we start with an auxiliary result.

Lemma 7. *Suppose for $i = 0, 1$ that $Q^i \in \mathcal{Q}_f$ with density processes $Z^i = Z^{Q^i}$. Then*

$$\sup_{0 \le t \le T} E_P\big[(Z_t^1 \log Z_t^0)^+\big] \le 2 + e^{-1} + H(Q^1|P) < \infty. \tag{3.7}$$

Proof. This slightly sharpens a result obtained in the proof of Lemma 2.1 in Frittelli (2000). For completeness we give details. If we set $\psi(x) := x \log x$, $Z^x := xZ^1 + (1-x)Z^0$ and

$$H(x; t) := \tfrac{1}{x}\big(\psi(Z_t^x) - \psi(Z_t^0)\big) \qquad \text{for } x \in (0, 1] \text{ and } t \text{ fixed,} \tag{3.8}$$

the random function $x \mapsto H(x; t)$ is increasing because ψ is convex, and so

$$H(1; t) \ge \lim_{x \searrow 0} \frac{\psi(Z_t^x) - \psi(Z_t^0)}{x} = \frac{d}{dx}\psi(Z_t^x)\Big|_{x=0} = \psi'(Z_t^0)(Z_t^1 - Z_t^0) = (\log Z_t^0 + 1)(Z_t^1 - Z_t^0).$$

Rearranging terms gives

$$Z_t^1 \log Z_t^0 \le \psi(Z_t^1) - \psi(Z_t^0) + Z_t^0 \log Z_t^0 + Z_t^0 - Z_t^1 \le \psi(Z_t^1) + e^{-1} + Z_t^0 + Z_t^1, \tag{3.9}$$

and the right-hand side is by (2.1) nonnegative with

$$E_P[\psi(Z_t^1)] \le E_P[\psi(Z_T^1)] = H(Q^1|P)$$

by Jensen's inequality. Hence (3.7) follows from (3.9). **q.e.d.**

Now we are ready to prove the second main result of this section.

Theorem 8. *Assume (A1) – (A3). Then the optimal measure Q^* from Theorem 6 is equivalent to P.*

Proof. 1) As in the proof of Lemma 7, we take $Q^0, Q^1 \in \mathcal{Q}_f$, set $Q^x := xQ^1 + (1-x)Q^0$ for $x \in [0, 1]$ and denote by Z^x the density process of Q^x with respect to P. With $\psi(x) = x \log x$ and H as in (3.8), we then obtain

$$\begin{aligned}
\tfrac{1}{x}\big(\Gamma(Q^x) - \Gamma(Q^0)\big) &= E_P[(Z_T^1 - Z_T^0)\mathcal{U}_{0,T}^\delta] \\
&\quad + \frac{1}{x}\beta E_P\bigg[\int_0^T \delta_s S_s^\delta\big(\psi(Z_s^x) - \psi(Z_s^0)\big)\,ds + S_T^\delta\big(\psi(Z_T^x) - \psi(Z_T^0)\big)\bigg] \\
&= E_P[(Z_T^1 - Z_T^0)\mathcal{U}_{0,T}^\delta] + \beta E_P\bigg[\int_0^T \delta_s S_s^\delta H(x; s)\,ds + S_T^\delta H(x; T)\bigg].
\end{aligned}$$

As x decreases to 0, $H(x;s)$ decreases as in the proof of Lemma 7 to $(\log Z^0_s + 1)(Z^1_s - Z^0_s)$, and

$$H(x;s) \le H(1;s) = \psi(Z^1_s) - \psi(Z^0_s) \le \psi(Z^1_s) + e^{-1}$$

shows that we have an integrable upper bound. Hence we can use monotone convergence to conclude that

$$\begin{aligned}(3.10)\quad \frac{d}{dx}\Gamma(Q^x)\Big|_{x=0} &= E_P[(Z^1_T - Z^0_T)\mathcal{U}^\delta_{0,T}] \\ &\quad + \beta E_P\left[\int_0^T \delta_s S^\delta_s (\log Z^0_s + 1)(Z^1_s - Z^0_s)\,ds + S^\delta_T(\log Z^0_T + 1)(Z^1_T - Z^0_T)\right] \\ &=: E_P[Y_1] + E_P[Y_2].\end{aligned}$$

As in the proof of Lemma 1, (A1) – (A3) imply that $Y_1 \in L^1(P)$, and since $x \mapsto H(x;s)$ is increasing, Y_2 is majorized by

$$\int_0^T \delta_s S^\delta_s H(1;s)\,ds + S^\delta_T H(1;T) \le \int_0^T \delta_s S^\delta_s \big(\psi(Z^1_s) + e^{-1}\big)\,ds + S^\delta_T\big(\psi(Z^1_T) + e^{-1}\big)$$

which is P-integrable because $Q^1 \in \mathcal{Q}_f$. Hence $Y_2^+ \in L^1(P)$ and so the right-hand side of (3.10) is well-defined in $[-\infty, +\infty)$.

2) Now take $Q^0 = Q^*$ and any $Q^1 \in \mathcal{Q}_f$ which is equivalent to P; this is possible since $\mathcal{Q}_f$ contains P. The optimality of Q^* yields $\Gamma(Q^x) - \Gamma(Q^*) \ge 0$ for all $x \in (0,1]$, hence also

$$(3.11)\qquad \frac{d}{dx}\Gamma(Q^x)\Big|_{x=0} \ge 0.$$

Therefore the right-hand side of (3.10) is nonnegative which implies that Y_2 must be in $L^1(P)$. This allows us to rearrange terms and rewrite (3.11) by using (3.10) as

$$\begin{aligned}\beta E_P\left[\int_0^T \delta_s S^\delta_s Z^1_s \log Z^*_s\,ds + S^\delta_T Z^1_T \log Z^*_T\right] &\ge -E_P[(Z^1_T - Z^*_T)\mathcal{U}^\delta_{0,T}] \\ (3.12)\qquad &\quad + \beta E_P\left[\int_0^T \delta_s S^\delta_s Z^*_s \log Z^*_s\,ds + S^\delta_T Z^*_T \log Z^*_T\right] \\ &\quad - \beta E_P\left[(Z^1_T - Z^*_T)\int_0^T \delta_s S^\delta_s\,ds + S^\delta_T\right].\end{aligned}$$

But the right-hand side of (3.12) is $> -\infty$ and the first term on the left-hand side is $< +\infty$ due to (A1) and Lemma 7. Moreover, (A1) gives $S^\delta_T \ge e^{-\|\delta\|_\infty T} > 0$. So if we have $Q^* \not\approx P$, we get $(\log Z^*_T)^- = \infty$ on the set $A := \{Z^*_T = 0\}$ and $P[A] > 0$. This gives $(Z^1_T \log Z^*_T)^- = \infty$ on A because $Z^1_T > 0$ since $Q^1 \approx P$. But since we know from Lemma 7 that $(Z^1_T \log Z^*_T)^+ \in L^1(P)$, we then conclude that $E_P[S^\delta_T Z^1_T \log Z^*_T] = -\infty$, and this gives a contradiction to (3.12). Therefore $Q^* \approx P$. **q.e.d.**

4. A BSDE description for the dynamic value process

In this section, we use stochastic control techniques to study the dynamic value process V associated to the optimization problem (1.1) or (1.3). We show that V is the unique solution of a backward stochastic differential equation (BSDE) with a quadratic driver, if the underlying filtration is continuous. This extends earlier work by Skiadas (2003), Schroder/Skiadas (1999) and Lazrak/Quenez (2003).

We first recall from Section 1 the conditional cost $\Gamma(\tau, Q) = E_Q[c(\,\cdot\,, Q) \,|\, \mathcal{F}_\tau]$ and the minimal conditional cost

$$J(\tau, Q) = Q\text{ - }\operatorname*{ess\,inf}_{Z^{Q'} \in \mathcal{D}(Q,\tau)} \Gamma(\tau, Q') \qquad \text{for } \tau \in \mathcal{S} \text{ and } Q \in \mathcal{Q}_f.$$

A measure $\tilde{Q} \in \mathcal{Q}_f$ is called *optimal* if it minimizes $Q \mapsto \Gamma(Q) = E_Q[c(\,\cdot\,, Q)]$ over $Q \in \mathcal{Q}_f$. Then we have the following *martingale optimality principle* from stochastic control.

Proposition 9. *Assume (A1) – (A3). Then:*

1) *The family $\{J(\tau, Q) \,|\, \tau \in \mathcal{S}, Q \in \mathcal{Q}_f\}$ is a submartingale system; this implies that for any $Q \in \mathcal{Q}_f$, we have for any stopping times $\sigma \le \tau$ the Q-submartingale property*

$$E_Q[J(\tau, Q) \,|\, \mathcal{F}_\sigma] \ge J(\sigma, Q) \qquad Q\text{-a.s.} \tag{4.1}$$

2) *$\tilde{Q} \in \mathcal{Q}_f$ is optimal if and only if $\{J(\tau, \tilde{Q}) \,|\, \tau \in \mathcal{S}\}$ is a $\tilde{Q}$-martingale system; this means that instead of (4.1), we have for any stopping times $\sigma \le \tau$*

$$E_Q[J(\tau, \tilde{Q}) \,|\, \mathcal{F}_\sigma] = J(\sigma, \tilde{Q}) \qquad \tilde{Q}\text{-a.s.}$$

3) *For each $Q \in \mathcal{Q}_f$, there exists an adapted RCLL process $J^Q = (J^Q_t)_{0 \le t \le T}$ which is a right closed Q-submartingale such that*

$$J^Q_\tau = J(\tau, Q) \qquad Q\text{-a.s. for each stopping time } \tau.$$

Proof. This is almost a direct consequence of Theorems 1.15 (for 1)), 1.17 (for 2)) and 1.21 (for 3)) in El Karoui (1981). It is straightforward (but a little tedious; see Bordigoni (2005) for details) to check that our control problem satisfies all the assumptions required for these results, with just one exception; we have neither $c \ge 0$ nor $\inf_{Z^{Q'} \in \mathcal{D}(Q,\tau)} E_{Q'}[|c(\,\cdot\,, Q')|] < \infty$ for all $\tau \in \mathcal{S}$ and $Q \in \mathcal{Q}_f$ as required in El Karoui (1981). However, closer inspection of the proofs in El Karoui (1981) shows that all the assertions there still hold true if one can show that $E_Q[|J(\tau, Q)|] < \infty$ for each $Q \in \mathcal{Q}_f$ and $\tau \in \mathcal{S}$. Because we have proved this in Lemma 5, our assertion follows. **q.e.d.**

We already know from Theorem 6 that there exists an optimal $Q^* \in \mathcal{Q}_f$, and we even have $Q^* \in \mathcal{Q}^e_f$ by Theorem 8. Hence we may equally well minimize $Q \mapsto \Gamma(Q)$ only over $Q \in \mathcal{Q}^e_f$ without losing any generality. For each $Q \in \mathcal{Q}^e_f$ and $\tau \in \mathcal{S}$, we now define

$$\tilde{V}(\tau, Q') := E_{Q'}[\mathcal{U}^\delta_{\tau,T} | \mathcal{F}_\tau] + \beta E_{Q'}[\mathcal{R}^\delta_{\tau,T}(Q') | \mathcal{F}_\tau]$$

and

$$V(\tau, Q) := Q\text{ - }\operatorname*{ess\,inf}_{Z^{Q'} \in \mathcal{D}(Q,\tau)} \tilde{V}(\tau, Q').$$

The latter is the *value* of the control problem started at time τ instead of 0 and assuming one has used the model Q up to time τ. By using the Bayes formula and the definition of $\mathcal{R}^\delta_{\tau,T}(Q')$, one easily sees that each $\tilde{V}(\tau, Q')$ depends only on the values of $Z^{Q'}$ on $]\!]\tau, T]\!]$ and therefore not on Q, since $Z^{Q'} \in \mathcal{D}(Q, \tau)$ only says that $Z^{Q'} = Z^Q$ on $[\![0, \tau]\!]$. So we can equally well take the ess inf under $P \approx Q$ and over all $Q' \in \mathcal{Q}_f$ and call the result $V(\tau)$ since it does not depend on $Q \in \mathcal{Q}^e_f$.

From the definition of $\mathcal{R}^\delta_{\tau,T}(Q')$, we have for Q' with $Z^{Q'} \in \mathcal{D}(Q, \tau)$ that

$$\begin{aligned}
\mathcal{R}^\delta_{0,T}(Q') &= \int_0^\tau \delta_s S^\delta_s \log Z^{Q'}_s \, ds + S^\delta_\tau \mathcal{R}^\delta_{\tau,T}(Q') + \Big(\int_\tau^T \delta_s S^\delta_s \, ds + S^\delta_T \Big) \log Z^{Q'}_\tau \\
&= S^\delta_\tau \mathcal{R}^\delta_{\tau,T}(Q') + \int_0^\tau \delta_s S^\delta_s \log Z^Q_s \, ds + S^\delta_\tau \log Z^Q_\tau.
\end{aligned}$$

Comparing the definitions of $V(\tau) = V(\tau, Q)$ and $J(\tau, Q)$ therefore yields for $Q \in \mathcal{Q}^e_f$

$$J(\tau, Q) = S^\delta_\tau V(\tau) + \alpha \int_0^\tau S^\delta_s U_s \, ds + \beta \int_0^\tau \delta_s S^\delta_s \log Z^Q_s \, ds + \beta S^\delta_\tau \log Z^Q_\tau,$$

because we can also take the ess inf for $J(\tau, Q)$ under $P \approx Q$. Since each $J(\,\cdot\,, Q)$ admits an RCLL version by Proposition 9, we can choose an adapted RCLL process $V = (V_t)_{0 \le t \le T}$ such that

$$V_\tau = V(\tau) = V(\tau, Q) \qquad P\text{-a.s. for } \tau \in \mathcal{S} \text{ and } Q \in \mathcal{Q}^e_f,$$

and then we have for each $Q \in \mathcal{Q}^e_f$

$$J^Q = S^\delta V + \alpha \int S^\delta_s U_s \, ds + \beta \int \delta_s S^\delta_s \log Z^Q_s \, ds + \beta S^\delta \log Z^Q. \tag{4.2}$$

As P is in $\mathcal{Q}^e_f$ and J^P is a P-submartingale by Proposition 9, (4.2) yields via $J^P = S^\delta V + \alpha \int S^\delta_s U_s \, ds$ that V is a P-special semimartingale. We write its canonical decomposition as

$$V = V_0 + M^V + A^V$$

and want to know more about M^V and A^V. Since S^δ is uniformly bounded from below and J^P is a P-submartingale, (A2) implies that M^V is a P-martingale. In a continuous filtration, we even obtain much stronger results a bit later.

Consider now the semimartingale backward equation

$$\begin{aligned} dY_t &= (\delta_t Y_t - \alpha U_t)\, dt + \frac{1}{2\beta} d\langle M\rangle_t + dM_t, \\ Y_T &= \alpha' U'_T. \end{aligned} \tag{4.3}$$

A *solution* of (4.3) is a pair (Y, M) satisfying (4.3), where Y is a P-semimartingale and M is a locally square-integrable local P-martingale null at 0. Note that Y is then automatically P-special, and that if M is continuous, so is Y.

Remark. Like the optimization problem (1.1), the BSDE (4.3) becomes much simpler when $\delta \equiv 0$; in fact, one can explicitly write down its solution. This has already been observed in Schroder/Skiadas (1999), and we come back to this point at the end of this section. ◇

Our main result in this section shows that (V, M^V) is the unique solution of (4.3) if the filtration $I\!\!F$ is continuous. As a preliminary, we first establish some auxiliary results about the structure and uniqueness of solutions to (4.3).

Lemma 10. *Assume (A1), (A2) and let (Y, M) be a solution of (4.3) with M continuous. Assume either $Y \in D_0^{\exp}$ or that $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is a true P-martingale. For any pair of stopping times $\sigma \le \tau$, we then have the recursive relation*

$$Y_\sigma = -\beta \log E_P\left[\exp\left(\tfrac{1}{\beta}\int_\sigma^\tau (\delta_s Y_s - \alpha U_s)\, ds - \tfrac{1}{\beta} Y_\tau\right) \,\middle|\, \mathcal{F}_\sigma\right]. \tag{4.4}$$

Proof. From (4.3), we have

$$Y_\tau - Y_\sigma = \int_\sigma^\tau dY_s = \int_\sigma^\tau (\delta_s Y_s - \alpha U_s)\, ds + M_\tau - M_\sigma + \tfrac{1}{2\beta}\big(\langle M\rangle_\tau - \langle M\rangle_\sigma\big).$$

Divide by $-\beta$, exponentiate and use continuity of M to obtain

$$\frac{\mathcal{E}\big(-\frac{1}{\beta}M\big)_\tau}{\mathcal{E}\big(-\frac{1}{\beta}M\big)_\sigma} = \exp\left(\tfrac{1}{\beta} Y_\sigma + \tfrac{1}{\beta}\int_\sigma^\tau (\delta_s Y_s - \alpha U_s)\, ds - \tfrac{1}{\beta} Y_\tau\right). \tag{4.5}$$

If $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is a P-martingale, (4.4) follows directly by conditioning on $\mathcal{F}_\sigma$ and solving for Y_σ. In general, we stop $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ after σ by τ_n to have the P-martingale property and thus obtain (4.5) and (4.4) with $\tau_n \wedge \tau$ instead of τ. Then (A1), (A2) and the assumption that

$Y \in D_0^{\exp}$ yield a P-integrable majorant for the right-hand side of (4.5) and so we can use dominated convergence to let $n \to \infty$ and again get (4.4) for τ. **q.e.d.**

The argument for the next result is a simple adaptation of the proof for Lemma A2 in Schroder/Skiadas (1999).

Lemma 11. **1)** *For any semimartingale Y, there is at most one local P-martingale M such that (Y, M) solves (4.3).*

2) *Assume (A1), (A2). Then (4.3) has at most one solution (Y, M) with $Y \in D_0^{\exp}$ and M continuous.*

Proof. 1) For any solution (Y, M) of (4.3), Y is P-special, and its unique local P-martingale part is M by (4.3).

2) Let (Y, M) and $(\tilde{Y}, \tilde{M})$ be two solutions as stated. Suppose that for some $t \in [0, T]$, the event $A := \{Y_t > \tilde{Y}_t\}$ has $P[A] > 0$. Since $Y_T = \alpha' U'_T = \tilde{Y}_T$, the stopping time $\tau := \inf\{s \geq t \,|\, Y_s \leq \tilde{Y}_s\}$ has values in $[t, T]$, and since $Y, \tilde{Y}$ are both continuous, we have $Y_\tau = \tilde{Y}_\tau$ on A and $Y_s > \tilde{Y}_s$ on $A \cap \{t \leq s < \tau\}$. This implies that

$$\int_t^\tau (\delta_s Y_s - \alpha U_s)\, ds - Y_\tau > \int_t^\tau (\delta_s \tilde{Y}_s - \alpha U_s)\, ds - \tilde{Y}_\tau \qquad \text{on } A \in \mathcal{F}_t$$

so that Lemma 10 yields

$$\exp\big(-\tfrac{1}{\beta} Y_t\big) = E_P\left[\exp\Big(\tfrac{1}{\beta}\int_t^\tau (\delta_s Y_s - \alpha U_s)\, ds - \tfrac{1}{\beta} Y_\tau\Big) \,\Big|\, \mathcal{F}_t\right] > \exp\big(-\tfrac{1}{\beta}\tilde{Y}_t\big) \qquad \text{on } A.$$

Hence $Y_t < \tilde{Y}_t$ on A, in contradiction to the definition of A, and so Y and $\tilde{Y}$ must be indistinguishable. By part 1), M and $\tilde{M}$ must then coincide as well. **q.e.d.**

Armed with the above results, we can now prove the announced characterization of (V, M^V) as the unique solution of the generalized BSDE (4.3).

Theorem 12. *Assume (A1) – (A3). If $I\!\!F$ is continuous, the pair (V, M^V) is the unique solution in $D_0^{\exp} \times \mathcal{M}_{0,\mathrm{loc}}(P)$ of the BSDE*

$$dY_t = (\delta_t Y_t - \alpha U_t)\, dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \tag{4.3}$$

$$Y_T = \alpha' U'_T.$$

Moreover, $\mathcal{E}\big(-\frac{1}{\beta} M^V\big)$ is a true P-martingale.

Proof. 1) We first use the martingale optimality principle to show that (V, M^V) is indeed a solution. For each $Q \in \mathcal{Q}_f^e$, we have $Z^Q = \mathcal{E}(L^Q)$ for some continuous local P-martingale L^Q

null at 0 since $Q = P$ on $\mathcal{F}_0$. This implies that $d(\log Z^Q) = dL^Q - \frac{1}{2}\, d\langle L^Q\rangle$, and combining this with Itô's formula applied to (4.2) yields

$$dJ^Q = S^\delta(dM^V + dA^V) - \delta S^\delta V\, dt + \alpha S^\delta U\, dt + \beta S^\delta\big(dL^Q - \tfrac{1}{2}\, d\langle L^Q\rangle\big). \tag{4.6}$$

By Girsanov's theorem,

$$N^Q := M^V + \beta L^Q - \langle M^V + \beta L^Q, L^Q\rangle$$

is a local Q-martingale. Together with (4.6), this gives the Q-canonical decomposition

$$dJ^Q = S^\delta\, dN^Q + S^\delta\big(dA^V - \delta V\, dt + \alpha U\, dt + d\langle M^V, L^Q\rangle + \beta\, d\langle L^Q\rangle - \tfrac{\beta}{2}\, d\langle L^Q\rangle\big). \tag{4.7}$$

Because J^Q is by Proposition 9 a Q-submartingale for any $Q \in \mathcal{Q}_f^e$ and a Q^*-martingale for the optimal Q^* (which exists and is in $\mathcal{Q}_f^e$ by Theorem 6 and Theorem 8), the second term in (4.7) is increasing for any $Q \in \mathcal{Q}_f^e$ and constant (at 0) for $Q = Q^*$. Thus we have

$$A^V = \int (\delta V - \alpha U)\, dt - \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_f^e} \big(\langle M^V, L^Q\rangle + \tfrac{\beta}{2}\langle L^Q\rangle\big),$$

where the ess inf is taken with respect to the strong order $\preceq$ (so that $A \preceq B$ means that $B - A$ is increasing). Step 2) shows that the ess inf term equals $-\frac{1}{2\beta}\langle M^V\rangle$ so that we get

$$dV_t = dM_t^V + dA_t^V = (\delta_t V_t - \alpha U_t)\, dt + \tfrac{1}{2\beta}\, d\langle M^V\rangle_t + dM_t^V.$$

Since clearly from the definitions $V_T = V(T, Q) = \alpha' U_T'$, (4.3) follows with $M = M^V$.

2) We claim that

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_f^e} \big(\langle M^V, L^Q\rangle + \tfrac{\beta}{2}\langle L^Q\rangle\big) = -\tfrac{1}{2\beta}\langle M^V\rangle, \tag{4.8}$$

and that the ess inf is attained for $L^{Q^*} = -\frac{1}{\beta} M^V$. To prove this, choose localizing stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that for $L^n := -\frac{1}{\beta}(M^V)^{\tau_n}$, the process $Z^n := \mathcal{E}(L^n)$ is bounded. Then Z^n is a strictly positive P-martingale starting at 1 with $E_P[|Z_T^n \log Z_T^n|] < \infty$ so that $dQ^n := Z_T^n\, dP$ defines an element $Q^n \in \mathcal{Q}_f^e$. Moreover, the definition of L^n gives

$$\langle M^V, L^n\rangle_t + \tfrac{\beta}{2}\langle L^n\rangle_t = -\tfrac{1}{\beta}\langle M^V\rangle_{t\wedge\tau_n} + \tfrac{1}{2\beta}\langle M^V\rangle_{t\wedge\tau_n}$$

and so we get for $n \to \infty$ that

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_f^e} \big(\langle M^V, L^Q\rangle + \tfrac{\beta}{2}\langle L^Q\rangle\big) \le \lim_{n\to\infty} \big(\langle M^V, L^n\rangle + \tfrac{\beta}{2}\langle L^n\rangle\big) = -\tfrac{1}{2\beta}\langle M^V\rangle.$$

Because we also have

$$\langle M^V, L^Q\rangle + \tfrac{\beta}{2}\langle L^Q\rangle = \tfrac{\beta}{2}\big\langle L^Q + \tfrac{1}{\beta}M^V\big\rangle - \tfrac{1}{2\beta}\langle M^V\rangle \ge -\tfrac{1}{2\beta}\langle M^V\rangle, \tag{4.9}$$

(4.8) follows. Finally, since the ess inf in (4.8) is attained by Q^* due to Proposition 9, combining (4.8) with (4.9) for $Q = Q^*$ yields

$$-\tfrac{1}{2\beta}\langle M^V\rangle = \big(\langle M^V, L^{Q^*}\rangle + \tfrac{\beta}{2}\langle L^{Q^*}\rangle\big) = \tfrac{\beta}{2}\big\langle L^{Q^*} + \tfrac{1}{\beta}M^V\big\rangle - \tfrac{1}{2\beta}\langle M^V\rangle,$$

and this implies that $L^{Q^*} = -\frac{1}{\beta}M^V$.

3) By step 2), $\mathcal{E}\big(-\frac{1}{\beta}M^V\big) = \mathcal{E}\big(L^{Q^*}\big) = Z^{Q^*}$ is a true P-martingale.

4) Since $I\!\!F$ is continuous, so is M^V; hence uniqueness follows from Lemma 11 once we show that $V \in D_0^{\exp}$. This is done below in Proposition 15, and completes the proof. **q.e.d.**

A closer look at the proof of Theorem 12 shows that we have the following additional integrability property for the P-martingale M^V.

Corollary 13. *Assume (A1) – (A3). If $I\!\!F$ is continuous, the optimal measure Q^* is given by $Z^{Q^*} = \mathcal{E}\big(-\frac{1}{\beta}M^V\big)$, and $\mathcal{E}\big(-\frac{1}{\beta}M^V\big)$ is a P-martingale whose supremum is in $L^1(P)$.*

Proof. The first assertion is just step 3) from the preceding proof. Because $Q^* \in \mathcal{Q}_f^e$, we have $E_P\big[Z_T^{Q^*}\log Z_T^{Q^*}\big] = H(Q^*|P) < \infty$, and this implies that $\sup\limits_{0\le t\le T} Z_t^{Q^*}$ is in $L^1(P)$. **q.e.d.**

To finish the proof of Theorem 12, it remains to show that $V \in D_0^{\exp}$. We begin with

Lemma 14. *Assume (A1) – (A3). Then the process $(J^P)^+$ is in $D_0^{\exp}$.*

Proof. We have seen in the proof of Lemma 5 that

$$\big(J(\tau,P)\big)^+ \le E_P\big[|c(\,\cdot\,,P)|\,\big|\,\mathcal{F}_\tau\big].$$

Fix $\gamma > 0$ and choose an RCLL version of the P-martingale $N := E_P\big[e^{\gamma|c(\,\cdot\,,P)|}\,\big|\,I\!\!F\big]$. Then Proposition 9, right-continuity of J^P and Jensen's inequality imply that

$$\exp\Big(\gamma \operatorname*{ess\,sup}_{0\le t\le T}(J_t^P)^+\Big) = \exp\Big(\gamma \sup_{0\le t\le T}(J_t^P)^+\Big) \le \sup_{0\le t\le T} N_t. \tag{4.10}$$

Since $S^\delta \le 1$, we have $|c(\,\cdot\,,P)| = |\mathcal{U}_{0,T}^\delta| \le \alpha\int_0^T |U_s|\,ds + \alpha'|U_T'| =: R$, and since $e^{\gamma R} \in L^p(P)$ for every $p \in (1,\infty)$ by (A2) and (A3), Doob's inequality implies that $\sup\limits_{0\le t\le T} N_t$ is in $L^p(P)$ for every $p \in (1,\infty)$. Hence the assertion follows from (4.10). **q.e.d.**

We have already shown that (V, M^V) is a solution of (4.3) and that $\mathcal{E}\big(-\frac{1}{\beta}M^V\big)$ is a true P-martingale. This allows us now to use Lemma 10 and prove that V inherits the good integrability properties of U and U_T'.

Proposition 15. *Assume (A1) – (A3). If $I\!\!F$ is continuous, the process V is in $D_0^{\exp}$.*

Proof. Because $D_0^{\exp}$ is a vector space, it is enough to prove that V^+ and V^- lie both in it. Using (4.4) for V with $\sigma = t$, $\tau = T$ and Jensen's inequality gives

$$-V_t \ge E_P\left[\int_t^T (\delta_s V_s - \alpha U_s)\,ds - \alpha' U_T' \,\middle|\, \mathcal{F}_t\right]$$

and therefore

$$V_t^+ = V_t + V_t^- \le V_t^- + E_P\left[\|\delta\|_\infty T \sup_{0\le s\le T} V_s^- + \alpha\int_0^T |U_s|\,ds + \alpha' |U_T'| \,\middle|\, \mathcal{F}_t\right].$$

Due to (A2) and (A3), the same argument via Doob's inequality as in the proof of Lemma 14 shows that the last term is in $D_0^{\exp}$ as soon as V^- is, and this implies then in turn that V^+ is in $D_0^{\exp}$. Hence it only remains to prove that V^- is in $D_0^{\exp}$.

Now (4.2) for $Q = P$ gives

$$\delta_s V_s = \delta_s\Big(J_s^P - \alpha\int_0^s S_r^\delta U_r\,dr\Big)\Big/ S_s^\delta \le \|\delta\|_\infty \Big(\sup_{0\le t\le T} (J_t^P)^+ + \alpha\int_0^T |U_r|\,dr\Big) e^{\|\delta\|_\infty T},$$

and combining this with (4.4) for V with $\sigma = t$, $\tau = T$ yields

$$\begin{aligned}(4.11)\qquad -V_t &\le \beta\log\left(1 + E_P\left[\exp\Big(\tfrac{1}{\beta}\int_t^T(\delta_s V_s - \alpha U_s)\,ds - \tfrac{1}{\beta}\alpha' U_T'\Big)\,\middle|\,\mathcal{F}_t\right]\right)\\ &\le \beta\log\left(1 + E_P\left[\exp\Big(\tfrac{1}{\beta}\|\delta\|_\infty e^{\|\delta\|_\infty T}\Big(\sup_{0\le t\le T}(J_t^P)^+ + \alpha\int_0^T |U_r|\,dr\Big)\right.\right.\\ &\qquad\left.\left. + \tfrac{1}{\beta}\alpha\int_0^T |U_s|\,ds + \tfrac{1}{\beta}\alpha' |U_T'|\Big)\,\middle|\,\mathcal{F}_t\right]\right)\\ &=: \beta\log\big(1 + E_P[e^B|\mathcal{F}_t]\big).\end{aligned}$$

Due to (A2), (A3) and Lemma 14, the random variable B satisfies $E_P[e^{\gamma|B|}] < \infty$ for all $\gamma > 0$. Hence the martingale $E_P[e^B|I\!\!F]$ has its supremum in $L^p(P)$ for every $p \in (1,\infty)$ by Doob's inequality, and this implies by (4.11) that V^- is in $D_0^{\exp}$. **q.e.d.**

Remarks. 1) The above argument needs continuity of $I\!\!F$ because we exploit via Lemma 10 the BSDE for V. However, one feels that the integrability of V should be a general result, and this raises the question if there is an alternative proof for Proposition 15 which works for general $I\!\!F$.

2) The BSDE (4.3) is very similar to an equation studied in detail in Mania/Schweizer (2005), but has a crucial difference: If the final value $Y_T = \alpha' U_T'$ is unbounded, there is no evident way in which the results from Mania/Schweizer (2005) could be used or adapted. $\diamond$

By exploiting the BSDE for (V, M^V), we can show that the P-martingale M^V has very good integrability properties. This adapts an argument in the proof of Lemma A1 from Schroder/Skiadas (1999).

Proposition 16. *Assume (A1) – (A3). If $I\!F$ is continuous, then M^V lies in the martingale space $\mathcal{M}_0^p(P)$ for every $p \in [1, \infty)$.*

Proof. Because $V \in D_0^{\exp}$ by Proposition 15, (A1) – (A3) imply via Doob's inequality that the (continuous) P-martingale

$$N := E_P\left[\exp\Big(\tfrac{1}{\beta}\int_0^T (\delta_s V_s - \alpha U_s)\,ds - \tfrac{1}{\beta}\alpha' U'_T\Big)\,\Big|\, I\!F\right]$$

lies in every $\mathcal{M}_0^p(P)$, and so $\langle N\rangle_T \in L^p(P)$ for every p by the BDG inequalities. Moreover, Lemma 10 applied to (V, M^V) with $\sigma = t$, $\tau = T$ yields

$$V_t = -\beta \log N_t + \int_0^t (\delta_s V_s - \alpha U_s)\,ds \qquad , \qquad 0 \le t \le T \tag{4.12}$$

which implies that

$$\frac{1}{N_t} = \exp\Big(\tfrac{1}{\beta} V_t - \tfrac{1}{\beta}\int_0^t (\delta_s V_s - \alpha U_s)\,ds\Big) \qquad , \qquad 0 \le t \le T. \tag{4.13}$$

Using (4.3) for (V, M^V) and comparing the local P-martingale parts in (4.12) gives via Itô's formula that $M^V = -\beta \int \frac{1}{N}\,dN$. Combining this with (4.13), we get

$$\begin{aligned}
\langle M^V\rangle_T &= \beta^2 \int_0^T \tfrac{1}{N_t^2}\,d\langle N\rangle_t \\
&\le \beta^2 \langle N\rangle_T \sup_{0\le t\le T} \frac{1}{N_t^2} \\
&\le \beta^2 \langle N\rangle_T \exp\Big(\tfrac{2}{\beta}\sup_{0\le t\le T}|V_t|(1 + \|\delta\|_\infty T) + \tfrac{2}{\beta}\alpha\int_0^T |U_s|\,ds\Big).
\end{aligned}$$

Due to (A1), (A2) and $V \in D_0^{\exp}$, all the terms on the right-hand side are in $L^p(P)$ for every $p \in [1, \infty)$, and hence so is $\langle M^V\rangle_T$ by Hölder's inequality. So the assertion follows by the BDG inequalities. **q.e.d.**

We have formulated Theorem 12 as a result on the characterization of the dynamic value process V for the stochastic control problem (1.3). If we want to restate our results in pure BSDE terms, we also have shown

Theorem 17. *Let δ and ϱ be progressively measurable processes and B an $\mathcal{F}_T$-measurable random variable. Assume that δ is nonnegative and uniformly bounded, that $\varrho \in D_1^{\exp}$ and that $\exp(\gamma|B|) \in L^1(P)$ for every $\gamma > 0$. If the filtration $I\!F$ is continuous, there exists for every $\beta > 0$ a unique solution $(Y, M) \in D_0^{\exp} \times \mathcal{M}_{0,\mathrm{loc}}(P)$ to the BSDE*

$$\begin{aligned} dY_t &= (\delta_t Y_t + \varrho_t)\, dt + \frac{1}{2\beta}\, d\langle M\rangle_t + dM_t, \\ Y_T &= B. \end{aligned} \tag{4.14}$$

For this solution, we have $M \in \mathcal{M}_0^p(P)$ for every $p \in [1, \infty)$.

Remark. As mentioned above, the BSDE (4.3) or (4.14) can be explicitly solved for $\delta \equiv 0$. This has already been observed in Schroder/Skiadas (1999), Appendix A; in fact, it follows immediately from Lemma 10 which gives for $\sigma = t$ and $\tau = T$ the representation

$$Y_t = -\beta \log E_P\left[\exp\Big(-\tfrac{1}{\beta} \int_t^T \varrho_s\, ds - \tfrac{1}{\beta} B \Big) \,\Big|\, \mathcal{F}_t \right]$$

for the solution of (4.14). Choosing $\varrho = \alpha U$ and $B = \alpha' U'_T$ gives the solution to (4.3). $\diamond$

5. A comparison with related results

This section is an attempt to position the results of the present paper in relation to other work in the area. Such a comparison naturally cannot be complete, but we have made an effort to include at least some of the most relevant papers.

5.1. Skiadas (2003) and Schroder/Skiadas (1999)

Our primary inspiration clearly comes from the two papers Skiadas (2003) and Schroder/Skiadas (1999) ([S03] and [SS99], for short). In [S03], Skiadas studies essentially the same optimization problem as (1.1) or (1.3), and proves that its dynamic value process V can be described by the BSDE

$$\begin{aligned} dV_t &= (\delta_t V_t - \alpha U_t)\, dt + \frac{1}{2\beta} |z_t|^2\, dt + z_t\, dW_t, \\ V_T &= \alpha' U'_T. \end{aligned} \tag{5.1}$$

This is clearly our BSDE (4.3) specialized to the case of a filtration $I\!F = I\!F^W$ generated by a P-Brownian motion W. It is a minor point that [S03] only treats the case $\alpha' = 0$. The important differences to our work lie in the interpretation and in the way that [S03] derives its results. The main point Skiadas wants to make is that the BSDE (5.1) coincides with one describing a stochastic differential utility; hence working with a standard expected utility under (a particular form of) model uncertainty is observationally equivalent to working with a corresponding stochastic differential utility under one fixed model. For the derivation, Skiadas argues in a first step that (5.1) does have a solution (V^*, z^*), since this is proved in [SS99]. In a second step, he uses explicit computations to show that z^* induces an optimal measure Q^*: in our terminology, he proves for every $\tau \in \mathcal{S}$ that

$$V_\tau = \tilde{V}(\tau, Q^*) \le \tilde{V}(\tau, Q') \qquad \text{for every } Q' \text{ with } Z^{Q'} \in \mathcal{D}(Q^*, \tau).$$

However, this approach has a disadvantage. The existence proof for (V^*, z^*) relies on a fixed point argument in [SS99], and thus from the beginning uses the assumption that $\mathbb{F} = \mathbb{F}^W$. (One could slightly generalize this fixed point method to a continuous filtration; see forthcoming work by G. Bordigoni.) In contrast, our method first shows for a general filtration the existence of an optimal measure Q^*. Only then do we assume and use continuity of $\mathbb{F}$ to deduce via the martingale optimality principle that V satisfies a BSDE. As a further minor point, the integrability of M^V in Proposition 16 is not given in [SS99].

An alternative proof for the result in [S03] can be found in Lazrak/Quenez (2003). They also assume $\mathbb{F} = \mathbb{F}^W$ and in addition impose the severe condition that U and U'_T are bounded. The argument then uses a comparison result for BSDEs from Kobylanski (2000).

5.2. Robustness, control and portfolio choice

Our second important source of inspiration has been provided by the work of L. P. Hansen and T. Sargent with coauthors; see for instance the homepage of Hansen at the URL `http://home.uchicago.edu/~lhansen`. We explicitly mention here the two papers Anderson/Hansen/Sargent (2003) and Hansen et al. (2006) which also contain more references. They both introduce and discuss (in slightly different ways) the basic problem of robust utility maximization when model uncertainty is penalized by a relative entropy term. Both papers are cast in Markovian settings and use mainly formal manipulations of Hamilton-Jacobi-Bellman (HJB) equations to provide insights about the optimal investment behaviour in these situations. While Hansen et al. (2006) find that "One Hamilton-Jacobi-Bellman (HJB) equation is worth a thousand words", our (still partial) analysis here is driven by a desire to obtain rigorous results in a general setting by stochastic methods.

A related paper by Maenhout (2004) studies (also via formal HJB analysis) a problem where the penalization parameter β is allowed to depend on V; this is also briefly discussed in Skiadas (2003). And when the present paper was almost finished, we discovered that A. Schied has also been working on the problem (1.1) with a fairly general penalization term for Q; see Schied (2007). However, his (static) results do not contain ours even without the dynamic parts in Section 4 — they only cover as one example the simple case $\delta \equiv 0$.

5.3. BSDEs with quadratic drivers

In the setting of a Brownian filtration $I\!F = I\!F^W$, the pure BSDE (4.14) takes the form

$$\begin{aligned} dY_t &= \left(\delta_t Y_t + \varrho_t + \frac{1}{2\beta}|z_t|^2 \right) dt + z_t\, dW_t, \\ Y_T &= B. \end{aligned} \tag{5.2}$$

This is one particular BSDE with a driver (dt-term) which is quadratic in the z-variable. Such BSDEs have been much studied recently and typically appear in problems from mathematical finance; see Duffie/Epstein (1992) for probably the first appearance of such a BSDE (derived in the context of stochastic differential utility), and for instance El Karoui/Peng/Quenez (2001), El Karoui/Hamadène (2003) or Schroder/Skiadas (2005) for some recent references. However, almost all (existence and comparison) results for these equations (with nonvanishing quadratic term) assume that the terminal value B is bounded. This condition is too restrictive for our purposes and seems very difficult to get rid of. A class of BSDEs with quadratic growth and unbounded terminal value has recently been studied by Briand/Hu (2006), but (5.2) does not satisfy their assumptions as soon as ϱ is unbounded.

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