Closedness in the semimartingale topology for spaces of stochastic integrals with constrained integrands

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Summary. Let S be an $\mathbb{R}^d$-valued semimartingale and (ψ^n) a sequence of C-valued integrands, i.e. predictable, S-integrable processes taking values in some given closed set $C(\omega, t) \subseteq \mathbb{R}^d$ which may depend on the state ω and time t in a predictable way. Suppose that the stochastic integrals $(\psi^n \cdot S)$ converge to X in the semimartingale topology. When can X be represented as a stochastic integral with respect to S of some C-valued integrand? We answer this with a necessary and sufficient condition (on S and C), and explain the relation to the sufficient conditions introduced earlier in (Czichowsky, Westray, Zheng, Convergence in the semimartingale topology and constrained portfolios, 2010; Mnif and Pham, Stochastic Process Appl 93:149–180, 2001; Pham, Ann Appl Probab 12:143-172, 2002). The existence of such representations is equivalent to the closedness (in the semimartingale topology) of the space of all stochastic integrals of C-valued integrands, which is crucial in mathematical finance for the existence of solutions to most optimisation problems under trading constraints. Moreover, we show that a predictably convex space of stochastic integrals is closed in the semimartingale topology if and only if it is a space of stochastic integrals of C-valued integrands, where each $C(\omega, t)$ is convex.

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1 Introduction

In mathematical finance, proving the existence of a solution to optimisation problems like superreplication, utility maximisation or quadratic hedging usually boils down to the same abstract problem: One must show that a subsequence of (predictably) convex combinations of an optimising sequence of wealth processes, i.e. stochastic integrals with respect to the underlying price process S, converges and that the limit is again a wealth process, i.e. can be represented as a stochastic integral with respect to S. As the space of *all* stochastic integrals is closed in the semimartingale topology, this is the suitable topology to work with.

For applications, it is natural to include trading constraints by requiring the strategy (integrand) to lie pointwise in some set C; this set is usually convex to keep the above procedure applicable, and one would like it to depend on the state and time as well. Examples of interest include no shortselling, no borrowing or nonnegative wealth constraints; see e.g. [4, 16]. As pointed out by Delbaen [8] and Karatzas and Kardaras [16], a natural and convenient formulation of constraints is in terms of *correspondences*, i.e. set-valued functions. This is the approach we also advocate and use here.

For motivation, consider a sequence of (predictably convex combinations of) strategies and suppose (as usually happens by the convexification trick) that this converges pointwise. Each strategy is predictable, so constraints should also be "predictable" in some sense. To have the limit still satisfy the same restrictions as the sequence, the constraints should moreover be of the form "closure of a sequence $(\psi^n(\omega, t))$ of random points", since this is where the limit will lie. But if each $\psi^n(\omega, t)$ is a predictable process, the above closure is then a predictable correspondence by the Castaing representation (see Proposition 2.3). This explains why correspondences come up naturally.

In our constrained optimisation problem, assuming that we have predictable, convex, closed constraints, the same procedure as in the unconstrained case yields a sequence of wealth processes (integrals) converging to some limit which is a candidate for the solution of our problem. (We have cheated a little in the motivation — the integrals usually converge, not the integrands.) This limit process is again a stochastic integral, but it still remains to check that the corresponding trading strategy also satisfies the constraints. In abstract terms, one asks whether the limit of a sequence of stochastic integrals of constrained integrands can again be represented as a stochastic integral of some constrained integrand or, equivalently, if the space of stochastic integrals of constrained integrands is closed in the semimartingale topology. We illustrate by a *counterexample* that this is not true in general, since it might happen that some assets become redundant, i.e. can be replicated on some predictable set by trading in the remaining ones. This phenomenon occurs when there is linear dependence between the components of S.

As in [4, 3, 19, 21], one could resolve this issue by simply assuming that there are no redundant assets; then the closedness result is true for all constraints formulated via closed (and convex) sets. Especially in Itô process models with a Brownian filtration, such a non-redundancy condition is useful (e.g. when working with artificial market completions), but it can be restrictive. Alternatively, as in [15, 25, 6], one can study only constraints given by polyhedral or continuous convex sets. While most constraints of practical interest are indeed polyhedral, this is conceptually unsatisfactory as one does not recover all results from the case when there are no redundant assets. A good formulation should thus account for the interplay between the constraints C and redundancies in the assets S.

To realise this idea, we use the *projection on the predictable range* of S. This is a predictable process taking values in the orthogonal projections in $\mathbb{R}^d$; it has been introduced in [24, 9, 8], and allows us to uniquely decompose each integrand into one part containing all relevant information for its stochastic integral and another part having stochastic integral zero. This reduces our problem to the question whether or not the projection of the constraints on the predictable range is closed. Convexity is not relevant for that aspect. Since that approach turns out to give a necessary and sufficient condition, we recover all previous results in [4, 19, 21, 15, 6] as special cases; and in addition, we obtain for constant constraints $C(\omega, t) \equiv C$ that closedness of the space of C-constrained integrands holds for *all* semimartingales if and only if all projections of C in $\mathbb{R}^d$ are closed. The well-known characterisation of polyhedral cones thus implies in particular that the closedness result for *constant* convex cone constraints is true for arbitrary semimartingales if and only if the constraints are polyhedral.

For a general constraint set $C(\omega, t)$ which is closed and convex, the set of stochastic integrals of C-constrained integrands is the prime example of a predictably convex space of stochastic integrals. By adapting arguments from [8], we show that this is in fact the *only* class of predictably convex spaces of stochastic integrals which are closed in the semimartingale topology. So this paper makes both mathematical contributions to stochastic calculus and financial contributions in the modelling and handling of trading constraints for optimisation problems from mathematical finance.

The remainder of the article is organised as follows. In Section 2, we formulate the problem in the terminology of stochastic processes and provide some results on measurable correspondences and measurable selectors. These are needed to introduce and handle the constraints. Section 3 contains a counterexample which illustrates where the difficulties arise and motivates in a simple setting the definition of the projection on the predictable range. The main results discussed above are established in Section 4. Section 5 gives the construction of the projection on the predictable range as well as two proofs omitted in Section 4. Finally, Section 6 briefly discusses some related work.

2 Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t < \infty}$ satisfying the usual conditions of completeness and right-continuity. For all notation concerning stochastic integration, we refer to the book of Jacod and Shiryaev [14].

Set $\overline{\Omega} := \Omega \times [0, \infty)$. The space of all $\mathbb{R}^d$-valued semimartingales is denoted by $\mathcal{S}^{0,d}(P) := \mathcal{S}^0(P; \mathbb{R}^d)$, or simply $\mathcal{S}(P)$ if the dimension is clear. The *Émery distance* (see [10]) of two semimartingales X and Y is $d(X,Y) = \sup_{|\vartheta| \le 1} \big(\sum_{n \in \mathbb{N}} 2^{-n} E\big[1 \wedge |(\vartheta \cdot (X - Y))_n|\big]\big)$, where $(\vartheta \cdot X)_t := \int_0^t \vartheta_s dX_s$ stands for the *vector stochastic integral*, which is by construction a real-valued semimartingale, and the supremum is taken over all $\mathbb{R}^d$-valued predictable processes ϑ bounded by 1. With this metric, $\mathcal{S}(P)$ is a complete topological vector space, and the corresponding topology is called the *semimartingale topology*. For brevity, we say "in $\mathcal{S}(P)$" for "in the semimartingale topology". For a given $\mathbb{R}^d$-valued semimartingale S, we write $\mathcal{L}(S)$ for the space of $\mathbb{R}^d$-valued, S-integrable, predictable processes ϑ and $L(S)$ for the space of equivalence classes $[\vartheta] = [\vartheta]^S = \{\varphi \in \mathcal{L}(S) \mid \varphi \cdot S = \vartheta \cdot S\}$ of processes in $\mathcal{L}(S)$ which yield the same stochastic integral with respect to S, identifying processes equal up to P-indistinguishability. By Theorem V.4 in [20], the space of stochastic integrals $\{\vartheta \cdot S \mid \vartheta \in \mathcal{L}(S)\}$ is closed in $\mathcal{S}(P)$. Equivalently, $L(S)$ is a complete topological vector space with respect to $d_S\big([\vartheta], [\varphi]\big) = d(\vartheta \cdot S, \varphi \cdot S)$, where ϑ and φ are representatives of the equivalence classes $[\vartheta]$ and $[\varphi]$.

In this paper, we generalise the above closedness result from [20] to integrands restricted to lie in a given closed set, in the following sense.

Let $C(\omega, t)$ be a non-empty, closed subset of $\mathbb{R}^d$ which may depend on ω and t in a predictably measurable way. Definition 2.2 below makes this precise: C should be a *predictable correspondence with closed values.* Denote by

$$\mathcal{C} := \mathcal{C}^S := \{\psi \in \mathcal{L}(S) \mid \psi(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t)\} \tag{2.1}$$

the set of C*-valued* or C*-constrained integrands* for S. If (ψ^n) is a sequence in $\mathcal{C}^S$ such that $(\psi^n \cdot S)$ converges to some X in the semimartingale topology, does there exist a ψ in $\mathcal{C}^S$ such that $X = \psi \cdot S$? In general, the answer is negative, as a simple counterexample in the next section illustrates, and so we ask under which conditions the above is true. By the closedness in $\mathcal{S}(P)$ of the space of all stochastic integrals, the limit X can always be represented as some stochastic integral $\vartheta \cdot S$. Thus it is enough to decide whether or not there exists for the limit class $[\vartheta]$ a representative ψ which is C-valued. Equivalently, one can ask whether $\mathcal{C}^S \cdot S$ is closed in $\mathcal{S}(P)$ or if the corresponding set

$$[\mathcal{C}] := [\mathcal{C}]^S := \{[\vartheta] \in L(S) \mid [\vartheta] \cap \mathcal{C} \neq \emptyset\}$$

of equivalence classes of elements of $\mathcal{C}^S$ is closed in $(L(S), d_S)$.

As already explained, this question arises naturally in mathematical finance for various optimisation problems under trading constraints; see [11], [21], [22], [19], [15] and [5]. But not all papers make it equally clear whether the procedure outlined in the introduction can be or is being used. For [19] and [15], this is clarified in [5]. Under additional assumptions, the closedness of $\mathcal{C}^S \cdot S$ in the semimartingale topology is sufficient to apply the results of Föllmer and Kramkov [11] on the optional decomposition under constraints, which give a dual characterisation of payoffs that can be superreplicated by constrained trading strategies. This is used in [21], [22] and [17] to prove the existence of solutions to constrained utility maximisation problems. The results in [11] are formulated more generally for sets of (special) semimartingales which are predictably convex.

Definition 2.1. *A set* $\mathfrak{S}$ *of semimartingales is called* predictably convex *if* $h \cdot X + (1-h) \cdot Y \in \mathfrak{S}$ *for all* X *and* Y *in* $\mathfrak{S}$ *and all* $[0,1]$*-valued predictable processes* h*. Analogously, a set* $\mathfrak{C} \subseteq \mathcal{L}(S)$ *of integrands is* predictably convex *if* $h\vartheta + (1-h)\varphi \in \mathfrak{C}$ *for all* ϑ *and* φ *in* $\mathfrak{C}$ *and all* $[0,1]$*-valued predictable processes* h*.*

The prime example of predictably convex sets of integrands is given by C-constrained integrands when C is convex-valued. Theorem 4.11

below shows that *all* predictably convex spaces $\mathfrak{C}$ of integrands must be of this form if $\mathfrak{C} \cdot S$ is in addition closed in $\mathcal{S}(P)$.

To formulate precisely the assumptions on the (random and time-dependent) set C, we adapt the language of measurable correspondences to our framework of predictable measurability and recall for later use some of the results in this context. Note that the general results we exploit do not depend on special properties of the predictable σ-field on $\overline{\Omega}$. However, we do use that the range space $\mathbb{R}^d$ is metric and σ-compact; this ensures by Proposition 1A in [23] or the proof of Lemma 18.2 in [1] that weak measurability and measurability for a closed-valued correspondence coincide in our setting.

Definition 2.2. *A mapping $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is called an* ($\mathbb{R}^d$-valued) correspondence*. Its* domain *is* $\mathrm{dom}(C) := \big\{(\omega,t) \;\big|\; C(\omega,t) \neq \emptyset\big\}$*. We call a correspondence C* predictable *if $C^{-1}(F) := \big\{(\omega,t) \;\big|\; C(\omega,t) \cap F \neq \emptyset\big\}$ is a predictable set for each closed $F \subseteq \mathbb{R}^d$. A correspondence has* predictable graph *if its graph* $\mathrm{gr}(C) := \big\{(\omega,t,x) \in \overline{\Omega} \times \mathbb{R}^d \;\big|\; x \in C(\omega,t)\big\}$ *is in $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$. A* predictable selector *of a predictable correspondence C is a predictable process ψ which satisfies $\psi(\omega,t) \in C(\omega,t)$ for all $(\omega,t) \in \mathrm{dom}(C)$.*

The following results ensure the existence of predictable selectors in all situations relevant for us.

Proposition 2.3 (Castaing). *For a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ with closed values, the following are equivalent:*

1) C is predictable.
2) $\mathrm{dom}(C)$ is predictable and there exists a Castaing representation *of C, i.e. a sequence (ψ^n) of predictable selectors of C such that*

$$C(\omega,t) = \overline{\{\psi^1(\omega,t), \psi^2(\omega,t), \dots\}} \qquad \textit{for each } (\omega,t) \in \mathrm{dom}(C).$$

Proof. See Corollary 18.14 in [1] or Theorem 1B in [23].

Proposition 2.4 (Aumann). *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a correspondence with non-empty values and predictable graph and μ a finite measure on $\big(\overline{\Omega}, \mathcal{P}\big)$. Then there exists a predictable process ψ with $\psi(\omega,t) \in C(\omega,t)$ μ-a.e.*

Proof. See Corollary 18.27 in [1]. $\square$

The proof of Proposition 2.4 is based on the following result on projections to which we refer later.

Proposition 2.5. *Let $(R, \mathcal{R}, \mu)$ be a σ-finite measure space, $\mathcal{R}_\mu$ the σ-field of μ-measurable sets and A in $\mathcal{R}_\mu \otimes \mathcal{B}(\mathbb{R}^d)$. Then the projection $\pi_R(A)$ of A on R belongs to $\mathcal{R}_\mu$.*

Proof. See Theorem 18.25 in [1]. $\square$

Measurability and graph measurability of a correspondence are linked as follows.

Proposition 2.6. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a correspondence. If C is predictable, its closure correspondence $\overline{C}$ given by $\overline{C}(\omega, t) := \overline{C(\omega, t)}$ has a predictable graph.*

Proof. See Theorem 18.6 in [1]. $\square$

Since we require in (2.1) for our integrands ψ that $\psi(\omega, t) \in C(\omega, t)$ for all (ω, t), we shall assume, as motivated in the introduction, that *C is predictable and has closed values.* Then Proposition 2.3 guarantees the existence of predictable selectors. Moreover, we shall use that predictable measurability of a correspondence is preserved under transformations by Carathéodory functions and is stable under countable unions and intersections. Recall that a function $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^m$ is called *Carathéodory* if $f(\omega, t, x)$ is predictable with respect to (ω, t) and continuous in x.

Proposition 2.7. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ be a predictable correspondence with closed values and $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}^d$ and $g : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}^m$ Carathéodory functions. Then C' and C'' given by*

$$C'(\omega, t) = \{y \in \mathbb{R}^m \mid f(\omega, t, y) \in C(\omega, t)\}$$

and

$$C''(\omega, t) = \overline{\{g(\omega, t, x) \mid x \in C(\omega, t)\}}$$

are predictable correspondences with closed values.

Proof. See Corollaries 1P and 1Q in [23]. $\square$

Proposition 2.8. *Let $C^n : \overline{\Omega} \to 2^{\mathbb{R}^d}$ for each $n \in \mathbb{N}$ be a predictable correspondence with closed values and define the correspondences C' and C'' by $C'(\omega, t) = \bigcap_{n \in \mathbb{N}} C^n(\omega, t)$ and $C''(\omega, t) = \bigcup_{n \in \mathbb{N}} C^n(\omega, t)$. Then C' and C'' are predictable and C' is closed-valued.*

Proof. See Theorem 1M in [23] and Lemma 18.4 in [1]. $\square$

To establish a relation between predictably convex spaces of integrands and C-valued integrands, we later use the following result, which is a reformulation of the contents of Theorem 5 in [8]. We view an $\mathbb{R}^d$-valued predictable process on Ω as a $\mathcal{P}$-measurable $\mathbb{R}^d$-valued mapping on $\overline{\Omega}$, take some probability μ on $(\overline{\Omega}, \mathcal{P})$ and denote by $\overline{B(0,r)}^{L^\infty}$ and $\overline{B(0,r)}$ the closures of a ball of radius r in $L^\infty(\overline{\Omega}, \mathcal{P}, \mu; \mathbb{R}^d)$ and in $\mathbb{R}^d$, respectively. Predictable convexity is understood as in the second part of Definition 2.1.

Proposition 2.9. *Let $\mathfrak{K}$ be a predictably convex and μ-weak*-compact subset of $\overline{B(0,r)}^{L^\infty}$ with $0 \in \mathfrak{K}$. Then there exists a predictable correspondence $K : \overline{\Omega} \to 2^{\overline{B(0,r)}} \setminus \{\emptyset\}$, whose values are convex and compact and contain zero, such that*

$$\mathfrak{K} = \Big\{ \vartheta \in L^\infty\big(\overline{\Omega}, \mathcal{P}, \mu; \mathbb{R}^d\big) \;\Big|\; \vartheta(\omega,t) \in K(\omega,t) \ \mu\text{-a.e.} \Big\}.$$

Proof. In the proof of Theorem 5 in [8], the set $\mathcal{C}^\lambda$ defined there for $\lambda > 0$ contains zero and is by Lemmas 10 and 11 in [8] a predictably convex and weak*-compact subset of $\overline{B(0,\lambda)}^{L^\infty}$. No other properties of $\mathcal{C}^\lambda$ are used. So we can modify the proof of Theorem 5 in [8] by replacing the use of the Radon–Nikodým theorem of Debreu and Schmeidler (Theorem 2 in [7]) with that of Artstein (Theorem 9.1 in [2]). This yields that $K := \Phi^r$ constructed in that proof is predictably measurable and has not only (as argued in [8]) predictable graph. Replacing the correspondence K coming from this construction by $K \cap \overline{B(0,r)}$ then gives that K is valued in $2^{\overline{B(0,r)}}$. $\square$

3 A motivating example

In this section, we give a simple example of a semimartingale Y and a predictable correspondence C with non-empty, closed, convex cones as values such that $\mathcal{C}^Y \cdot Y$ is not closed in $\mathcal{S}(P)$. This illustrates where the problems with our basic question arise and suggests a way to overcome them. The example is the same as Example 2.2 in [6], but we use it here for a different purpose and with different emphasis.

Let $W = (W^1, W^2, W^3)^\top$ be a 3-dimensional Brownian motion and $Y = \sigma \cdot W$, where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The matrix σ and hence $\hat{c} = \sigma\sigma^\top$ have a non-trivial kernel spanned by $w = \frac{1}{\sqrt{2}}(0,1,1)^\top$, i.e. $\mathrm{Ker}(\hat{c}) = \mathrm{Ker}(\sigma) = \mathbb{R}w = \mathrm{span}\{w\}$. By construction, the stochastic integral of each $\mathbb{R}^3$-valued predictable process valued in $\mathrm{Ker}(\hat{c})$ $dP \otimes dt$-a.e. is zero, and vice versa. Thus the equivalence class $[\vartheta]^Y$ of any given $\vartheta \in \mathcal{L}(Y)$ is given by

$$[\vartheta]^Y = \{\vartheta + hw \mid h \text{ is a real-valued predictable process}\}$$

up to $dP \otimes dt$-a.e. equality, since adding a representative of 0 to some element of $\mathcal{L}(Y)$ does not change its equivalence class. Let K be the closed and convex cone

$$K = \big\{(x,y,z)^\top \in \mathbb{R}^3 \bigm| x^2 + y^2 \leq z^2,\ z \geq 0\big\}$$

and C the (constant) predictable correspondence with non-empty and closed values given by $C(\omega,t) = K$ for all $(\omega,t) \in \overline{\Omega}$. Define the sequence of (constant) processes (ψ^n) by $\psi^n = (1, \sqrt{n^2-1}, n)^\top$ for each $n \in \mathbb{N}$. In geometric terms, K is a circular cone around the z-axis, and (ψ^n) is a sequence of points on its surface going to infinity. (Instead of n, any sequence $z_n \to \infty$ in $[1,\infty)$ would do as well.) Each ψ^n is C-valued, and we compute $\psi^n \cdot Y = (\sigma\psi^n) \cdot W = W^1 + \big(\sqrt{n^2-1} - n\big)(W^2 - W^3)$. Using this explicit expression yields by a simple calculation that $\psi^n \cdot Y \to W^1$ locally in $\mathcal{M}^2(P)$ and therefore in $\mathcal{S}(P)$; see [6] for details. However, the (constant) process $e_1 := (1,0,0)^\top$ leading to the limiting stochastic integral $e_1 \cdot Y = W^1$ does not have values in C, and since its equivalence class is $\big\{e_1 + hw \bigm| h \text{ is a real-valued predictable process}\big\}$, also no other integrand equivalent to e_1 does. Thus $\mathcal{C}^Y \cdot Y$ is not closed in $\mathcal{S}(P)$.

To see why this causes problems, define $\tau := \inf\big\{t > 0 \bigm| |W_t| = 1\big\}$ and set $S := Y^\tau$. The arguments above then imply that the sequence $(\psi^n \cdot Y^\tau)$ is bounded from below (uniformly in n, t, ω) and converges in $\mathcal{S}(P)$ to $(W^1)^\tau$, which cannot be represented as $\psi \cdot S$ for any C-valued integrand ψ. Thus the set $\mathcal{C}^S \cdot S$ does not satisfy Assumption 3.1 of the optional decomposition theorem under constraints in [11]. But for instance the proof of Proposition 2.13 in [17] (see p. 1835) explicitly uses that result of [11] in a setting where constrained integrands could be given by C-valued integrands as above. So technically, the argument in [17] is not valid without further assumptions (and Theorem 4.5 and Corollary 4.9 below show ways to fix this).

What can we learn from the counterexample? The key point is that *the convergence of stochastic integrals $\psi^n \cdot Y$ need not imply the pointwise convergence of their integrands.* Without constraints, this causes no problems; by Mémin's theorem, the limit is still *some* stochastic integral of Y, here $e_1 \cdot Y$. But if we insist on having C-valued integrands, the example shows that we ask for too much. Since K is closed, we can deduce above that $(|\psi^n|)$ must diverge (otherwise we should get along a subsequence a limit, which would be C-valued by closedness), and in fact $|\psi^n| = \sqrt{2}\,n \to \infty$. But at the same time, $(\sigma\psi^n)$ converges to $e_1 = (1,0,0)^\top$ — and this observation brings up the key idea of not looking at ψ^n, but at suitable projections of ψ^n linked (via σ) to the integrator Y.

To make this precise, denote the orthogonal projection on $\mathrm{Im}(\sigma\sigma^\top)$ by

$$\Pi^Y = \mathbb{1}_{d\times d} - ww^\top = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then $\Pi^Y\psi^n = \big(1, \frac{1}{2}(\sqrt{n^2-1}-n), -\frac{1}{2}(\sqrt{n^2-1}-n)\big)^\top$ converges to the limit integrand $(1,0,0)^\top = e_1$. We might worry about the obvious fact that $\Pi^Y\psi^n$ does not take values in C; but for the stochastic integrals, this does not matter because $(\Pi^Y\psi^n)\cdot Y = \psi^n \cdot Y$. Indeed, any $\vartheta \in \mathcal{L}(Y)$ can be written as a sum $\vartheta = \Pi^Y\vartheta + (ww^\top)\vartheta$ of one part with values in $\mathrm{Im}(\sigma\sigma^\top)$ and another part orthogonal to the first one; and since $\sigma^\top w = 0$ implies that $\big((ww^\top)\vartheta\big)\cdot Y = (\vartheta^\top ww^\top\sigma)^\top \cdot W = 0$, the claim follows. Going a little further, we even have for any $\vartheta \in \mathcal{L}(Y)$ and any $\mathbb{R}^d$-valued predictable process φ that

$$\varphi \in \mathcal{L}(Y) \text{ with } \varphi\cdot Y = \vartheta\cdot Y \iff \Pi^Y\varphi = \Pi^Y\vartheta \ dP\otimes dt\text{-a.e.}, \tag{3.1}$$

by using that $\mathrm{Ker}(\sigma\sigma^\top) \cap \mathrm{Im}(\sigma\sigma^\top) = \{0\}$ and that $\sigma^\top(\Pi^Y v) = \sigma^\top v$ for all $v \in \mathbb{R}^d$ to check the Y-integrability of φ. The significance of (3.1) is that the stochastic integral $\vartheta\cdot Y$ is uniquely determined by $\Pi^Y\vartheta$, and so $\Pi^Y\vartheta$ gives a "minimal" choice of a representative of the equivalence class $[\vartheta]^Y$. Moreover, Π^Y gives via (3.1) a simple way to decide whether or not a given $\mathbb{R}^d$-valued predictable process φ belongs to the equivalence class $[\vartheta]^Y$.

Coming back to the set K, we observe that

$$\Pi^Y K = \left\{ \left(x, \frac{1}{2}(y-z), -\frac{1}{2}(y-z)\right)^\top \,\middle|\, x^2+y^2 \le z^2,\ z \ge 0 \right\}$$

is the projection of the cone K on the plane through the origin and with the normal vector $(0,1,1)^\top$. In geometric terms, the projection of each horizontal slice of the cone transforms the circle above the x-y-plane into an ellipse in the projection plane having the origin as a point of its boundary. As we move up along the z-axis, the circles become larger, and so do the ellipses which in addition flatten out towards the line through the origin and the point $e_1 = (1, 0, 0)^\top$. But since they never reach that line although they come arbitrarily close, $\Pi^Y K$ is not closed in $\mathbb{R}^d$ — and this is the source of all problems in our counterexample. It explains why the limit $e_1 = \lim_{n\to\infty} \Pi^Y \psi^n$ is not in $\Pi^Y K$, which implies by (3.1) that there cannot exist any C-valued integrand ψ such that $\Pi^Y \psi = e_1$. But the insight about $\Pi^Y K$ also suggests that if we assume for a predictable correspondence C that

$$\Pi^Y C(\omega, t) \text{ is closed } dP \otimes dt\text{-a.e.}, \tag{3.2}$$

we ought to get that $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$. This indeed works (see Theorem 4.5), and it turns out that condition (3.2) is not only sufficient, but also necessary.

The above explicit computations rely on the specific structure of Y, but they nevertheless motivate the approach for a general semimartingale S. We are going to define a predictable process Π^S taking values in the orthogonal projections in $\mathbb{R}^d$ and satisfying (3.1) with $dP \otimes dt$ replaced by a suitable measure on $\big(\overline{\Omega}, \mathcal{P}\big)$ to control the stochastic integrals with respect to S. The process Π^S will be called the *projection on the predictable range* and will allow us to formulate and prove our main results in the next section.

4 Main results

This section contains the main results (Theorems 4.5 and 4.11) as well as some consequences and auxiliary results. Before we can formulate and prove them, we need some facts and results about the projection on the predictable range of S. For the reader's convenience, the actual construction of Π^S is postponed to Section 5.

As in [14], Theorem II.2.34, each semimartingale S has the *canonical representation*

$$S = S^c + \widetilde{A} + [x \mathbb{1}_{\{|x|\le 1\}}] * (\mu - \nu) + [x \mathbb{1}_{\{|x|>1\}}] * \mu$$

with the jump measure μ of S and its predictable compensator ν. Then the triplet (b, c, F) of predictable characteristics of S consists of a predictable $\mathbb{R}^d$-valued process b, a predictable nonnegative-definite matrix-valued process c and a predictable process F with values in the set of Lévy measures such that

$$\widetilde{A} = b \cdot B, \qquad [S^c, S^c] = c \cdot B \qquad \text{and} \qquad \nu = F \cdot B, \tag{4.1}$$

where $B := \sum_{i=1}^d \big([S^c, S^c]^{i,i} + \mathrm{Var}(\widetilde{A}^i)\big) + (|x|^2 \wedge 1) * \nu$.

Note that B is locally bounded since it is predictable and increasing. Therefore $P \otimes B$ is σ-finite on $\big(\overline{\Omega}, \mathcal{P}\big)$ and there exists a probability measure P_B equivalent to $P \otimes B$. By the construction of the stochastic integral, S-integrable, predictable processes which are P_B-a.e. equal yield the same stochastic integral with respect to S (up to P-indistinguishability). Put differently, $\varphi = \vartheta$ P_B-a.e. implies for the equivalence classes in $L(S)$ that $[\varphi] = [\vartheta]$. But the converse is not true; a sufficient and necessary condition involves the projection Π^S on the predictable range of S, as we shall see below. Because S is now (in contrast to Section 3) a general semimartingale, the actual construction of Π^S and the proof of its properties become more technical and are postponed to the next section. We give here merely the definition and two auxiliary results.

Definition 4.1. *The* projection on the predictable range of S *is a predictable process* $\Pi^S : \overline{\Omega} \to \mathbb{R}^{d \times d}$ *which takes values in the orthogonal projections in* $\mathbb{R}^d$ *and has the following property: If* $\vartheta \in \mathcal{L}(S)$ *and* φ *is predictable, then* φ *is in* $\mathcal{L}(S)$ *with* $\varphi \cdot S = \vartheta \cdot S$ *if and only if* $\Pi^S\vartheta = \Pi^S\varphi$ P_B*-a.e. We choose and fix one version of* Π^S.

Remark 4.2. There are many possible choices for a process B satisfying (4.1). However, the definition of Π^S is independent of the choice of B in the sense that (with obvious notation) $\Pi^{S,B}\vartheta = \Pi^{S,B}\varphi$ P_B-a.e. if and only if $\Pi^{S,B'}\vartheta = \Pi^{S,B'}\varphi$ $P_{B'}$-a.e. This is because stochastic integrals of S do not depend on the choice of B.

As illustrated by the example in Section 3, the convergence in $\mathcal{S}(P)$ of stochastic integrals does not imply in general that the integrands converge P_B-a.e. But like in the example, a subsequence of the projections of the integrands on the predictable range does.

Lemma 4.3. *Let* (ϑ^n) *be a sequence in* $\mathcal{L}(S)$ *such that* $\vartheta^n \cdot S \to \vartheta \cdot S$ *in* $\mathcal{S}(P)$*. Then there exists a subsequence* (n_k) *such that* $\Pi^S\vartheta^{n_k} \to \Pi^S\vartheta$ P_B*-a.e.*

Lemma 4.4. *Let* $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ *be a predictable correspondence with closed values and such that the projection on the predictable range of* S *is* not closed*, i.e.*

$$\widetilde{F} = \big\{(\omega, t) \in \overline{\Omega} \;\big|\; \Pi^S(\omega, t)C(\omega, t) \text{ is not closed}\big\}$$

has outer P_B-measure > 0. Then there exist $\vartheta \in \mathcal{L}(S)$ and a sequence (ψ^n) of C-valued integrands such that $\psi^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$, but there is no C-valued integrand ψ such that $\psi \cdot S = \vartheta \cdot S$. Equivalently, there exists a sequence $\big([\psi^n]\big)$ in $[\mathcal{C}]^S$ such that $[\psi^n] \overset{L(S)}{\to} [\vartheta]$ but $[\vartheta] \notin [\mathcal{C}]^S$, i.e. $[\mathcal{C}]^S$ is not closed in $L(S)$.

Lemmas 4.3 and 4.4 as well as the existence of Π^S will be shown in Section 5. Admitting that, we can now prove our first main result; related work in [16] is discussed in Section 6. Recall the definition of $\mathcal{C} := \mathcal{C}^S$ from (2.1).

Theorem 4.5. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed values. Then $\mathcal{C}^S \cdot S$ is closed in $\mathcal{S}(P)$ if and only if the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e. Equivalently: There exists a C-valued integrand ψ with $X = \psi \cdot S$ for any sequence (ψ^n) of C-valued integrands with $\psi^n \cdot S \to X$ in $\mathcal{S}(P)$ if and only if the projection of C on the predictable range of S is closed.*

Proof. "$\Rightarrow$": This implication follows immediately from Lemma 4.4.
"$\Leftarrow$": Let (ψ^n) be a sequence in $\mathcal{C}$ with $\psi^n \cdot S \to X$ in $\mathcal{S}(P)$. Then there exist by Mémin's theorem $\vartheta \in \mathcal{L}(S)$ with $X = \vartheta \cdot S$ and by Lemma 4.3 a subsequence, again indexed by n, with $\Pi^S \psi^n \to \Pi^S \vartheta$ P_B-a.e. So it remains to show that we can find a C-valued representative ψ of the limit class $[\vartheta] = [\Pi^S \vartheta]$. To that end, we observe that the P_B-a.e. closedness of $\Pi^S(\omega,t)C(\omega,t)$ implies that $\Pi^S \vartheta = \lim_{n\to\infty} \Pi^S \psi^n \in \Pi^S C$ P_B-a.e. By Proposition 2.7, the correspondences given by $\{\Pi^S(\omega,t)\vartheta(\omega,t)\}$, $C'(\omega,t) = \{\Pi^S(\omega,t)\vartheta(\omega,t)\} \cap \Pi^S(\omega,t)C(\omega,t)$ and

$$C''(\omega,t) = \big\{z \in \mathbb{R}^d \;\big|\; \Pi^S(\omega,t)z \in C'(\omega,t)\big\} \cap C(\omega,t)$$

are predictable and closed-valued. Indeed, $\Pi^S \vartheta$ is a predictable process, and $\big\{z \in \mathbb{R}^d \;\big|\; \Pi^S(\omega,t)z \in C'(\omega,t)\big\}$ and $\Pi^S C = \overline{\Pi^S C}$ are the preimage and (the closure of) the image of a closed-valued correspondence under a Carathéodory function, respectively. Thus C' and C'' are the intersections of two predictable and closed-valued correspondences and therefore predictable by Proposition 2.8. So there exists by Proposition 2.3 a predictable selector ψ of C'' on

$$\mathrm{dom}(C'') = \big\{(\omega,t) \;\big|\; \Pi^S(\omega,t)\vartheta(\omega,t) \in \Pi^S(\omega,t)C(\omega,t)\big\}.$$

This ψ can be extended to a C-valued integrand by using any predictable selector on the P_B-nullset $\big(\mathrm{dom}(C'')\big)^c$. By construction, ψ is then in $\mathcal{C}$ and satisfies $\Pi^S \psi = \Pi^S \vartheta$ P_B-a.e., so that $\psi \in [\vartheta]$ by the definition of Π^S. This completes the proof. $\square$

Theorem 4.5 gives as necessary and sufficient condition for the closedness of the space of C-constrained integrals of S that the projection of the constraint set C on the predictable range of S is closed. This uses information from both the semimartingale S and the constraints C, as well as their interplay. We shall see below how this allows to recapture several earlier results as special cases.

Corollary 4.6. *Suppose that $S = S_0 + M + A$ is in $\mathcal{S}^2_{loc}(P)$ and define the process a via $A = a \cdot B$. If*

$$[0]^M = \{ha \mid h \text{ is real-valued and predictable}\} \tag{4.2}$$

up to P_B-a.e. equality, then $\mathcal{C}^S \cdot S$ is closed in $\mathcal{S}(P)$ for all predictable correspondences $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with closed values.

Proof. By Lemma 5.1 below, (4.2) implies $[0]^S = [0]^M \cap [0]^A = \{0\}$ and therefore $\Pi^S = \mathbb{1}_{d\times d}$ by (5.2) below. So the projection of any closed-valued correspondence C on the predictable range of S is closed, which gives the assertion by Theorem 4.5. $\square$

In applications from mathematical finance, S often satisfies the so-called *structure condition (SC)*, i.e. $S = S_0 + M + A$ is in $\mathcal{S}^2_{loc}(P)$ and there exists an $\mathbb{R}^d$-valued predictable process $\lambda \in \mathcal{L}^2_{loc}(M)$ such that $A = \lambda \cdot \langle M, M\rangle$ or, equivalently, $a = \hat{c}\lambda$ P_B-a.e.; this is a weak no-arbitrage type condition. In this situation, Lemma 5.1 below gives $[0]^M \subseteq [0]^A$, and thus condition (4.2) holds if and only if $[0]^M = \{0\}$ (up to P_B-a.e. equality), which means that $\hat{c}$ is P_B-a.e. invertible. This is the case covered in Lemma 3.1 in [21], where one has conditions only on S but not on C. Basically this ensures that there are no redundant assets, i.e. every stochastic integral is realised by exactly one integrand (up to P_B-a.e. equality).

The opposite extreme is to place conditions only on C that ensure closedness of $\mathcal{C}^S \cdot S$ for arbitrary semimartingales S, as in Theorem 3.5 of [6]. We recover this as a special case in the following corollary; note that in a slight extension over [6], the constraints need not be convex. Recall that a closed convex set $K \subseteq \mathbb{R}^d$ is called *continuous* if its support function $\delta(v|K) = \sup_{w\in K} w^\top v$ is continuous for all vectors $v \in \mathbb{R}^d$ with $|v| = 1$; see [13].

Corollary 4.7. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ be a predictable correspondence with closed values. Then $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ for all semimartingales Y if with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega, t)$ of $C(\omega, t)$ are closed in $\mathbb{R}^d$.*

In particular, if with probability 1*, every* $C(\omega,t)$*,* $t \geq 0$*, is compact, or polyhedral, or a continuous and convex set, then* $\mathcal{C}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all semimartingales* Y*.*

Proof. If a set is compact or polyhedral, all its projections have the same property (see Corollary 2.15 in [18]) and are thus closed. For a continuous convex set, every projection is closed by Theorem 1.3 in [13]. Now if with probability 1, for all $t \geq 0$ all projections $\Pi C(\omega,t)$ of $C(\omega,t)$ are closed, the projection $\Pi^Y C$ of C on the predictable range of every semimartingale Y is closed $P \otimes B^Y$-a.e. So $\mathcal{C}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ by Theorem 4.5. □

Combining Theorem 4.5 with the example in Section 3, we obtain the following corollary. It is formulated for fixed sets K, but can probably be generalised to predictable correspondences C by using measurable selections.

Corollary 4.8. *Suppose* $(\Omega,\mathcal{F},P)$ *is sufficiently rich. Fix* $K \subseteq \mathbb{R}^d$ *and define as in* (2.1) $\mathcal{K}^Y = \big\{\psi \in \mathcal{L}(Y) \;\big|\; \psi(\omega,t) \in K \text{ for all } (\omega,t)\big\}$*. Then* $\mathcal{K}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all* $\mathbb{R}^d$*-valued semimartingales* Y *if and only if all projections* ΠK *of* K *in* $\mathbb{R}^d$ *are closed.*

Proof. The "if" part follows immediately from Theorem 4.5. For the converse, assume by way of contradiction that there is a projection Π in $\mathbb{R}^d$ such that ΠK is not closed. Let W be a d-dimensional Brownian motion and set $Y = \Pi^\top \cdot W$. Then Π is the projection on the predictable range of Y, and therefore $\mathcal{K}^Y \cdot Y$ is not closed by Theorem 4.5. □

If the constraints are not only convex, but also cones, a characterisation of convex polyhedra due to Klee [18] gives an even sharper result.

Corollary 4.9. *Let* $K \subseteq \mathbb{R}^d$ *be a closed convex cone. Then* $\mathcal{K}^Y \cdot Y$ *is closed in* $\mathcal{S}(P)$ *for all* $\mathbb{R}^d$*-valued semimartingales* Y *if and only if* K *is polyhedral.*

Proof. By Corollary 4.8, $\mathcal{K}^Y \cdot Y$ is closed in $\mathcal{S}(P)$ if and only if all projections ΠK are closed in $\mathbb{R}^d$. But Theorem 4.11 in [18] says that all projections of a convex cone are closed in $\mathbb{R}^d$ if and only if that cone is polyhedral. □

Remark 4.10. Armed with the last result, we can briefly come back to the proof of Proposition 2.13 in [17]. We have already pointed out in Section 3 that the argument in [17] uses the optional decomposition under constraints from [11], without verifying its Assumption 3.1. In view of Corollary 4.9, we can now be more precise: The argument in [17] as it stands (i.e. without assumptions on S) only works for *polyhedral* cone constraints; for others, one could by Corollary 4.9 construct a semimartingale S giving a contradiction.

We now turn to our second main result. Recall again the definition of $\mathcal{C}$ from (2.1) and note that for a correspondence C with convex values, $\mathcal{C}$ is the prime example of a predictably convex space of integrands. The next theorem shows that this is actually the only class of predictably convex integrands if we assume in addition that the resulting space $\mathcal{C}\cdot S$ of stochastic integrals is closed in $\mathcal{S}(P)$. The result and its proof are inspired from Theorems 3 and 4 in [8], but require quite a number of modifications.

Theorem 4.11. *Let $\mathfrak{C}\subseteq\mathcal{L}(S)$ be non-empty. Then $\mathfrak{C}\cdot S$ is predictably convex and closed in the semimartingale topology if and only if there exists a predictable correspondence $C:\overline{\Omega}\to 2^{\mathbb{R}^d}\setminus\{\emptyset\}$ with closed convex values such that the projection of C on the predictable range of S is closed, i.e. $\Pi^S(\omega,t)C(\omega,t)$ is closed P_B-a.e., and such that we have $\mathfrak{C}\cdot S=\mathcal{C}^S\cdot S$, i.e.*

$$\begin{aligned}\mathfrak{C}\cdot S&=\{\psi\cdot S\mid\psi\in\mathfrak{C}\}\\&=\{\psi\cdot S\mid\psi\in\mathcal{L}(S)\text{ and }\psi(\omega,t)\in C(\omega,t)\text{ for all }(\omega,t)\}.\end{aligned}$$

Proof. "$\Leftarrow$": The pointwise convexity of C immediately implies that $\mathcal{C}^S\cdot S$ is predictably convex, and closedness follows from Theorem 4.5. "$\Rightarrow$": Like at the end of Section 2, we view predictable processes on Ω as $\mathcal{P}$-measurable random variables on $\overline{\Omega}=\Omega\times[0,\infty)$. Since we are only interested in a non-empty space of stochastic integrals with respect to S, we lose no generality if we replace $\mathfrak{C}$ by $\{\vartheta-\varphi\in\mathcal{L}(S)\mid\vartheta\in[\mathfrak{C}]\}$ for some $\varphi\in\mathfrak{C}$ and identify this with a subspace of $L^0\big(\overline{\Omega},\mathcal{P},P_B;\mathbb{R}^d\big)$ which contains zero. Indeed, if the assertion is true for $\mathfrak{C}-\varphi$ with a correspondence $\widetilde{C}$, it is also true for $\mathfrak{C}$ with $C=\widetilde{C}+\varphi$, which is a predictable correspondence by Proposition 2.7. In order to apply Proposition 2.9, we truncate $\mathfrak{C}$ to get

$$\mathfrak{C}^q=\big\{\psi\in\mathfrak{C}\;\big|\;\|\psi\|_{L^\infty}\le q\big\}=\mathfrak{C}\cap\overline{B(0,q)}^{L^\infty}\quad\text{for }q\in\mathbb{Q}_+.$$

Then $\mathfrak{C}^q$ inherits predictable convexity from $\mathfrak{C}$ and is thus a convex subset of $\overline{B(0,q)}^{L^\infty}$. Moreover, $\mathfrak{C}^q$ is closed with respect to convergence in P_B-measure since its elements are uniformly bounded by q and $\mathfrak{C}\cdot S$

is closed in $\mathcal{S}(P)$; this uses the fact, easily proved via dominated convergence separately for the M- and A-integrals, that for any uniformly bounded sequence of integrands (ψ^n) converging pointwise, the stochastic integrals converge in $\mathcal{S}(P)$. By a well-known application of the Krein–Šmulian and Banach–Alaoglu theorems (see Theorems A.62 and A.63 and Lemma A.64 in [12]), $\mathfrak{C}^q$ is thus weak*-compact, and Proposition 2.9 gives a predictable correspondence $C^q : \overline{\Omega} \to 2^{\overline{B(0,q)}} \setminus \{\emptyset\}$ with convex compact values containing zero such that

$$\mathfrak{C}^q = \left\{\psi \in L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big) \;\middle|\; \psi(\omega,t) \in C^q(\omega,t) \ P_B\text{-a.e.}\right\}.$$

By the definition of $\mathfrak{C}^q$ we obtain, after possibly modifying the sets on a P_B-nullset, that

$$C^{q_2}(\omega,t) \cap \overline{B(0,q_1)} = C^{q_1}(\omega,t) \qquad \text{for all } (\omega,t) \in \overline{\Omega} \tag{4.3}$$

for $0 < q_1 \leq q_2 < \infty$ by Lemma 12 in [8], since the graph of each C^q is predictable by Proposition 2.6. Using the characterisation of closed sets in metric spaces as limit points of converging sequences implies with (4.3) that the correspondence C given by

$$C(\omega,t) := \bigcup_{q \in \mathbb{Q}_+} C^q(\omega,t)$$

has closed values. Moreover, each $C(\omega,t)$ is convex as the union of an increasing sequence of convex sets, and it only remains to show that $\mathfrak{C} \cdot S = \mathcal{C} \cdot S$.

Suppose first that ψ is in $\mathfrak{C}$. By predictable convexity and since $0 \in \mathfrak{C}$, $\psi^n := \mathbb{1}_{\{|\psi| \leq n\}}\psi$ is in $\mathfrak{C}^n$ and therefore C^n- and hence C-valued. Since (ψ^n) converges pointwise to ψ, the closedness of C implies that ψ is C-valued, so that $\psi \in \mathcal{C}$ and $\mathfrak{C} \cdot S \subseteq \mathcal{C} \cdot S$. Conversely, if ψ is in $\mathcal{C}$, then $\psi^n := \mathbb{1}_{\{|\psi| \leq n\}}\psi$ is C^n-valued and hence in $\mathfrak{C}^n \subseteq \mathfrak{C}$. But $(\psi^n \cdot S)$ converges to $\psi \cdot S$ in $\mathcal{S}(P)$ and $\mathfrak{C} \cdot S$ is closed in $\mathcal{S}(P)$. So the limit $\psi \cdot S$ is in $\mathfrak{C} \cdot S$ and hence $\psi \in \mathfrak{C}$ and $\mathcal{C} \cdot S \subseteq \mathfrak{C} \cdot S$. Finally, $\mathcal{C} \cdot S = \mathfrak{C} \cdot S$ is closed in $\mathcal{S}(P)$, and therefore $\Pi^S C$ is closed P_B-a.e. by Theorem 4.5. This completes the proof. □

Remark 4.12. 1) Theorem 4.11 can be used as follows. Start with any convex-valued correspondence C, form the space $\mathcal{C} \cdot S$ of corresponding stochastic integrals and take its closure in $\mathcal{S}(P)$. Then Theorem 4.11 tells us that we can realise this closure as a space of stochastic integrals from $\widetilde{C}$-constrained integrands, for some predictable correspondence $\widetilde{C}$ with convex and closed values. In other words, $\overline{\mathcal{C} \cdot S}^{\mathcal{S}(P)} = \widetilde{\mathcal{C}} \cdot S$; and one possible choice of $\widetilde{C}$ is $\widetilde{C} = \big(\Pi^S\big)^{-1}(\overline{C})$. Another possible choice would be $\widetilde{C} = \overline{\overline{C} + \mathfrak{N}}$, where $\mathfrak{N}$ denotes the correspondence of null investments for S; see Section 6.

2) If we assume in Theorem 4.11 that $\mathfrak{C} \subseteq \mathcal{L}^p_{loc}(S)$ for $p \in [1, \infty)$, then $\mathfrak{C} \cdot S \subseteq \mathcal{S}^p_{loc}(P)$, and $\mathfrak{C} \cdot S$ is closed in $\mathcal{S}^p(P)$ if and only if there exists C as in the theorem. This can be useful for applications (e.g., mean-variance hedging under constraints, with $p = 2$).

5 Projection on the predictable range

In this section, we construct the projection Π^S on the predictable range of a general semimartingale S in continuous time. The idea to introduce such a projection comes from [24] and [9], where it was used to prove the fundamental theorem of asset pricing in discrete time. It was also used for a continuous local martingale in [8] to investigate the structure of m-stable sets and in particular the set of risk-neutral measures.

As already explained before Definition 4.1, a *sufficient* condition for $\varphi \cdot S = \vartheta \cdot S$ (up to P-indistinguishability) or, equivalently, $\varphi = \vartheta$ in $L(S)$ or $[\varphi] = [\vartheta]$, is that $\varphi = \vartheta$ P_B-a.e. If we again view predictable processes on Ω as $\mathcal{P}$-measurable random variables on $\overline{\Omega} = \Omega \times [0, \infty)$, i.e. elements of $\mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big)$, then $\varphi = \vartheta$ P_B-a.e. is the same as saying that $\varphi = \vartheta$ in $L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big)$. But to get a *necessary and sufficient* condition for $[\vartheta] = [\varphi]$, we need to understand not only what $0 \in \mathcal{L}(S)$ looks like, but rather the precise structure of (the equivalence class) $[0]$. This is achieved by Π^S.

The construction of Π^S basically proceeds by generalising that of Π^Y in the example in Section 3 and adapting the steps in [9] to continuous time. The idea is as follows. We start by characterising the equivalence class $[0]$ as a linear subspace of $L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big)$. Since this subspace satisfies a certain stability property, we can construct predictable processes $e^1, \dots, e^d$ which form an "orthonormal basis" of $[0]$ in the sense that $[0]$ equals up to P_B-a.e. equality their linear combinations with predictable coefficients, i.e.

$$[0] = \left\{ \sum_{j=1}^{d} h^j e^j \;\middle|\; h^1, \dots, h^d \text{ are real-valued predictable} \right\} \tag{5.1}$$

up to P_B-a.e. equality. But these linear combinations contribute 0 to the integral with respect to S; so we filter them out to obtain the part of the integrand which determines the stochastic integral, by defining

$$\Pi^S := \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j (e^j)^\top. \tag{5.2}$$

This construction then yields the projection on the predictable range as in Definition 4.1.

To describe $[0] = [0]^S$ as a linear subspace of $L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big)$, we exploit that although we work with a general semimartingale S, we can by Lemma I.3 in [20] switch to an equivalent probability Q under which S is locally square-integrable. Since the stochastic integral and hence $[0]^S$ are invariant under a change to an equivalent measure, any representation we obtain $Q \otimes B$-a.e. also holds P_B-a.e., as $P_B \sim P \otimes B \sim Q \otimes B$. Let $S = S_0 + M^Q + A^Q$ be the canonical decomposition of S under Q into an $\mathbb{R}^d$-valued square-integrable Q-martingale $M^Q \in \mathcal{M}_0^{2,d}(Q)$ null at 0 and an $\mathbb{R}^d$-valued predictable process $A^Q \in \mathcal{A}^{1,d}(Q)$ of Q-integrable variation $\mathrm{Var}(A^Q)$ also null at 0. By Propositions II.2.9 and II.2.29 in [14], there exist an increasing, locally Q-integrable, predictable process B^Q, an $\mathbb{R}^d$-valued process a^Q and a predictable $\mathbb{R}^{d\times d}$-valued process $\hat{c}^Q$ whose values are positive semidefinite symmetric matrices such that

$$(A^Q)^i = (a^Q)^i \cdot B^Q \quad \text{and} \quad \big\langle (M^Q)^i, (M^Q)^j \big\rangle^Q = (\hat{c}^Q)^{ij} \cdot B^Q \tag{5.3}$$

for $i, j = 1, \dots, d$. By expressing the semimartingale characteristics of S under Q by those under P via Girsanov's theorem, writing A^Q and $\big\langle M^Q, M^Q \big\rangle^Q$ in terms of semimartingale characteristics and then passing to differential characteristics with B as predictable increasing process, we obtain that we can and do choose $B^Q = B$ in (5.3); see Theorem III.3.24 and Propositions II.2.29 and II.2.9 in [14]. Using the canonical decomposition of S under Q as auxiliary tool then allows us to give the following characterisation of $[0]^S$.

Lemma 5.1. *Let $Q \sim P$ such that $S = S_0 + M^Q + A^Q \in \mathcal{S}^2_{loc}(Q)$. Then*

1) $[0]^{M^Q} = \big\{\varphi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\big|\; \hat{c}^Q \varphi = 0 \ P_B\text{-a.e.}\big\}$.
2) $[0]^{A^Q} = \big\{\varphi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\big|\; (a^Q)^\top \varphi = 0 \ P_B\text{-a.e.}\big\}$.
3) $[0]^S = [0]^{M^Q} \cap [0]^{A^Q}$.

Moreover, $[0]^{M^Q}$, $[0]^{A^Q}$ and $[0]^S$ all do not depend on Q.

Proof. The last assertion is clear since the stochastic integral of a semimartingale (like M^Q, A^Q, S) is invariant under a change to an equivalent measure. Because also $P_B \sim Q \otimes B$, we can argue for the rest of the proof under the measure Q. Then the inclusions "$\supseteq$" follow immediately from the definition of the stochastic integral with respect to a square-integrable martingale and a finite variation process, since the conditions on the right-hand side ensure that φ is in $\mathcal{L}^2(M^Q)$ and $\mathcal{L}^1(A^Q)$. For the converse, we start with $\varphi \in [0]^S$ and set $\varphi^n := \mathbb{1}_{\{|\varphi|\leq n\}}\varphi$. Then $\varphi^n \cdot S = 0$ implies that $\varphi^n \cdot M^Q = 0$ and $\varphi^n \cdot A^Q = 0$ by the uniqueness of the Q-canonical decomposition of $\varphi^n \cdot S$; this uses that φ^n is bounded. Therefore we can reduce the proof of "$\subseteq$" for 3) to that for 1) and 2). So assume now that φ is in either $[0]^{M^Q}$ or $[0]^{A^Q}$ so that $\varphi^n \cdot M^Q = 0$ or $\varphi^n \cdot A^Q = 0$. But φ^n is bounded, hence in $\mathcal{L}^2(M^Q)$ or $\mathcal{L}^1(A^Q)$, for each n, and by the construction of the stochastic integral, we obtain that $\hat{c}^Q\,\varphi^n = 0$ or $(a^Q)^\top\varphi^n = 0$ $Q\otimes B$-a.e. and hence P_B-a.e. Since (φ^n) converges pointwise to φ, the inclusions "$\subseteq$" for 1) and 2) follow by passing to the limit. □

The following technical lemma, which is a modification of Lemma 6.2.1 in [9], gives the announced "orthonormal basis" of $[0]^S$ in the sense of (5.1).

Lemma 5.2. *Let $U \subseteq L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big)$ be a linear subspace which is closed with respect to convergence in P_B-measure and satisfies the following stability property:*

$$\varphi^1\mathbb{1}_F + \varphi^2\mathbb{1}_{F^c} \in U \qquad \text{for all } \varphi^1 \text{ and } \varphi^2 \text{ in } U \text{ and } F \in \mathcal{P}.$$

Then there exist $e^j \in L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d\big)$ for $j = 1, \dots, d$ such that

1) $\{e^{j+1} \neq 0\} \subseteq \{e^j \neq 0\}$ for $j = 1, \dots, d-1$;
2) $|e^j(\omega, t)| = 1$ or $|e^j(\omega, t)| = 0$;
3) $(e^j)^\top e^k = 0$ for $j \neq k$;
4) $\varphi \in U$ if and only if there are $h^1, \dots, h^d$ in $L^0\big(\overline{\Omega}, \mathcal{P}, P_B; \mathbb{R}\big)$ with $\varphi = \sum_{j=1}^d h^j e^j$, i.e.

$$U = \left\{ \sum_{j=1}^d h^j e^j \;\middle|\; h^1, \dots, h^d \text{ are real-valued predictable} \right\}.$$

Proof. The predictable processes $e^1, \dots, e^d$ with the properties 1)–4) are the column vectors of the measurable projection-valued mapping constructed in Lemma 6.2.1 in [9]. Therefore their existence follows immediately from the construction given there. □

By Lemma I.3 in [20], there always exists a probability measure Q as in Lemma 5.1, and therefore the space $[0]^S$ satisfies the assumptions of Lemma 5.2. So we take a “basis” $e^1, \dots, e^d$ as in the latter result and define $\varPi^S$ as in (5.2) by

$$\varPi^S := \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j (e^j)^\top .$$

Then $\varPi^S(\omega,t)$ is the projection on the orthogonal complement of the linear space spanned in $\mathbb{R}^d$ by $e^1(\omega,t),\dots,e^d(\omega,t)$ so that $\varPi^S(\omega,t)\gamma$ is orthogonal to all $e^i(\omega,t)$ for each $\gamma\in\mathbb{R}^d$; and Lemma 5.2 says that each element of $[0]^S$ is a (random and time-dependent) linear combination of $e^1,\dots,e^d$, and vice versa. In particular, $\vartheta - \varPi^S\vartheta$ is in $[0]^S$ for every predictable $\mathbb{R}^d$-valued ϑ. The next result shows that $\varPi^S$ satisfies the properties required in Definition 4.1. Note that $\varPi^S$ is only defined up to P_B-nullsets since the e^j are; so we have to choose one version for $\varPi^S$ to be specific.

Lemma 5.3 (Projection on the predictable range of S). *For a semimartingale S, the projection $\varPi^S$ on the predictable range of S exists, i.e. there exists a predictable process $\varPi^S : \overline{\Omega} \to \mathbb{R}^{d\times d}$ which takes values in the orthogonal projections in $\mathbb{R}^d$ and has the following property: If $\vartheta\in\mathcal{L}(S)$ and ψ is an $\mathbb{R}^d$-valued predictable process, then*

$$\psi\in\mathcal{L}(S) \text{ with } \psi\cdot S = \vartheta\cdot S \qquad\Longleftrightarrow\qquad \varPi^S\psi = \varPi^S\vartheta \ \ P_B\text{-a.e.} \tag{5.4}$$

Proof. If we define $\varPi^S$ as above, Lemma 5.2 implies that $\varPi^S$ is predictable and valued in the orthogonal projections in $\mathbb{R}^d$, and it only remains to check (5.4). So take $\vartheta\in\mathcal{L}(S)$ and assume first that $\varPi^S\vartheta = \varPi^S\psi$ P_B-a.e. The definition of $\varPi^S$ and Lemma 5.1 then yield that $\vartheta-\varPi^S\vartheta$ and $\varPi^S\vartheta-\varPi^S\psi$ are in $[0]^S$, which implies that $\varPi^S\vartheta = \vartheta - \big(\vartheta-\varPi^S\vartheta\big)$ and $\varPi^S\psi$ are in $\mathcal{L}(S)$ and also that $\vartheta\cdot S = (\varPi^S\vartheta)\cdot S = (\varPi^S\psi)\cdot S$. Because also $\psi-\varPi^S\psi$ is in $[0]^S\subseteq\mathcal{L}(S)$, we conclude that $\psi\in\mathcal{L}(S)$ with $\vartheta\cdot S=\psi\cdot S$. Conversely, if $\psi\cdot S=\vartheta\cdot S$, then $\psi-\vartheta\in[0]^S$, and we always have $(\psi-\vartheta)-\varPi^S(\psi-\vartheta)\in[0]^S$. Therefore $\varPi^S(\psi-\vartheta)\in[0]^S$ which says by Lemma 5.2 that for P_B-a.e. (ω,t), $\varPi^S(\psi-\vartheta)(\omega,t)$ is a linear combination of the $e^i(\omega,t)$. But the column vectors of $\varPi^S$ are orthogonal to $e^1,\dots,e^d$ for each fixed (ω,t), and so we obtain $\varPi^S(\psi-\vartheta)=0$ P_B-a.e., which completes the proof. □

With the existence of the projection on the predictable range established, it remains to prove Lemmas 4.3 and 4.4, which we recall for convenience.

Lemma 4.3. *Let* (ϑ^n) *be a sequence in* $\mathcal{L}(S)$ *such that* $\vartheta^n \cdot S \to \vartheta \cdot S$ *in* $\mathcal{S}(P)$. *Then there exists a subsequence* (n_k) *such that* $\Pi^S \vartheta^{n_k} \to \Pi^S \vartheta$ P_B*-a.e.*

Proof. As in the proof of Theorem V.4 in [20], we can switch to a probability measure $Q \sim P$ such that $\frac{dQ}{dP}$ is bounded, $S - S_0 = M^Q + A^Q$ is in $\mathcal{M}^{2,d}(Q) \oplus \mathcal{A}^{1,d}(Q)$ and $\vartheta^n \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2,d}(Q) \oplus \mathcal{A}^{1,d}(Q)$ along a subsequence, again indexed by n. Since $\vartheta^n \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2,1}(Q) \oplus \mathcal{A}^{1,1}(Q)$, we obtain by using (4.1) with $B^Q = B$ that

$$E_Q\left[\int_0^\infty (\vartheta^n_s - \vartheta_s)^\top \hat{c}^Q_s (\vartheta^n_s - \vartheta_s) dB_s + \int_0^\infty \left|(\vartheta^n_s - \vartheta_s)^\top a^Q_s\right| dB_s\right] \longrightarrow 0$$

as $n \to \infty$, which implies that there exists a subsequence, again indexed by n, such that

$$(\vartheta^n - \vartheta)^\top \hat{c}^Q (\vartheta^n - \vartheta) \to 0 \quad \text{and} \quad |(\vartheta^n - \vartheta)^\top a^Q| \to 0 \quad Q \otimes B\text{-a.e.} \tag{5.5}$$

Since $P_B \sim Q \otimes B$, Lemma 5.1 gives

$$[0]^S = \left\{\varphi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \hat{c}^Q \varphi = 0 \text{ and } (a^Q)^\top \varphi = 0 \quad Q \otimes B\text{-a.e.}\right\}.$$

Let $e^1, \dots, e^d$ be predictable processes from Lemma 5.2 which satisfy properties 1)–4) for $[0]^S$ and set

$$U = \left\{\psi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \psi^\top \varphi = 0 \quad Q \otimes B\text{-a.e. for all } \varphi \in [0]^S\right\},$$
$$V = \left\{\psi \in \mathcal{L}^0\big(\overline{\Omega}, \mathcal{P}; \mathbb{R}^d\big) \;\middle|\; \psi^\top \varphi = 0 \quad Q \otimes B\text{-a.e. for all } \varphi \in [0]^{M^Q}\right\}$$

so that loosely speaking, $U^\perp = [0]^S$ and $V^\perp = [0]^{M^Q}$. Then $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$ satisfy the assumptions of Lemma 5.2 and thus there exist predictable processes $u^1, \dots, u^d$ and $v^1, \dots, v^d$ with the properties 1)–4) for $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$, respectively. By the definition of U and V we also obtain, using $[0]^S = [0]^{M^Q} \cap [0]^{A^Q}$, that

$$(e^j)^\top u^k = (e^j)^\top v^k = (u^j)^\top v^k = 0 \quad Q \otimes B\text{-a.e. for } j, k = 1, \dots, d$$

and

$$[0]^{M^Q} = \left\{\sum_{j=1}^d h^j e^j + \sum_{k=1}^d h^{d+k} u^k \;\middle|\; h^1, \dots, h^{2d} \text{ real-valued predictable}\right\},$$
$$[0]^{A^Q} = \left\{\sum_{j=1}^d h^j e^j + \sum_{k=1}^d h^{d+k} v^k \;\middle|\; h^1, \dots, h^{2d} \text{ real-valued predictable}\right\}$$

up to $Q \otimes B$-a.e. equality. Therefore $\mathit{\Pi}^{M^Q}$ and $\mathit{\Pi}^{A^Q}$ can be written as

$$\begin{aligned}\mathit{\Pi}^{M^Q} &= \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j(e^j)^\top - \sum_{k=1}^{d} u^k(u^k)^\top,\\ \mathit{\Pi}^{A^Q} &= \mathbb{1}_{d\times d} - \sum_{j=1}^{d} e^j(e^j)^\top - \sum_{k=1}^{d} v^k(v^k)^\top,\end{aligned}$$

and we have

$$\left(\sum_{k=1}^{d} v^k(v^k)^\top\right)\mathit{\Pi}^{A^Q}\vartheta^n = \left(\sum_{k=1}^{d} v^k(v^k)^\top\right)\vartheta^n, \tag{5.6}$$

all up to $Q\otimes B$-a.e. equality. Since $\mathit{\Pi}^{M^Q}(\vartheta^n-\vartheta)$ and $\mathit{\Pi}^{A^Q}(\vartheta^n-\vartheta)$ are by Lemma 5.1 $Q\otimes B$-a.e. valued in $\mathrm{Im}(\hat{c}^Q)$ and $\mathrm{Im}\big((a^Q)^\top\big)$, respectively, (5.5) yields $\mathit{\Pi}^{M^Q}\vartheta^n \to \mathit{\Pi}^{M^Q}\vartheta$ and $\mathit{\Pi}^{A^Q}\vartheta^n \to \mathit{\Pi}^{A^Q}\vartheta$ $Q\otimes B$-a.e. From the latter convergence and (5.6), it follows that

$$\left(\sum_{k=1}^{d} v^k(v^k)^\top\right)\vartheta^n \to \left(\sum_{k=1}^{d} v^k(v^k)^\top\right)\vartheta \quad Q\otimes B\text{-a.e.},$$

and since $Q\otimes B \sim P_B$ and

$$\mathit{\Pi}^S = \mathit{\Pi}^{M^Q} + \sum_{k=1}^{d} v^k(v^k)^\top \qquad Q\otimes B\text{-a.e.},$$

we obtain that $\mathit{\Pi}^S\vartheta^n \to \mathit{\Pi}^S\vartheta$ P_B-a.e. by combining everything. □

The only result whose proof is now still open is Lemma 4.4. This provides the general (and fairly abstract) version of the counterexample in Section 3, as well as the necessity part for the equivalence in Theorem 4.5.

Lemma 4.4. *Let $C : \overline{\Omega} \to 2^{\mathbb{R}^d}\setminus\{\emptyset\}$ be a predictable correspondence with closed values and such that the projection on the predictable range of S is* not closed, *i.e.*

$$\widetilde{F} = \big\{(\omega,t)\in\overline{\Omega}\ \big|\ \mathit{\Pi}^S(\omega,t)C(\omega,t) \text{ is not closed}\big\}$$

has outer P_B-measure >0. Then there exist $\vartheta\in\mathcal{L}(S)$ and a sequence (ψ^n) of C-valued integrands such that $\psi^n\cdot S\to\vartheta\cdot S$ in $\mathcal{S}(P)$, but there is no C-valued integrand ψ such that $\psi\cdot S=\vartheta\cdot S$. Equivalently, there exists a sequence $\big([\psi^n]\big)$ in $[\mathcal{C}]^S$ such that $[\psi^n]\overset{L(S)}{\to}[\vartheta]$ but $[\vartheta]\notin[\mathcal{C}]^S$, i.e. $[\mathcal{C}]^S$ is not closed in $L(S)$.

Proof. The basic idea is to construct a $\vartheta \in \mathcal{L}(S)$ which is valued in $\overline{\Pi^S C} \setminus \Pi^S C$ on some $F \in \mathcal{P}$ with $F \subseteq \widetilde{F}$ and $P_B(F) > 0$, and in C on F^c. Then there exists no C-valued integrand $\psi \in [\vartheta]$ by the definition of Π^S since $\Pi^S \vartheta \notin \Pi^S C$ on F; but one can construct a sequence (ψ^n) of C-valued integrands with $\Pi^S \psi^n \to \Pi^S \psi$ pointwise since $\Pi^S \vartheta \in \overline{\Pi^S C}$. However, this is technically a bit more involved for several reasons: While C, $\Pi^S C$ and $\overline{\Pi^S C}$ are all predictable, $(\Pi^S C)^c$ need not be; so $\widetilde{F}$ need not be predictable, and one cannot use Proposition 2.3 to obtain a predictable selector. In addition, $\overline{\Pi^S C} \setminus \Pi^S C$ need not be closed-valued.

We first argue that $\widetilde{F}$ is $\mathcal{P}_{P_B}$-measurable. Let $\overline{B(0,n)}$ be a closed ball of radius n in $\mathbb{R}^d$. Then $\Pi^S\big(C \cap \overline{B(0,n)}\big)$ is compact-valued as C is closed-valued. Since C is predictable and $\Pi^S(\omega,t)x$ with $x \in \mathbb{R}^d$ is a Carathéodory function, $\overline{\Pi^S C}$ is predictable by Proposition 2.7. By the same argument, $\Pi^S\big(C \cap \overline{B(0,n)}\big) = \overline{\Pi^S\big(C \cap \overline{B(0,n)}\big)}$ is predictable since $C \cap \overline{B(0,n)}$ is, and then so is $\Pi^S C = \bigcup_{n=1}^{\infty} \Pi^S\big(C \cap \overline{B(0,n)}\big)$ as a countable union of predictable correspondences; see Proposition 2.8. Then Proposition 2.6 implies that $\overline{\Pi^S C}$ and $\Pi^S\big(C \cap \overline{B(0,n)}\big)$ have predictable graph; hence so does $\Pi^S C$. Therefore $\mathrm{gr}(\overline{\Pi^S C}) \cap \big(\mathrm{gr}(\Pi^S C)\big)^c$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and so by Proposition 2.5,

$$
\begin{aligned}
\widetilde{F} &= \big\{(\omega,t) \in \overline{\Omega} \;\big|\; \Pi^S(\omega,t)C(\omega,t) \text{ is not closed}\big\} \\
&= \big\{(\omega,t) \in \overline{\Omega} \;\big|\; \overline{\Pi^S(\omega,t)C(\omega,t)} \setminus \Pi^S(\omega,t)C(\omega,t) \neq \emptyset\big\} \\
&= \pi_{\overline{\Omega}}\Big(\mathrm{gr}(\overline{\Pi^S C}) \cap \big(\mathrm{gr}(\Pi^S C)\big)^c\Big)
\end{aligned}
$$

is indeed $\mathcal{P}_{P_B}$-measurable. Thus there exists a predictable set $F \subseteq \widetilde{F}$ with $P_B(F) > 0$.

Now fix some C-valued integrand $\widetilde{\psi} \in \mathcal{L}(S)$ and define the correspondence C' by

$$
C'(\omega,t) = \begin{cases} \overline{\Pi^S(\omega,t)C(\omega,t)} \setminus \Pi^S(\omega,t)C(\omega,t) & \text{for } (\omega,t) \in F, \\ \widetilde{\psi}(\omega,t) & \text{else.} \end{cases}
$$

Then C' has non-empty values and predictable graph and therefore admits a P_B-a.e. predictable selector ϑ by Proposition 2.4. By possibly subtracting a predictable P_B-nullset from F, we can without loss of generality assume that ϑ takes values in C'. Moreover, the predictable sets $F_n := F \cap \{|\vartheta| \leq n\}$ increase to F and so we can, by shrinking

F to some F_n if necessary, assume that ϑ is uniformly bounded in (ω,t) on F. Let $\{\varphi^m \mid m \in \mathbb{N}\}$ be a Castaing representation of C as in Proposition 2.3. Then $\overline{\Pi^S C} = \overline{\{\Pi^S \varphi^m \mid m \in \mathbb{N}\}}$, and because $\vartheta \in \overline{\Pi^S C}$, we can find for each $n \in \mathbb{N}$ a predictable process ψ^n such that $\Pi^S(\omega,t)\psi^n(\omega,t) \in \vartheta(\omega,t) + \overline{B(0,\frac{1}{n})}$ on F and $\psi^n = \widetilde{\psi}$ on F^c. Note that on F, we have $\vartheta \in \overline{\Pi^S C} \subseteq \Pi^S \mathbb{R}^d$ and therefore $\Pi^S\vartheta = \vartheta$; so $\Pi^S\vartheta = \mathbb{1}_F\vartheta + \mathbb{1}_{F^c}\Pi^S\widetilde{\psi}$ and this shows that $\Pi^S\psi^n \to \Pi^S\vartheta$ uniformly in (ω,t) by construction. Since $\Pi^S\vartheta \in \mathcal{L}(S)$ because ϑ is bounded on F, we thus first get $\Pi^S\psi^n \in \mathcal{L}(S)$, hence $\psi^n \in \mathcal{L}(S)$, and then also that $\psi^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$ by dominated convergence. But now $\{\Pi^S\vartheta\} \cap \Pi^S C = \emptyset$ on F shows by Lemma 5.3 that there exists no C-valued integrand $\psi \in [\vartheta]$ and therefore $[\vartheta] \notin [\mathcal{C}]$. This ends the proof. $\square$

6 Related work

We have already explained how our results generalise most of the existing literature on optimisation problems under constraints. In this section, we discuss the relation to the work of Karatzas and Kardaras [16].

We start by introducing the terminology of [16]. For a given S with triplet (b,c,F), the linear subspace of *null investments* $\mathfrak{N}$ is given by the predictable correspondence

$$\mathfrak{N}(\omega,t) := \big\{z \in \mathbb{R}^d \;\big|\; z^\top c(\omega,t) = 0,\ z^\top b(\omega,t) = 0 \\ \text{and } F(\omega,t)(\{x \mid z^\top x \neq 0\}) = 0\big\}$$

(see Definition 3.6 in [16]). Note that we use F instead of ν and that our B is slightly different than in [16]. But this does not affect the definition of $\mathfrak{N}$. As in Definition 3.7 in [16], a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is said to *impose predictable closed convex constraints* if

0) $\mathfrak{N}(\omega,t) \subseteq C(\omega,t)$ for all $(\omega,t) \in \overline{\Omega}$,
1) $C(\omega,t)$ is a closed and convex set for all $(\omega,t) \in \overline{\Omega}$, and
2) C is predictable.

To avoid confusion, we call constraints with 0)–2) *KK-constraints* in the sequel.

In the comment following their Theorem 4.4 on p. 467 in [16], Karatzas and Kardaras (KK) remark that $\mathcal{C} \cdot S$ is closed in $\mathcal{S}(P)$ if C describes KK-constraints. For comparison, our Theorem 4.5 starts with C which is predictable and has closed values, and shows that $\mathcal{C} \cdot S$ is then closed in $\mathcal{S}(P)$ if and only if $\Pi^S C$ is closed P_B-a.e. So we do not need convexity of C, and our condition on C and S is not only sufficient, but also necessary.

Before explaining the connections in more detail, we make the simple but important observation that

$$\text{0) plus 1) imply that } C + \mathfrak{N} = C \text{ (for all } (\omega, t) \in \overline{\Omega}). \tag{6.1}$$

Indeed, each $\mathfrak{N}(\omega, t)$ is a linear subspace, hence contains 0, and so $C \subseteq C + \mathfrak{N}$. Conversely, $\frac{1}{\varepsilon} z \in \mathfrak{N} \subseteq C$ for every $z \in \mathfrak{N}$ and $\varepsilon > 0$ due to 0); so for every $c \in C$, $(1 - \varepsilon)c + z \in C$ by convexity and hence $c + z = \lim_{\varepsilon \searrow 0} (1 - \varepsilon)c + z$ is in C by closedness, giving $C + \mathfrak{N} \subseteq C$.

As a matter of fact, KK say, but do not explicitly prove, that $\mathcal{C} \cdot S$ is closed in $\mathcal{S}(P)$. However, the clear hint they give suggests the following reasoning. Let (ϑ^n) be a sequence in $\mathcal{C}$ such that $(\vartheta^n \cdot S) \to X$ in $\mathcal{S}(P)$. By the proof of Theorem V.4 in [20], there exist $\widetilde{\vartheta}^n \in [\vartheta^n]$ and $\vartheta \in \mathcal{L}(S)$ such that $\vartheta \cdot S = X$ and $\widetilde{\vartheta}^n \to \vartheta$ P_B-a.e. From the description of $\mathfrak{N}$ in Section 3.3 in [16], $\widetilde{\vartheta}^n \in [\vartheta^n]$ translates into $\widetilde{\vartheta}^n - \vartheta^n \in \mathfrak{N}$ P_B-a.e. or $\widetilde{\vartheta}^n \in \vartheta^n + \mathfrak{N}$ P_B-a.e. Because each ϑ^n has values in C, (6.1) thus shows that each $\widetilde{\vartheta}^n$ can be chosen to be C-valued, and by the closedness of C, the same is then true for the limit ϑ of $(\widetilde{\vartheta}^n)$. Hence we are done.

In order to relate the KK result to our work, we now observe that

$$\text{0) plus 1) imply that } \Pi^S C \text{ is closed } P_B\text{-a.e.}$$

To see this, we start with the fact that the null investments $\mathfrak{N}$ and $[0]^S$ are linked by

$$[0]^S = \{\varphi \mid \varphi \text{ is } \mathbb{R}^d\text{-valued predictable with } \varphi \in \mathfrak{N} \ P_B\text{-a.e.}\}; \tag{6.2}$$

see Section 3.3 in [16]. Recalling that Π^S is the projection on the orthogonal complement of $[0]^S$, we see from (6.2) that the column vectors of Π^S are P_B-a.e. a generating system of $\mathfrak{N}^\perp$ so that the projection of $\vartheta \in \mathcal{L}(S)$ on the predictable range of S can be alternatively defined P_B-a.e. as a predictable selector of the closed-valued predictable correspondence $\{\vartheta + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ or P_B-a.e. as the pointwise projection $\Pi^{\mathfrak{N}(\omega,t)} \vartheta(\omega, t)$ in $\mathbb{R}^d$ of $\vartheta(\omega, t)$ on $\mathfrak{N}(\omega, t)$, which is always a predictable process. This yields $\Pi^S C = \{C + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ P_B-a.e.; but by (6.1), $C + \mathfrak{N} = C$ due to 0) and 1), and so $\Pi^S C$ is P_B-a.e. closed like C and $\mathfrak{N}^\perp$.

In the KK notation, we could reformulate our Theorem 4.5 as saying that for a predictable and closed-valued C, the space $\mathcal{C} \cdot S$ is closed in

$\mathcal{S}(P)$ if and only if $C + \mathfrak{N}$ is closed P_B-a.e. This is easily seen from the argument above showing that $\Pi^S C = \{C + \mathfrak{N}\} \cap \mathfrak{N}^\perp$ P_B-a.e. If C is also convex-valued, 0) is a simple and intuitive sufficient condition; it seems however more difficult to find an elegant formulation without convexity.

The difference between our constraints and the KK formulation in [16] is as follows. We fix a set C of constraints and demand that the strategies should lie in C pointwise, so that $\vartheta(\omega, t) \in C(\omega, t)$ for all (ω, t). KK in contrast only stipulate that $\vartheta(\omega, t) \in C(\omega, t) + \mathfrak{N}(\omega, t)$ or, equivalently, that $[\vartheta] \in [\mathcal{C}]$. At the level of wealth (which is as usual in mathematical finance modelled by the stochastic integral $\vartheta \cdot S$), this makes no difference since all $\mathfrak{N}$-valued processes have integral zero. But for practical checking and risk management, it is much simpler if one can just look at the strategy ϑ and tick off pointwise whether or not it lies in C. If S has complicated redundancy properties, it may be quite difficult to see whether one can bring ϑ into C by adding something from $\mathfrak{N}$. Of course, when discussing the closedness of the space of integrals $\vartheta \cdot S$, we face the same level of difficulty when we have to check whether $\Pi^S C$ is closed P_B-a.e. But for actually working with given strategies, we believe that our formulation of constraints is more natural and simpler to handle.

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