Locally Φ-integrable σ-martingale densities for general semimartingales

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Abstract: A P-σ-martingale density for a given stochastic process S is a local P-martingale $Z > 0$ starting at 1 such that the product ZS is a P-σ-martingale. Existence of a P-σ-martingale density is equivalent to a classic absence-of-arbitrage property of S, and it is invariant if we replace the reference measure P with a locally equivalent measure Q. Now suppose that there exists a P-σ-martingale density for S. Can we find another P-σ-martingale density for S having some extra local integrability $\mathcal{I}_{\mathrm{loc}}(P)$ under P? We show that the answer is always positive for one part of S that we identify, and we show that the complete answer depends in a precise quantitative way on the local integrability of the drift-to-jump ratio of the remaining "jumpy" part of S. Our proofs provide in addition new ideas and results in infinite-dimensional spaces.

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1. Introduction

Consider a stochastic process $S = (S_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, I\!\!F, P)$. Assume S is arbitrage-free in the sense that there exists an equivalent martingale measure (EMM) R for S, i.e. a probability measure R equivalent to P such that S is a (local or maybe σ-)martingale under R. Let Q be another probability measure which is (maybe locally) equivalent to P and think of Q as an alternative reference model. We ask the following two questions:

1) Knowing only that S admits an EMM R, can we find another EMM R' whose density $\frac{dR'}{dP}$ has some additional integrability $\mathcal{I}(P)$ under P? (This question does not need the introduction of Q.)

2) Suppose S admits an EMM R' whose density $\frac{dR'}{dP}$ with respect to P has the integrability $\mathcal{I}(P)$ under P. Can we then also find an EMM R'' whose density $\frac{dR''}{dQ}$ with respect to Q has the (same) integrability $\mathcal{I}(Q)$ under Q?

It is well known from the classic Dalang–Morton–Willinger (DMW) theorem that the answer to both questions is positive in finite discrete time — we can even find an EMM with a bounded density. In contrast, both questions in the above form have negative answers in continuous time, as the following simple example illustrates.

Example 1.1. Let $N = (N_t)_{t \geq 0}$ be a standard Poisson process with intensity 1 and $I\!\!F$ the P-augmentation of the filtration generated by N. Consider the process

$$S_t = N_t - f(t) = N_t - t + \int_0^t \big(1 - \dot{f}(s)\big)\, ds, \qquad t \geq 0,$$

for a deterministic function f. Since the Poisson martingale $M_t = N_t - t$, $t \geq 0$, has the predictable representation property in $I\!\!F$, there is at most one candidate for an EMM R for S, and it can be formally obtained via the Girsanov transformation removing the instantaneous drift $1 - \dot{f}$ from S. But if $\dot{f}$ lacks sufficient integrability, the density $\frac{dR}{dP}$ of that single candidate will fail to have any good integrability, so that 1) has a negative answer. Moreover, if we take as new reference measure the EMM $Q = R$, then $R'' = Q$ has a density $\frac{dR''}{dQ} \equiv 1$ with arbitrary integrability $\mathcal{I}(Q)$, while the only possible choice $R' = Q$ gives $\frac{dR'}{dP}$ without nice integrability if $\dot{f}$ is bad. So also 2) (with P and Q interchanged) has a negative answer. $\square$

One key insight from our results in this paper is that in some sense (made precise below), Example 1.1 already contains the ingredients for everything that can go wrong in continuous time. To explain this rigorously, we need to introduce some concepts. A crucial point for these is that we formulate all our results locally, i.e. by means of localisation, as follows.

We now start properly with a probability space $(\Omega, \mathcal{F}, P)$ with a right-continuous filtration $I\!\!F^0 = (\mathcal{F}^0_t)_{t \geq 0}$ and $\mathcal{F}^0_\infty = \bigvee_{t \geq 0} \mathcal{F}^0_t$, and an $I\!\!R^d$-valued adapted RCLL process $S = (S_t)_{t \geq 0}$.

For any probability R on $(\Omega, \mathcal{F})$, we write as usual $R \overset{\text{loc}}{\approx} P$ if $R|_{\mathcal{F}_t}$ is equivalent to $P|_{\mathcal{F}_t}$ for any $t \geq 0$, where $\mathbb{F} = \mathbb{F}^R = (\mathcal{F}_t)_{0 \leq t \leq \infty}$ is the augmentation of $\mathbb{F}^0$ with respect to $\frac{1}{2}(R+P)$. For $R \overset{\text{loc}}{\approx} P$, we introduce the set

$$\mathcal{D}_{\mathrm{e},\sigma}(S,R) := \{\text{local } R\text{-martingales } Z > 0 \text{ with } Z_0 = 1 \mid ZS \text{ is an } R\text{-}\sigma\text{-martingale}\}$$

of *R-σ-martingale densities for S*. We recall that an $\mathbb{R}^d$-valued σ-martingale Y is a stochastic process of the form $Y = \int \psi \, dM = \psi \boldsymbol{\cdot} M$ with an $\mathbb{R}^d$-valued local martingale M and a one-dimensional predictable M-integrable process $\psi > 0$. Equivalently, there exists a bounded one-dimensional predictable process $\varphi > 0$ such that $\varphi \boldsymbol{\cdot} Y$ is an $\mathbb{R}^d$-valued local martingale. If $Y = ZS$ and $Z > 0$ is a local martingale, then $\varphi \boldsymbol{\cdot} Y = \varphi \boldsymbol{\cdot} (ZS)$ is a local martingale if and only if $Z(\varphi \boldsymbol{\cdot} S)$ is, and this property then extends to all one-dimensional bounded predictable $\varphi > 0$; see Lemma 2.9 below. Basic references for notations and results from stochastic calculus are Dellacherie/Meyer (1982) and Jacod/Shiryaev (2003), DM and JS from now on.

If S describes the discounted prices of d risky assets in a financial market containing also a bank account with constant discounted price 1, the set $\mathcal{D}_{\mathrm{e},\sigma}(S,R)$ appears naturally via arbitrage considerations from mathematical finance; see for instance Stricker/Yan (1998). Indeed, if S is a P-semimartingale and satisfies the classic no-arbitrage condition NFLVR, the fundamental theorem of asset pricing (FTAP) states that there exists a probability P^* equivalent to P with $P^* = P$ on $\mathcal{F}_0$ such that S is a P^*-σ-martingale; see Delbaen/Schachermayer (1998) or Chapter 14 of Delbaen/Schachermayer (2006). By the Bayes rule, the density process Z^* with respect to P of this equivalent σ-martingale measure P^* for S is then in $\mathcal{D}_{\mathrm{e},\sigma}(S,P)$; see Proposition 5.1 of Kallsen (2004). Note that if $\mathcal{D}_{\mathrm{e},\sigma}(S,R) \neq \emptyset$, then S is an R-semimartingale; this follows by applying Itô's formula to $S = (ZS)/Z$ with $Z \in \mathcal{D}_{\mathrm{e},\sigma}(S,R)$.

Because the condition NFLVR is formulated only in terms of semimartingales and the spaces L^0, L^∞, it is invariant under an equivalent change of measure. The same holds for the condition $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$: If Q is locally equivalent to P, then $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$ holds iff $\mathcal{D}_{\mathrm{e},\sigma}(S,Q) \neq \emptyset$. The proof is straightforward: If $D^{Q;P}$ is the density process of Q with respect to P and Z^P is in $\mathcal{D}_{\mathrm{e},\sigma}(S,P)$ so that $Z^P S$ is a P-σ-martingale, the Bayes rule directly yields that $Z^Q := Z^P D^{Q;P}_0 / D^{Q;P}$ is in $\mathcal{D}_{\mathrm{e},\sigma}(S,Q)$. The converse is symmetric. Moreover, the condition $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$ is also equivalent to saying that S satisfies the condition NUPBR (which is strictly weaker than NFLVR); see for instance Kardaras (2012) for $d = 1$ or Takaoka/Schweizer (2014).

Our first main result is

Theorem 1.2. *Let $S = (S_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued adapted RCLL process and assume that $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$. Then there exist $Z^P \in \mathcal{D}_{\mathrm{e},\sigma}(S,P)$ and $Z^{(1)}$ with the following property: For any one-dimensional bounded predictable $\varphi > 0$ such that $Z^P(\varphi \boldsymbol{\cdot} S)$ is a local P-martingale, we have a decomposition*

$$\varphi \boldsymbol{\cdot} S = X^{(1)} + X^{(0)}$$

such that $Z^{(1)}$ *is a locally bounded* P-σ*-martingale density for* $X^{(1)}$*, and such that* $X^{(0)}$ *admits a* P-σ*-martingale density, is quasi-left-continuous and has the form*

$$X^{(0)} = \sum_{n=1}^{\infty} (\widetilde{B}^n + \varphi_{\tau_n} \Delta X^{(0)}_{\tau_n} I_{[\![\tau_n,\infty[\![}) = \widetilde{B} + \sum_{n=1}^{\infty} \varphi_{\tau_n} \Delta X^{(0)}_{\tau_n} I_{[\![\tau_n,\infty[\![}, \tag{1.1}$$

where $\widetilde{B}$ *and each* $\widetilde{B}^n = I_{]\!]\tau_{n-1},\tau_n]\!]} \cdot \widetilde{B}^n$ *are continuous and of finite variation and* $(\tau_n)_{n\in I\!N}$ *is a sequence of stopping times with* $\tau_n \nearrow \infty$ P*-a.s. Note that* $Z^{(1)}$ *does not depend on* φ*.*

A more detailed version of Theorem 1.2 is given below in Theorem 8.1, but the above version is sufficient to convey the first main message. If a process S admits a P-σ-martingale density, then at least one of its integrals $\varphi \cdot S$ with $\varphi > 0$ bounded predictable admits a local P-martingale $Z > 0$ such that $Z(\varphi \cdot S)$ is also a local P-martingale. As pointed out above, the same is then true for all such integrals $\varphi \cdot S$. According to Theorem 1.2, any such integral $\varphi \cdot S$ has one part $X^{(1)}$ which admits a locally bounded P-σ-martingale density $Z^{(1)}$ — which can even be chosen to work simultaneously for all φ —, and a second part $X^{(0)}$ which is an at most countable sum of quasi-left-continuous single-jump processes each with a continuous drift part. This shows that for question 1), at least in localised form, the only possible difficulties come from those parts of S that have the same basic structure as Example 1.1.

Our second main result deals with the part $X^{(0)}$ from Theorem 1.2. To explain this, we need to look ahead a bit. In (4.3) in Section 4, we decompose $S - S_0 = S^a + S^i$, where S^i is quasi-left-continuous, and the precise version (in Theorem 8.1) of Theorem 1.2 shows that

$$\varphi \cdot S = (\varphi \cdot S^a + \bar{X}^{(1)}) + X^{(0)}$$

so that $X^{(0)}$ is quasi-left-continuous like S^i and comes from S^i only.

Theorem 1.3. *In the setting of Theorem 1.2, suppose that the quasi-left-continuous process* $X^{(0)}$ *has as in (1.1) the form*

$$X^{(0)} = \sum_{n=1}^{\infty} (\widetilde{B}^n + \varphi_{\tau_n} \Delta X^{(0)}_{\tau_n} I_{[\![\tau_n,\infty[\![}) = (\varphi \widetilde{b}) \cdot A + \varphi \cdot (x * \mu^{(0)}), \tag{1.2}$$

where $\mu^{(0)}$ *is the jump measure of* $X^{(0)}$*,* $\nu^{(0)}(dt,dx) = F^{(0)}_t(dx)\,dA_t$ *its compensator and* A *controls the characteristics of* S*. Define, with* $\psi(x) = \frac{x}{1+|x|}$ *on* $I\!R^d$*, the predictable process*

$$\widetilde{\mathcal{R}}^{(0)} := I_{\{\widetilde{b}\neq 0\}} F^{(0)}\text{-}\operatorname*{ess\,sup}_{z\in I\!R^d} \frac{(-z^\top \widetilde{b})^-}{\int_{I\!R^d} (z^\top \psi(x))^- \, F^{(0)}(dx)}. \tag{1.3}$$

(This is well defined, as remarked below in Sections 6 and 8, and argued in detail in Section 9.) Recall that $X^{(0)}$ *admits a* P-σ*-martingale density and let* $\Phi : [0,\infty) \to I\!R$ *with* $\Phi(0) = 0$ *be strictly convex, in* C^1*, of at least linear growth, and uniformly bounded from below. If*

$$\int_0^T F^{(0)}_t(I\!R^d) \big|\Phi\big(\alpha(t)\widetilde{\mathcal{R}}^{(0)}_t\big)\big| \, dA_t < \infty \qquad P\text{-a.s. for each } T \in (0,\infty) \tag{1.4}$$

for a measurable function $\alpha : [0,\infty) \to I\!R$ which is uniformly strictly positive on compact intervals, then $X^{(0)}$ admits a P-σ-martingale density Z in $L^\Phi_{\rm loc}(P)$, i.e. with $\sup_{0\le t\le T_n} Z_t \in L^\Phi(P)$ for each n, for a sequence $(T_n)_{n\in I\!N}$ of stopping times with $T_n \nearrow \infty$ P-a.s.

Again, a more detailed version is given below in Theorem 8.3, and again the current version is enough to convey the key message of the result. If we have (a φ-integral of) a single-jump process with a continuous drift part (or even a countable sum of such processes) and if that process admits a P-σ-martingale density, then we can find another P-σ-martingale density with local integrability in L^Φ if we have a local control in L^Φ on the *drift-to-jump ratio* of the process, as measured by the quantity $\widetilde{\mathcal{R}}^{(0)}$ in (1.3), where the control is quantified by (1.4). (We believe that this result is essentially sharp, in view of Example 1.1 and Theorem 10.1 below. But we do not investigate this in detail here, for reasons of space.)

In combination, Theorems 1.2 and 1.3 show clearly what can be said about nicely integrable σ-martingale densities in continuous time. Always assuming that the basic process S admits at least one P-σ-martingale density, one can split any integral $\varphi\cdot S$ into an unproblematic part $X^{(1)} = \varphi\cdot S^a + \bar{X}^{(1)}$ (which even admits a locally bounded P-σ-martingale density) and a part $X^{(0)}$ consisting of single jumps with continuous drifts. The drift-to-jump behaviour of $X^{(0)}$ then completely determines, in a very precise quantitative sense, how much (local) integrability we can expect for a P-σ-martingale density for $X^{(0)}$ or S.

Remark 1.4. If we assume instead of (1.4) that $\widetilde{\mathcal{R}}^{(0)}$ is locally bounded, then $X^{(0)}$ admits a P-σ-martingale density Z which is even locally bounded. For more details, we refer to Remark 8.5 below. $\diamond$

Remark 1.5. If S is *continuous*, the whole situation is much simpler: If $\mathcal{D}_{{\rm e},\sigma}(S,P) \neq \emptyset$, then S even admits a P-σ-martingale density which is locally bounded away from 0 and ∞.

We can actually give two different arguments. For the first, in terms of the present paper, we start with some $\bar{Z} = \mathcal{E}(\bar{N})$ in $\mathcal{D}_{{\rm e},\sigma}(S,P)$ and use the Jacod decomposition in Theorem 2.4 below to write $\bar{N} = \beta\cdot S + N'$ with some S-integrable predictable process β and some local P-martingale N' null at 0 with $[S,N'] \equiv 0$. Then the product rule readily shows that also $\bar{\bar{Z}} := \mathcal{E}(\beta\cdot S)$ is in $\mathcal{D}_{{\rm e},\sigma}(S,P)$, and locally bounded away from 0 and ∞, as it is continuous.

Using different terminology, we can alternatively argue as follows. If S is continuous and admits a P-σ-martingale density, it is well known that S satisfies the structure condition (SC) under P, and so the P-minimal martingale density $\widehat{Z}^{(P)}$ exists. Since this is continuous like S, we obtain the same conclusion as above. For more details on the above results and terminologies, we refer to Hulley/Schweizer (2010). $\diamond$

To the best of our knowledge, the existing literature has no general results similar to Theorems 1.2 or 1.3. At best, we are aware of some remotely related work. Under additional integrability or continuity assumptions, Stricker (1990) has given necessary and sufficient arbitrage-type conditions on S such that there exists an equivalent martingale measure R for S with $\frac{dR}{dP} \in L^p(P)$. In finite discrete time, the classic result of Dalang/Morton/Willinger (1990) shows that absence of arbitrage for S is equivalent to the existence of an EMM R for S with $\frac{dR}{dP} \in L^\infty$. Also in finite discrete time, Rokhlin (2010) has given a necessary and sufficient condition for the existence of an EMM R for S with $\frac{dR}{dP}$ bounded below by some given random variable $f > 0$; an abstract variant of this result can be found in Rokhlin/Schachermayer (2006). Rásonyi (2001) (for finite discrete time) and Kabanov/Stricker (2001) have shown that the set of equivalent (σ-)martingale measures R for S with bounded density $\frac{dR}{dP} \in L^\infty$ is dense, for the total variation norm, in the set of all E(σ)MMs for S. (Earlier uses of this result without complete proofs appear in Jacka (1992) and Kramkov (1996).) The paper of Kabanov/Stricker (2001) also has a variant of the last denseness result with boundedness replaced by some integrability. However, almost all these papers either have extra conditions on S or work in and exploit a setting of finite discrete time. In addition, they also study different questions than we do here and mostly use quite different techniques.

Another distantly related result can be found in Theorem 1.4 of Prokaj/Rásonyi (2011); see also Theorem 2.2.2 of Kabanov/Safarian (2010). It shows that for a process S in discrete time, one can find martingale measures for S arbitrarily close (in the total variation norm) to the original measure P and such that they also make integrable an arbitrary (but a priori given) process. One referee has pointed out that some of our proofs and the above result have in common that they are based on the same idea as the key step in the proof of the DMW theorem: They exploit the one-step characterisation of absence of arbitrage to obtain a martingale measure with bounded density. However, neither Prokaj/Rásonyi (2011) nor Kabanov/Safarian (2010) give any continuous-time version of that result. Moreover, one key aspect of our paper is that we must go substantially beyond a DMW framework since our single-jump processes have an additional nonzero drift. As discussed in detail in Section 9, this seemingly small difference is at the root of many technical and conceptual difficulties we encounter.

Apart from the two main theorems presented above, our paper involves several major and innovative contributions. One is Theorem 7.1 which shows a continuous-time analogue of the classic DMW theorem: If S is a process consisting of at most countably many jumps at predictable times, then S admits a P-σ-martingale density if and only if it admits a locally bounded P-σ-martingale density. This can be viewed as a generalisation of Theorem 5 in Stricker (1990). The proof uses with Theorem 2.1 from Choulli/Schweizer (2015) a result of independent interest — we extend a theorem of Borwein/Lewis (1991) on integral functionals subject to linear equality constraints from finitely to (maybe uncountably) infinitely many constraints. Next, the key step for the proof of Theorem 1.3 or 8.3, given in Theorem 9.2, uses novel ideas and techniques. To construct a nicely integrable P-σ-martingale density for a single-jump process $X^{(0)}$ with continuous drift, we first provide an infinite-dimensional extension of a key lemma in Kabanov/Stricker (2001). In addition, we construct the desired

P-σ-martingale density $Z^{(0)}$ by solving an optimisation problem (whose objective functional is linked to the function Φ specifying the desired integrability) under constraints (given by the σ-martingale condition for the product $Z^{(0)}X^{(0)}$). This innovative construction is crucial in helping us to circumvent issues of measurable selection which otherwise become quite delicate.

The paper is structured as follows. Section 2 presents a classic decomposition, going back to Grigelionis (1973) and Jacod (1979), to describe any local P-martingale null at 0 in terms of four stochastic processes (called Jacod parameters) computed with respect to some given semimartingale (here, usually S). It also contains other auxiliary results of general interest. Section 3 recalls concepts like L^Φ and proves corresponding results on local integrability of exponential local martingales. Section 4 prepares the ground by showing how one can break up our general process S into simpler pieces with extra properties. One decomposition splits S as $S - S_0 = S^a + S^i$ into an accessible part S^a and a quasi-left-continuous part S^i. Another is to simply write

$$S - S_0 = S - S^0 = \sum_{n=1}^{\infty} (S^{\tau_n} - S^{\tau_{n-1}}) = \sum_{n=1}^{\infty} I_{]\!]\tau_{n-1},\tau_n]\!]} \cdot S \tag{1.5}$$

for a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\tau_0 := 0$ and $\tau_n \nearrow \infty$ P-a.s.

The work on S starts in earnest in Section 5. For a predictable bounded $\varphi > 0$ such that $X := \varphi \cdot S$ has a P-local martingale density, we study $X^\tau - X^\sigma = I_{]\!]\sigma,\tau]\!]} \cdot X$, as representative for $X^{\tau_n} - X^{\tau_{n-1}}$, and we show how to split off from this a single-jump process $X^{(0)}$ with continuous drift such that the remaining part of $X^\tau - X^\sigma$ admits a locally bounded P-σ-martingale density; see Theorem 5.1. Moreover, all ingredients "live" only on $]\!]\sigma,\tau]\!]$ so that we can later piece things together for X via (1.5). Section 6 shows how to construct a P-σ-martingale density in L^Φ_{loc} for $X^\tau - X^\sigma$ if S is quasi-left-continuous and if we have an L^Φ-control on $X^{(0)}$; this is done for $X^{(0)}$ via Theorem 6.1 which is formulated, but not yet proved there. Section 7 constructs in Theorem 7.1 a locally bounded P-σ-martingale density for the accessible part S^a of S. In Section 8, we combine the previous work to establish in Theorems 8.1 and 8.3 the general versions of the main results presented in Theorems 1.2 and 1.3. Sections 9 and 10 contain the proof of Theorem 6.1 (which is restated with more details in Theorem 9.2). As already mentioned, this part contains two major conceptual innovations, discussed in detail after Theorem 9.2.

2. Positive local martingales and the Jacod decomposition

Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d$-valued semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the usual conditions. Our goal here is to describe all local P-martingales $N = (N_t)_{t\geq 0}$ in terms of certain parameters relative to X, and to make explicit what this entails if $Z = \mathcal{E}(N)$ is strictly positive and ZX is a σ-martingale. Many of the results presented in this section can be found in the literature; but they are less well known than they deserve to be, and the proofs one can find are not always very detailed. Since our arguments rely heavily on these techniques, we have therefore opted for a largely self-contained exposition.

We first fix notations, following JS. The jump measure of X is denoted by $\mu = \mu^X$, and its P-compensator by $\nu = \nu^P$. The characteristics of X with respect to a fixed truncation function h are $B = b \bullet A = \int b\,dA$, $C = c \bullet A$, $\nu^P(dt,dx) = F_t^P(dx)\,dA_t$, where the dominating process A is null at 0, increasing, predictable and RCLL (hence locally bounded); B, b are $I\!R^d$-valued, C, c are $d \times d$-matrix-valued processes, and each F_t^P is a random measure on $(I\!R^d, \mathcal{B}(I\!R^d))$ with $F_t^P(\{0\}) = 0$. For a product-measurable function $W \geq 0$ on $\Omega \times [0,\infty) \times I\!R^d$, we need the processes $W * \mu_t := \int_0^t \int_{I\!R^d} W_s(x)\,\mu(ds,dx)$, $t \geq 0$, and

$$\widehat{W}_t := \int_{I\!R^d} W_t(x)\,\nu^P(\{t\},dx), \qquad t \geq 0, \tag{2.1}$$

with values in $[0,+\infty]$; see JS, II.1.24. If W is measurable with respect to $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(I\!R^d)$, then $\widehat{W}$ is by JS, Lemma II.1.25, predictable and given by

$$\widehat{W} = {}^{\mathbf{p}}\big(W(\Delta X) I_{\{\Delta X \neq 0\}}\big), \tag{2.2}$$

where ${}^{\mathbf{p}}Y$ is the predictable projection of the process Y. In particular, for $W \equiv 1$, we have

$$a_t := \widehat{1}_t = \nu^P(\{t\}, I\!R^d) = {}^{\mathbf{p}}\big(I_{\{\Delta X \neq 0\}}\big)_t, \qquad t \geq 0. \tag{2.3}$$

We also need the measure $M_\mu^P = P \otimes \mu$ on $\Omega \times [0,\infty) \times I\!R^d$ given by $\int W\,dM_\mu^P = E[W * \mu_\infty]$. Other unexplained notations can be found in JS.

Lemma 2.1. *We have, up to an evanescent predictable set,*

$$\{a = 1\} \subseteq \{\Delta X \neq 0\}. \tag{2.4}$$

If $1 * \mu$ is P-a.s. finite-valued (which means that for P-almost all ω, $X_.(\omega)$ has on any compact interval at most finitely many jumps), then the (predictable) process

$$\frac{1}{1-a} I_{\{a<1\}} = (1-a)^{-1} I_{\{a<1\}} \text{ is locally bounded,} \tag{2.5}$$

and therefore also $(1 - a + \widehat{f})^{-1} I_{\{a<1\}}$ is locally bounded for any measurable $f \geq 0$ on $\Omega \times [0,\infty) \times I\!R^d$. More generally, for any $\widetilde{\mathcal{P}}$-measurable function W such that $|W| * \nu^P$ is P-a.s. finite-valued, the predictable process $(1 - \widehat{W})^{-1} I_{\{\widehat{W}<1\}}$ is well defined and locally bounded. In particular, this is true for any $\widetilde{\mathcal{P}}$-measurable and finite-valued function W if $1 * \mu$ is P-a.s. finite-valued.

Proof. This partly follows the proof of Theorem 11.14 in He/Wang/Yan (1992). By (2.3), we have $0 \le a \le 1$ and the process a is thin. For any predictable stopping time T with $[\![T]\!] \subseteq \{a = 1\}$, (2.3) yields

$$E[\mu([\![T]\!] \times I\!R^d)] = E\big[I_{\{\Delta X_T \neq 0\}} I_{\{T<\infty\}}\big] = E\big[a_T I_{\{T<\infty\}}\big] = P[T < \infty]$$

so that we get $\mu([\![T]\!] \times I\!R^d) = 1$ P-a.s. on $\{T < \infty\}$. Since T was arbitrary and a is thin, this implies (2.4). For (2.5), we first note that for any $\delta \in (0, 1)$,

$$(1-a)^{-1} I_{\{a<1\}} \le (1-\delta)^{-1} I_{\{a\le\delta\}} + (1-a)^{-1} I_{\{\delta<a<1\}}.$$

If the process $1 * \mu$ is P-a.s. finite-valued, then so is its compensator $V := 1 * \nu^P$, and V is RCLL. For any $\delta > 0$, the set $\{s \le t \mid \Delta V_s > \delta\}$ is therefore finite P-a.s. for each fixed t. But $\Delta V_s = \nu^P(\{s\}, I\!R^d) = a_s$ by (2.3), and so for P-almost all ω, any compact interval contains at most finitely many s with $a_s(\omega) > \delta$. Therefore the process

$$\sup_{0\le s\le t} (1-a_s)^{-1} I_{\{a_s<1\}} \le (1-\delta)^{-1} + \sup_{0\le s\le t} (1-a_s)^{-1} I_{\{\delta<a_s<1\}}, \qquad t \ge 0,$$

is increasing and P-a.s. finite-valued, because the last supremum is actually a maximum over those finitely many s. So $(1-a)^{-1} I_{\{a<1\}}$ is prelocally bounded, and since it is predictable like a, it is therefore locally bounded; see DM, VIII.11. This proves (2.5). The argument for W is analogous; we simply replace the process $1 * \nu^P$ by $W * \nu^P$, which has finite variation by assumption, and we use (2.1) instead of (2.3). In particular, if $1 * \mu$ is P-a.s. finite-valued, then so is $|W| * \mu$ and hence also $|W| * \nu^P$. **q.e.d.**

The next result is classic.

Proposition 2.2. [Jacod (1979), Théorème 3.75] *Recall that the $I\!R^d$-valued semimartingale X is fixed. Every (real-valued) local P-martingale N null at 0 can be written as*

$$N = W * (\mu - \nu^P) + g * \mu + \bar{N}.$$

More explicitly, with

$$U := M^P_\mu(\Delta N \,|\, \widetilde{\mathcal{P}}),$$

we have that

$$g := \Delta N - U \text{ is in } \mathcal{H}^1_{\mathrm{loc}}(\mu) \tag{2.6}$$

(which means that $g : \Omega \times [0, \infty) \times I\!R^d \to I\!R$ is measurable, the signed measure $g.M^P_\mu$ is $\widetilde{\mathcal{P}}$-σ-finite, $M^P_\mu(g \,|\, \widetilde{\mathcal{P}}) = 0$, and $(g^2 * \mu)^{1/2}$ is locally integrable), that

$$W := U + \frac{\widehat{U}}{1-a} I_{\{a<1\}} \text{ is in } \mathcal{G}^1_{\mathrm{loc}}(\mu), \tag{2.7}$$

which means that W is $\widetilde{\mathcal{P}}$-measurable and

$$\Big(\sum_{0<s\le\cdot}\big(W_s(\Delta X_s)I_{\{\Delta X_s\neq 0\}}-\widehat{W}_s\big)^2\Big)^{1/2} \text{ is locally integrable,}$$

and that

$$\bar{N} \text{ is a local } P\text{-martingale null at 0 with } \{\Delta\bar{N}\neq 0\}\subseteq\{\Delta X=0\}. \tag{2.8}$$

Moreover, we can and do choose a version of U with

$$\{\widehat{U}\neq 0\}\subseteq\{a<1\}, \qquad \text{so that } \widehat{U}=0 \text{ on } \{a=1\}. \tag{2.9}$$

If we know that $\Delta N>-1$, then also $U>-1$ M^P_μ-a.e., and setting

$$f:=U+1>0 \tag{2.10}$$

then gives by (2.3)

$$W=f-1+\frac{\widehat{f}-a}{1-a}I_{\{a<1\}}. \tag{2.11}$$

The next result collects some additional properties that we need later.

Lemma 2.3. *In the setting of Proposition 2.2, the process $\widehat{U}$ is locally bounded and the processes $\sup_{0<s\le\cdot}|f_s(\Delta X_s)|I_{\{\Delta X_s\neq 0\}}$ and $\sup_{0<s\le\cdot}|g_s(\Delta X_s)|I_{\{\Delta X_s\neq 0\}}$ are locally integrable. If $1*\mu$ is finite-valued P-a.s., then also $\widehat{W}$ is locally bounded.*

Proof. In view of (2.7), (2.1), (2.3) and (2.9), we have

$$\widehat{W}=\widehat{U}+a\frac{\widehat{U}}{1-a}I_{\{a<1\}}=\widehat{U}\frac{1}{1-a}I_{\{a<1\}} \tag{2.12}$$

and therefore

$$W(\Delta X)I_{\{\Delta X\neq 0\}}-\widehat{W}=U(\Delta X)I_{\{\Delta X\neq 0\}}-\widehat{W}I_{\{\Delta X=0\}}. \tag{2.13}$$

Due to (2.5) in Lemma 2.1, the last assertion follows if we prove that $\widehat{U}$ is locally bounded. Set $Y_t:=U_t(\Delta X_t)I_{\{\Delta X_t\neq 0\}}$; then $\widehat{U}={}^{\mathbf{p}}Y$ by (2.2) and hence $|\widehat{U}|=|{}^{\mathbf{p}}Y|\le{}^{\mathbf{p}}|Y|$ by Jensen's inequality. Moreover, (2.13) gives for any $s\le t$ that

$$\begin{aligned}|Y_s| &\le\Big(\sum_{0<r\le t}\big(U_r(\Delta X_r)\big)^2 I_{\{\Delta X_r\neq 0\}}\Big)^{1/2}=(U^2*\mu_t)^{1/2}\\ &=\Big(\sum_{0<r\le t}\big(W_r(\Delta X_r)-\widehat{W}_r\big)^2 I_{\{\Delta X_r\neq 0\}}\Big)^{1/2}\\ &\le\Big(\sum_{0<r\le t}\big(W_r(\Delta X_r)I_{\{\Delta X_r\neq 0\}}-\widehat{W}_r\big)^2\Big)^{1/2}=:V_t.\end{aligned} \tag{2.14}$$

By Proposition 2.2, the increasing process V is locally integrable because W is in $\mathcal{G}^1_{\mathrm{loc}}(\mu)$. As $f = U + 1$, this gives the first half of the second assertion, and moreover V has a compensator $\widetilde{V}$. Because $|Y| \le V$, we get $|\widehat{U}| \le {}^{\mathbf{p}}|Y| \le {}^{\mathbf{p}}V = V_- + {}^{\mathbf{p}}(\Delta V) = V_- + \Delta\widetilde{V} \le V_- + \widetilde{V}$, where the second equality uses DM, Theorem VI.76. But V_- is locally bounded, and so is $\widetilde{V}$ since it is predictable and RCLL. Hence $\widehat{U}$ is locally bounded as well. Finally, the assertion about g is proved in exactly the same way as that for f, exploiting instead of $W \in \mathcal{G}^1_{\mathrm{loc}}(\mu)$ that g is in $\mathcal{H}^1_{\mathrm{loc}}(\mu)$ so that $(g^2 * \mu)^{1/2}$ is locally integrable. **q.e.d.**

Now apply the Kunita–Watanabe decomposition under P to $\bar{N}$ and the continuous local martingale part X^c of X to write $\bar{N} = \beta \mathbin{\bullet} X^c + N'$, where β is a (predictable) X^c-integrable process and N' a local P-martingale null at 0 and strongly P-orthogonal to X^c. By the continuity of X^c, this means that $\langle X^c, N' \rangle \equiv 0$; so also $[X, N'] = \langle X^c, N' \rangle + \sum \Delta X \Delta N' \equiv 0$ because $\Delta N' = \Delta \bar{N}$ vanishes on $\{\Delta X \ne 0\}$ by (2.8). In summary, we have proved

Theorem 2.4. (Jacod decomposition) *Recall that the $I\!\!R^d$-valued semimartingale X is fixed. Every local P-martingale N null at 0 can be written as*

$$N = \beta \mathbin{\bullet} X^c + W * (\mu - \nu^P) + g * \mu + N', \tag{2.15}$$

where β is X^c-integrable, $W = U + \frac{\widehat{U}}{1-a} I_{\{a<1\}} \in \mathcal{G}^1_{\mathrm{loc}}(\mu)$, $g \in \mathcal{H}^1_{\mathrm{loc}}(\mu)$, and N' is a local P-martingale null at 0 with

$$[X, N'] \equiv 0. \tag{2.16}$$

If $\Delta N > -1$, then $W = f - 1 + \frac{\widehat{f} - a}{1-a} I_{\{a<1\}}$, where $f > 0$ is $\widetilde{\mathcal{P}}$-measurable, and then

$$f + g > 0 \qquad M^P_\mu\text{-a.e.} \tag{2.17}$$

Proof. The only statement left to prove is (2.17). But we have M^P_μ-a.e. that $g = \Delta N - U$ by (2.6) and $f = U + 1$ by (2.10), so $f + g = \Delta N + 1 > 0$. **q.e.d.**

In the sequel, we call (β, W, g, N') or (β, U, g, N'), or (β, f, g, N') for $\Delta N > -1$, the *Jacod parameters* of N (under P, with respect to X).

Remark 2.5. The decomposition (2.15) in Theorem 2.4 has already been given (without details) in Choulli/Stricker (2006), Theorem 2.1, or in Choulli/Stricker/Li (2007), Theorem 2.1. Both contain the additional integrability assertion that (for every $T \in (0, \infty)$)

$$\int_0^T \int_{I\!\!R^d \setminus \{0\}} |U_s(x)| \, \nu^P(ds, dx) < \infty \qquad P\text{-a.s.}, \tag{2.18}$$

but do not give a proof for this. Our subsequent results do not need (2.18). ◇

For later use, we explicitly compute the jumps of N in terms of its Jacod parameters.

Lemma 2.6. *If the local P-martingale N null at 0 is given by the Jacod decomposition (2.15), then its jumps are*

$$\Delta N = \big(f(\Delta X) + g(\Delta X) - 1\big) I_{\{\Delta X \neq 0\}} + \big(\Delta N' - \widehat{W}\big) I_{\{\Delta X = 0\}} \tag{2.19}$$

with $f = U + 1$. As a consequence, $\Delta N > -1$ if and only if

$$\begin{cases} f(\Delta X) + g(\Delta X) > 0 & \text{on } \{\Delta X \neq 0\}, \\ 1 - \widehat{W} + \Delta N' > 0 & \text{on } \{\Delta X = 0\}. \end{cases}$$

Proof. The second assertion clearly follows from (2.19). By (2.6), $\Delta N = g + U = f + g - 1$ M^P_μ-a.e. On the other hand, (2.15) gives

$$\begin{aligned} \Delta N &= W(\Delta X) I_{\{\Delta X \neq 0\}} - \widehat{W} + g(\Delta X) I_{\{\Delta X \neq 0\}} + \Delta N' \\ &= \big(W(\Delta X) - \widehat{W} + g(\Delta X)\big) I_{\{\Delta X \neq 0\}} + \big(\Delta N' - \widehat{W}\big) I_{\{\Delta X = 0\}}, \end{aligned} \tag{2.20}$$

because $\Delta X \Delta N' = \Delta[X, N'] = 0$ by (2.16) and hence $\Delta N' = 0$ on $\{\Delta X \neq 0\}$. But we have seen in (2.7) and (2.12) that $W - \widehat{W} = U = f - 1$ M^P_μ-a.e., and so (2.19) follows. **q.e.d.**

Our next result tells us how information about $\mathcal{E}(N)$ gives us information about the Jacod parameters of N. We say that a measurable function $\Psi : \Omega \times [0, \infty) \times I\!R^d \to I\!R$ is *μ-locally bounded* if there exist stopping times $(T_n)_{n \in I\!N}$ increasing to $+\infty$ P-a.s. such that for each n, $\Psi I_{]\!]0,T_n]\!]}$ is bounded (uniformly in (ω, t, x)) μ-almost everywhere P-a.s. (M^P_μ-a.e., in other words). If Ψ does not depend on $x \in I\!R^d$, this reduces to the usual notion of local boundedness; the point here is that the stopping times T_n do not depend on x.

Proposition 2.7. *Suppose that the local P-martingale N null at 0 is given by the Jacod decomposition (2.15) and $f = U + 1$.*

1) *If $\mathcal{E}(N) > 0$, then*

$$1 - \widehat{W} > 0 \quad \text{and} \quad 1 + \frac{\Delta N'}{1 - \widehat{W}} > 0, \qquad \text{both on } \{\Delta X = 0\}. \tag{2.21}$$

2) *If $\mathcal{E}(N) > 0$, then $f > 0$ and $f + g > 0$, M^P_μ-a.e. If $\Delta N \geq -1 + \delta > -1$ with a constant δ, we even have $f + g \geq \delta > 0$ and $f \geq \delta > 0$, M^P_μ-a.e. If $\mathcal{E}(N) > 0$ is even locally bounded, then N is locally bounded, and f and $f + g$ are both μ-locally bounded.*

Proof. 1) This is similar to the argument for (2.4) in Lemma 2.1. Since $\mathcal{E}(N) > 0$ implies $\Delta N > -1$, we get $1-\widehat{W}+\Delta N' > 0$ on $\{\Delta X = 0\}$ or, since $\Delta N' = 0$ on $\{\Delta X \neq 0\}$ by (2.16),

$$(1-\widehat{W})I_{\{\Delta X=0\}} + \Delta N' \geq 0. \tag{2.22}$$

Taking predictable projections and using (2.3) and the fact that N' is a local P-martingale (so that ${}^{\mathbf{p}}\Delta N' = 0$) gives $(1-\widehat{W})(1-a) \geq 0$. Because $a < 1$ on $\{\Delta X = 0\}$ by Lemma 2.1, we get $1-\widehat{W} \geq 0$ on $\{\Delta X = 0\}$. Now $I_{\{\widehat{W}=1\}}\Delta N' = I_{\{\widehat{W}=1\}}I_{\{\Delta X=0\}}\Delta N' \geq 0$ by (2.16) and (2.22), and since ${}^{\mathbf{p}}(I_{\{\widehat{W}=1\}}\Delta N') = I_{\{\widehat{W}=1\}}{}^{\mathbf{p}}\Delta N' = 0$, we must have $\widehat{W} < 1$ on $\{\Delta N' \neq 0\}$ up to an evanescent predictable set. So $\Delta N' = 0$ on $\{\widehat{W} = 1\}$, and since $1-\widehat{W}+\Delta N' > 0$ on $\{\Delta X = 0\}$, we must have $1-\widehat{W} > 0$ on $\{\Delta X = 0\}$ to avoid a contradiction. Dividing by $1-\widehat{W}$ then yields (2.21).

2) Clearly $N = \frac{1}{\mathcal{E}(N)_-} \cdot \mathcal{E}(N)$ is locally bounded if $\mathcal{E}(N) > 0$ is. Then ΔN is locally bounded as well, and we can by localisation assume that $1+\Delta N \leq C$ P-a.s. for some constant $C < \infty$. Using only $\mathcal{E}(N) > 0$ to get $\Delta N > -1$ then yields by (2.19) that $0 < f+g \leq C$ P-a.s. on $\{\Delta X \neq 0\}$, i.e. M^P_μ-a.e., with even a uniform lower bound $\delta > 0$ if $1+\Delta N \geq \delta$. Because f is $\widetilde{\mathcal{P}}$-measurable and $M^P_\mu(g \,|\, \widetilde{\mathcal{P}}) = 0$ since $g \in \mathcal{H}^1_{\mathrm{loc}}(\mu)$, we also get by "conditioning on $\widetilde{\mathcal{P}}$ under M^P_μ" that $0 < f \leq C$ M^P_μ-a.e., again with a uniform lower bound $\delta > 0$ if $1+\Delta N \geq \delta$. Finally, the proof makes it clear that the lower bounds on f and $f+g$ do not need local boundedness of $\mathcal{E}(N)$. **q.e.d.**

We next refine the proof technique from Lemma 2.3 to improve a part of Lemma 2.1. Note that $\bar{U}$ is just an abstract function not related to X or N.

Lemma 2.8. **1)** *Let $\bar{U}$ be a $\widetilde{\mathcal{P}}$-measurable function on $\Omega \times [0,\infty) \times I\!R^d$ such that $\widehat{\bar{U}} < 1$ and*

$$V := \Big(\sum_{0<s\leq \cdot} \big(\bar{U}_s(\Delta X_s)\big)^2 I_{\{\Delta X_s \neq 0\}} \Big)^{1/2} = (\bar{U}^2 * \mu)^{1/2} \text{ is locally integrable.} \tag{2.23}$$

Then the predictable process

$$(1-\widehat{\bar{U}})^{-1} = \frac{1}{1-\widehat{\bar{U}}} \text{ is locally bounded.} \tag{2.24}$$

2) *Let N be any local P-martingale null at 0 with $\mathcal{E}(N) > 0$. For the Jacod parameters (β, f, g, N') of N, we then have that*

$$\text{the process } (1-a+\widehat{f})^{-1} \text{ is locally bounded.} \tag{2.25}$$

3) *Suppose that $1 * \mu$ is finite-valued P-a.s. and $f > 0$ is any $\widetilde{\mathcal{P}}$-measurable function which is μ-locally bounded. Then*

$$\text{the process } (1-a+\widehat{f})^{-1} \text{ is locally bounded.} \tag{2.26}$$

Proof. 2) We know from Theorem 2.4 and Proposition 2.7 that f is $\widetilde{\mathcal{P}}$-measurable and $f > 0$ M_μ^P-a.e. So $\bar{U} := -f + 1 < 1$ M_μ^P-a.e. and therefore $\widehat{\bar{U}} = -\widehat{f} + a < a \le 1$. Moreover, (2.7) and (2.12) give $W - \widehat{W} = U = f - 1 = -\bar{U}$ M_μ^P-a.e. so that we get (2.23) with the same estimate as in (2.14). Hence (2.25) follows from (2.24).

3) We again choose $\bar{U} := -f + 1$ and write $\|f\|_{x,\infty}$ for the essential supremum of f with respect to the variable x. Then we get $\bar{U}^2 * \mu \le (1 + \|f\|_{x,\infty})^2 * \mu$, and $1 * \mu$ is RCLL, adapted, finite-valued by assumption and has bounded jumps. So it is locally bounded, and the same is true for $(1 + \|f\|_{x,\infty})^2 * \mu$ because f is μ-locally bounded. So the process V resulting in (2.23) from this $\bar{U}$ is even locally bounded, and we can again use part 1) to get (2.26).

1) For $\delta \in (0,1)$, define an increasing sequence $(\tau_n)_{n \in I\!N}$ of stopping times by $\tau_0 := 0$ and

$$\tau_{n+1} := \inf\left\{t > \tau_n \,\middle|\, {}^{\tau_n}(\bar{U}^2 * \mu)_t = \sum_{\tau_n < s \le t} \big(\bar{U}_s(\Delta X_s)\big)^2 I_{\{\Delta X_s \ne 0\}} > \delta^2\right\}.$$

Moreover, define processes V^n, $n \in I\!N_0$, by

$$V_t^n := {}^{\tau_n}(\bar{U}^2 * \mu)_t = \Big(\sum_{\tau_n < s \le t} \big(\bar{U}_s(\Delta X_s)\big)^2 I_{\{\Delta X_s \ne 0\}}\Big)^{1/2} \le V_t.$$

By definition, each V^n is adapted, RCLL, increasing, and finite-valued due to (2.23). Therefore $V^n_{\tau_{n+1}} \ge \delta^2$ on $\{\tau_{n+1} < \infty\}$, and so we must have $\tau_n \nearrow \infty$ P-a.s., again due to (2.23), because $V_t \ge \sum_{n=0}^{\infty} V^n_{\tau_{n+1}} = +\infty$ on $\{\sup_{n \in I\!N} \tau_n \le t\}$. From the definitions, we also have $V^n_{t-} \le \delta$ P-a.s. for $t \le \tau_{n+1}$.

Now fix $\eta \in (0,1)$ and write $(1 - \widehat{\bar{U}})^{-1} = (1 - \widehat{\bar{U}})^{-1} I_{\{\hat{\bar{U}} \ge 1-\eta\}} + (1 - \widehat{\bar{U}})^{-1} I_{\{\hat{\bar{U}} < 1-\eta\}}$. The second summand is bounded by $1/\eta$ so that we focus on $Y := (1 - \widehat{\bar{U}})^{-1} I_{\{\hat{\bar{U}} \ge 1-\eta\}}$. Because Y is predictable, it will by DM, VIII.11 be locally bounded if we show that it is prelocally bounded, which is equivalent to

$$\sup_{0 \le s \le t} Y_s < \infty \qquad P\text{-a.s. for all } t \ge 0. \tag{2.27}$$

Because $\tau_n \nearrow \infty$ P-a.s., (2.27) will follow if we show that

$$\sup_{\tau_n < s \le t \wedge \tau_{n+1}} Y_s < \infty \qquad P\text{-a.s. on } \{t > \tau_n\}. \tag{2.28}$$

Denoting by $\widetilde{V^n}$ the compensator of V^n, which exists thanks to (2.23) since $V^n \le V$, we obtain as in the proof of Lemma 2.3 that $\widehat{\bar{U}}_s \le |\widehat{\bar{U}}_s| \le V^n_{s-} + {}^{\mathbf{P}}(\Delta V^n)_s = V^n_{s-} + \Delta \widetilde{V^n_s}$ for $\tau_n < s \le \tau_{n+1}$. This implies that

$$\begin{aligned}\left\{s \in (\tau_n, \tau_{n+1}] \,\middle|\, \widehat{\bar{U}}_s \ge 1 - \eta\right\} &\subseteq \left\{s \in (\tau_n, \tau_{n+1}] \,\middle|\, \Delta\widetilde{V^n_s} \ge 1 - \eta - V^n_{s-}\right\} \\ &\subseteq \left\{s \in (\tau_n, \tau_{n+1}] \,\middle|\, \Delta\widetilde{V^n_s} \ge 1 - \eta - \delta\right\} =: \Gamma_n.\end{aligned}$$

But each set $\Gamma_n \cap [0,t]$ is P-a.s. finite since $\widetilde{V^n}$ is RCLL, and so (2.28) holds because

$$\sup_{\tau_n < s \le \tau_{n+1}} Y_s = \max_{\tau_n < s \le \tau_{n+1}} (1 - \widehat{\widehat{U}}_s)^{-1} I_{\{\hat{\hat{U}}_s \ge 1-\eta\}} < \infty \qquad P\text{-a.s.}$$

Thus Y is locally bounded and therefore $(1-\widehat{\widehat{U}})^{-1}$ is so, too. **q.e.d.**

Much of our work below depends on describing when a product $Y\mathcal{E}(N)$ is a σ-martingale. For ease of reference, we formulate this as a lemma. Note that the equivalence of 1) and 3) is just the definition of a σ-martingale.

Lemma 2.9 *Let $Y = (Y_t)_{t\ge 0}$ be an $I\!R^d$-valued semimartingale and $Z = \mathcal{E}(N) > 0$, where N is a local martingale null at 0 with $\Delta N > -1$. Then the following are equivalent:*

1) *$YZ = Y\mathcal{E}(N)$ is a σ-martingale.*

2) *$Y + [Y,N]$ is a σ-martingale.*

3) *There is a bounded predictable process $\varphi > 0$ such that $\varphi \cdot (YZ)$ is a local martingale.*

4) *There is a bounded predictable process $\varphi > 0$ such that $(\varphi \cdot Y)Z$ is a local martingale.*

5) *$\psi \cdot (YZ)$ is a σ-martingale for every bounded predictable process $\psi > 0$.*

6) *$(\psi \cdot Y)Z$ is a σ-martingale for every bounded predictable process $\psi > 0$.*

Proof. First of all, 1) is equivalent to 3) by the definition of a σ-martingale. Next, the product rule and $Z = \mathcal{E}(N) = 1 + Z_- \cdot N$ give

$$YZ = Y\mathcal{E}(N) = Y_- \cdot Z + Z_- \cdot Y + Z_- \cdot [N,Y] = Y_- \cdot Z + Z_- \cdot (Y + [Y,N]). \tag{2.29}$$

Since $Y_- \cdot Z$ is like Z a local martingale and $Z_-, 1/Z_-$ are both predictable, > 0 and locally bounded, we see that 1) and 2) are equivalent. Moreover, 1) implies 5) by Proposition III.6.42 in JS, and of course 5) implies 1). In the same way, 2) is equivalent to saying that $\psi \cdot (Y + [Y,N]) = \psi \cdot Y + [\psi \cdot Y, N]$ is a σ-martingale for every bounded predictable process $\psi > 0$, which is in turn equivalent to 6) by using the equivalence of 1) and 2) with $\psi \cdot Y$ instead of Y. Finally, the same computation as in (2.29) yields

$$\varphi \cdot (YZ) - (\varphi \cdot Y)Z = (\varphi Y_-) \cdot Z + (\varphi Z_-) \cdot Y + \varphi \cdot [Y,Z] - (\varphi \cdot Y)_- \cdot Z - (Z_- \varphi) \cdot Y - \varphi \cdot [Y,Z].$$

Because this is a local martingale like Z, we see that 3) and 4) are equivalent. **q.e.d.**

To characterise the properties in Lemma 2.9 further via Jacod parameters, we choose A such that it dominates the characteristics of both X and Y and write $\mu^Y, \nu^{Y,P}, b^Y, F^{Y,P}$ etc. to distinguish quantities for Y from those for X. We first need an explicit expression for $Y + [Y,N]$. To formulate that, we recall from Theorem II.2.34 in JS the canonical representation of Y as

$$Y = Y_0 + Y^c + h * (\mu^Y - \nu^{Y,P}) + (x-h) * \mu^Y + B^Y \tag{2.30}$$

with $B^Y = b^Y \cdot A$, and introduce the $d \times d$-matrix-valued process $c^{Y,X}$ via $\langle Y^c, X^c \rangle = c^{Y,X} \cdot A$. The typical example for Y in the next result is $Y = I_\Gamma \cdot X$ for some predictable set Γ.

Lemma 2.10. *Recall that the $\mathbb{R}^d$-valued semimartingale X is fixed and consider a local P-martingale N null at 0 given by the Jacod decomposition (2.15). Suppose that $\mu^Y \ll \mu$, so the $\mathbb{R}^d$-valued semimartingale Y can only jump if X does, that $\frac{\Delta Y}{\Delta X} \in \{0,1\}$ on $\{\Delta X \neq 0\}$, and that $(N')^c$ from (2.15) is strongly P-orthogonal to Y^c. If we set $f = U + 1$, then*

$$(2.31) \quad Y + [Y,N] = Y_0 + Y^c + h * (\mu^Y - \nu^{Y,P}) + (b^Y + c^{Y,X}\beta) \cdot A + \big(x(f+g) - h\big) * \mu^Y .$$

Proof. By assumption, we have $\Delta X = \Delta Y \neq 0$ on $\{\Delta Y \neq 0\}$. Hence (2.19) and (2.15) give

$$(2.32) \qquad \begin{aligned} [Y,N] &= \langle Y^c, N^c \rangle + \sum \Delta Y \Delta N \\ &= \langle Y^c, \beta \cdot X^c \rangle + \sum \big(f(\Delta X) + g(\Delta X) - 1\big) \Delta Y I_{\{\Delta X \neq 0\}} \\ &= (c^{Y,X}\beta) \cdot A + \big(x(f+g-1)\big) * \mu^Y , \end{aligned}$$

and adding (2.30) and (2.32) yields (2.31). **q.e.d.**

The next result is a slight variation of Lemma 2.4 in Choulli/Stricker/Li (2007). It deals with σ-martingales instead of local martingales. In the arguments below and also later, we use several times the following simple fact: For any product-measurable $W \geq 0$, the process $C := W * \mu^Y$ is locally integrable if and only if $C' := M^P_{\mu^Y}(W \,|\, \widetilde{\mathcal{P}}) * \nu^{Y,P}$ is, and $C' = C^{\mathbf{p}}$ is then the compensator of C.

Lemma 2.11. *Recall that the $\mathbb{R}^d$-valued semimartingale X is fixed. Suppose that the local P-martingale N null at 0 is given by the Jacod decomposition (2.15) with $f = U + 1$ and that $Z = \mathcal{E}(N) > 0$. Let Y be an $\mathbb{R}^d$-valued semimartingale with $\mu^Y, b^Y, c^{Y,X}, F^{Y,P}$ and suppose that $\mu^Y \ll \mu$, that $\frac{\Delta Y}{\Delta X} \in \{0,1\}$ on $\{\Delta X \neq 0\}$, and that $(N')^c$ from (2.15) is strongly P-orthogonal to Y^c. Then ZY is a P-σ-martingale if and only if for $P \otimes A$-almost all (ω, t), we have one of the three equivalent properties*

$$(2.33) \qquad V^{(1)} := \int_{\mathbb{R}^d} \Big| x\big(f_t(x) + M^P_{\mu^Y}(g \,|\, \widetilde{\mathcal{P}})(t,x)\big) - h(x) \Big| F^{Y,P}_t(dx) < \infty,$$

$$(2.34) \qquad V^{(11)} := \int_{\{|x|>1\}} |x| \big(f_t(x) + M^P_{\mu^Y}(g \,|\, \widetilde{\mathcal{P}})(t,x)\big) F^{Y,P}_t(dx) < \infty,$$

$$(2.35) \qquad V^{(2)} := \int_{\mathbb{R}^d} M^P_{\mu^Y}\big(|x(f+g) - h| \,\big|\, \widetilde{\mathcal{P}}\big)(t,x) F^{Y,P}_t(dx) < \infty,$$

as well as

$$(2.36) \qquad b^Y_t + c^{Y,X}_t \beta_t + \int_{\mathbb{R}^d} \Big(x\big(f_t(x) + M^P_{\mu^Y}(g \,|\, \widetilde{\mathcal{P}})(t,x)\big) - h(x) \Big) F^{Y,P}_t(dx) = 0.$$

Proof. By definition, ZY is a P-σ-martingale iff there is a bounded predictable $\varphi > 0$ such that $\varphi\cdot(ZY)$ is a local P-martingale. Using $Z = 1 + Z_-\cdot N$ and comparing the expressions

$$\varphi\cdot(ZY) = (\varphi Y_-)\cdot Z + (Z_-\varphi)\cdot Y + \varphi\cdot[Z,Y],$$
$$Z(\varphi\cdot Y) = (\varphi\cdot Y)_-\cdot Z + (Z_-\varphi)\cdot Y + \varphi\cdot[Z,Y] = (\varphi\cdot Y)_-\cdot Z + Z_-\cdot(\varphi\cdot Y + \varphi\cdot[N,Y])$$

shows that

$$\varphi\cdot(ZY) \text{ is a local } P\text{-martingale iff } Z(\varphi\cdot Y) \text{ or, equivalently, } \varphi\cdot Y + [N,\varphi\cdot Y] \text{ is.}$$

By Lemma 2.10 and its proof, $\varphi\cdot Y + [N,\varphi\cdot Y] = \varphi\cdot(Y^c + h * (\mu^Y - \nu^{Y,P})) + D^\varphi$ with

$$(2.37)\quad D^\varphi := \big(\varphi(b^Y + c^{Y,X}\beta)\big)\cdot A + \Big(\varphi\big(x(f+g)-h\big)\Big) * \mu^Y = \varphi\cdot\big((x-h)*\mu^Y + B^Y + [Y,N]\big),$$

and so ZY is a P-σ-martingale iff D^φ is a local P-martingale for some bounded predictable $\varphi > 0$. Note in (2.37) that in the first representation, the first summand in D^φ is predictable and locally P-integrable, and that both summands are of finite variation due to the second representation. We also point out that the property $g \in \mathcal{H}^1_{\text{loc}}(\mu)$ and the assumptions on μ^Y and $\frac{\Delta Y}{\Delta X}$ imply that $g.M^P_{\mu^Y}$ is $\widetilde{\mathcal{P}}$-σ-finite so that $M^P_{\mu^Y}(g\,|\,\widetilde{\mathcal{P}})$ is well defined.

1) Suppose first that ZY is a P-σ-martingale and take some φ as above. Then D^φ is a local P-martingale and $\bar{D} := (\varphi(x(f+g)-h)) * \mu^Y$ is a P-special semimartingale and of finite variation, hence of locally P-integrable variation $D' := |\bar{D}|_{\text{var}} = (\varphi|x(f+g)-h|) * \mu^Y$. Therefore its P-compensator exists, is also of locally P-integrable variation, and is given by

$$\begin{aligned}\bar{D}^{\mathbf{p}} &= M^P_{\mu^Y}\big(\varphi(x(f+g)-h)\,\big|\,\widetilde{\mathcal{P}}\big) * \nu^{Y,P}\\ &= \varphi\big(xM^P_{\mu^Y}(f+g\,|\,\widetilde{\mathcal{P}}) - h\big) * \nu^{Y,P}\\ &= \int \varphi_t \int_{\mathbb{R}^d} \big(xM^P_{\mu^Y}(f+g\,|\,\widetilde{\mathcal{P}})(t,x) - h(x)\big)F^{Y,P}_t(dx)\,dA_t.\end{aligned}$$

Note above that $M^P_{\mu^Y}(f+g\,|\,\widetilde{\mathcal{P}})$ is well defined in $[0,+\infty]$ since $f+g \geq 0$ M^P_μ-a.e., hence also $M^P_{\mu^Y}$-a.e. Next, the compensator of D' exists as well and is of locally integrable variation, which implies (2.35) because $\varphi > 0$. Finally, (2.36) follows by noting that the process $(\varphi(b^Y + c^{Y,X}\beta))\cdot A + \bar{D}^{\mathbf{p}} = D^\varphi - (\bar{D} - \bar{D}^{\mathbf{p}})$ is predictable, of finite variation and a local P-martingale, hence constant. We also use for (2.36) that f is $\widetilde{\mathcal{P}}$-measurable, and we point out that $M^P_{\mu^Y}(g\,|\,\widetilde{\mathcal{P}})$ need not vanish though $M^P_\mu(g\,|\,\widetilde{\mathcal{P}})$ does.

2) Conversely, (2.35) implies that the bounded predictable process $\varphi > 0$ given by

$$1/\varphi_t := 1 + |b^Y_t| + |c^{Y,X}_t\beta_t| + \int_{\mathbb{R}^d} M^P_{\mu^Y}\big(|x(f+g)-h|\,\big|\,\widetilde{\mathcal{P}}\big)(t,x)F^{Y,P}_t(dx)$$

is well defined, and $(\varphi b^Y)\cdot A$, $(\varphi c^{Y,X}\beta)\cdot A$, $(\varphi M^P_{\mu^Y}(|x(f+g)-h|\,|\,\widetilde{\mathcal{P}})) * \nu^{Y,P}$ all have locally P-integrable variation since their variations are bounded above by A. As a consequence, also

$\bar{D} = \varphi(x(f+g)-h)*\mu^Y$ has locally P-integrable variation and $\bar{D}-\bar{D}^{\mathbf{p}}$ is a local P-martingale. Since $\bar{D} - \bar{D}^{\mathbf{p}} = D^{\varphi}$ by (2.37) and (2.36), this shows that ZY is a P-σ-martingale.

3) It remains to prove that (2.33)–(2.35) are all equivalent. To that end, we first note that $h(x) = xI_{\{|x|\le 1\}}$ together with positivity of f and $f+g$ yields that

$$
\begin{aligned}
V^{(1)} &= V^{(11)} + \int_{\{|x|\le 1\}} |x| \big| f_t(x) + M^P_{\mu^Y}(g\,|\,\widetilde{\mathcal{P}})(t,x) - 1 \big| F^{Y,P}_t(dx) \\
&\le V^{(11)} + \int_{\{|x|\le 1\}} |x| M^P_{\mu^Y}\big(|f+g-1|\,\big|\,\widetilde{\mathcal{P}}\big)(t,x) F^{Y,P}_t(dx) \\
&=: V^{(11)} + V^{(22)} \\
&= V^{(2)}.
\end{aligned}
\tag{2.38}
$$

This already shows the implications "(2.35) $\Longrightarrow$ (2.33) $\Longrightarrow$ (2.34)" since all V-expressions above are nonnegative. But in view of (2.19) and our assumption on $\frac{\Delta Y}{\Delta X}$, we also have

$$
\begin{aligned}
C &:= \big| x(f+g-1)I_{\{|x|\le 1\}} \big| * \mu^Y \\
&= \sum |\Delta Y||\Delta N| I_{\{|\Delta Y|\le 1\}} \\
&\le \Big(\sum (\Delta Y)^2 I_{\{|\Delta Y|\le 1\}} \Big)^{1/2} \Big(\sum (\Delta N)^2 \Big)^{1/2} \\
&\le \big(I_{\{|\Delta Y|\le 1\}} \cdot [Y] \big)^{1/2} \Big(\sum (\Delta N)^2 \Big)^{1/2}.
\end{aligned}
$$

On the right-hand side above, the first factor is locally bounded since it is a finite-valued increasing adapted process with bounded jumps, and the second factor is locally P-integrable because N is a local P-martingale; see JS, Corollary I.4.55. So C is locally P-integrable, and hence so is its compensator $C^{\mathbf{p}}$, which then equals $V^{(22)} \cdot A$. Therefore we have $V^{(22)} < \infty$ for $P \otimes A$-almost all (ω, t), and hence we also have the implication "(2.34) $\Longrightarrow$ (2.35)" due to (2.38). This completes the proof. **q.e.d.**

If Y has a simpler structure, we also obtain a simpler description for the σ-martingale property of $Y\mathcal{E}(N)$. We spell this out here since it will be used later.

Corollary 2.12. *Let X be an $\mathbb{R}^d$-valued semimartingale and Y of the form*

$$
Y = Y_0 + x * \bar{\mu} + \bar{b} \cdot \bar{A},
\tag{2.39}
$$

where $\bar{\mu} \ll \mu^X$ and $\frac{\Delta Y}{\Delta X} \in \{0,1\}$ and $\bar{A} \ll A$ is continuous. Suppose that the P-compensator $\bar{\nu}^P$ of $\bar{\mu}$ has the form $\bar{\nu}^P(dt,dx) = \bar{F}^P_t(dx)\,d\bar{A}_t$. Let N be a local P-martingale null at 0 with Jacod parameters $(\beta, f, 0, 0)$ with respect to X and suppose that $Z = \mathcal{E}(N) > 0$. Then ZY is a P-σ-martingale if and only if we have for $P \otimes \bar{A}$-almost all (ω, t)

$$
\int_{\mathbb{R}^d} |x| f_t(x) \bar{F}^P_t(dx) < \infty
\tag{2.40}
$$

and the zero-drift condition

$$\bar{b}_t + \int_{\mathbb{R}^d} x f_t(x) \bar{F}_t^P(dx) = 0. \tag{2.41}$$

Proof. As in the proof of Lemma 2.10, but using $Y^c \equiv 0$, $g \equiv 0$ and (2.39) instead of (2.30), we get first $[Y, N] = \sum \Delta Y \Delta N = \sum \Delta Y (f(\Delta X) - 1) I_{\{\Delta X \neq 0\}} = (x(f-1)) * \bar{\mu}$ and then $Y + [Y, N] = Y_0 + \bar{b} \cdot \bar{A} + (xf) * \bar{\mu}$. Now we argue analogously as in the proof of Lemma 2.11 to get the result. **q.e.d.**

By choosing $Y \equiv X$ in Lemma 2.11, we obtain in particular a criterion, in terms of the Jacod parameters of N, for $Z = \mathcal{E}(N)$ to be a P-σ-martingale density for X.

Corollary 2.13. *Let X be an $\mathbb{R}^d$-valued semimartingale with characteristics (b, c, F^P) with respect to A and N a local P-martingale null at 0 with Jacod parameters (β, f, g, N') with respect to X. Suppose also that $Z = \mathcal{E}(N) > 0$. Then ZX is a P-σ-martingale if and only if we have for $P \otimes A$-almost all (ω, t)*

$$\int_{\mathbb{R}^d} |x f_t(x) - h(x)| F_t^P(dx) < \infty \tag{2.42}$$

and the zero-drift condition

$$b_t + c_t \beta_t + \int_{\mathbb{R}^d} \big(x f_t(x) - h(x)\big) F_t^P(dx) = 0. \tag{2.43}$$

Proof. Because $M_\mu^P(g \,|\, \widetilde{\mathcal{P}}) = 0$ by Proposition 2.2, this follows immediately from Lemma 2.11 for $Y \equiv X$, using (2.33). **q.e.d.**

A closer look at Corollary 2.13 shows that the σ-martingale property of ZX is not influenced by g or N'. The simplest way to construct a P-σ-martingale density is therefore to set these two parameters to 0. More formally, we have

Corollary 2.14. *For an $\mathbb{R}^d$-valued adapted RCLL process X, the following are equivalent:*

1) *$\mathcal{D}_{e,\sigma}(X, P) \neq \emptyset$, i.e. there exists a P-σ-martingale density $Z = \mathcal{E}(N)$ for X.*

2) *There exists a P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ for X such that N^P has Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to X.*

3) *X is a semimartingale (with characteristics (b, c, F^P) with respect to A) and there exist a predictable X^c-integrable process β and a $\widetilde{\mathcal{P}}$-measurable function $f > 0$ M_μ^P-a.e. satisfying $W := f - 1 + \widehat{\frac{f-a}{1-a}} I_{\{a<1\}} \in \mathcal{G}^1_{\text{loc}}(\mu)$ and $1 - \widehat{W} > 0$ on $\{\Delta X = 0\}$ and such that we have for $P \otimes A$-almost all (ω, t)*

$$\int_{\mathbb{R}^d} |x f_t(x) - h(x)| F_t^P(dx) < \infty \tag{2.42}$$

and the zero-drift condition

$$b_t + c_t\beta_t + \int_{\mathbb{R}^d} \big(xf_t(x) - h(x)\big)F_t^P(dx) = 0. \tag{2.43}$$

Proof. Clearly 2) gives 1). Conversely, if 1) holds, take N with Jacod parameters (β, f, g, N') and set $\beta^P := \beta$, $f^P := f$, $g^P \equiv 0$ (which is in $\mathcal{H}^1_{\text{loc}}(\mu)$), $N'^P \equiv 0$ and as in (2.15)

$$N^P := \beta^P \boldsymbol{\cdot} X^c + W^P * (\mu - \nu^P) + g^P * \mu + N'^P = \beta \boldsymbol{\cdot} X^c + W * (\mu - \nu^P), \tag{2.44}$$

where $\mu := \mu^X$ is the jump measure of X and $W^P := W = f - 1 + \frac{\widehat{f-a}}{1-a} I_{\{a<1\}}$ as in (2.11). Since $Z > 0$, we have $\Delta N > -1$ and hence $f > 0$ M^P_μ-a.e. and $1 - \widehat{W} > 0$ on $\{\Delta X = 0\}$ by Lemma 2.6. Moreover, ZX is a P-σ-martingale because Z is in $\mathcal{D}_{\text{e},\sigma}(X, P)$. By Lemma 2.6 again, we have $\Delta N^P > -1$ and hence $Z^P := \mathcal{E}(N^P) > 0$, and so Corollary 2.13 directly implies that also $Z^P X$ is a P-σ-martingale, by the construction of N^P. So we get 2).

If we have 2), then $\mathcal{D}_{\text{e},\sigma}(X, P) \neq \emptyset$ implies that X is a semimartingale. If we then take $\beta = \beta^P$, $f = f^P$, we get the first three claimed properties in 3) from Theorem 2.4 and Lemma 2.6, and the last two from Corollary 2.13. So 2) implies 3). Conversely, defining N^P as in (2.44), with $\beta^P = \beta$ and $f^P = f$, gives a local P-martingale N^P with Jacod parameters $(\beta^P, f^P, 0, 0)$ and $Z^P = \mathcal{E}(N^P) > 0$ because $\Delta N^P > -1$ due to $f > 0$ M^P_μ-a.e. and $1 - \widehat{W} > 0$ on $\{\Delta X = 0\}$, by Lemma 2.6. By Corollary 2.13, (2.42) and (2.43) then imply that $Z^P X$ is a P-σ-martingale, and so 3) implies 2). This ends the proof. **q.e.d.**

Remark 2.15. If X is quasi-left-continuous, then $\widehat{\cdot} \equiv 0$ and hence $a = \widehat{1} \equiv 0$. The condition $1 - \widehat{W} > 0$ on $\{\Delta X = 0\}$ is then always satisfied, and W simplifies to $W = f - 1$. ◇

3. Integrability issues

One of our main goals is to study suitably integrable σ-martingale densities for a given process. In this section, we recall some concepts and prove some results on local integrability of exponential local martingales. Like Section 2, these are of independent interest.

We start with an $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the usual conditions and recall the jump measure $\mu = \mu^X$ of X and the corresponding measure $M^P_\mu = P \otimes \mu$ on $\Omega \times [0, \infty) \times \mathbb{R}^d$. We fix throughout a convex function $\Phi : [0, \infty) \to \mathbb{R}$ with $\Phi(0) = 0$. So Φ is finite and hence continuous on $[0, \infty)$.

Definition 3.1. A random variable Y is in $L^\Phi := L^\Phi(P)$ if $E[|\Phi(\alpha|Y|)|] < \infty$ for some $\alpha > 0$. An RCLL stochastic process $Y = (Y_t)_{t \geq 0}$ is in L^Φ if the random variable $Y^*_\infty := \sup_{t \geq 0} |Y_t|$ is in L^Φ. We say that $Y = (Y_t)_{t \geq 0}$ is locally in L^Φ, written as $Y \in L^\Phi_{\text{loc}}$ or $Y \in L^\Phi_{\text{loc}}(P)$, if there are stopping times $T_n \nearrow \infty$ P-a.s. such that for each $n \in I\!N$, the stopped process Y^{T_n} is in L^Φ. This means that there are constants $\alpha_n > 0$ such that $E[|\Phi(\alpha_n Y^*_{T_n})|] < \infty$ for all $n \in I\!N$. A product-measurable function W on $\Omega \times [0,\infty) \times I\!R^d$ is in $L^\Phi(M^P_\mu)$ if

$$E\Big[\int_0^\infty \int_{I\!R^d} \big|\Phi\big(\alpha|W(\omega,t,x)|\big)\big|\,\mu(\omega,dt,dx)\Big] < \infty \qquad \text{for some } \alpha > 0.$$

We say that W is locally in $L^\Phi(M^P_\mu)$, written as $W \in L^\Phi_{\rm loc}(M^P_\mu)$, if there are stopping times $T_n \nearrow \infty$ P-a.s. such that for each $n \in I\!N$, the process $|W| I_{[\![0,T_n]\!]\times I\!R^d}$ is in $L^\Phi(M^P_\mu)$; this means that there are constants $\alpha_n > 0$ such that

$$E\Big[\int_0^{T_n} \int_{I\!R^d} \big|\Phi\big(\alpha|W(\omega,t,x)|\big)\big|\,\mu(\omega,dt,dx)\Big] = E\big[\big|\Phi(\alpha_n|W|)\big| * \mu_{T_n}\big] < \infty. \tag{3.1}$$

Remark 3.2. $W \in L^\Phi(M^P_\mu)$ is not equivalent to $|W| * \mu_\infty \in L^\Phi$, nor does $W \in L^\Phi_{\rm loc}(M^P_\mu)$ mean that the process $|W| * \mu$ is in $L^\Phi_{\rm loc}$. Note also that the definition of $L^\Phi_{\rm loc}(M^P_\mu)$ uses that $\Phi(0) = 0$. ◇

Now let $N = (N_t)_{t\geq 0}$ be a local P-martingale null at 0 and such that $Z = \mathcal{E}(N) > 0$. In most of our applications, Z will be a P-σ-martingale density for some process, but we do not need this here. Our goal is to characterise the property $Z \in L^\Phi_{\rm loc}$ in terms of Φ-integrability properties of the Jacod parameters (β, f, g, N') of N with respect to X.

We start with a simple but useful estimate for Φ. On the one hand, convexity of Φ gives for $0 \leq x \leq y \leq z$ that $\Phi(y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(z) \leq |\Phi(x)| + |\Phi(z)|$ for some $\lambda \in [0,1]$. On the other hand, we can always minorise the convex function Φ by an affine function ℓ with $\ell(1) = \Phi(1)$, say; so for any $y \geq 0$, we have $\Phi(y) \geq a(1)y + b(1) \geq -|a(1)|y - |b(1)|$ with constants $a(1), b(1)$; see Ekeland/Témam (1999), Proposition I.3.1. Hence we obtain that

$$|\Phi(y)| \leq |\Phi(x)| + |\Phi(z)| + |a(1)|y + |b(1)| \qquad \text{for all } 0 \leq x \leq y \leq z. \tag{3.2}$$

For later use, we first give some extra properties of L^Φ under an extra condition on Φ.

Lemma 3.3. *Suppose that Φ grows at least linearly for large x, i.e., there is some constant $D > 0$ such that $|\Phi(x)| \geq Dx$ for $x \geq x_0$. Then $L^\Phi \subseteq L^1 := L^1(P)$ and L^Φ_+ is solid, meaning that $0 \leq X \leq Y$ with $Y \in L^\Phi$ implies that $X \in L^\Phi$ as well.*

Proof. Take $0 \leq x \leq y$ and use the linear growth assumption to write, for $\alpha > 0$,

$$\alpha x \leq \alpha y \leq x_0 I_{\{\alpha y < x_0\}} + \alpha y I_{\{\alpha y \geq x_0\}} \leq x_0 + |\Phi(\alpha y)|/D. \tag{3.3}$$

Combine this with (3.2), for $0 \leq \alpha x \leq \alpha y$, to get

$$\begin{aligned} |\Phi(\alpha x)| &\leq |\Phi(0)| + |\Phi(\alpha y)| + |a(1)|\alpha x + |b(1)| \\ &\leq |\Phi(0)| + |\Phi(\alpha y)|\big(1 + |a(1)|/D\big) + |b(1)| + |a(1)|x_0. \end{aligned} \tag{3.4}$$

So if $Y \geq 0$ and $\Phi(\alpha Y) \in L^1$, then (3.4) implies that also $\Phi(\alpha X) \in L^1$ if $0 \leq X \leq Y$, giving solidity of L^Φ_+. Moreover, applying (3.3) to $y = |Y|$ gives $\alpha|Y| \leq x_0 + |\Phi(\alpha|Y|)|/D$, and this readily shows that $Y \in L^\Phi$ yields $Y \in L^1$. **q.e.d.**

Lemma 3.4. *Let N be a local P-martingale null at 0 with $Z = \mathcal{E}(N) > 0$. Then Z is in L^Φ_{loc} if and only if there are constants $\alpha_n > 0$, $n \in I\!N$, and stopping times $T_n \nearrow \infty$ P-a.s. such that each $\Phi(\alpha_n Z)$ is a semimartingale with $(\Phi(\alpha_n Z))^*_{T_n} \in L^1$.*

Proof. By definition, $Z \in L^\Phi_{\mathrm{loc}}$ is equivalent to the existence of constants $\alpha_n > 0$ and stopping times $T'_n \nearrow \infty$ P-a.s. such that $\Phi(\alpha_n Z^*_{T'_n}) \in L^1$ for all n; so we have a control on $\Phi(\alpha_n Z^*)$. On the other hand, by Theorem VII.25 of DM, the semimartingale $\Phi(\alpha_n Z)$ is special if and only if $(\Phi(\alpha_n Z))^*$ is locally integrable. To relate the two running suprema, define $\varrho_n := \inf\{t \geq 0 \mid Z_t > n\}$, note that $Z^*_s \leq n + Z_s$ for $s \leq \varrho_n$ because $Z > 0$, and take stopping times $S_n \nearrow \infty$ P-a.s. with $Z^*_{S_n} \in L^1$ for all n, using that every local P-martingale is locally in $\mathcal{H}^1$. Because $Z_0 = 1$, we have $1 \leq Z^*_s \leq Z^*_t$ for $s \leq t$ and hence from (3.2)

$$|\Phi(Z^*_s)| \leq |\Phi(1)| + |\Phi(Z^*_t)| + |a(1)|Z^*_t + |b(1)| \qquad \text{for } s \leq t. \tag{3.5}$$

Now if $Z \in L^\Phi_{\mathrm{loc}}$, we define the stopping times $T_n := T'_n \wedge S_n \nearrow \infty$ P-a.s. and apply (3.2), for $0 \leq \alpha_n Z_s \leq \alpha_n Z^*_{T_n}$ for $s \leq T_n \leq \min(T'_n, S_n)$, and (3.5) to obtain

$$\begin{aligned}\sup_{0 \leq s \leq T_n} |\Phi(\alpha_n Z_s)| &\leq |\Phi(0)| + |\Phi(\alpha_n Z^*_{T_n})| + |a(1)|\alpha_n Z^*_{T_n} + |b(1)| \\ &\leq |\Phi(0)| + |\Phi(\alpha_n)| + |\Phi(\alpha_n Z^*_{T'_n})| + 2|a(1)|\alpha_n Z^*_{S_n} + 2|b(1)|.\end{aligned}$$

Because the right-hand side is in L^1 by assumption, we have the "only if" part.

Conversely, if $(\Phi(2\alpha_n Z))^*_{T_n} \in L^1$ for all n, set $T'_n := \varrho_n \wedge S_n \wedge T_n \nearrow \infty$ P-a.s. and use $\alpha_n \leq \alpha_n Z^*_{T'_n} \leq \alpha_n n + \alpha_n Z_{T'_n} = \frac{1}{2}(2\alpha_n n + 2\alpha_n Z_{T'_n})$ together with (3.2) applied twice to obtain

$$\begin{aligned}|\Phi(\alpha_n Z^*_{T'_n})| &\leq |\Phi(\alpha_n)| + |\Phi(\alpha_n n + \alpha_n Z_{T'_n})| + |a(1)|Z^*_{S_n} + |b(1)| \\ &\leq |\Phi(\alpha_n)| + |\Phi(2\alpha_n n)| + |\Phi(2\alpha_n Z_{T'_n})| + |a(1)|(\alpha_n n + \alpha_n Z_{T'_n}) + |b(1)| \\ &\quad + |a(1)|Z^*_{S_n} + |b(1)| \\ &\leq |\Phi(\alpha_n)| + |\Phi(2\alpha_n n)| + \big(\Phi(2\alpha_n Z)\big)^*_{T_n} \\ &\quad + \alpha_n n|a(1)| + (\alpha_n + 1)|a(1)|Z^*_{S_n} + 2|b(1)|.\end{aligned}$$

By assumption, the right-hand side is in L^1; so each $Z^{T'_n}$ is in L^Φ, giving $Z \in L^\Phi_{\mathrm{loc}}$. **q.e.d.**

Lemma 3.5. *Fix $\alpha > 0$ and a local P-martingale N null at 0 with $Z = \mathcal{E}(N) > 0$. Then $\Phi(\alpha Z)$ is a P-special semimartingale if and only if there are constants $\beta_n > 0$, $n \in I\!N$, and $b > 0$ and stopping times $T_n \nearrow \infty$ P-a.s. such that*

$$\sup_{0 < s \leq T_n} \big|\Phi\big(\beta_n(1 + \Delta N_s)\big)\big| I_{\{1 + \Delta N_s > b\}} \text{ is in } L^1, \text{ for each } n \in I\!N.$$

Proof. By Theorem VII.25 of DM, the semimartingale $\Phi(\alpha Z)$ is special if and only if the process $J^1_\cdot := \sup_{0<s\le\cdot} |\Delta\Phi(\alpha Z)_s|$ is locally integrable. Because $Z = \mathcal{E}(N)$ gives $\Delta Z = Z_- \Delta N$ and hence $Z = Z_-(1 + \Delta N)$, we have

$$\Delta\Phi(\alpha Z) = \Phi(\alpha Z) - \Phi(\alpha Z_-) = \Phi\big(\alpha Z_-(1 + \Delta N)\big) - \Phi(\alpha Z_-).$$

Due to $Z > 0$, the process αZ_- is locally bounded away from 0 and ∞; so $\Phi(\alpha Z_-)$ is locally bounded. Since $1 + \Delta N > 0$ and $\Phi(0) = 0$ is finite, the process $\Phi(\alpha Z_-(1 + \Delta N))I_{\{1+\Delta N\le b\}}$ for any $b > 0$ is also locally bounded. Thus J^1 is locally integrable if and only if

$$J^2_\cdot := \sup_{0<s\le\cdot} \big|\Phi\big(\alpha Z_{s-}(1 + \Delta N_s)\big)\big| I_{\{1+\Delta N_s>b\}} \text{ is locally integrable.}$$

Note that b does not depend on α in any way.

Now recall again that αZ_- is locally bounded away from 0 and ∞ and that N as a local P-martingale is locally in $\mathcal{H}^1$. So for stopping times $T_k \nearrow \infty$ P-a.s., we have

$$0 < \delta_k \le \alpha Z_- \le \delta_k^{-1} < \infty \qquad \text{on } [\![0, T_k]\!]$$

and $N^*_{T_k} \in L^1$. This implies

$$\delta_k b \le \delta_k(1 + \Delta N) \le \alpha Z_-(1 + \Delta N) \le \delta_k^{-1}(1 + \Delta N) \qquad \text{on } [\![0, T_k]\!] \cap \{1 + \Delta N > b\},$$

and applying (3.2) therefore yields

$$\begin{aligned}\Big(\Phi\big(\delta_k(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \le |\Phi(\delta_k b)| &+ \Big(\Phi\big(\alpha Z_-(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \\ &+ |a(1)|\delta_k(1 + 2N^*_{T_k}) + |b(1)|\end{aligned}$$

as well as

$$\begin{aligned}\Big(\Phi\big(\alpha Z_-(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \le |\Phi(\delta_k b)| &+ \Big(\Phi\big(\delta_k^{-1}(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \\ &+ |a(1)|\delta_k^{-1}(1 + 2N^*_{T_k}) + |b(1)|.\end{aligned}$$

So we see that

$$\Big(\Phi\big(\alpha Z_-(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \in L^1 \qquad \text{if and only if}$$

$$\Big(\Phi\big(\beta_k(1 + \Delta N)\big)I_{\{1+\Delta N>b\}}\Big)^*_{T_k} \in L^1 \qquad \text{for some } \beta_k \in (0, \infty).$$

Note that the β_k depend on α. The assertion of Lemma 3.5 follows. **q.e.d.**

Proposition 3.6. *Suppose that N is a local P-martingale null at 0 with $Z = \mathcal{E}(N) > 0$ and having Jacod parameters (β, f, g, N') with respect to X. Recall from (2.11) the quantity $W = f - 1 + \widehat{\frac{f-a}{1-a}} I_{\{a<1\}}$. Then Z is in $L^\Phi_{\rm loc}$ if and only if we have both*

$$(3.6) \qquad (f+g) I_{\{f+g>b\}} \in L^\Phi_{\rm loc}(M^P_\mu) \qquad \text{for some } b > 0,$$

(3.7) *there are stopping times $T_n \nearrow \infty$ P-a.s. and constants $\beta_n > 0$ such that*

$$E\Big[\sum_{0<s\le T_n} \big|\Phi\big(\beta_n(1 - \widehat{W}_s + \Delta N'_s)\big)\big| I_{\{1-\widehat{W}_s+\Delta N'_s>b,\, \Delta X_s=0\}} \Big] < \infty \qquad \text{for all } n \in I\!N.$$

Proof. Combining Lemmas 3.4 and 3.5 with a diagonal procedure shows that $Z \in L^\Phi_{\rm loc}$ is equivalent to $\sup\limits_{0<s\le T'_n} |\Phi(\beta_n(1+\Delta N_s))| I_{\{1+\Delta N_s>b\}}$ being in L^1 for each $n \in I\!N$, for constants $\beta_n > 0$, $n \in I\!N$, and $b > 0$. By Theorem VII.25 of DM, the last property holds if and only if $J^n := \sum |\Phi(\beta_n(1+\Delta N))| I_{\{1+\Delta N>b\}}$ has $J^n_{T_n} \in L^1$ for each $n \in I\!N$, with stopping times $T_n \nearrow \infty$ P-a.s. Lemma 2.6 gives

$$1 + \Delta N = \big(f(\Delta X) + g(\Delta X)\big) I_{\{\Delta X \neq 0\}} + (1 - \widehat{W} + \Delta N') I_{\{\Delta X = 0\}},$$

and so

$$\begin{aligned} J^n &= \big|\Phi\big(\beta_n(f+g)\big)\big| I_{\{f+g>b\}} * \mu + \sum \big|\Phi\big(\beta_n(1-\widehat{W}+\Delta N')\big)\big| I_{\{1-\widehat{W}+\Delta N'>b,\, \Delta X=0\}} \\ &=: J^{n,1} + J^{n,2}. \end{aligned}$$

By using $\Phi(0) = 0$, we can write $J^{n,1} = |\Phi(\beta_n(f+g)I_{\{f+g>b\}})| * \mu$, and this has $J^{n,1}_{T_n} \in L^1$ if and only if $E[|\Phi(\beta_n(f+g)I_{\{f+g>b\}})| * \mu_{T_n}] < \infty$ for stopping times $T_n \nearrow \infty$ P-a.s. Since the latter is equivalent to $(f+g)I_{\{f+g>b\}} \in L^\Phi_{\rm loc}(M^P_\mu)$, the claim follows. **q.e.d.**

For our subsequent applications, we only need a special case of Proposition 3.6, namely

Lemma 3.7. *Suppose that N is a local P-martingale null at 0 with $Z = \mathcal{E}(N) > 0$ and having Jacod parameters $(\beta, f, 0, 0)$ with respect to X. Suppose also that X is quasi-left-continuous. Then Z is in $L^\Phi_{\rm loc}$ if and only if $f I_{\{f>b\}} \in L^\Phi_{\rm loc}(M^P_\mu)$ for some $b \ge 1$.*

Proof. Because X is quasi-left-continuous, $\widehat{\cdot} \equiv 0$. Moreover, $N' \equiv 0$ by assumption, and so the sum in (3.7) is always 0 for $b \ge 1$. Because also $g \equiv 0$ by assumption, (3.6) reduces to $f I_{\{f>b\}} \in L^\Phi_{\rm loc}(M^P_\mu)$, and so the assertion follows from Proposition 3.6. **q.e.d.**

4. Some preliminary steps for simplification

Before we start working on our problem, we reduce its complexity by some preliminary work. One idea is to split a general process S into parts each having extra properties, and to deal with these parts separately. A second idea is to chop up S into pieces concentrated on suitable pairwise disjoint stochastic intervals and to work on each piece separately. The present section makes this more precise.

For the first idea and decomposition, we start with an $\mathbb{R}^d$-valued adapted RCLL process $S = (S_t)_{t \geq 0}$. We exhaust the jumps of S by a sequence of stopping times (as in JS, Proposition I.1.32), split each of these into an accessible and a totally inaccessible part (see JS, Theorem I.2.22), and cover the graph of each accessible time with pairwise disjoint graphs of predictable times. Renumbering then yields a sequence $(R_n)_{n \in \mathbb{N}}$ of predictable stopping times with pairwise disjoint graphs such that $\Delta S_T I_{\{T<\infty\}} = 0$ P-a.s. for each predictable stopping time T satisfying $P[T = R_n < \infty] = 0$ for all $n \in \mathbb{N}$. If we set

$$D := \bigcup_{n=1}^{\infty} [\![R_n]\!] \in \mathcal{P},$$

the above condition on T simply means that $[\![T]\!] \cap D$ is evanescent. Setting

$$S^a := I_D \cdot S = \sum_{n=1}^{\infty} I_{[\![R_n]\!]} \cdot S, \tag{4.1}$$

$$S^i := I_{D^c} \cdot S = S - S_0 - S^a, \tag{4.2}$$

we then obtain

$$S - S_0 = S^a + S^i, \tag{4.3}$$

and by construction, $\{\Delta S^i \neq 0\} \cap [\![T]\!] = \{\Delta S \neq 0\} \cap D^c \cap [\![T]\!]$ is evanescent for every predictable stopping time T so that $\Delta S_T = 0$ P-a.s. on $\{T < \infty\}$ for every predictable T, i.e.

S^i is *quasi-left-continuous.*

(This is the same construction as in Delbaen/Schachermayer (2006) in the proof of their main Theorem 14.1.1, pages 302/303.)

Since all our results assume that $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$, we can assume (as mentioned in Section 1) that S is a semimartingale, and then so are S^a and S^i. In view of (4.1) and (4.2), the following simple result therefore allows us to treat S^a and S^i separately.

Lemma 4.1. *Suppose that $X^{(1)}$ and $X^{(2)}$ are semimartingales of the form $X^{(1)} = I_D \cdot X$, $X^{(2)} = I_{D^c} \cdot X$ for some semimartingale X and some predictable set D. Then X admits a σ-martingale density $Z = \mathcal{E}(N)$ if and only if $X^{(1)}, X^{(2)}$ admit σ-martingale densities $Z^{(1)} = \mathcal{E}(N^{(1)})$, $Z^{(2)} = \mathcal{E}(N^{(2)})$, and we can even choose these to satisfy $N^{(1)} = I_D \cdot N$, $N^{(2)} = I_{D^c} \cdot N$ so that $N = N^{(1)} + N^{(2)}$ and $[N^{(i)}, N^{(k)}] \equiv 0$ as well as $[N^{(i)}, X^{(k)}] \equiv 0$ for $i \neq k$. In particular, we can always arrange that $Z = Z^{(1)} Z^{(2)}$.*

Proof. We first start with a σ-martingale density $Z = \mathcal{E}(N)$ for X and recall from Lemma 2.9 that ZX is a σ-martingale if and only if $X + [X, N]$ is. Setting $N^{(1)} := I_D \cdot N$, $N^{(2)} := I_{D^c} \cdot N$

then gives $[N^{(1)}, N^{(2)}] \equiv 0$ and so $Z^{(1)}Z^{(2)} = \mathcal{E}(N^{(1)})\mathcal{E}(N^{(2)}) = \mathcal{E}(N^{(1)} + N^{(2)}) = \mathcal{E}(N) = Z$ by Yor's formula. Moreover, $X^{(1)} + [X^{(1)}, N^{(1)}] = I_D \bullet (X + [X, N])$ is a σ-martingale like $X + [X, N]$; so $Z^{(1)}$ is a σ-martingale density for $X^{(1)}$ and of the claimed form. The argument for $X^{(2)}$ and $Z^{(2)}$ is analogous, and it is clear by construction that $[N^{(i)}, X^{(k)}] \equiv 0$ for $i \neq k$.

Conversely, let $\bar{Z}^{(i)} = \mathcal{E}(\bar{N}^{(i)})$ be a σ-martingale density for $X^{(i)}$ so that $X^{(i)} + [X^{(i)}, \bar{N}^{(i)}]$ is a σ-martingale by Lemma 2.9 . Then so is $I_D \bullet (X^{(1)} + [X^{(1)}, \bar{N}^{(1)}]) = X^{(1)} + [X^{(1)}, I_D \bullet \bar{N}^{(1)}]$ because $I_D \bullet X^{(1)} = X^{(1)}$, and thus $Z^{(1)} = \mathcal{E}(N^{(1)})$ with $N^{(1)} := I_D \bullet \bar{N}^{(1)}$ is by Lemma 2.9 a σ-martingale density for $X^{(1)}$. In the same way, $Z^{(2)} = \mathcal{E}(N^{(2)})$ with $N^{(2)} := I_{D^c} \bullet \bar{N}^{(2)}$ is a σ-martingale density for $X^{(2)}$, and $[N^{(i)}, X^{(k)}] \equiv 0$ for $i \neq k$ by construction. Setting $N := N^{(1)} + N^{(2)}$, we thus get $Z = \mathcal{E}(N) = Z^{(1)}Z^{(2)}$ by Yor's formula, since $[N^{(1)}, N^{(2)}] \equiv 0$, and $X + [X, N] = X^{(1)} + X^{(2)} + [X^{(1)}, N^{(1)}] + [X^{(2)}, N^{(2)}]$ is like $X^{(i)} + [X^{(i)}, N^{(i)}]$ a σ-martingale, so that Z is a σ-martingale density for X. This completes the proof. **q.e.d.**

For later use, we also provide the following simple result. Its proof is almost identical to the second half of the preceding argument for the converse part and therefore omitted.

Lemma 4.2. *Suppose that $Z^{(i)} = \mathcal{E}(N^{(i)})$ is a σ-martingale density for the ($\mathbb{R}^d$-valued adapted RCLL) process $X^{(i)}$ for $i = 1, 2$. Suppose also that we have $[N^{(i)}, N^{(k)}] \equiv 0$ and $[N^{(i)}, X^{(k)}] \equiv 0$ for $i \neq k$. Then $Z := Z^{(1)}Z^{(2)} = \mathcal{E}(N)$ with $N := N^{(1)} + N^{(2)}$ is a σ-martingale density for $X := X^{(1)} + X^{(2)}$.*

From Lemma 4.1, it is clear that finding a σ-martingale density for S is equivalent to finding separately σ-martingale densities for S^a and S^i and then simply taking their product. Moreover, we can exploit as extra properties that S^a is intuitively a process that consists only of jumps at predictable stopping times, and that S^i is quasi-left-continuous. One crucial consequence of the latter fact is that the process dominating the characteristics of S^i (under P or any $Q \overset{\text{loc}}{\approx} P$, or both at the same time) can be chosen *continuous*; see JS, Proposition II.2.9. We thus get in (2.1) for S^i that $\widehat{W} \equiv 0$ for any $W \geq 0$ and hence in (2.3) also that $a \equiv 0$; see JS, Corollary II.1.19. This allows to simplify many of the general expressions from Section 2.

The second idea and decomposition is very simple. Since we can view $S - S_0$ as a process on $(0, \infty)$, we take a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_n \nearrow \infty$ P-a.s. and write

$$\Omega \times (0, \infty) = \bigcup_{n=1}^{\infty}]\!]\tau_{n-1}, \tau_n]\!] \qquad (\text{with } \tau_0 := 0)$$

and

$$S - S_0 = S - S^0 = \sum_{n=1}^{\infty} (S^{\tau_n} - S^{\tau_{n-1}}) = \sum_{n=1}^{\infty} I_{]\!]\tau_{n-1}, \tau_n]\!]} \bullet S.$$

We study S on each interval $]\!]\tau_{n-1}, \tau_n]\!]$ by looking at $S^{\tau_n} - S^{\tau_{n-1}}$, and piece things together with the subsequent minor extension of Lemma 4.2. Typical examples are $D_n :=]\!]\tau_{n-1}, \tau_n]\!]$

for $\tau_0 := 0$ and a sequence $(\tau_n)_{n\in I\!N}$ of stopping times with $\tau_n \nearrow \infty$ P-a.s., or $D_n := [\![R_n]\!]$ for a sequence $(R_n)_{n\in I\!N}$ of predictable stopping times with pairwise disjoint graphs. In the former case, the absence of accumulation points required in Lemma 4.3 is clearly satisfied; in the latter, it imposes an extra condition on the sequence (R_n).

Lemma 4.3. *Let $(D_n)_{n\in I\!N}$ be a sequence of pairwise disjoint predictable sets. For each $n \in I\!N$, let X^n be an $I\!R^d$-valued adapted RCLL process with $X^n = I_{D_n} \cdot X^n$, and suppose that $Z^n = \mathcal{E}(N^n)$ is a σ-martingale density for X^n with $N^n = I_{D_n} \cdot N^n$. Suppose also that the sequence (D_n) has P-a.s. no accumulation point. Then*

$$Z := \prod_{n=1}^{\infty} Z^n = \prod_{n=1}^{\infty} \mathcal{E}(N^n) = \mathcal{E}\Big(\sum_{n=1}^{\infty} N^n\Big) =: \mathcal{E}(N) \tag{4.4}$$

is a σ-martingale density for $X := \sum_{n=1}^{\infty} X^n$.

Proof. First of all, $N := \sum_{n=1}^{\infty} N^n = \sum_{n=1}^{\infty} I_{D_n} \cdot N^n$ is well defined because the D_n are pairwise disjoint and have no accumulation point. The same applies to X. Moreover, the assumptions and Lemma 2.9 imply that $X^n + [X^n, N^n]$ is a σ-martingale for each n, and $[N^m, N^n] \equiv 0$ and $[X^m, N^n] \equiv 0$ for $m \neq n$ because the D_n are pairwise disjoint. So the third equality in (4.4) follows from Yor's formula, and $X + [X, N] = \sum_{n=1}^{\infty} (X^n + [X^n, N^n])$ is a σ-martingale, which proves the assertion again via Lemma 2.9 . **q.e.d.**

In the sequel, we want to work with σ-martingale densities for S, and it will be useful and important to do this in a simple way that also matches up well with the stopping times $(\tau_n)_{n\in I\!N}$ from above. In more detail, this goes as follows.

Assume that $\mathcal{D}_{e,\sigma}(S, P) \neq \emptyset$ (and of course this could be done under some $Q \overset{\text{loc}}{\approx} P$ as well). By Corollary 2.14, we can then choose a P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ for S such that N^P has Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to S. Since the τ_n above will be constructed recursively, we can do this in a way that ensures that

$$\tau_n \leq \inf\big\{t > \tau_{n-1} \,\big|\, |N^P_t| > \tfrac{n}{2}\big\}.$$

Then we clearly get $|N^P| \leq \frac{n}{2}$ on $]\!]\tau_{n-1}, \tau_n[\![$ and therefore

$$|\Delta N^P| \leq n \qquad \text{on }]\!]\tau_{n-1}, \tau_n[\![.$$

So we can always find a σ-martingale density Z^P for S and a sequence $(\tau_n)_{n\in I\!N}$ of stopping times $\tau_n \nearrow \infty$ P-a.s. such that the stochastic logarithm N^P of Z^P has bounded jumps on each *open* stochastic interval $]\!]\tau_{n-1}, \tau_n[\![$, and this will be exploited later. Note, however, that we cannot control the jumps of N^P at the stopping times τ_n.

Remark 4.4. If we start above not with S, but with $S^a = I_D \boldsymbol{\cdot} S$, then S^a has clearly no continuous local martingale part. As a consequence, the zero-drift conditions (2.42), (2.43) in Corollary 2.14 simplify to (2.40), (2.41) in Corollary 2.12 because $c \equiv 0$; so we do not need the Jacod parameter β^P and can even choose a P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ with an N^P which has Jacod parameters $(0, f^P, 0, 0)$ with respect to S^a. ◇

5. Reducing a stopped process to a single-jump process

In view of Lemma 4.3, it is natural to analyse S on a stochastic interval $\rrbracket\sigma,\tau\rrbracket$ with $\sigma \le \tau$, and we do this in this section by looking separately at $\rrbracket\sigma,\tau\llbracket$ and $\llbracket\tau\rrbracket$. Starting with an $I\!R^d$-valued adapted RCLL process S and stopping times $\sigma \le \tau$, we write

(5.1) $I_{\rrbracket\sigma,\tau\rrbracket} \boldsymbol{\cdot} S = S^\tau - S^\sigma = S^{\tau-} - S^\sigma + \Delta S_\tau I_{\llbracket\tau,\infty\llbracket} =: S^{\tau-} - S^\sigma + J^{(\tau)} = I_{\rrbracket\sigma,\tau\llbracket} \boldsymbol{\cdot} S + I_{\llbracket\tau\rrbracket} \boldsymbol{\cdot} S.$

In (5.1), $S^{\tau-}$ denotes the process S pre-stopped at τ, and $J^{(\tau)}$ is clearly a single-jump process. Our main result in this section shows that under a suitable condition on τ, the pre-stopped process $S^{\tau-} - S^\sigma$ can be controlled fairly well. More precisely, we have

Theorem 5.1. *Suppose S is an $I\!R^d$-valued adapted RCLL process and $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$. Let $Z^P = \mathcal{E}(N^P)$ be a P-σ-martingale density for S with N^P having Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to S. Let $\sigma \le \tau$ be stopping times such that ΔN^P is bounded by a constant on $\rrbracket\sigma,\tau\llbracket$. Using Lemma 2.9, choose and fix a bounded predictable process $\varphi > 0$ such that $Z^P(\varphi \boldsymbol{\cdot} S)$ is a local P-martingale, and define $X := I_{\rrbracket\sigma,\infty\llbracket} \boldsymbol{\cdot} (\varphi \boldsymbol{\cdot} S)$. Then there exist an $I\!R^d$-valued predictable RCLL process $\widetilde{B}$ of finite variation and null on $\llbracket 0,\sigma\rrbracket$, an $I\!R^d$-valued semimartingale $\widetilde{M}$ null on $\llbracket 0,\sigma\rrbracket$ and a strictly positive local P-martingale $Z^{(1)} = \mathcal{E}(N^{(1)})$ with $Z^{(1)}_0 = 1$ such that $Z^{(1)}$ is locally bounded and the pre-stopped process $X^{\tau-}$ satisfies*

1) *$X^{\tau-} = X^{\tau-} - X^\sigma = \widetilde{M} + \widetilde{B} = \widetilde{M}^\tau + \widetilde{B}^\tau = I_{\rrbracket\sigma,\tau\rrbracket} \boldsymbol{\cdot} \widetilde{M} + I_{\rrbracket\sigma,\tau\rrbracket} \boldsymbol{\cdot} \widetilde{B}$.*

2) *Both $(Z^P)^\tau X^\tau = (Z^P)^\tau (X^\tau - X^\sigma)$ and $Z^{(1)}\widetilde{M} = (Z^{(1)})^\tau (\widetilde{M}^\tau - \widetilde{M}^\sigma)$ are local P-martingales.*

Moreover, we can also assume or impose that

$$N^{(1)} = I_{\rrbracket\sigma,\tau\rrbracket} \boldsymbol{\cdot} N^{(1)}.$$

If S is quasi-left-continuous, then we have in addition that

3) *$\widetilde{B}$ is continuous.*

4) *$(Z^P)^\tau \widetilde{M} = (Z^P)^\tau (\widetilde{M}^\tau - \widetilde{M}^\sigma)$ is also a local P-martingale.*

Before we start proving Theorem 5.1, let us explain its use. Recall that a local P-martingale $Z > 0$ with $Z_0 = 1$ is a P-σ-martingale density for Y if ZY is a P-σ-martingale. We call Z a *P-local martingale density* for Y if ZY is a local P-martingale. In view of 1) and since $\Delta X_\tau = \Delta(\varphi{\cdot}S)_\tau = \varphi_\tau \Delta S_\tau$, we can write

$$
\text{(5.2)} \qquad I_{]\!]\sigma,\tau]\!]}{\cdot}X = X^\tau - X^\sigma = X^{\tau-} - X^\sigma + \Delta X_\tau I_{[\![\tau,\infty[\![} \\
= \widetilde{M} + (\widetilde{B} + \Delta X_\tau I_{[\![\tau,\infty[\![}) = \widetilde{M} + (\widetilde{B} + \varphi_\tau \Delta S_\tau I_{[\![\tau,\infty[\![}),
$$

and $(Z^P)^\tau$ is by 2) a P-local martingale density for the left-hand side of (5.2). If S is quasi-left-continuous, then $(Z^P)^\tau$ is by 4) also a P-local martingale density for the first term on the right-hand side and therefore also for the second summand on the right-hand side, which is a single-jump process whose "drift part" $\widetilde{B}$ is continuous by 3).

Remark 5.2. 1) In our later applications of Theorem 5.1, the process S will be quasi-left-continuous. However, some of the techniques used to prove Theorem 5.1 for general S will also appear in other arguments below. For this reason, we state and prove Theorem 5.1 for general, not necessarily quasi-left-continuous S.

2) The basic object of our analysis is the process S, and we want to formulate our main results and especially our conditions in terms of S. So we also have to keep track of how the results in Theorem 5.1 depend on the choice of φ. ◇

For ease of notation, we prove Theorem 5.1 for the case $\sigma \equiv 0$; the argument for general $\sigma \le \tau$ is completely analogous. So in (5.1), we look at $I_{]\!]0,\tau]\!]}{\cdot}S = S^\tau - S_0$, and we first study the process $S^{\tau-} - S_0$. This is a semimartingale, and we denote the associated quantities by $\mu_1, \nu_1^P, B^1, C^1, A^1$, using for $S^{\tau-}$ the same truncation function as for S. In particular, we have $d\mu_1 = I_{]\!]0,\tau[\![}\, d\mu$. We also denote by μ_0 the jump measure of the single-jump process $J^{(\tau)} = \Delta S_\tau I_{[\![\tau,\infty[\![}$, so that we get $S^{\tau-} - S_0 = S^\tau - S_0 - J^{(\tau)} = I_{]\!]0,\tau]\!]}{\cdot}S - x * \mu_0$ and $d\mu_0 + d\mu_1 = I_{[\![0,\tau]\!]}\, d\mu$. *Note that our subsequent results from Lemma 5.3 to Corollary 5.6 all assume that the conditions of Theorem 5.1 are satisfied with $\sigma \equiv 0$.*

Lemma 5.3. *The process A^1 dominating the characteristics of $S^{\tau-} - S_0$ under P (and also under some $Q \overset{\text{loc}}{\approx} P$, if that is needed) can be chosen such that $A^1 \ll A$. In particular, if A is continuous, then so is A^1.*

Proof. As seen above, $S^{\tau-} = S^\tau - x * \mu_0$. The characteristics of S^τ under P (as well as Q, if needed) can be dominated by A, and since $\mu_0 \ll \mu$ implies $\nu_0^P \ll \nu^P$, the same is true for the characteristics of $x * \mu_0$, so that $A^1 \ll A$. The same arguments apply under Q. **q.e.d.**

By the assumption in Theorem 5.1, $Z^P S$ is a P-σ-martingale. So Lemma 2.9 implies that whenever we choose a bounded predictable process $\varphi > 0$ and set $X := \varphi{\cdot}S$, the product $Z^P X$ is a local P-martingale, and so is then $(Z^P)^\tau X^\tau$. For later use, we note that combining $S^\tau = S^{\tau-} + J^{(\tau)}$ and $\Delta X_\tau = \Delta(\varphi{\cdot}S)_\tau = \varphi_\tau \Delta S_\tau = \Delta(\varphi{\cdot}J^{(\tau)})_\tau$ implies that

$\varphi\raisebox{0.5ex}{\textbullet}(S^{\tau-}) = (\varphi\raisebox{0.5ex}{\textbullet}S)^{\tau-} = X^{\tau-}$; thus

$$\{\Delta X^{\tau-} \neq 0\} = \{\Delta S^{\tau-} \neq 0\} =]\!]0,\tau[\![\cap \{\Delta S \neq 0\}. \tag{5.3}$$

Recalling that the characteristics of $S^{\tau-}$ are denoted by a sub- or superscript 1, we now define the process

$$N^{(1)} := \big(\beta^P\cdot S^c + W^{(1)} * (\mu_1 - \nu_1^P)\big)^\tau = (N^{(1)})^\tau = I_{]\!]0,\tau]\!]}\cdot N^{(1)} \tag{5.4}$$

(which is a local P-martingale as argued in Proposition 5.4 below), with

$$W_t^{(1)} := \frac{f_t^P - 1}{\mathcal{D}_t^{(1)}} := \frac{f_t^P - 1}{1 - a_t^1 + \widehat{f_t^P}^1} = \frac{f_t^P - 1}{1 - \int_{I\!R^d} 1\,\nu_1^P(\{t\},dx) + \int_{I\!R^d} f_t^P(x)\,\nu_1^P(\{t\},dx)}; \tag{5.5}$$

compare (2.1) and note the superscripts 1 on $\widehat{\cdot}^{\,1}$ and a^1 here since we work with μ_1 and ν_1^P. Note that $W^{(1)}$ is well defined since $a^1 \leq 1$ and $f^P > 0$, hence $\widehat{f^P}^1 > 0$ on $\{a^1 > 0\}$. We also point out that the main difference between $N^{(1)}$ from (5.4) and N^P from (2.44) is that $N^{(1)}$ "does not involve the jump of S at τ" since we work with μ_1 instead of μ.

Proposition 5.4. *The process $Z^{(1)} := \mathcal{E}(N^{(1)})$ is a strictly positive local P-martingale with $Z_0^{(1)} = 1$. It is locally bounded, the product $Z^{(1)}X^{\tau-}$ is a special P-semimartingale, and*

$$X^{\tau-} = \widetilde{M} + \widetilde{B}, \tag{5.6}$$

where $Z^{(1)}\widetilde{M}$ is a local P-martingale and $\widetilde{B}$ is predictable and of finite variation. Moreover, we have $Y = Y^\tau = I_{]\!]0,\tau]\!]}\cdot Y$ for $Y \in \{N^{(1)}, \widetilde{M}, \widetilde{B}\}$ and $(Z^{(1)})^\tau = Z^{(1)}$.

If $Z^{(1)}$ were a uniformly integrable true P-martingale, we could use it as a density process to define a probability $Q^{(1)}$ equivalent to P. Then Proposition 5.4 would say that $X^{\tau-}$ is a special $Q^{(1)}$-semimartingale with $Q^{(1)}$-canonical decomposition (5.6). In general, we have these properties only "locally". This is the content of Proposition 5.4.

Proof of Proposition 5.4. 1) First of all, part 2) of Lemma 2.8 for the process $S^{\tau-}$ yields that $1/\mathcal{D}^{(1)}$ is locally bounded. More precisely, we prove (2.23) for $\bar{U} := -f^P + 1$ via (2.14) by estimating from above with S, using (5.3). Moreover, $N^P = \beta^P\cdot S^c + W^P * (\mu - \nu^P)$ has Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to S; so (2.19) in Lemma 2.6 and (2.20) yield

$$\Delta N^P = \big(f^P(\Delta S) - 1\big)I_{\{\Delta S\neq 0\}} - \widehat{W^P}I_{\{\Delta S=0\}} = W^P(\Delta S)I_{\{\Delta S\neq 0\}} - \widehat{W^P}. \tag{5.7}$$

Because W^P is in $\mathcal{G}^1_{\mathrm{loc}}(\mu)$ for P, the process $\big(\sum_{0<s\leq\cdot}(W_s^P(\Delta S_s)I_{\{\Delta S_s\neq 0\}} - \widehat{W_s^P})^2\big)^{1/2}$ is locally P-integrable. This is by (5.7) equivalent to saying that both $\big(\sum_{0<s\leq\cdot}(\widehat{W_s^P})^2 I_{\{\Delta S_s=0\}}\big)^{1/2}$ and

$\big(\sum_{0<s\le\cdot}(f_s^P(\Delta S_s)-1)^2 I_{\{\Delta S_s\neq 0\}}\big)^{1/2}=((f^P-1)^2*\mu)^{1/2}$ are locally P-integrable, and hence in particular finite-valued.

2) To show that $N^{(1)}$ is well defined and a local P-martingale, we need to argue that $W^{(1)}$ is in $\mathcal{G}^1_{\mathrm{loc}}(\mu_1)$ for P. So we first compute with the help of (5.5) that

$$(5.8)\qquad W^{(1)}(\Delta S^{\tau-})I_{\{\Delta S^{\tau-}\neq 0\}}-\widehat{W^{(1)}}^{\,1}=\frac{f^P(\Delta S^{\tau-})-1}{\mathcal{D}^{(1)}}I_{\{\Delta S^{\tau-}\neq 0\}}-\frac{\widehat{f^P}^{\,1}-a^1}{\mathcal{D}^{(1)}}.$$

Using $(a-b)^2\le 2a^2+2b^2$ and (5.3), we thus obtain

$$K:=\sum_{0<s\le\cdot}\Big(W_s^{(1)}(\Delta S_s^{\tau-})I_{\{\Delta S_s^{\tau-}\neq 0\}}-\widehat{W_s^{(1)}}^{\,1}\Big)^2\le\frac{2}{(\mathcal{D}^{(1)})^2}\centerdot J^1+\frac{2}{(\mathcal{D}^{(1)})^2}\centerdot J^2$$

with

$$J^1:=\sum_{0<s\le\cdot}\big(f_s^P(\Delta S_s)-1\big)^2 I_{\{\Delta S_s\neq 0\}}I_{]\!]0,\tau[\![}=(f^P-1)^2*\mu_1\le(f^P-1)^2*\mu,$$

$$J^2:=\sum_{0<s\le\cdot}(\widehat{f_s^P}^{\,1}-a_s^1)^2.$$

We claim that J^1 and J^2 are both locally bounded. Because also $1/\mathcal{D}^{(1)}$ is locally bounded, this will imply that K is locally bounded, which of course implies that $K^{1/2}$ is locally P-integrable. The latter precisely means that $W^{(1)}$ is in $\mathcal{G}^1_{\mathrm{loc}}(\mu_1)$ for P.

Clearly, J^1 and J^2 are adapted and RCLL where they are finite. Moreover, J^1 is finite-valued due to Step 1), because W^P is in $\mathcal{G}^1_{\mathrm{loc}}(\mu)$ for P, and (5.7) yields

$$\Delta J^1=\big(f^P(\Delta S)-1\big)^2 I_{\{\Delta S\neq 0\}}I_{]\!]0,\tau[\![}=(\Delta N^P)^2 I_{\{\Delta S\neq 0\}}I_{]\!]0,\tau[\![}.$$

But this is bounded because ΔN^P is bounded on $]\!]0,\tau[\![$ by assumption; so J^1 has bounded jumps and is therefore also locally bounded.

For J^2, we first estimate by Jensen's inequality that

$$\begin{aligned}(\widehat{f_t^P}^{\,1}-a_t^1)^2&=\Big(\int_{\mathbb{R}^d}\big(f_t^P(x)-1\big)\,\nu_1^P(\{t\},dx)\Big)^2\\&=(a_t^1)^2\Big(\int_{\mathbb{R}^d}\big(f_t^P(x)-1\big)\tfrac{\nu_1^P(\{t\},dx)}{\nu_1^P(\{t\},\mathbb{R}^d)}\Big)^2\\&\le(a_t^1)^2\int_{\mathbb{R}^d}\big(f_t^P(x)-1\big)^2\tfrac{\nu_1^P(\{t\},dx)}{a_t^1}\\&=a_t^1\int_{\mathbb{R}^d}\big(f_t^P(x)-1\big)^2\,\nu_1^P(\{t\},dx).\end{aligned}$$

(If $a_t^1=\nu_1^P(\{t\},\mathbb{R}^d)=0$, then the left-hand side is also zero.) Since $0\le a^1\le 1$, we thus get

$$(\widehat{f_t^P}^{\,1}-a_t^1)^2\le\int_{\mathbb{R}^d}\big(f_t^P(x)-1)\big)^2\,\nu_1^P(\{t\},dx)={}^{\mathbf{P}}(\Delta J^1)_t=\Delta\big((J^1)^{\mathbf{P}}\big)_t$$

by (2.2), the definition of J^1 and Theorem VI.76 of DM, and so $J^2 \leq \sum \Delta((J^1)^{\mathbf{p}}) \leq (J^1)^{\mathbf{p}}$. Because J^1 is locally bounded, so is its compensator $(J^1)^{\mathbf{p}}$ and hence also J^2. This completes the argument for Step 2).

3) In analogy to (5.7), we can now compute the jumps of $N^{(1)}$ as

$$
\begin{aligned}
\Delta N^{(1)} &= W^{(1)}(\Delta S^{\tau-}) I_{\{\Delta S^{\tau-}\neq 0\}} - \widehat{W^{(1)}}^{\,1} \\
&= \frac{f^P(\Delta S) - 1}{\mathcal{D}^{(1)}} I_{\{\Delta S \neq 0\}} I_{]\!]0,\tau[\![} - \frac{\widehat{f^P}^{\,1} - a^1}{\mathcal{D}^{(1)}} \\
&= \Big(\frac{f^P(\Delta S)}{\mathcal{D}^{(1)}} - 1\Big) I_{\{\Delta S \neq 0\}} I_{]\!]0,\tau[\![} + \Big(\frac{1}{\mathcal{D}^{(1)}} - 1\Big) I_{\{\Delta S^{\tau-} = 0\}},
\end{aligned}
\tag{5.9}
$$

using (5.3) and (5.8). As f^P and $\mathcal{D}^{(1)}$ are both strictly positive, we see that $\Delta N^{(1)} > -1$, and so $Z^{(1)} = \mathcal{E}(N^{(1)})$ is a strictly positive local P-martingale with $Z_0^{(1)} = 1$. Moreover, (5.9) together with (5.7) and (5.3) yields

$$
\Delta N^{(1)} = \frac{\Delta N^P}{\mathcal{D}^{(1)}} I_{\{\Delta S \neq 0\}} I_{]\!]0,\tau[\![} + \Big(\frac{1}{\mathcal{D}^{(1)}} - 1\Big)
\tag{5.10}
$$

and together with $N^P = (N^P)^\tau$ on $]\!]0,\tau[\![$ also

$$
(1 + \Delta N^{(1)}) I_{\{\Delta X^{\tau-}\neq 0\}} = \frac{f^P(\Delta S)}{\mathcal{D}^{(1)}} I_{\{\Delta S \neq 0\}} I_{]\!]0,\tau[\![} = \frac{1}{\mathcal{D}^{(1)}} \big(1 + \Delta (N^P)^\tau\big) I_{\{\Delta X^{\tau-}\neq 0\}}.
\tag{5.11}
$$

But ΔN^P is bounded on $]\!]0,\tau[\![$ by assumption and $1/\mathcal{D}^{(1)}$ is locally bounded; so (5.11) shows that $N^{(1)}$ has locally bounded jumps, and so $Z^{(1)}$ is locally bounded.

4) For the final assertion, recall first that $(Z^P X)^\tau$ is a local P-martingale. By the product rule, so is then $(X + [N^P, X])^\tau$. So $\sup_{0<t\leq \cdot} |\Delta(X_{t\wedge\tau} + [N^P, X]_{t\wedge\tau})| = \sup_{0<t\leq\cdot} |\Delta X_t^\tau (1 + \Delta(N^P)_t^\tau)|$ is locally P-integrable, and hence so is $\sup_{0<t\leq\cdot} |\Delta X_t^{\tau-}(1 + \Delta(N^P)_t^\tau)|$. Using (5.11) and the fact that $1/\mathcal{D}^{(1)}$ is locally bounded thus shows that $\sup_{0<t\leq\cdot} |\Delta X_t^{\tau-}(1 + \Delta N_t^{(1)})|$ is locally P-integrable as well. By Theorem VII.25 of DM, the semimartingale $X' := X^{\tau-} + [N^{(1)}, X^{\tau-}]$ is therefore P-special, and then so is $Z^{(1)} X^{\tau-}$ by the product rule again.

To obtain (5.6), we start with the P-canonical decomposition

$$
X^{\tau-} + [N^{(1)}, X^{\tau-}] = X' = M' + B'
\tag{5.12}
$$

and set $\widetilde{B} := B'$, $\widetilde{M} := X^{\tau-} - B'$. So we have to show that $Z^{(1)}(X^{\tau-} - B')$ is a local P-martingale or equivalently that $X^{\tau-} - B' + [N^{(1)}, X^{\tau-} - B']$ is a local P-martingale. But by (5.12), this process equals $M' - [N^{(1)}, B']$, and this is a local P-martingale by Yoeurp's lemma since B' is predictable and of finite variation; see Theorem VII.36 of DM. The last assertion is clear from the definitions and the fact that $(X^{\tau-})^\tau = X^{\tau-} = I_{]\!]0,\tau]\!]} \cdot X^{\tau-}$. **q.e.d.**

To obtain the properties claimed in Theorem 5.1, we next take a closer look at $\widetilde{B}$ from Proposition 5.4. Recall that A^1 is the process dominating the characteristics of $S^{\tau-}$.

Lemma 5.5. *The process $\widetilde{B}$ from Proposition 5.4 has the form $\widetilde{B} = (\varphi\widetilde{b}) \cdot A$ for an $\mathbb{R}^d$-valued predictable process $\widetilde{b}$ which does not depend on φ. In particular, if A is continuous, so is $\widetilde{B}$.*

Proof. The second assertion is clear from the first one. To prove the first, we recall from the proof of Proposition 5.4 that $\widetilde{B} = B'$ is the predictable FV part of $X' = X^{\tau-} + [N^{(1)}, X^{\tau-}]$; see (5.12). As in Lemma 2.10, we compute from (5.4) and $X^{\tau-} = \varphi \cdot (S^{\tau-})$ that

$$
\begin{aligned}
X' &= X^{\tau-} + [N^{(1)}, X^{\tau-}] \\
&= \varphi \cdot (S^{\tau-})^c + (\varphi h) * (\mu_1 - \nu_1^P) + \big(\varphi(b^1 + c^1 \beta^P)\big) \cdot A^1 + \big(\varphi(x f^{(1)} - h)\big) * \mu_1,
\end{aligned}
$$

where $f^{(1)}$ is associated to $W^{(1)}$ as in (2.11). (We could write out the formula for $f^{(1)}$, but it is not needed here.) Now the first three summands on the right-hand side are all locally P-integrable, and so is X' since it is P-special. Thus the μ_1-integral process on the right-hand side is also locally P-integrable and therefore admits a P-compensator, which is $(\varphi(x f^{(1)} - h)) * \nu_1^P$. Therefore $\widetilde{B} = (\varphi(b^1 + c^1 \beta^P)) \cdot A^1 + (\varphi(x f^{(1)} - h)) * \nu_1^P$, and since $\nu_1^P(dt, dx) = F_{1,t}^P(dx)\, dA_t^1$ and $A^1 \ll A$ by Lemma 5.3, the process

$$
\widetilde{b} := \Big((b^1 + c^1 \beta^P) + \int_{\mathbb{R}^d} \big(x f^{(1)}(x) - h(x) \big) F_1^P(dx) \Big) \alpha^1
$$

with $dA^1 = \alpha^1\, dA$ gives one representation of $\widetilde{B}$ as desired.

It remains to show that although $\widetilde{B}$ depends on the choice of φ, the process $\widetilde{b}$ does not. So start with $\bar{\varphi}$ instead of φ to get $\bar{X} = \bar{\varphi} \cdot S$ and go through the above arguments again. Instead of (5.12), we then obtain $\bar{X}' = \bar{X}^{\tau-} + [N^{(1)}, \bar{X}^{\tau-}] = \bar{M}' + \bar{B}'$, and we have $\bar{X}^{\tau-} = \bar{\varphi} \cdot S^{\tau-}$ and $X^{\tau-} = \varphi \cdot S^{\tau-}$. Because φ and $\bar{\varphi}$ are both $S^{\tau-}$-integrable, we obtain from Theorem 4.7 in Cherny/Shiryaev (2002) that the ratio $\bar{\varphi}/\varphi$ is first $X^{\tau-}$-integrable and then also X'-integrable, and since X' is P-special with P-canonical decomposition (5.12), Lemma 4.2 of Cherny/Shiryaev (2002) implies that $\bar{X}' = (\bar{\varphi}/\varphi) \cdot X' = (\bar{\varphi}/\varphi) \cdot M' + (\bar{\varphi}/\varphi) \cdot B'$ is the P-canonical decomposition of $\bar{X}'$. So we obtain from $B' = \widetilde{B} = (\varphi \widetilde{b}) \cdot A$ that

$$
\bar{B}' = (\bar{\varphi}/\varphi) \cdot B' = (\bar{\varphi}/\varphi) \cdot \widetilde{B} = (\bar{\varphi} \widetilde{b}) \cdot A,
$$

with the same process $\widetilde{b}$ that we have explicitly constructed above. **q.e.d.**

With the above preparations, we are now ready for the

Proof of Theorem 5.1. Recall that $(Z^P)^\tau X^\tau$ is a local P-martingale from the choice of φ and that $X^\tau = (\varphi \cdot S)^\tau = \varphi \cdot S^\tau$. If we write $X^{\tau-} = \widetilde{M} + \widetilde{B}$ as in (5.6), then $\widetilde{M}^\tau = \widetilde{M}$ and $\widetilde{B}^\tau = \widetilde{B}$, and $Z^{(1)} \widetilde{M}$ is a local P-martingale, all by Proposition 5.4; so we have 1) and 2).

If in addition S is quasi-left-continuous, then A can be chosen continuous (see JS, Proposition II.2.9) so that $\widetilde{B}$ is continuous by Lemma 5.5. From (5.6), we thus obtain

$$\{\Delta \widetilde{M} \neq 0\} = \{\Delta X^{\tau-} \neq 0\}. \tag{5.13}$$

Moreover, A^1 is also continuous by Lemma 5.3, and so $\widehat{\cdot}^{1} \equiv 0$ by JS, Proposition II.2.9. Thus we get $\mathcal{D}^{(1)} \equiv 1$ via (5.5), and combining this with (5.13) and (5.11) therefore yields

$$\Delta N^{(1)} I_{\{\Delta \widetilde{M} \neq 0\}} = \Delta (N^P)^\tau I_{\{\Delta \widetilde{M} \neq 0\}}.$$

So $\Delta \widetilde{M} \Delta N^{(1)} = \Delta \widetilde{M} \Delta (N^P)^\tau$, and clearly $(N^{(1)})^c = ((N^P)^\tau)^c$ by (2.44) and (5.4). Therefore $\widetilde{M} + [\widetilde{M}, (N^P)^\tau] = \widetilde{M} + [\widetilde{M}, N^{(1)}]$ is a local P-martingale, because $Z^{(1)} \widetilde{M}$ is one by Proposition 5.4, and this implies by the product rule that $(Z^P)^\tau \widetilde{M}$ is a local P-martingale. This ends the proof. **q.e.d.**

For later use, we record a result already proved above.

Corollary 5.6. **1)** *With $W^{(1)}$ from (5.5) and*

$$N^{(1)} := I_{\rrbracket \sigma, \tau \rrbracket} \centerdot \big(\beta^P \centerdot S^c + W^{(1)} * (\mu_1 - \nu_1^P) \big) = I_{\rrbracket \sigma, \tau \rrbracket} \centerdot N^{(1)}, \tag{5.14}$$

the process $Z^{(1)} := \mathcal{E}(N^{(1)})$ is a locally bounded P-local martingale density for the process $\widetilde{M} = \widetilde{M}^\tau = I_{\rrbracket \sigma, \tau \rrbracket} \centerdot \widetilde{M}$.

2) *If S is quasi-left-continuous, then the process $(Z^P)^\tau = \mathcal{E}(I_{\rrbracket 0, \tau \rrbracket} \centerdot N^P)$ is a P-local martingale density for $\widetilde{B} + \Delta S_\tau I_{\llbracket \tau, \infty \llbracket}$.*

Moreover, $Z^{(1)}$ does not depend on the choice of φ in Theorem 5.1.

Proof. Part 1) is contained in Proposition 5.4; we just need to adjust (5.4) from $\sigma \equiv 0$ to a general $\sigma \leq \tau$. For part 2), we use (5.2) to write

$$(Z^P)^\tau (\widetilde{B} + \Delta X_\tau I_{\llbracket \tau, \infty \llbracket}) = (Z^P)^\tau (X^\tau - X^\sigma) - (Z^P)^\tau \widetilde{M}$$

and note that both terms on the right-hand side are local P-martingales by Theorem 5.1. Finally, $N^{(1)}$ in (5.14) clearly does not depend on φ, and so neither does $Z^{(1)} = \mathcal{E}(N^{(1)})$.

q.e.d.

6. The key construction for the quasi-left-continuous part of S

In Section 5 in (5.2), we have, for a fixed bounded predictable process $\varphi > 0$, decomposed the process $X^\tau - X^\sigma = (\varphi \centerdot S)^\tau - (\varphi \centerdot S)^\sigma = \varphi \centerdot (S^\tau - S^\sigma)$ into two summands, and we have constructed in Corollary 5.6 a local martingale density for $\widetilde{M}$, the first of these. The second summand is the process

$$X^{(0)} := \widetilde{B} + \Delta X_\tau I_{[\![\tau,\infty[\![} = I_{]\!]\sigma,\tau]\!]} \cdot X^{(0)}, \tag{6.1}$$

and if S is quasi-left-continuous, $X^{(0)}$ is continuous on $]\!]\sigma,\tau]\!]$ except for at most one single jump at τ, by Theorem 5.1. This section's goal is to construct for $X^{(0)}$ a local martingale density $Z^{(0)}$ with good local integrability properties. This is more than we get from Theorem 5.1 — all we know about $(Z^P)^\tau$ from there is that it is locally in $\mathcal{H}^1$ like every local martingale.

In view of (4.3) and because we shall deal with S^a separately later, we assume throughout this section that S is quasi-left-continuous.

To formulate our results, we need a bit of notation. We denote again by μ_0 the jump measure of the single-jump process $J^{(\tau)} := \Delta S_\tau I_{[\![\tau,\infty[\![}$, by ν_0^P its P-compensator, and write

$$\widehat{W_t}^{0,P} := \int_{\mathbb{R}^d} W_t(x)\, \nu_0^P(\{t\}, dx)$$

as in (2.1). For a probability $Q \overset{\text{loc}}{\approx} P$, we also use ν_0^Q and $\widehat{W_t}^{0,Q}$. (We explain in Remark 6.4 at the end of this section why we work here with Q and not only P.) Recall that A dominates the characteristics of S under both P and Q. Note that $d\mu_0 + d\mu_1 = I_{]\!]\sigma,\tau]\!]}\, d\mu$ implies for $R \in \{P, Q\}$ that $\nu_0^R + \nu_1^R = \nu^R$ on the predictable set $]\!]\sigma,\tau]\!]$ and therefore

$$\widehat{W_t}^{0,R} = \int_{\mathbb{R}^d} W_t(x)\, \nu_0^R(\{t\}, dx) = \int_{\mathbb{R}^d} W_t(x)\, \nu^R(\{t\}, dx) = \widehat{W_t}^{R} \equiv 0 \qquad \text{on }]\!]\sigma,\tau]\!] \tag{6.2}$$

because S is quasi-left-continuous. Finally, the definition (6.1) of $X^{(0)}$, $\Delta X_\tau = \varphi_\tau \Delta S_\tau$ and Lemma 5.5 give

$$X^{(0)} = (\varphi\widetilde{b}) \cdot A + \varphi \cdot (x * \mu_0), \tag{6.3}$$

and we know from parts 2) and 4) of Theorem 5.1 that $(Z^P)^\tau \in \mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, P) \neq \emptyset$. In fact, $(Z^P)^\tau$ is even a P-local martingale density for $X^{(0)}$.

Now take $Q \overset{\text{loc}}{\approx} P$ so that also $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, Q) \neq \emptyset$. We denote the Q-compensator of μ_0 by ν_0^Q and write as usual $\nu_0^Q(dt, dx) = F_{0,t}^Q(dx)\, dA_t$. Define a bijection ψ from $\mathbb{R}^d$ to the open unit ball $U_1(0, \mathbb{R}^d)$ in $\mathbb{R}^d$ by $\psi(x) := \frac{x}{1+|x|}$ and introduce the auxiliary predictable process

$$\widetilde{\mathcal{R}}(Q) := I_{\{\widetilde{b}\neq 0\}} \operatorname*{ess\,sup}_{z\in\mathbb{R}^d} \frac{(-z^\top \widetilde{b})^-}{\int_{\mathbb{R}^d} (z^\top \psi(x))^- \, F_0^Q(dx)}, \tag{6.4}$$

where the essential supremum for $\widetilde{\mathcal{R}}_t(Q)(\omega)$ is taken with respect to the (random) measure $F_{0,t}^Q(\omega, \cdot)$ on $\mathbb{R}^d$. (We argue in Section 9, at the end of Step 1, that the ratio in (6.4) is well defined, with the convention $0/0 := 0$.) We introduce on $(\Omega \times [0,\infty) \times I\!R^d, \widetilde{\mathcal{P}})$ and $(\Omega \times [0,\infty), \mathcal{P})$ the probability measures

$$\begin{aligned} m_Q(d\omega, dt, dx) &:= C_Q F^Q_{0,t}(\omega, dx)\, dA_t(\omega)\, Q(d\omega), \\ m(d\omega, dt) &:= C_Q F^Q_{0,t}(\omega, I\!R^d)\, dA_t(\omega)\, Q(d\omega), \end{aligned} \tag{6.5}$$

where C_Q is a normalising constant. (It will be part of the proof in Section 9 below that m_Q, m are null or well defined and that $C_Q \in (0,\infty)$ if $m_Q \not\equiv 0$.) We point out that all the above quantities depend only on S, but not on the choice of φ, due to Lemma 5.5.

Finally let $\Phi : [0,\infty) \to I\!R$ be a function satisfying the following properties:

(6.6) Φ is strictly convex and in C^1, and $\Phi(0) = 0$.

(6.7) Φ grows at least linearly for large x, i.e., there is some constant $D > 0$ such that $|\Phi(x)| \geq Dx$ for $x \geq x_0$.

(6.8) Φ is bounded from below by a constant, i.e., $\Phi(x) \geq$ const. for all $x \geq 0$.

For the definition of the spaces L^Φ and L^Φ_{loc}, we refer to Section 3. In particular, Lemma 3.3 shows that L^Φ_+ is a cone due to the linear growth condition (6.7) on Φ.

The main result of this section is then

Theorem 6.1. *Suppose that S is quasi-left-continuous and the process $X^{(0)}$ in (6.1) satisfies $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, P) \neq \emptyset$. If $Q \overset{\mathrm{loc}}{\approx} P$ and if*

$$\widetilde{\mathcal{R}}(Q) \in L^\Phi_{\mathrm{loc}}(m, \mathcal{P}), \tag{6.9}$$

then we can find for $X^{(0)}$ a Q-σ-martingale density $Z^{(0)}$ with $Z^{(0)} \in L^\Phi_{\mathrm{loc}}(Q)$.

Remark 6.2. If we assume instead of (6.9) that $\widetilde{\mathcal{R}}(Q)$ is locally bounded, then we can find for $X^{(0)}$ a Q-σ-martingale density $Z^{(0)}$ which is also locally bounded. For more details, we refer to Remark 9.3 below. ◇

We postpone the proof of Theorem 6.1 to Sections 9 and 10 and proceed directly with our main line of argument. A look at (9.53) in Step 12 of the proof in Section 9, with 0 there replaced by σ, shows that $Z^{(0)}$ is given by $Z^{(0)} = \mathcal{E}(N^{(0)})$ with

$$N^{(0)} := \widetilde{W}^Q * (\widehat{\mu}_0 - \widehat{\nu}^Q_0) = I_{]\!]\sigma,\tau]\!]} \cdot N^{(0)}, \tag{6.10}$$

where $\widetilde{W}^Q = \widetilde{f}^Q - 1$ for a process $\widetilde{f}^Q$ constructed in the proof of Theorem 6.1, $\widehat{\mu}_0$ is the jump measure of $\widehat{X}^{(0)} := \frac{1}{1+H} \cdot X^{(0)}$ for a finite-valued predictable process $H \geq 0$ (which is also constructed in the course of the proof), and $d\widehat{\nu}^Q_0 = \frac{1}{1+H}\, d\nu^Q_0$ is the Q-compensator of $\widehat{\mu}_0$. We next apply Corollary 5.6 under Q instead of P to obtain the Q-local martingale density $Z^{(1)} = \mathcal{E}(N^{(1)})$ for $\widetilde{M} = X^\tau - X^\sigma - X^{(0)}$, due to (5.2). Recall from (5.14) in Corollary 5.6 (applied with Q) that $N^{(1)}$ is explicitly constructed and given by

$$N^{(1)} := I_{]\!]\sigma,\tau]\!]} \cdot \big(\beta^Q \cdot S^c + W^Q * (\mu_1 - \nu_1^Q)\big) = I_{]\!]\sigma,\tau]\!]} \cdot N^{(1)}. \tag{6.11}$$

Proposition 6.3. *Suppose that S is quasi-left-continuous. For $Z^{(1)} = \mathcal{E}(N^{(1)})$ with $N^{(1)}$ as in (6.11), the process*

$$Z := Z^{(0)} Z^{(1)} = \mathcal{E}(N^{(0)} + N^{(1)}) =: \mathcal{E}(N) \tag{6.12}$$

is a Q-σ-martingale density for $S^\tau - S^\sigma = I_{]\!]\sigma,\tau]\!]} \cdot S$, and $N = I_{]\!]\sigma,\tau]\!]} \cdot N$. Moreover, if τ is chosen such that $Z^{(1)}$ is locally bounded, then the process Z is in $L^\Phi_{\mathrm{loc}}(Q)$.

Proof. Because $X^\tau - X^\sigma = \varphi \cdot (S^\tau - S^\sigma)$, it is by Lemma 2.9 enough to show that $Z^{(1)}$ is a Q-σ-martingale density for $X^\tau - X^\sigma$. For symmetry, set $X^{(1)} := \widetilde{M}$ so that we have $X^\tau - X^\sigma = X^{(1)} + X^{(0)}$.

1) We shall argue below that

$$[N^{(i)}, N^{(k)}] = 0 \qquad \text{for } i \neq k, \tag{6.13}$$

$$[N^{(i)}, X^{(k)}] = 0 \qquad \text{for } i \neq k, \tag{6.14}$$

and we already know that

$$Z^{(i)} = \mathcal{E}(N^{(i)}) \text{ is a } Q\text{-}\sigma\text{-martingale density for } X^{(i)}, \quad \text{and } N^{(i)} = I_{]\!]\sigma,\tau]\!]} \cdot N^{(i)}, \text{ for } i = 0, 1. \tag{6.15}$$

The equality in (6.12) then follows from Yor's formula and (6.13), and we claim that also the middle assertion then holds. (This repeats a small part of the proof of Lemma 4.1.) In fact, to argue that $Z^{(0)} Z^{(1)}$ is a Q-σ-martingale density for $X := X^{(0)} + X^{(1)}$, it is by Lemma 2.9 equivalent to show that $X + [X, N^{(0)} + N^{(1)}]$ is a Q-σ-martingale. But (6.14) yields

$$X + [X, N^{(0)} + N^{(1)}] = X^{(0)} + [X^{(0)}, N^{(0)}] + X^{(1)} + [X^{(1)}, N^{(1)}],$$

and this is a Q-σ-martingale due to (6.15). That $N = I_{]\!]\sigma,\tau]\!]} \cdot N$ is also clear from (6.15).

2) To obtain (6.13), note first that $(N^{(0)})^c \equiv 0$ by (6.10) so that $\langle (N^{(0)})^c, (N^{(1)})^c \rangle = 0$ and therefore $[N^{(0)}, N^{(1)}] = \sum \Delta N^{(0)} \Delta N^{(1)}$. But by (6.11), $N^{(1)}$ can jump at most on $\operatorname{supp} \mu_1 = (\operatorname{supp} \mu) \cap]\!]\sigma, \tau[\![$ by the definition of μ_1 in Section 5 (for general $\sigma \leq \tau$, not for $\sigma \equiv 0$), and (6.10) implies that $N^{(0)}$ can only jump on $\operatorname{supp} \widehat{\mu}_0 \subseteq [\![\tau]\!]$, by the definition of $\widehat{\mu}_0$. Thus $N^{(0)}$ and $N^{(1)}$ have no common jumps and so $[N^{(0)}, N^{(1)}] = 0$. This yields (6.13).

3) Next we prove (6.14). By its definition, $X^{(0)}$ is of finite variation with at most one jump at τ by Theorem 5.1, and $N^{(1)}$ at most jumps on $]\!]\sigma, \tau[\![$ as just seen in Step 2) so

that $[X^{(0)}, N^{(1)}] = \sum \Delta X^{(0)} \Delta N^{(1)} = 0$. On the other hand, $X^{(1)} = \widetilde{M} = X^{\tau-} - \widetilde{B}$ can only jump on $]\!]\sigma, \tau[\![$ since $\widetilde{B}$ is continuous by Theorem 5.1. So exactly like in Step 2), we obtain $[X^{(1)}, N^{(0)}] = \langle (X^{(1)})^c, (N^{(0)})^c \rangle + \sum \Delta X^{(1)} \Delta N^{(0)} = 0$ because both summands vanish. This gives (6.14).

4) Finally, suppose that $Z^{(1)}$ is locally bounded (which can indeed be achieved for a suitable choice of τ by Theorem 5.1). We know that $Z^{(0)}$ is in $L^{\Phi}_{\mathrm{loc}}(Q)$ by Theorem 6.1, and this means that for a localising sequence $(\sigma_n)_{n \in I\!N}$ of stopping times, each random variable $(Z^{(0)})^*_{\sigma_n} = \sup_{0 \le t \le \sigma_n} Z^{(0)}_t$ is in $L^{\Phi}_{+}(Q)$. As the latter is a cone, the product $Z^{(1)} Z^{(0)}$ is therefore also in $L^{\Phi}_{\mathrm{loc}}(Q)$. This ends the proof. **q.e.d.**

Remark 6.4. 1) In Section 9 below, we actually prove a more detailed version of Theorem 6.1. We have kept the formulation here deliberately short for ease of reading.

2) Let us return to questions 1) and 2) from Section 1, rephrased in localised form. If we take $Q = P$, Theorem 6.1 provides sufficient conditions on the jumpy part of S for question 1) to have a positive answer. As mentioned in Section 1 after Theorem 1.3, we believe these conditions are essentially sharp, i.e. more or less also necessary. For question 2), we can thus expect to have as given some property like (6.9) for P. If we now apply Theorem 6.1 with Q, it remains to check how we can also verify the condition (6.9) for Q. We can either try to study this directly under Q, after working out the structure of S under Q; or we can start from (6.9) under P and try to derive (6.9) under Q, by using the structure of the density process $D^{Q;P}$ of Q with respect to P. However, we do not embark on that here. $\diamond$

7. Handling the accessible part of S

In Section 4, we have seen how $S - S_0$ can be decomposed as $S^a + S^i$, and the results in Section 6 will allow us to deal with the quasi-left-continuous part S^i. In this section, we show that if S admits a P-σ-martingale density, then its "accessible part" S^a even admits a locally bounded P-σ-martingale density. This can be viewed as a generalisation of Theorem 5 in Stricker (1990). We actually prove a more general result, namely

Theorem 7.1. *Suppose S is an $I\!R^d$-valued adapted RCLL process, $(R_n)_{n \in I\!N}$ is a sequence of predictable stopping times with pairwise disjoint graphs and $D := \bigcup_{n=1}^{\infty} [\![R_n]\!]$. Then the following are equivalent:*

1) *$\mathcal{D}_{\mathrm{e},\sigma}(I_D \cdot S, P) \neq \emptyset$, i.e. $I_D \cdot S$ admits a P-σ-martingale density.*

2) *$\mathcal{D}_{\mathrm{e},\sigma}(I_D \cdot S, P) \cap \{$locally bounded processes$\} \neq \emptyset$, i.e. $I_D \cdot S$ admits a P-σ-martingale density which is locally bounded.*

3) *For any $Q \overset{\text{loc}}{\approx} P$, $I_D \cdot S$ admits a Q-σ-martingale density which is locally bounded. The σ-martingale density $Z = \mathcal{E}(N)$ for $I_D \cdot S$ can then be chosen such that $N = I_D \cdot N$.*

Proof. Clearly, 2) implies 1) and 3) implies 2). Moreover, as already pointed out in Section 1, 1) is equivalent to $\mathcal{D}_{\mathrm{e},\sigma}(I_D \cdot S, Q) \neq \emptyset$ for any $Q \overset{\text{loc}}{\approx} P$, and so 1) will imply 3) as soon as we show that it implies 2). Proving the latter is quite difficult, even if the basic idea looks simple. We start with a P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ for $\bar{S} = I_D \cdot S$ and write the P-σ-martingale property of $Z^P \bar{S}$ as in (2.41) in Corollary 2.12 as the statement that the Jacod parameter f^P of N^P satisfies a zero-drift equation (ZDE for short). So the ZDE has a positive solution f^P, and we should like to show that it then even has a positive and (locally) bounded solution $\tilde{f}^P$. Using this as Jacod parameter for a new local P-martingale $\tilde{N}^P$, we get that $\tilde{N}^P$ is locally bounded, and hence so is then $\tilde{Z}^P = \mathcal{E}(\tilde{N}^P)$. But $\tilde{Z}^P$ is also a P-σ-martingale density for $\bar{S}$ because $\tilde{f}^P$ satisfies the ZDE, and so we get 2).

The problem with the above argument is that things do not exactly work like this for technical reasons. The ZDE can be seen as stating that a linear operator attains a value (here zero) in a positive function, and we essentially want to deduce that the operator then attains that value also in a positive and bounded function. Such a result can be found in Theorem 2.9 of Borwein/Lewis (1991) for the case where the range of the operator is finite-dimensional (in fact, $\mathbb{R}^n$). We need here an extension to a range in an infinite-dimensional space, and so we need some extra properties of the operator. For our setting, these can be achieved partly by localisation and partly by imposing some integrability properties on S or $\bar{S}$. The latter can be achieved by means of a measure change, and so we must first work under a different measure $Q^{(m)}$, for each m, and then go back to P. Let us now make this more precise.

a) Because $\bar{S} := I_D \cdot S$ is a semimartingale since $\mathcal{D}_{\mathrm{e},\sigma}(\bar{S}, P) \neq \emptyset$, the increasing processes $\bar{S}^* := \sup_{0<s\leq \cdot} |\bar{S}_s|$ and $V := \sum_{i=1}^d [\bar{S}^i]$ are finite-valued adapted RCLL processes. For each $m \in \mathbb{N}$, we can therefore define a probability measure $Q^{(m)} \approx P$ by

$$\frac{dQ^{(m)}}{dP} := \text{const.}(m) \exp(-\bar{S}^*_m - V_m).$$

The density process $D^{Q^{(m)};P}$ of $Q^{(m)}$ with respect to P is bounded, and both $\bar{S}$ and its optional quadratic variation $[\bar{S}]$ have on $[\![0, m]\!]$ all $Q^{(m)}$-moments (for each coordinate of $\bar{S}$ or each entry of the matrix $[\bar{S}]$). The assumption $\mathcal{D}_{\mathrm{e},\sigma}(\bar{S}, P) \neq \emptyset$ also yields $\mathcal{D}_{\mathrm{e},\sigma}(\bar{S}, Q^{(m)}) \neq \emptyset$ since $Q^{(m)} \approx P$, and so Corollary 2.14 allows us to choose for $\bar{S}$ a $Q^{(m)}$-σ-martingale density $Z^{Q^{(m)}} = \mathcal{E}(N^{Q^{(m)}})$ where $N^{Q^{(m)}}$ has Jacod parameters $(\beta^{Q^{(m)}}, f^{Q^{(m)}}, 0, 0)$ with respect to $\bar{S}$.

b) Denote by $\mu^{\bar{S}}$ and $\nu^{Q^{(m)}}$ the jump measure of $\bar{S}$ and its $Q^{(m)}$-compensator. As in Section 2, we can write $\nu^{Q^{(m)}}(dt, dx) = F_t^{Q^{(m)}}(dx)\, dA_t^{Q^{(m)}}$, but we exploit here the structure of $\bar{S} = I_D \cdot S$ to choose a better version for this decomposition. In fact, because $D = \bigcup_{n=1}^{\infty} [\![R_n]\!]$

is predictable like the R_n, we have $d\nu^{Q^{(m)}} = I_D\, d\nu^{Q^{(m)}}$ from $d\mu^{\bar S} = I_D\, d\mu^{\bar S}$, and so

$$\nu^{Q^{(m)}}(dt, dx) = \sum_{n=1}^{\infty} I_{\{R_n<\infty\}}\delta_{R_n}(dt)\,\nu^{Q^{(m)}}(\{t\}, dx) = \bar F_t^{Q^{(m)}}(dx)\, dA_t^{(m)}$$

with

$$\begin{aligned}\bar F_t^{Q^{(m)}}(dx) &:= \nu^{Q^{(m)}}(\{t\}, dx),\\ dA_t^{(m)} &:= \sum_{n=1}^{\infty} I_{\{R_n<\infty\}}\delta_{R_n}(dt).\end{aligned} \tag{7.1}$$

Because $\bar S = I_D \centerdot S = x * \mu^{\bar S}$, we then obtain by Corollary 2.12 for the $Q^{(m)}$-σ-martingale density $Z^{Q^{(m)}}$ from Step a) that

$$\int_{\mathbb{R}^d} |x| f_t^{Q^{(m)}}(x)\bar F_t^{Q^{(m)}}(dx) < \infty \qquad Q^{(m)}\otimes A^{(m)}\text{-a.e.}, \tag{7.2}$$

$$\int_{\mathbb{R}^d} x f_t^{Q^{(m)}}(x)\bar F_t^{Q^{(m)}}(dx) = 0 \qquad Q^{(m)}\otimes A^{(m)}\text{-a.e.} \tag{7.3}$$

c) At this point, it looks tempting to prove directly with a DMW argument that we can find a bounded positive solution $\widetilde f^Q$ to (7.2), (7.3); it could seem that a proof like for example in Step 2 or Step 10 in the proof of Theorem 9.2 below might work. However, this is not possible because the measure $Q^{(m)}\otimes A^{(m)}\otimes \bar F^{Q^{(m)}}$ is not finite in general and therefore cannot be normalised to a probability measure. So we have to split $\bar S$ into a pre-stopped part and a single-jump part and deal with these two separately.

For technical reasons, we need some additional properties of $N^{Q^{(m)}}$ and $f^{Q^{(m)}}$ that we can achieve by suitable stopping. First of all, note that $V = \sum_{i=1}^{d} [\bar S^i]$ is $Q^{(m)}$-integrable on $[\![0, m]\!]$ by the definition of $Q^{(m)}$ so that it has a $Q^{(m)}$-compensator $\widetilde V$ on $[\![0, m]\!]$. Because $\widetilde V$ is predictable and RCLL, it is locally bounded on $[\![0, m]\!]$, and we have

$$\Delta \widetilde V = \Delta(V^{\mathbf{p}}) = {}^{\mathbf{p}}(\Delta V) = {}^{\mathbf{p}}\Big(\sum_{i=1}^{d}(\Delta \bar S^i)^2\Big) = {}^{\mathbf{p}}(|\Delta \bar S|^2) \qquad \text{on } [\![0, m]\!]$$

by DM, Theorem VI.76, where ${}^{\mathbf{p}}C$ denotes here the $Q^{(m)}$-predictable projection and $C^{\mathbf{p}}$ the $Q^{(m)}$-compensator of C. By Jensen's inequality, $({}^{\mathbf{p}}(|\Delta \bar S|))^2 \le {}^{\mathbf{p}}(|\Delta \bar S|^2)$, and so we obtain that ${}^{\mathbf{p}}(|\Delta \bar S|)$ is locally bounded on $[\![0, m]\!]$. But as in Section 2,

$${}^{\mathbf{p}}(|\Delta \bar S|)_t = \int_{\mathbb{R}^d} |x| \nu^{Q^{(m)}}(\{t\}, dx) = \int_{\mathbb{R}^d} |x| \bar F_t^{Q^{(m)}}(dx)$$

by (2.2), (2.1) and (7.1), and so

(7.4) the process $\int_{\mathbb{R}^d} |x| \bar F_t^{Q^{(m)}}(dx)$, $t \ge 0$, is locally bounded on $[\![0, m]\!]$.

We next look at $\bar{S} + [\bar{S}, N^{Q^{(m)}}]$. This is a $Q^{(m)}$-σ-martingale by Lemma 2.9 because $Z^{Q^{(m)}}$ is a $Q^{(m)}$-σ-martingale density for $\bar{S}$, and as in the proof of Corollary 2.12, we have

$$\bar{S} + [\bar{S}, N^{Q^{(m)}}] = (x f^{Q^{(m)}}) * \mu^{\bar{S}}.$$

As in the proof of Lemma 2.11, the predictable process $\varphi^{(m)} > 0$ defined by

$$1/\varphi_t^{(m)} := 1 + \int_{I\!R^d} |x| f_t^{Q^{(m)}}(x) \bar{F}_t^{Q^{(m)}}(dx), \qquad t \geq 0,$$

is well defined due to (7.2) and bounded, and by construction,

(7.5) $\qquad$ the process $\varphi_t^{(m)} \int_{I\!R^d} |x| f_t^{Q^{(m)}}(x) \bar{F}_t^{Q^{(m)}}(dx)$, $t \geq 0$, is bounded.

Moreover, again as in the proof of Lemma 2.11, the zero-drift condition (7.3) implies that $(\varphi^{(m)} x f^{Q^{(m)}}) * \mu^{\bar{S}} = \varphi^{(m)} \cdot (\bar{S} + [\bar{S}, N^{Q^{(m)}}])$ is a local $Q^{(m)}$-martingale.

Now fix m, take a localising sequence $(\tilde{\varrho}_k^{(m)})_{k \in I\!N}$ for the processes in (7.4), (7.5) and set

$$\varrho_k^{(m)} := \tilde{\varrho}_k^{(m)} \wedge \inf \big\{ t \geq 0 \,\big|\, |N_t^{Q^{(m)}}| > \tfrac{k}{2} \big\}.$$

We then choose k_m large enough so that $P[\varrho_{k_m}^{(m)} < m] \leq 2^{-m}$. This is possible since for each m, $\varrho_k^{(m)} \nearrow \infty$ $Q^{(m)}$-a.s. as $k \to \infty$, hence also P-a.s. and in $L^0(P)$. In addition, we choose k_m recursively in m in such a way that the sequence $(\varrho_{k_m}^{(m)})_{m \in I\!N}$ is increasing, and we set

$$\sigma_m := \varrho_{k_m}^{(m)} \wedge m.$$

Then $\sum_{m=1}^{\infty} P[\sigma_m < m] = \sum_{m=1}^{\infty} P[\varrho_{k_m}^{(m)} < m] < \infty$ and therefore $\sigma_m \nearrow \infty$ P-a.s. by Borel–Cantelli. Moreover, we also have by construction that

(7.6) $\qquad$ $|\Delta N^{Q^{(m)}}|$ is bounded by a constant on $]\!]0, \sigma_m[\![$.

d) For each $m \in I\!N$, we now look at the stopped process $\bar{S}^{\sigma_m}$ and write this as

$$\bar{S}^{\sigma_m} = \bar{S}^{\sigma_m -} + J^{(m)}$$

like in (5.1). We denote by $\mu^{(m)}$, $\mu_1^{(m)}$ and $\mu_0^{(m)}$ the jump measures of $\bar{S}^{\sigma_m}$, $\bar{S}^{\sigma_m -}$ and $J^{(m)}$, respectively, so that obviously $d\mu_1^{(m)} + d\mu_0^{(m)} = d\mu^{(m)} = I_{[\![0, \sigma_m]\!]} \, d\mu^{\bar{S}}$. The $Q^{(m)}$-compensators are denoted by $\nu^{(m)}$, $\nu_1^{(m)}$ and $\nu_0^{(m)}$, can be written like in Step b) as

$$\nu_i^{(m)}(dt, dx) = F_{i,t}^{(m)}(dx) \, dA_t^{(m)}$$

with

$$F^{(m)}_{i,t}(dx) = \nu^{(m)}_i(\{t\}, dx), \tag{7.7}$$

and satisfy $d\nu^{(m)}_1 + d\nu^{(m)}_0 = d\nu^{(m)} = I_{[\![0,\sigma_m]\!]}\, d\nu^{Q^{(m)}}$ and

$$F^{(m)}_{1,t} + F^{(m)}_{0,t} = I_{[\![0,\sigma_m]\!]} \bar{F}^{Q^{(m)}}_t =: F^{(m)}_t. \tag{7.8}$$

For each $m \in I\!N$, we then define a measure $\kappa^{(m)}(\omega, t, dx)$ on $I\!R^d$ by

$$\kappa^{(m)}(\omega, t, dx) := I_{D\cap[\![0,\sigma_m]\!]}(\omega, t)\varphi^{(m)}_t(\omega)\, \nu^{(m)}_0(\omega, \{t\}, dx). \tag{7.9}$$

This is $\mathcal{P}$-measurable like D, $\varphi^{(m)}$ and $\nu^{(m)}_0$. Moreover, the measure

$$\pi^{(m)}(d\omega, dt, dx) := \kappa^{(m)}(\omega, t, dx)\, dA^{(m)}_t(\omega)\, Q^{(m)}(d\omega) \tag{7.10}$$

on $\Omega \times [0, \infty) \times I\!R^d$ is finite, because $\varphi^{(m)}$ is bounded and

$$\begin{aligned} E_{Q^{(m)}}\Big[\int_0^\infty \nu^{(m)}_0(\{t\}, I\!R^d)\, dA^{(m)}_t \Big] &= E_{Q^{(m)}}[(1 * \nu^{(m)}_0)_\infty] \\ &= E_{Q^{(m)}}[(1 * \mu^{(m)}_0)_\infty] \\ &= Q^{(m)}\Big[\bigcup_{n=1}^{\infty} \{R_n < \infty, \Delta S_{R_n} \neq 0, R_n = \sigma_m\} \Big] < \infty. \end{aligned}$$

The marginal $\bar{\pi}^{(m)}$ of $\pi^{(m)}$ on $\bar{\Omega} := \Omega \times [0, \infty)$ is also finite and obviously equivalent to $Q^{(m)} \otimes A^{(m)}$, by the definition (7.10) of $\pi^{(m)}$. Thus (7.9), (7.7) and (7.8), (7.4) give

$$\begin{aligned} \Big\| \int_{I\!R^d} |x| \kappa^{(m)}(dx) \Big\|_{L^\infty(\bar{\pi}^{(m)})} &= (Q^{(m)} \otimes A^{(m)})\text{- ess sup} \int_{I\!R^d} |x| \kappa^{(m)}(dx) \\ &\leq \|\varphi^{(m)}\|_{L^\infty(\bar{\pi}^{(m)})} \Big\| I_{[\![0,\sigma_m]\!]} \int_{I\!R^d} |x| F^{(m)}_0(dx) \Big\|_{L^\infty(\bar{\pi}^{(m)})} < \infty, \end{aligned} \tag{7.11}$$

and in the same way, (7.5) yields that

$$\begin{aligned} &\Big\| \int_{I\!R^d} |x| f^{Q^{(m)}}(x) \kappa^{(m)}(dx) \Big\|_{L^\infty(\bar{\pi}^{(m)})} \\ &= (Q^{(m)} \otimes A^{(m)})\text{- ess sup} \Big(I_{D\cap[\![0,\sigma_m]\!]} \varphi^{(m)} \int_{I\!R^d} |x| f^{Q^{(m)}}(x) F^{(m)}_0(dx) \Big) < \infty. \end{aligned} \tag{7.12}$$

Moreover, the $\widetilde{\mathcal{P}}$-measurable function $f^{Q^{(m)}}$ on $\Omega \times [0, \infty) \times I\!R^d$ is > 0 $\pi^{(m)}$-a.e. and satisfies

$$\int_{I\!R^d} x f^{Q^{(m)}}(x) \kappa^{(m)}(dx) = b^{(m)} := -I_{D\cap[\![0,\sigma_m]\!]} \varphi^{(m)} \int_{I\!R^d} x f^{Q^{(m)}}(x) F^{(m)}_1(dx) \tag{7.13}$$

by (7.9), (7.7), (7.8) and the zero-drift property (7.3). Moreover, by (7.8) and (7.5), the process $b^{(m)}$ is bounded by some constant. Note that (7.11)–(7.13) correspond precisely to the conditions (2.2), (2.3) and (2.5) in Choulli/Schweizer (2015). By Theorem 2.1 in Choulli/Schweizer (2015), there hence exists a $\widetilde{\mathcal{P}}$-measurable function $\tilde{f}_0^{(m)}$ on $\Omega \times [0,\infty) \times I\!R^d$ which is strictly positive $\pi^{(m)}$-a.e., bounded by a constant c_m (say) $\pi^{(m)}$-a.e. and which satisfies

$$\int_{I\!R^d} x\tilde{f}_0^{(m)}(x)\kappa^{(m)}(dx) = b^{(m)} \qquad Q^{(m)} \otimes A^{(m)}\text{-a.e.} \tag{7.14}$$

Because $\varphi^{(m)} > 0$ and

$$-b^{(m)} + I_{D\cap[\![0,\sigma_m]\!]}\varphi^{(m)} \int_{I\!R^d} x f^{Q^{(m)}}(x)F_0^{(m)}(dx) = 0$$

due to (7.8) and the zero-drift condition (7.3), we can rewrite (7.14) as

$$\int_{I\!R^d} x\tilde{f}_0^{(m)}(x)F_0^{(m)}(dx) = \int_{I\!R^d} x f^{Q^{(m)}}(x)F_0^{(m)}(dx) \qquad \bar{\pi}^{(m)}\text{-a.e. (on } D\cap[\![0,\sigma_m]\!]). \tag{7.15}$$

Moreover, because $|\tilde{f}_0^{(m)}| \le c_m\ \pi^{(m)}$-a.e., we also get from (7.4) that

$$\int_{I\!R^d} |x|\tilde{f}_0^{(m)}(x)F_0^{(m)}(dx) \le c_m \int_{I\!R^d} |x|\bar{F}^{Q^{(m)}}(dx) < \infty \qquad \bar{\pi}^{(m)}\text{-a.e. (on } D\cap[\![0,\sigma_m]\!]). \tag{7.16}$$

So essentially, we have started with a positive solution $f^{Q^{(m)}}$ to the zero-drift condition in (7.2), (7.3) and have been able to find in (7.15), (7.16) for the single-jump part $J^{(m)}$ of $\bar{S}^{\sigma_m}$ even a positive and bounded solution $\tilde{f}_0^{(m)}$.

e) Now define the $\widetilde{\mathcal{P}}$-measurable functions $W^{(m),1}$ and $W^{(m),0}$ by

$$W^{(m),1} := \frac{f^{Q^{(m)}} - 1}{\mathcal{D}^{(1)}} := \frac{f^{Q^{(m)}} - 1}{1 - a^1 + \widehat{f^{Q^{(m)}}}^{\,1}},$$

$$W^{(m),0} := \frac{\tilde{f}_0^{(m)} - 1}{\mathcal{D}^{(0)}} := \frac{\tilde{f}_0^{(m)} - 1}{1 - a^0 + \widehat{\tilde{f}_0^{(m)}}^{\,0}},$$

where $a^i := \widehat{1}^{\,i}$ and $\widehat{\cdot}^{\,i}$ are taken here with respect to $Q^{(m)}$ and for the measures $\nu_i^{(m)}$, $i = 0, 1$. We also define

$$\mathcal{D} := 1 - a + \widehat{f^{Q^{(m)}}}^{\,1} + \widehat{\tilde{f}_0^{(m)}}^{\,0} = \mathcal{D}^{(1)} + \mathcal{D}^{(0)} - 1.$$

As in the proof of Proposition 5.4 (with S replaced by $\bar{S}$, τ by σ_m and P by $Q^{(m)}$), we then first argue that $1/\mathcal{D}^{(1)}$ is locally bounded and $W^{(m),1}$ is in $\mathcal{G}^1_{\mathrm{loc}}(\mu_1^{(m)})$ for $Q^{(m)}$. As a consequence, the process

$$N^{(m),1} := W^{(m),1} * (\mu_1^{(m)} - \nu_1^{(m)})$$

is well defined and a local $Q^{(m)}$-martingale null at 0. In analogy to (5.9), its jumps are

$$\Delta N^{(m),1} = \Big(\frac{f^{Q^{(m)}}(\Delta\bar{S})}{\mathcal{D}^{(1)}} - 1\Big) I_{\Gamma_1} + \Big(\frac{1}{\mathcal{D}^{(1)}} - 1\Big) I_{\Gamma_1^c}, \tag{7.17}$$

where $\Gamma_1 := \{\Delta\bar{S}^{\sigma_m -} \neq 0\}$ is the support of the measure $\mu_1^{(m)}$. Moreover, recalling that $N^{Q^{(m)}}$ has Jacod parameters $(\beta^{Q^{(m)}}, f^{Q^{(m)}}, 0, 0)$, we get

$$\Delta N^{Q^{(m)}} = \big(f^{Q^{(m)}}(\Delta\bar{S}) - 1\big) I_{\{\Delta\bar{S}^{\sigma_m} \neq 0\}} - \frac{\widehat{f^{Q^{(m)}}} - a}{1-a} I_{\{\Delta\bar{S}^{\sigma_m} = 0\}}. \tag{7.18}$$

Now let us consider $W^{(m),0}$. By part 3) of Lemma 2.8, $1/\mathcal{D}^{(0)}$ is locally bounded. Moreover, with $\Gamma_0 := \{\Delta J^{(m)} \neq 0\} = \operatorname{supp}\mu_0^{(m)} \subseteq \rrbracket 0, \sigma_m \rrbracket$, we have

$$\begin{aligned}
\sum \big(W^{(m),0}(\Delta\bar{S}) I_{\{\Delta J^{(m)} \neq 0\}} - \widehat{W^{(m),0}}^{\,0}\big)^2 &= \sum \Big(\frac{\tilde{f}_0^{(m)}(\Delta\bar{S}) - 1}{\mathcal{D}^{(0)}} I_{\Gamma_0} - \frac{1}{\mathcal{D}^{(0)}}\big(\widehat{\tilde{f}_0^{(m)}}^{\,0} - a^0\big)\Big)^2 \\
&\le \frac{2}{(\mathcal{D}^{(0)})^2}\big(\tilde{f}_0^{(m)}(\Delta\bar{S}) - 1\big)^2 I_{\Gamma_0} \\
&\quad + \sum \frac{2}{(\mathcal{D}^{(0)})^2}\big(\widehat{\tilde{f}_0^{(m)}}^{\,0} - a^0\big)^2,
\end{aligned}$$

using that $J^{(m)}$ is a single-jump process. But the first term on the right-hand side is locally bounded like $1/\mathcal{D}^{(0)}$ because $\tilde{f}_0^{(m)}$ is bounded, and the second can be estimated from above by the compensator of the first term, exactly as in the proof of Proposition 5.4, Step 2). Thus we obtain that $W^{(m),0}$ is in $\mathcal{G}^1_{\mathrm{loc}}(\mu_0^{(m)})$ for $Q^{(m)}$ so that the process

$$N^{(m),0} := W^{(m),0} * (\mu_0^{(m)} - \nu_0^{(m)})$$

is well defined and a local $Q^{(m)}$-martingale null at 0. Its jumps, in analogy to (7.17), are

$$\Delta N^{(m),0} = \Big(\frac{\tilde{f}_0^{(m)}(\Delta\bar{S})}{\mathcal{D}^{(0)}} - 1\Big) I_{\Gamma_0} + \Big(\frac{1}{\mathcal{D}^{(0)}} - 1\Big) I_{\Gamma_0^c}. \tag{7.19}$$

Finally, we need to argue that $1/\mathcal{D}$ is also locally bounded. This is done as in the proof of Lemma 2.8 with minor modifications, as follows. We first replace everywhere $\sum_{0<s\le\cdot} (\bar{U}_s(\Delta X_s))^2 I_{\{\Delta X_s \neq 0\}} = \bar{U}^2 * \mu$ by $(f^{Q^{(m)}} - 1)^2 * \mu_1^{(m)} + (\tilde{f}_0^{(m)} - 1)^2 * \mu_0^{(m)} =: V^{(1)} + V^{(0)}$. The latter sum is locally $Q^{(m)}$-integrable; this is argued separately for the two summands, exactly as in the proof of Proposition 5.4. Then we replace all the terms $\widehat{\bar{U}}$ by the sum $\widehat{\bar{U}}^{(1)} + \widehat{\bar{U}}^{(0)} := (\widehat{f^{Q^{(m)}}}^{\,1} - a^1) + (\widehat{\tilde{f}_0^{(m)}}^{\,0} - a^0) = \mathcal{D} - 1$, and we apply the estimates from the proof of Lemma 2.3 separately for $\widehat{\bar{U}}^{(1)}$, $\widehat{\bar{U}}^{(0)}$ and for $V^{n,(1)}, V^{n,(0)}$ constructed like in the proof of Lemma 2.8. Then everything goes through and the local boundedness of $1/\mathcal{D}$ follows.

f) In view of the results in Step e), the process

$$\widetilde{N}^{(m)} := \frac{\mathcal{D}^{(1)}}{\mathcal{D}} \cdot N^{(m),1} + \frac{\mathcal{D}^{(0)}}{\mathcal{D}} \cdot N^{(m),0} \tag{7.20}$$

is well defined and a local $Q^{(m)}$-martingale null at 0. Indeed, $\mathcal{D}^{(0)}$ is bounded since $\tilde{f}_0^{(m)}$ is bounded, and due to (7.18),

$$0 \le \mathcal{D}^{(1)} = 1 + (\widehat{f^{Q^{(m)}}}^1 - \widehat{1}^1) = 1 + {}^{\mathbf{P}}\Big(\big(f^{Q^{(m)}}(\Delta \bar{S}) - 1\big) I_{\{\Delta \bar{S}^{\sigma_m -} \neq 0\}}\Big) = 1 + {}^{\mathbf{P}}(\Delta N^{Q^{(m)}} I_{\Gamma_1})$$

is also bounded because of $\Gamma_1 \subseteq]\!]0, \sigma_m[\![$ and (7.6). Now note that Γ_0, Γ_1 and $\Gamma_0^c \cap \Gamma_1^c$ are pairwise disjoint with union $\Omega \times (0, \infty)$, and that $\Gamma_0 \subseteq \Gamma_1^c$ and $\Gamma_1 \subseteq \Gamma_0^c$. Combining this with (7.20), (7.17) and (7.19) yields the jumps of $\widetilde{N}^{(m)}$ as

$$\begin{aligned} \Delta \widetilde{N}^{(m)} &= \Big(\frac{f^{Q^{(m)}}(\Delta \bar{S})}{\mathcal{D}} - \frac{\mathcal{D}^{(1)}}{\mathcal{D}}\Big) I_{\Gamma_1} + \Big(\frac{1}{\mathcal{D}} - \frac{\mathcal{D}^{(1)}}{\mathcal{D}}\Big) I_{\Gamma_1^c} \\ &\quad + \Big(\frac{\tilde{f}_0^{(m)}(\Delta \bar{S})}{\mathcal{D}} - \frac{\mathcal{D}^{(0)}}{\mathcal{D}}\Big) I_{\Gamma_0} + \Big(\frac{1}{\mathcal{D}} - \frac{\mathcal{D}^{(0)}}{\mathcal{D}}\Big) I_{\Gamma_0^c} \\ &= \frac{1}{\mathcal{D}}\big(f^{Q^{(m)}}(\Delta \bar{S}) - \mathcal{D}^{(1)} + 1 - \mathcal{D}^{(0)}\big) I_{\Gamma_1} \\ &\quad + \frac{1}{\mathcal{D}}\big(\tilde{f}_0^{(m)}(\Delta \bar{S}) - \mathcal{D}^{(0)} + 1 - \mathcal{D}^{(1)}\big) I_{\Gamma_0} \\ &\quad + \frac{1}{\mathcal{D}}(1 - \mathcal{D}^{(1)} + 1 - \mathcal{D}^{(0)}) I_{\Gamma_0^c \cap \Gamma_1^c} \\ &= \Big(\frac{f^{Q^{(m)}}(\Delta \bar{S})}{\mathcal{D}} - 1\Big) I_{\Gamma_1} + \Big(\frac{\tilde{f}_0^{(m)}(\Delta \bar{S})}{\mathcal{D}} - 1\Big) I_{\Gamma_0} + \Big(\frac{1}{\mathcal{D}} - 1\Big) I_{\Gamma_0^c \cap \Gamma_1^c}, \end{aligned} \tag{7.21}$$

where we have used that $1 - \mathcal{D}^{(0)} - \mathcal{D}^{(1)} = -\mathcal{D}$. Because $f^{Q^{(m)}}$, $\tilde{f}_0^{(m)}$ and $\mathcal{D}$ are all > 0, (7.21) shows that $\Delta \widetilde{N}^{(m)} > -1$ so that $\widetilde{Z}^{(m)} := \mathcal{E}(\widetilde{N}^{(m)})$ is a local $Q^{(m)}$-martingale and > 0, with $\widetilde{Z}_0^{(m)} = 1$. Next, $\Delta \widetilde{N}^{(m)}$ is locally bounded like $1/\mathcal{D}$, because $\tilde{f}_0^{(m)}$ is bounded and $I_{\Gamma_1}(1 + \Delta \widetilde{N}^{(m)})\mathcal{D} = I_{\Gamma_1}(1 + \Delta N^{Q^{(m)}})$ by (7.18) is bounded due to $\Gamma_1 \subseteq]\!]0, \sigma_m[\![$ and (7.6). So $\widetilde{Z}^{(m)} = \mathcal{E}(\widetilde{N}^{(m)})$ is also locally bounded. Moreover, $\Gamma_0 \cup \Gamma_1 \subseteq D \cap [\![0, \sigma_m]\!]$ implies that

$$\widetilde{N}^{(m)} = I_{]\!]0,\sigma_m]\!]} \cdot \widetilde{N}^{(m)} = I_D \cdot \widetilde{N}^{(m)}.$$

Finally, using $\bar{S}^{\sigma_m} = x * \mu^{(m)}$ and (7.21) gives

$$\bar{S}^{\sigma_m} + [\bar{S}^{\sigma_m}, \widetilde{N}^{(m)}] = x * \mu^{(m)} + \sum \Delta \bar{S}^{\sigma_m} \Delta \widetilde{N}^{(m)} = \frac{1}{\mathcal{D}} \cdot \big((x f^{Q^{(m)}}) * \mu_1^{(m)} + (x \tilde{f}_0^{(m)}) * \mu_0^{(m)}\big).$$

Due to (7.8), (7.5), (7.4) and boundedness of $\tilde{f}_0^{(m)}$, the process

$$\frac{1}{\mathcal{D}} \varphi^{(m)} \Big(\int_{\mathbb{R}^d} |x| f^{Q^{(m)}}(x) F_1^{(m)}(dx) + \int_{\mathbb{R}^d} |x| \tilde{f}_0^{(m)}(x) F_0^{(m)}(dx) \Big)$$

is locally bounded, hence locally $Q^{(m)}$-integrable, on $[\![0,\sigma_m]\!]$. In view of (7.7), the process

$$\varphi^{(m)}\bullet(\bar S^{\sigma_m}+[\bar S^{\sigma_m},\widetilde N^{(m)}]) =: L^{(m)} = I_{D\cap[\![0,\sigma_m]\!]}\bullet L^{(m)}$$

therefore has a compensator which equals

$$\frac{1}{\mathcal{D}}\varphi^{(m)}\Big(\int_{I\!R^d} x f^{Q^{(m)}}(x)F_1^{(m)}(dx)+\int_{I\!R^d} x\tilde f_0^{(m)}(x)F_0^{(m)}(dx)\Big)=0 \qquad \text{on } D\cap[\![0,\sigma_m]\!]$$

thanks to (7.15), (7.8) and (7.3). So $L^{(m)}$ is a local $Q^{(m)}$-martingale, which means that $\bar S^{\sigma_m}+[\bar S^{\sigma_m},\widetilde N^{(m)}]$ is a $Q^{(m)}$-σ-martingale, and Lemma 2.9 thus shows that $\widetilde Z^{(m)}=\mathcal{E}(\widetilde N^{(m)})$ is a $Q^{(m)}$-σ-martingale density for $\bar S^{\sigma_m}$ and locally bounded.

g) Now we can finally go back to P. For each m, we have a $Q^{(m)}$-σ-martingale density $\widetilde Z^{(m)}$ for $\bar S^{\sigma_m}$, and we recall that the density process of $Q^{(m)}$ with respect to P is $D^{Q^{(m)};P}$ and bounded. By the Bayes rule, $Z^{(m)}:=\widetilde Z^{(m)}D^{Q^{(m)};P}$ is therefore a P-σ-martingale density for $\bar S^{\sigma_m}$ and locally bounded like $\widetilde Z^{(m)}$ and $D^{Q^{(m)};P}$. Write $Z^{(m)}=\mathcal{E}(N^{(m)})$. Because $Z^{(m)}\bar S^{\sigma_m}$ and $(Z^{(m)}\bar S^{\sigma_m})^{\sigma_{m-1}}$ are both P-σ-martingales, so is

$$Z^{(m)}(\bar S^{\sigma_m}-\bar S^{\sigma_{m-1}})=Z^{(m)}\bar S^{\sigma_m}-(Z^{(m)}\bar S^{\sigma_m})^{\sigma_{m-1}}-\bar S^{\sigma_{m-1}}\big(Z^{(m)}-(Z^{(m)})^{\sigma_{m-1}}\big).$$

So $Z^{(m)}=\mathcal{E}(N^{(m)})$ is a P-σ-martingale density for $\bar S^{\sigma_m}-\bar S^{\sigma_{m-1}}=I_{]\!]\sigma_{m-1},\sigma_m]\!]}\bullet(\bar S^{\sigma_m}-\bar S^{\sigma_{m-1}})$, and thus also $\bar Z^{(m)}:=\mathcal{E}(\bar N^{(m)})$ with $\bar N^{(m)}:=I_{]\!]\sigma_{m-1},\sigma_m]\!]}\bullet N^{(m)}$ is a P-σ-martingale density for $\bar S^{\sigma_m}-\bar S^{\sigma_{m-1}}$. But now Lemma 4.3 with D_m replaced by $]\!]\sigma_{m-1},\sigma_m]\!]$ implies that

$$\bar Z:=\prod_{m=1}^{\infty}\bar Z^{(m)}=\mathcal{E}\Big(\sum_{m=1}^{\infty}\bar N^{(m)}\Big)=:\mathcal{E}(\bar N)$$

is a P-σ-martingale density for $\sum_{m=1}^{\infty}(\bar S^{\sigma_m}-\bar S^{\sigma_{m-1}})=\bar S$, and because all the $\bar Z^{(m)}$ are locally bounded like $Z^{(m)}$, so is $\bar Z$; this uses $\bar N^{(m)}=I_{]\!]\sigma_{m-1},\sigma_m]\!]}\bullet\bar N^{(m)}$. Hence we have 2). **q.e.d.**

For our purposes, the following corollary of Theorem 7.1 is sufficient.

Corollary 7.2. *Suppose that S is an $I\!R^d$-valued adapted RCLL process and decompose $S=S_0+S^a+S^i$ as in (4.1), (4.2). If $\mathcal{D}_{\mathrm{e},\sigma}(S,P)\neq\emptyset$, then there exists a P-σ-martingale density $Z^a=\mathcal{E}(N^a)$ for S^a such that Z^a is locally bounded. Moreover, N^a can be chosen such that $N^a=I_D\bullet N^a$.*

Proof. Because Lemma 4.1 implies that also $\mathcal{D}_{\mathrm{e},\sigma}(S^a,P)\neq\emptyset$, the assertion follows immediately from (4.1) and Theorem 7.1. **q.e.d.**

As explained in Lemma 4.1, we can construct σ-martingale densities Z^a and Z^i separately for S^a and S^i, and their product will be a σ-martingale density for S. Since Z^a can be constructed to be locally bounded, it will be enough to obtain local integrabilities for Z^i.

8. Putting everything together

We now have everything in place for formulating and proving our main results. We do this in two steps having both their own intrinsic interest. The first step decomposes an integral $\varphi \cdot S$ of a general process S admitting some σ-martingale density into a sum of two parts; the first summand there is so "nice" that it even admits a σ-martingale density which is locally bounded. The second summand is an at most countable sum of single-jump processes each having a continuous drift term; one could say that this part of $\varphi \cdot S$ collects and in some way compensates the "bad" jumps of S. In a second step, we then show that if the drifts of the single-jump terms are not too extreme in comparison to the jump behaviour, also this second part of $\varphi \cdot S$ admits a "nice" σ-martingale density. More precisely, there exists a σ-martingale density with integrability properties directly related to the quantitative control available on the drift-to-jump ratio.

We begin with the first step of the above scheme. In view of possible extensions later, we prove a bit more than we actually need here, and we comment on that after the result.

Theorem 8.1. *Let $S = (S_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued adapted RCLL process and $(\varrho_n)_{n \in \mathbb{N}}$ an increasing sequence of stopping times with $\varrho_n \nearrow \infty$ P-a.s. Recall from (4.2) the quasi-left-continuous part $S^i = S - S_0 - S^a$ of S, and assume that $\mathcal{D}_{\mathrm{e},\sigma}(S, P) \neq \emptyset$. Then there exist $Z^P \in \mathcal{D}_{\mathrm{e},\sigma}(S^i, P) \neq \emptyset$, a predictable set D and two locally bounded local P-martingales $Z^a = \mathcal{E}(N^a)$ and $\bar{Z}^{(1)} = \mathcal{E}(\bar{N}^{(1)})$ with the following property: For any one-dimensional bounded predictable process $\varphi > 0$ such that $Z^P(\varphi \cdot S^i)$ is a local P-martingale, we can write*

$$\varphi \cdot S = X = X^{(1)} + X^{(0)} \tag{8.1}$$

with

$$X^{(1)} = \varphi \cdot S^a + \bar{X}^{(1)} = I_D \cdot (\varphi \cdot S^a) + I_{D^c} \cdot \bar{X}^{(1)}$$

and with $X^{(0)}$ of the form

$$X^{(0)} = I_{D^c} \cdot X^{(0)} = \sum_{n=1}^{\infty} (\widetilde{B}^n + \varphi_{\tau_n} \Delta S^i_{\tau_n} I_{[\![\tau_n, \infty[\![}) =: \sum_{n=1}^{\infty} Y^{(n,0)}, \tag{8.2}$$

where $(\tau_n)_{n \in \mathbb{N}}$ is an increasing sequence of stopping times satisfying $\tau_n \nearrow \infty$ P-a.s. and $\tau_n \leq \min(n, \varrho_n)$ for all n, and each $\widetilde{B}^n = I_{]\!]\tau_{n-1}, \tau_n]\!]} \cdot \widetilde{B}^n$ is continuous and of finite variation. In particular, $X^{(0)}$ is quasi-left-continuous like S^i. The process $X^{(1)}$ admits a P-σ-martingale density $\widetilde{Z}^{(1)} = \mathcal{E}(\widetilde{N}^{(1)})$ which is locally bounded and has $\widetilde{N}^{(1)}$ of the form

$$\widetilde{N}^{(1)} = I_D \cdot N^a + I_{D^c} \cdot \bar{N}^{(1)},$$

and Z^a and $\bar{Z}^{(1)}$ are locally bounded P-σ-martingale densities for S^a and $\bar{X}^{(1)}$, respectively. The P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ for S^i has N^P with Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to S^i and $N^P = I_{D^c} \cdot N^P$, and Z^P is also a P-σ-martingale density for $X^{(0)}$.

Moreover, if $Q \overset{\text{loc}}{\approx} P$ *has the density process* $D^{Q;P} = D_0^{Q;P}\,\mathcal{E}(N)$*, then the sequence* $(\tau_n)_{n\in I\!N}$ *can be chosen such that for some* $\delta > 0$ *not depending on* n*,*

$$\Delta N \geq -1+\delta \qquad \text{on } \rrbracket \tau_{n-1}, \tau_n \llbracket, \text{ for all } n, \tag{8.3}$$

$$|\Delta N| \leq n \qquad \text{on } \rrbracket \tau_{n-1}, \tau_n \llbracket, \text{ for all } n. \tag{8.4}$$

Remark 8.2. Neither the sequence $(\varrho_n)_{n\in I\!N}$ nor (8.3), (8.4) is needed below in our application of Theorem 8.1; so let us explain why we include them. If we want to prove results via localisation or want to work under local assumptions, it may be useful or necessary to localise a number of quantities before using Theorem 8.1. This can be done via the sequence (ϱ_n). Next, it may also be of interest to study what happens if we change from the original measure P to another reference measure $Q \overset{\text{loc}}{\approx} P$. Because $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$ is equivalent to $\mathcal{D}_{\mathrm{e},\sigma}(S,Q) \neq \emptyset$, we can use Theorem 8.1 under Q as well and then also obtain for the corresponding (Q-dependent) $X^{(1)}$ a Q-σ-martingale density which is locally bounded. However, the σ-martingale density for the part $X^{(0)}$ is harder to control, and its properties may depend on whether we work under P or Q. To relate them under P and under Q, it is important to have a good control on the density process $D^{Q;P}$ of Q with respect to P, or more precisely on the jumps of its stochastic logarithm N. This is what we achieve, at least locally, with (8.3) and (8.4). To be fair, however, we should point out that we do not get a control over ΔN on the graphs $\llbracket \tau_n \rrbracket$ — and it is exactly there that things usually become most complicated. ⋄

Proof of Theorem 8.1. 1) Decompose first $S = S_0 + S^a + S^i$ as in (4.1), (4.2) and use Lemma 4.1 and Corollary 7.2 to obtain P-σ-martingale densities $Z^a = \mathcal{E}(N^a)$ for $S^a = I_D \mathbin{\bullet} S$ and $Z^i = \mathcal{E}(N^i)$ for $S^i = I_{D^c} \mathbin{\bullet} S$ such that $N^a = I_D \mathbin{\bullet} N^a$ and $N^i = I_{D^c} \mathbin{\bullet} N^i$. Moreover, Z^a can be chosen to be locally bounded, and we use Corollary 2.14 to choose for S^i a P-σ-martingale density $Z^P = \mathcal{E}(N^P)$ with N^P having Jacod parameters $(\beta^P, f^P, 0, 0)$ with respect to S^i. Moreover, we can and do also choose N^P to have $N^P = I_{D^c} \mathbin{\bullet} N^P$, by Lemma 4.1. Note that φ does not appear up to here.

2) Now choose and fix φ. We next want to decompose $X^i := \varphi \mathbin{\bullet} S^i$ along a sequence $(\tau_n)_{n\in I\!N}$ and apply Theorem 5.1 to each resulting piece. At the same time, we want to keep control over ΔN. So starting from the given sequence $(\varrho_n)_{n\in I\!N}$, we fix $\delta > 0$ and define $\tau_0 := 0$ and recursively

$$\tau_n := \inf\left\{t > \tau_{n-1} \,\middle|\, \Delta N_t < -1+\delta \text{ or } |N_t| > \tfrac{n}{2} \text{ or } |N_t^P| > \tfrac{n}{2}\right\} \wedge \varrho_n \wedge n \qquad \text{for } n \in I\!N. \tag{8.5}$$

Clearly $\tau_n \leq \min(n, \varrho_n)$. Because N is P-a.s. RCLL, we know for P-almost all ω that for each fixed $t \geq 0$, there can be at most finitely many $s \leq t$ with $\Delta N_s(\omega) < -1+\delta$, and so $(\tau_n)_{n\in I\!N}$ increases to $+\infty$ P-a.s. like $(\varrho_n)_{n\in I\!N}$. Moreover, (8.5) yields for each $n \in I\!N$ that

$$\Delta N \geq -1+\delta \qquad \text{on } \rrbracket \tau_{n-1}, \tau_n \llbracket, \tag{8.6}$$

$$|\Delta N| \leq n \qquad \text{on } \rrbracket \tau_{n-1}, \tau_n \llbracket, \tag{8.7}$$

$$|\Delta N^P| \leq n \qquad \text{on } \rrbracket \tau_{n-1}, \tau_n \llbracket, \tag{8.8}$$

which gives (8.3) and (8.4). Note that (8.8) is analogous to the condition $|\Delta N^P| \le$ const. on $]\!]\sigma,\tau[\![$ that we imposed in Theorem 5.1.

3) Now write $Y := X^i := \varphi \bullet S^i$ for brevity and observe that $Z^P Y$ is a local P-martingale by Lemma 2.9 since $Z^P \in \mathcal{D}_{\mathrm{e},\sigma}(S^i, P)$. Note also that Y is quasi-left-continuous like S^i. For each n, we consider the process

$$(8.9) \qquad Y^{\tau_n} - Y^{\tau_{n-1}} = \begin{cases} 0 & \text{on } [\![0,\tau_{n-1}]\!], \\ Y^{\tau_n} - Y_{\tau_{n-1}} & \text{on }]\!]\tau_{n-1},\tau_n]\!], \\ Y_{\tau_n} - Y_{\tau_{n-1}} & \text{on }]\!]\tau_n,\infty[\![. \end{cases}$$

Like $Z^P Y$, both $(Z^P Y)^{\tau_n}$ and $(Z^P Y)^{\tau_{n-1}}$ are local P-martingales, and thus so is

$$(Z^P)^{\tau_n}(Y^{\tau_n} - Y^{\tau_{n-1}}) = (Z^P Y)^{\tau_n} - (Z^P Y)^{\tau_{n-1}} - Y^{\tau_{n-1}}\big((Z^P)^{\tau_n} - (Z^P)^{\tau_{n-1}}\big).$$

So $(Z^P)^{\tau_n}$ is a P-local martingale density for $Y^{\tau_n} - Y^{\tau_{n-1}}$ and we can apply Theorem 5.1 to $Y^{\tau_n} - Y^{\tau_{n-1}}$ (which is quasi-left-continuous like Y) instead of $X^\tau - X^\sigma$, with $]\!]\tau_{n-1},\tau_n]\!]$ instead of $]\!]\sigma,\tau]\!]$, to write

$$(8.10) \qquad Y^{\tau_n} - Y^{\tau_{n-1}} = \widetilde{M}^n + (\widetilde{B}^n + \Delta Y_{\tau_n} I_{[\![\tau_n,\infty[\![}) =: Y^{(n,1)} + Y^{(n,0)},$$

where $\widetilde{B}^n$ is continuous. By Corollary 5.6, we then get for each n a P-local martingale density $Z^{(n,1)} = \mathcal{E}(N^{(n,1)})$ for $Y^{(n,1)}$, where $Z^{(n,1)}$ does not depend on φ, and we also have

$$(8.11) \qquad N^{(n,1)} = I_{]\!]\tau_{n-1},\tau_n]\!]} \bullet N^{(n,1)}.$$

As $Y = \varphi \bullet S^i = \varphi \bullet (I_{D^c} \bullet S) = I_{D^c} \bullet Y$, we can choose to have $N^{(n,1)} = I_{D^c} \bullet N^{(n,1)}$ by Lemma 4.1. Moreover, since $|\Delta N^P| \le n$ on $]\!]\tau_{n-1},\tau_n[\![$ by (8.8), Corollary 5.6 (for $]\!]\tau_{n-1},\tau_n]\!]$ instead of $]\!]\sigma,\tau]\!]$) also shows that $Z^{(n,1)}$ is locally bounded. Finally, by Theorem 5.1 and Corollary 5.6, $\widetilde{Z}^{(n)} = \mathcal{E}(\widetilde{N}^{(n)})$ with $\widetilde{N}^{(n)} := I_{]\!]\tau_{n-1},\tau_n]\!]} \bullet N^P$ is a P-local martingale density for $Y^{(n,0)}$, for all n. Of course, each P-local martingale density is also a P-σ-martingale density.

4) The next step is a standard argument of "piecing things together". First, Lemma 4.3 with $X^n := Y^{(n,1)}$, $D_n :=]\!]\tau_{n-1},\tau_n]\!]$, $N^n := N^{(n,1)} = I_{D^c} \bullet N^{(n,1)}$ implies that $\bar{Z}^{(1)} := \mathcal{E}(\bar{N}^{(1)})$ with $\bar{N}^{(1)} := \sum_{n=1}^\infty N^{(n,1)} = I_{D^c} \bullet \bar{N}^{(1)}$ is a P-σ-martingale density for $\bar{X}^{(1)} := \sum_{n=1}^\infty Y^{(n,1)}$. In the same way, but with $X^n := Y^{(n,0)}$, $N^n := \widetilde{N}^n = I_{]\!]\tau_{n-1},\tau_n]\!]} \bullet N^P$, we get that $Z^P = \mathcal{E}(N^P)$ is a P-σ-martingale density for $\bar{X}^{(0)} := \sum_{n=1}^\infty Y^{(n,0)}$, and since $Y_0 = (\varphi \bullet S^i)_0 = 0$, we have

$$(8.12) \qquad \varphi \bullet S^i = Y = \sum_{n=1}^\infty (Y^{\tau_n} - Y^{\tau_{n-1}}) = \sum_{n=1}^\infty Y^{(n,1)} + \sum_{n=1}^\infty Y^{(n,0)} = \bar{X}^{(1)} + \bar{X}^{(0)}.$$

Moreover, each $Z^{(n,1)} = \mathcal{E}(N^{(n,1)})$ is locally bounded; so each $N^{(n,1)}$ has locally bounded jumps, and in view of (8.11), so has then their sum $\bar{N}^{(1)}$. This implies that $\bar{Z}^{(1)}$ is locally bounded. Finally, like each $Z^{(n,1)}$, also $\bar{Z}^{(1)}$ does not depend on φ.

5) Now recall (8.12) and $S = S_0 + S^a + S^i$ and define

$$X^{(1)} := \varphi\cdot S^a + \bar{X}^{(1)}, \qquad X^{(0)} := \bar{X}^{(0)}$$

to get (8.1) as well as (8.2), due to (8.12) and (8.10). We have already seen in Step 4) that $Z^P = \mathcal{E}(N^P)$ is a P-σ-martingale density for $X^{(0)} = \bar{X}^{(0)}$. Moreover, the construction in (8.10) of $Y^{(n,0)}$ from $Y = \varphi\cdot S^i = I_{D^c}\cdot Y$ shows via (8.12) that also $X^{(0)} = \bar{X}^{(0)} = I_{D^c}\cdot X^{(0)}$.

It only remains to look at

$$X^{(1)} = \varphi\cdot S^a + \bar{X}^{(1)} = I_D\cdot(\varphi\cdot S^a) + I_{D^c}\cdot\bar{X}^{(1)}, \tag{8.13}$$

where the second equality is due to (8.12) and the construction of $Y^{(n,1)}$ in (8.10) again. But by Corollary 7.2 and Step 4), S^a and $\bar{X}^{(1)}$ each admit P-σ-martingale densities $Z^a = \mathcal{E}(N^a)$ and $\bar{Z}^{(1)} = \mathcal{E}(\bar{N}^{(1)})$ respectively which are locally bounded. Moreover, Lemma 4.1 and (8.13) allow us to have $N^a = I_D\cdot N^a$ and $\bar{N}^{(1)} = I_{D^c}\cdot\bar{N}^{(1)}$. Again using Lemma 4.1 shows that $\widetilde{Z}^{(1)} = Z^a\bar{Z}^{(1)} = \mathcal{E}(N^a + \bar{N}^{(1)}) =: \mathcal{E}(\widetilde{N}^{(1)})$ is a P-σ-martingale density for $X^{(1)}$; it is locally bounded and does not depend on φ, like Z^a and $\bar{Z}^{(1)}$, and this completes the proof. **q.e.d.**

For the first step in Theorem 8.1 of our programme, we did not need any extra assumptions apart from $\mathcal{D}_{\mathrm{e},\sigma}(S,P) \neq \emptyset$. This is different for the second step. We formulate this for $Q \overset{\text{loc}}{\approx} P$ like Theorem 6.1, and refer to Remark 6.4 for an explanation why this is useful.

Theorem 8.3. *Under the assumptions of Theorem 8.1, fix $Q \overset{\text{loc}}{\approx} P$. Apply Theorem 8.1 under Q and write the resulting quasi-left-continuous process $X^{(0)}$ from (8.2) as*

$$X^{(0)} = \widetilde{B} + \sum_{n=1}^{\infty} \varphi_{\tau_n}\Delta S^i_{\tau_n} I_{[\![\tau_n,\infty[\![} = (\varphi\widetilde{b})\cdot A + \varphi\cdot(x * \mu^{(0)}), \tag{8.14}$$

where $\mu^{(0)}$ denotes the jump measure of $X^{(0)}$ and $\nu^{(0),Q}(dt,dx) = F_t^{(0),Q}(dx)\,dA_t$ its Q-compensator, and where the $I\!R^d$-valued predictable process $\widetilde{b}$ does not depend on the choice of φ. Define, with $\psi(x) = \frac{x}{1+|x|}$ as in Section 6, the predictable process

$$\widetilde{\mathcal{R}}^{(0)}(Q) := I_{\{\widetilde{b}\neq 0\}} \operatorname*{ess\,sup}_{z\in I\!R^d} \frac{(-z^\top\widetilde{b})^-}{\int_{I\!R^d} (z^\top\psi(x))^- \,F^{(0),Q}(dx)}, \tag{8.15}$$

where the ess sup is with respect to $F_t^{(0),Q}(\omega,\cdot)$. (We explain below why this ratio is well defined.) Let $\Phi : [0,\infty) \to I\!R$ satisfy the properties (6.6)–(6.8). Finally, suppose there exists a measurable function $\alpha : [0,\infty) \to I\!R$ which is uniformly strictly positive on each compact interval and such that

$$\int_0^T F_t^{(0),Q}(I\!R^d)\big|\Phi\big(\alpha(t)\widetilde{\mathcal{R}}_t^{(0)}(Q)\big)\big|\,dA_t < \infty \qquad Q\text{-a.s. for each } T \in (0,\infty). \tag{8.16}$$

Then there exists a Q-σ-martingale density $Z=\mathcal{E}(N)$ for S with $Z\in L^\Phi_{\mathrm{loc}}(Q)$.

Proof. The (fairly obvious) idea is to start from (8.12), apply the results of Sections 5 and 6 to each summand $Y^{\tau_n}-Y^{\tau_{n-1}}$ from (8.9), and piece things together. The main work will be to verify the assumption (6.9) of Theorem 6.1.

1) We first go back to the proof of Theorem 8.1 and write $S-S_0=S^a+S^i$ as in (4.3). Since $Q\overset{\mathrm{loc}}{\approx}P$ gives $\mathcal{D}_{\mathrm{e},\sigma}(S,Q)\neq\emptyset$, we can and do work in the sequel under Q instead of P. As in Step 1) of the above proof, we get a locally bounded Q-σ-martingale density $Z^a=\mathcal{E}(N^a)$ for S^a with $N^a=I_D\cdot N^a$, and a Q-σ-martingale density $Z^Q=\mathcal{E}(N^Q)$ for S^i where $N^Q=I_{D^c}\cdot N^Q$ has Jacod parameters $(\beta^Q,f^Q,0,0)$. We construct the stopping times $(\tau_n)_{n\in\mathbb{N}}$ as in (8.5) and obtain as in Step 3) of the above proof for each $n\in\mathbb{N}$ a Q-σ-martingale density $Z^{(n,1)}=\mathcal{E}(N^{(n,1)})$ for $Y^{(n,1)}$ with $N^{(n,1)}=(I_{D^c}I_{]\!]\tau_{n-1},\tau_n]\!]})\cdot N^{(n,1)}$ and such that $Z^{(n,1)}$ is locally bounded. Moreover, for each process

$$Y^{(n,0)}=I_{]\!]\tau_{n-1},\tau_n]\!]}\cdot Y^{(n,0)}=\widetilde{B}^n+\varphi_{\tau_n}\Delta S^i_{\tau_n}I_{[\![\tau_n,\infty[\![},\tag{8.17}$$

we have $\mathcal{D}_{\mathrm{e},\sigma}(Y^{(n,0)},Q)\neq\emptyset$, again from Step 3) of that proof.

2) For each $n\in\mathbb{N}$, Theorem 5.1 and Lemma 5.5 show that we can write (8.17) as

$$Y^{(n,0)}=(\varphi\widetilde{b}^n)\cdot A+\varphi\cdot(x*\mu_0^{(n)}),\tag{8.18}$$

where $\mu_0^{(n)}$ is the jump measure of the single-jump process $J^{(\tau_n)}:=\Delta S^i_{\tau_n}I_{[\![\tau_n,\infty[\![}$ and $\widetilde{b}^n$ is $\mathbb{R}^d$-valued, predictable and does not depend on φ. As usual, the Q-compensator of $\mu_0^{(n)}$ is written as $\nu_0^{(Q,n)}(dt,dx)=F^{(Q,n)}_{0,t}(dx)\,dA_t$. Because $X^{(0)}=\bar{X}^{(0)}=\sum\limits_{n=1}^\infty Y^{(n,0)}$, comparing (8.18) and (8.14) and writing $\dot\cup$ for a pairwise disjoint union yields

$$\widetilde{b}=\sum_{n=1}^\infty\widetilde{b}^n=\sum_{n=1}^\infty\widetilde{b}^nI_{]\!]\tau_{n-1},\tau_n]\!]}\qquad\text{and}\qquad\{\widetilde{b}\neq0\}=\dot{\bigcup_{n=1}^\infty}\{\widetilde{b}^n\neq0\},\tag{8.19}$$

$$dF^{(0),Q}=\sum_{n=1}^\infty dF_0^{(Q,n)}=\sum_{n=1}^\infty I_{]\!]\tau_{n-1},\tau_n]\!]}\,dF_0^{(Q,n)}.\tag{8.20}$$

In analogy to (6.4) and (8.15), define

$$\widetilde{\mathcal{R}}^n(Q):=I_{\{\widetilde{b}^n\neq0\}}\operatorname*{ess\,sup}_{z\in\mathbb{R}^d}\frac{(-z^\top\widetilde{b}^n)^-}{\int_{\mathbb{R}^d}(z^\top\psi(x))^-\,F_0^{(Q,n)}(dx)},\tag{8.22}$$

with the ess sup taken with respect to $F_0^{(Q,n)}$. Note that (8.22) is well defined, as pointed out in Section 6 and argued in Section 9. But $\{\widetilde{b}^n\neq0\}\subseteq\,]\!]\tau_{n-1},\tau_n]\!]$ and $\widetilde{b}^n=\widetilde{b}$ on $]\!]\tau_{n-1},\tau_n]\!]$ by (8.19), and comparing (8.22), (8.15) therefore shows that

$$\widetilde{\mathcal{R}}^{(0)}(Q)=\sum_{n=1}^\infty\widetilde{\mathcal{R}}^n(Q)I_{]\!]\tau_{n-1},\tau_n]\!]}.\tag{8.23}$$

In particular, also (8.15) is well defined.

3) Now fix $n \in I\!N$ and look at $Y^{(n,0)}$. We want to apply Theorem 6.1 to $Y^{(n,0)}$ on $]\!]\tau_{n-1}, \tau_n]\!]$, instead of $X^{(0)}$ on $]\!]\sigma, \tau]\!]$ there, and so we need to check instead of (6.9) that

$$\widetilde{\mathcal{R}}^n(Q) \in L^\Phi_{\mathrm{loc}}(m^{(n)}, \mathcal{P}), \tag{8.24}$$

where $m^{(n)}$ is defined in analogy to m from (6.5). By looking at the definition (3.1) in Section 3, we see that (8.24) is equivalent to showing that

$$E_Q\Big[\int_0^{\sigma_k} F^{(Q,n)}_{0,t}(I\!R^d)\big|\Phi\big(\alpha_{n,k}\widetilde{\mathcal{R}}^n_t(Q)\big)\big|\,dA_t\Big] < \infty \tag{8.25}$$

for constants $\alpha_{n,k} > 0$ and a sequence of stopping times $\sigma_k \nearrow \infty$ Q-a.s. as $k \to \infty$. But as $\tau_n \le n$, we have $\alpha(t) \ge \alpha_n := \inf\limits_{0\le t\le n} \alpha(t) > 0$ for $t \le \tau_n$ and hence, due to (8.23), $0 \le \alpha_n\widetilde{\mathcal{R}}^n_t(Q) \le \alpha(t)\widetilde{\mathcal{R}}^{(0)}_t(Q)$ on $]\!]\tau_{n-1}, \tau_n]\!]$. Therefore (8.20) and (8.23) allow us to obtain

$$\begin{aligned}F^{(Q,n)}_{0,t}(I\!R^d)\big|\Phi\big(\alpha_n\widetilde{\mathcal{R}}^n_t(Q)\big)\big| &= I_{]\!]\tau_{n-1},\tau_n]\!]}F^{(Q,n)}_{0,t}(I\!R^d)\big|\Phi\big(\alpha_n\widetilde{\mathcal{R}}^{(0)}_t(Q)\big)\big| \\ &\le F^{(Q,n)}_{0,t}(I\!R^d)\Big(\big|\Phi\big(\alpha(t)\widetilde{\mathcal{R}}^{(0)}_t(Q)\big)\big|C_1 + C_2\Big)\end{aligned}$$

with finite constants C_1, C_2; the last inequality comes from (3.4) and uses the properties (6.6) and (6.7) of Φ. So again using (8.20) for the term with C_1, we get

$$\begin{aligned}\int F^{(Q,n)}_0(I\!R^d)\big|\Phi\big(\alpha_n\widetilde{\mathcal{R}}^n(Q)\big)\big|\,dA &\le C_1\int F^{(0),Q}(I\!R^d)\big|\Phi\big(\alpha(\,\cdot\,)\widetilde{\mathcal{R}}^{(0)}(Q)\big)\big|\,dA \\ &\quad + C_2\int F^{(Q,n)}_0(I\!R^d)\,dA.\end{aligned}$$

But the second summand on the right-hand side is Q-integrable because $F^{(Q,n)}_0$ comes from a single-jump process; see (9.15) in the proof of Theorem 9.2. Moreover, the first summand on the right-hand side is predictable and finite-valued by the assumption (8.16); so it is prelocally bounded and hence even locally bounded by VIII.11 in DM. So the left-hand side is locally Q-integrable, and this is exactly what we need for (8.25).

4) Thanks to (8.24), we can now apply Theorem 6.1 to $Y^{(n,0)}$ on $]\!]\tau_{n-1}, \tau_n]\!]$, instead of $X^{(0)}$ on $]\!]\sigma, \tau]\!]$ there, and obtain a Q-σ-martingale density $Z^{(n,0)} = \mathcal{E}(N^{(n,0)})$ for $Y^{(n,0)}$ with $Z^{(n,0)} \in L^\Phi_{\mathrm{loc}}(Q)$ and

$$N^{(n,0)} = I_{]\!]\tau_{n-1},\tau_n]\!]}\centerdot N^{(n,0)} = \big(I_{D^c}I_{]\!]\tau_{n-1},\tau_n]\!]}\big)\centerdot N^{(n,0)};$$

the last equality uses again that $Y^{(n,0)} = I_{D^c}\centerdot Y^{(n,0)}$ which holds because $Y^{(n,0)}$ comes from $Y = \varphi\centerdot S^i$. Combining this with Step 1) and Proposition 6.3 shows that the product

$$\bar{Z}^{(n,0)} := Z^{(n,1)}Z^{(n,0)} = \mathcal{E}(N^{(n,1)} + N^{(n,0)}) =: \mathcal{E}(\bar{N}^{(n,0)})$$

is a Q-σ-martingale density for the sum $Y^{(n,1)} + Y^{(n,0)} = Y^{\tau_n} - Y^{\tau_{n-1}}$, that we also have $\bar{N}^{(n,0)} = (I_{D^c} I_{]\!]\tau_{n-1},\tau_n]\!]}) \boldsymbol{\cdot} \bar{N}^{(n,0)}$, and that $\bar{Z}^{(n,0)} \in L^{\Phi}_{\mathrm{loc}}(Q)$. As in Step 4) of the proof of Theorem 8.1, we obtain from Lemma 4.3 that $\bar{Z}^i := \mathcal{E}(\bar{N}^i)$ with $\bar{N}^i = \sum_{n=1}^{\infty} \bar{N}^{(n,0)} = I_{D^c} \boldsymbol{\cdot} \bar{N}^i$ is a Q-σ-martingale density for the sum $\sum_{n=1}^{\infty} (Y^{\tau_n} - Y^{\tau_{n-1}}) = Y = \varphi \boldsymbol{\cdot} S^i$, and $\bar{Z}^i \in L^{\Phi}_{\mathrm{loc}}(Q)$. By Lemma 2.9, $\bar{Z}^i$ is then also a Q-σ-martingale density for S^i. Lemma 4.1 thus yields that $Z := Z^a \bar{Z}^i = \mathcal{E}(N^a + \bar{N}^i) =: \mathcal{E}(N)$ is a Q-σ-martingale density for $S^a + S^i = S - S_0$, hence also for S, and Z is like $\bar{Z}^i$ in $L^{\Phi}_{\mathrm{loc}}(Q)$ as Z^a is locally bounded. This ends the proof. **q.e.d.**

Remark 8.4. Theorem 8.3 answers the localised versions of both questions 1) and 2) from Section 1. It provides a sufficient condition on the drift-to-jump ratio of the "tricky" part $X^{(0)}$ of $\varphi \boldsymbol{\cdot} S$ for S itself to admit a locally Φ-integrable σ-martingale density. For $Q = P$, we get an answer for question 1), for $Q \overset{\mathrm{loc}}{\approx} P$ to question 2). As remarked earlier, we believe that this condition is essentially also necessary; Example 1.1 at least illustrates that things can and do go wrong in general. $\diamond$

Remark 8.5. If we assume instead of (8.16) that $\widetilde{\mathcal{R}}^{(0)}(Q)$ is locally bounded, then there exists a Q-σ-martingale density $Z = \mathcal{E}(N)$ for S such that Z is even locally bounded. To see this, note first from (8.23) that instead of (8.24), we get in the proof of Theorem 8.3 that $\widetilde{\mathcal{R}}^n(Q)$ is locally bounded for each n. Using then Remark 6.2 instead of Theorem 6.1 in Step 4) of that proof, we get $Z^{(n,0)}$ which is locally bounded, and then also $\bar{Z}^{(n,0)}$, $\bar{Z}^i$ and finally Z are locally bounded. $\diamond$

At this point, we have established the main results on the existence of nicely integrable σ-martingale densities. It remains to prove two auxiliary results, and both these results and the techniques used for their proofs are of independent interest.

9. The key result for the single-jump case

This section contains the most difficult result of our paper, despite the fact that it only deals with a process with one single jump. Let us first explain the setup we consider here.

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, let $S = (S_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued adapted RCLL process. We assume that $\mathcal{D}_{\mathrm{e},\sigma}(S, P) \neq \emptyset$ so that S is a semimartingale, and we decompose $S = S_0 + S^a + S^i$ as in (4.3), where S^i is quasi-left-continuous. Fix a stopping time τ and consider as in (5.2), (6.1) and (6.3) a process of the form

$$X^{(0)} = \widetilde{B} + \varphi \boldsymbol{\cdot} J^{(\tau)} = \widetilde{B} + \varphi_\tau \Delta S^i_\tau I_{[\![\tau,\infty[\![} = (\widetilde{\varphi b}) \boldsymbol{\cdot} A + \varphi \boldsymbol{\cdot} (x * \mu_0) \tag{9.1}$$

obtained from S^i as in Theorem 5.1, where $\widetilde{B} = \widetilde{B}^\tau = I_{]\!]\sigma,\tau]\!]} \cdot \widetilde{B}$ is continuous and null on $[\![0,\sigma]\!]$, A dominates the characteristics of S, and μ_0 denotes the jump measure for the single-jump process $J^{(\tau)} = \Delta S^i_\tau I_{[\![\tau,\infty[\![}$. Because we ensure in our applications that we are in the framework of Theorem 5.1, we can and do also assume $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, P) \neq \emptyset$. In (9.1), we also have a bounded predictable process $\varphi > 0$ as in Theorem 5.1, and as pointed out in Theorem 5.1 as well as in Lemma 5.5, what we do does not depend on the choice of φ.

Remark 9.1. For ease of notation, we take $\sigma \equiv 0$ and hence work below on the stochastic interval $]\!]0,\tau]\!]$; so our starting point is the process $X^\tau - X^\sigma = X^\tau - X^0$, with $X := \varphi \cdot S$. The arguments for $X^\tau - X^\sigma$ and $]\!]\sigma,\tau]\!]$ are completely analogous. In the applications, we start instead with $X^{\tau_n} - X^{\tau_{n-1}}$ and then use the corresponding results on $]\!]\tau_{n-1},\tau_n]\!]$. ◇

Now take $Q \overset{\mathrm{loc}}{\approx} P$ so that also $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, Q) \neq \emptyset$. We denote the Q-compensator of μ_0 by ν_0^Q and write as usual $\nu_0^Q(dt,dx) = F^Q_{0,t}(dx)\,dA_t$. Define a bijection ψ from $I\!R^d$ to the open unit ball $U_1(0, I\!R^d)$ in $I\!R^d$ by $\psi(x) := \frac{x}{1+|x|}$ and introduce the set

$$\Gamma := \{F^Q_0(I\!R^d) > 0\} \in \mathcal{P}. \tag{9.2}$$

Then we define the auxiliary predictable process

$$\widetilde{\mathcal{R}}(Q) := I_{\{\widetilde{b}\neq 0\}} \operatorname*{ess\,sup}_{z\in I\!R^d} \frac{(-z^\top \widetilde{b})^-}{\int_{I\!R^d} (z^\top \psi(x))^- \, F^Q_0(dx)}, \qquad \text{with } \{\widetilde{b} \neq 0\} \subseteq \Gamma \subseteq \Omega \times [0,\infty); \tag{9.3}$$

the essential supremum for $\widetilde{\mathcal{R}}_t(Q)(\omega)$ is taken with respect to the (random) measure $F^Q_{0,t}(\omega,\,\cdot\,)$ on $I\!R^d$. To be precise, we set $0/0 := 0$, and we argue at the end of Step 1 below that $\widetilde{\mathcal{R}}(Q)$ is well defined and the first inclusion above is justified. We introduce on the spaces $(\Omega \times [0,\infty) \times I\!R^d, \widetilde{\mathcal{P}})$ and $(\Omega \times [0,\infty), \mathcal{P})$ the probability measures

$$m_Q(d\omega, dt, dx) := C_Q F^Q_{0,t}(\omega, dx)\, dA_t(\omega)\, Q(d\omega), \tag{9.4}$$

$$m(d\omega, dt) := C_Q F^Q_{0,t}(\omega, I\!R^d)\, dA_t(\omega)\, Q(d\omega), \tag{9.5}$$

where C_Q is a normalising constant. (It will also be part of the proof below that m_Q, m are null or well defined and that $C_Q \in (0,\infty)$ if $m_Q \not\equiv 0$.)

Finally let $\Phi : [0,\infty) \to I\!R$ be a function satisfying the following properties:

(9.6) Φ is strictly convex and in C^1, and $\Phi(0) = 0$.

(9.7) Φ grows at least linearly for large x, i.e., there is some constant $D > 0$ such that $|\Phi(x)| \geq Dx$ for $x \geq x_0$.

(9.8) Φ is bounded from below by a constant, i.e., $\Phi(x) \geq$ const. for all $x \geq 0$.

Of course, (9.6)–(9.8) are the same conditions as (6.6)–(6.8) in Section 6.

The main result of this section is then

Theorem 9.2. *Suppose that $X^{(0)}$ in (9.1) satisfies $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, P) \neq \emptyset$. If $Q \overset{\mathrm{loc}}{\approx} P$ and*

$$\widetilde{\mathcal{R}}(Q) \in L^{\Phi}_{\mathrm{loc}}(m, \mathcal{P}), \tag{9.9}$$

then there exist a probability measure $\widehat{m}_Q \approx m_Q$ and a $\widetilde{\mathcal{P}}$-measurable function $\widetilde{f}^Q$ on $\Omega \times [0, \infty) \times I\!R^d$ with $\widetilde{f}^Q > 0$ $Q \otimes A \otimes F_0^Q$-a.e.,

$$\widetilde{f}^Q \in L^{\Phi}_{\mathrm{loc}}(\widehat{m}_Q, \widetilde{\mathcal{P}}) \tag{9.10}$$

and

$$\int_{I\!R^d} |x| \widetilde{f}^Q(x) F_0^Q(dx) < \infty \qquad Q \otimes A\text{-a.e.}, \tag{9.11}$$

$$\widetilde{b} + \int_{I\!R^d} x \widetilde{f}^Q(x) F_0^Q(dx) = 0 \qquad Q \otimes A\text{-a.e.} \tag{9.12}$$

As a consequence, we can find for $X^{(0)}$ a Q-σ-martingale density $Z^{(0)}$ with $Z^{(0)} \in L^{\Phi}_{\mathrm{loc}}(Q)$.

The proof of Theorem 9.2 is long and goes over several steps. Before embarking on it, we give a short overview as well as some comments on related work and ideas.

Remark 9.3. If we assume instead of (9.9) that $\widetilde{\mathcal{R}}(Q)$ is locally bounded, then also $\widetilde{f}^Q$ can be constructed to be locally bounded (in the sense that there exist stopping times $T_k \nearrow \infty$ Q-a.s. as $k \to \infty$, and not depending on $x \in I\!R^d$, such that

$$|\widetilde{f}^Q(\omega, t, x)| \leq \mathrm{const.}(n) \qquad Q \otimes A \otimes F_0^Q\text{-a.e. on } [\![0, T_k]\!] \times I\!R^d$$

for all $k \in I\!N$). The resulting Q-σ-martingale density $Z^{(0)}$ for $X^{(0)}$ is then also locally bounded. We explain how to get this alternative result in Section 9.7. ⋄

9.1. Proof overview

Our ultimate goal is to find for $X^{(0)}$ a Q-σ-martingale density $Z^{(0)}$ in $L^{\Phi}_{\mathrm{loc}}(Q)$. From Corollary 2.12, we see that finding a Q-σ-martingale density for $X^{(0)}$ boils down to finding a strictly positive solution to the Q-zero drift equation (9.12), (9.11). Lemma 3.7 then shows that we need (for that solution $\widetilde{f}^Q$, say) essentially also (9.10) if the resulting Q-σ-martingale density should be in $L^{\Phi}_{\mathrm{loc}}(Q)$. By assumption and since $Q \overset{\mathrm{loc}}{\approx} P$, we have $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, Q) \neq \emptyset$ and hence at least one strictly positive solution to (9.12), (9.11). So the challenge, and the contents of the subsequent proof, is to argue that there is another solution with the extra property (9.10).

At a conceptual level, Theorem 9.2 is similar to the classic DMW theorem from Dalang/ Morton/Willinger (1990). The latter states that for a one-period model, absence of arbitrage is equivalent to the existence of an equivalent martingale measure, and that this even implies that there exists an equivalent martingale measure with a bounded density. The setting in Theorem 9.2 looks similar to a one-period model because it deals with a process having one single jump, at a random (stopping) time. However, in contrast to the DMW setup, there is in addition a drift term $\widetilde{b}$ in the process $X^{(0)}$. (If we embed the DMW setup into continuous time, we have a process which is piecewise constant and hence has drift 0 between jumps.) This apparently small difference makes matters much more complicated. In fact, it is the presence of the (in general unavoidable) drift $\widetilde{b}$ that causes most of the mathematical difficulties and in particular makes it necessary to add the condition (9.9) that $\widetilde{\mathcal{R}}(Q)$ is in $L^{\Phi}_{\mathrm{loc}}(m, \mathcal{P})$. Example 1.1 has already illustrated this issue and the resulting problems.

Let us now outline the main steps and ideas of the proof. We first distinguish the cases $\widetilde{b} = 0$ and $\widetilde{b} \neq 0$. On the set $\{\widetilde{b} = 0\}$, we are essentially in a DMW setup, and we can use the classic DMW theorem to obtain a solution $\widetilde{f}^Q$ to (9.12), (9.11) which is even bounded (uniformly in (ω, t, x)). At the level of Q-σ-martingale densities, this means that we can find "on $\{\widetilde{b} = 0\}$" a Q-σ-martingale density which is locally bounded, and this can be viewed as a continuous-time version of the DMW result.

On $\{\widetilde{b} \neq 0\}$, things are more difficult. We could again start with the DMW theorem. But there we must first normalise $\widetilde{b}$ to some $\bar{b}$, and the subsequent (ω, t)-dependent un-normalisation back to $\widetilde{b}$ leaves us with a solution $\bar{Y}_0$ to (9.12), (9.11) which is bounded in x, but not uniformly so in (ω, t); see (9.23) below. To obtain a solution controlled in (ω, t, x), we therefore use a technique inspired from the proof of the key Lemma 4.1 in Kabanov/Stricker (2001). Their idea is to write an equation like (9.12) as the statement that a suitable linear functional Ψ has a zero, and then prove via a Hahn–Banach separation argument that the existence of some zero for Ψ, which we get from either $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, Q) \neq \emptyset$ or the above DMW argument, implies the existence of even a zero with better properties (like (9.10), in our case).

However, there are still two major extra steps to take, and this is where our main innovations come in. First, we cannot use the Kabanov–Stricker result (or a straightforward extension of it) since they argue for fixed (ω, t), whereas we need to produce a quantity which is controlled simultaneously in ω, t and x. For the same reason, we have not been able to combine the Kabanov–Stricker idea directly with a measurable selection argument since we could not manage to get the required control over (ω, t) in a sufficiently good form. (We can get an upper bound, as explained below in Remark 9.4, but only at the cost of relaxing the lower bound from strict positivity to nonnegativity.) We therefore extend the approach from Kabanov/Stricker (2001) to the case where the linear functional Ψ takes values in an infinite-dimensional Banach space (instead of $\mathbb{R}^m$), and this in turn leads us to use corresponding separation results and notions of interior for convex sets. This idea is of independent interest and may turn out to be fruitful in other aspects of arbitrage theory as well.

A second major innovation is related to our use of a separation argument connected to the mapping Ψ. This is also where the assumption (9.9) that $\widetilde{\mathcal{R}}(Q) \in L^{\Phi}_{\mathrm{loc}}(m, \mathcal{P})$ comes into play. For using the separation argument, we need to make sure that $\Psi(L^{\Phi}_{+}(\widehat{m}_Q, \widetilde{\mathcal{P}}))$ contains 0, or, put differently, that (9.12), (9.11) do have a nonnegative solution also satisfying (9.10). This (seemingly only small) point was inadvertently overlooked in an earlier version of this paper, and fixing the issue has prompted us to add the above extra assumption and has also required us to prove some new results. Again, we believe that these are of independent interest; let us explain in more detail what they are.

The assumption $\mathcal{D}_{\mathrm{e},\sigma}(X^{(0)}, Q) \neq \emptyset$ guarantees the existence of a (strictly positive) solution to (9.12), (9.11), but tells us not much more about the properties of that solution. A classic idea from optimisation theory is then to look for a solution which in addition optimises some functional. If that is feasible, the optimiser comes from a problem that includes (9.12) as a constraint. Hence it can be described more explicitly via the first order conditions for optimality, and this can be used to find bounds on that solution to (9.12). We have taken this idea and the basic approach for implementing it from a paper by Cole/Goodrich (1993), of course adjusting and extending it to our situation at hand. More precisely, we use this line of argument (with a functional given from our convex function Φ) with respect to x for fixed (ω, t), and then use a measurable selection argument to obtain a solution to (9.12), (9.11) controlled from above by $\widetilde{\mathcal{R}}(Q)$. To the best of our knowledge, such ideas for constructing "good martingale measures" have not been used before.

Remark 9.4. The Cole–Goodrich argument outlined above gives us a solution to (9.12), (9.11) which is bounded above by $\widetilde{\mathcal{R}}(Q)$ and nonnegative. If $\widetilde{\mathcal{R}}(Q) \in L^{\Phi}_{(\mathrm{loc})}(m, \mathcal{P})$, we readily obtain for that solution that it is in $L^{\Phi}_{(\mathrm{loc})}(\widehat{m}_Q, \widetilde{\mathcal{P}})$; see the proof of part 1) of Lemma 9.9 below. However, we are not able to guarantee from the Cole–Goodrich approach also a strictly positive lower bound on the (pointwise in (ω, t)) solution. This is the reason why we subsequently have to use the separation argument for the mapping Ψ. $\diamond$

The above outline explains the main ideas and steps of the proof. To actually implement this, we follow a slightly different logical order.

9.2. Preparations

Proof of Theorem 9.2. As announced, this goes over several steps. We first assume that

$$\widetilde{\mathcal{R}}(Q) \in L^{\Phi}(m, \mathcal{P}) \tag{9.13}$$

and relax this to (9.9) via localisation at the end in Step 11.

Step 1: Similarly as in part b) of the proof of Theorem 7.1, we first deduce from the assumption $\mathcal{D}_{e,\sigma}(X^{(0)}, Q) \neq \emptyset$ in Theorem 9.2 a certain equation, namely (9.12) and (9.11). For that, we need to deal with some technical issues. First of all, we use Corollaries 2.12 and 2.14 to choose for $X^{(0)}$ a Q-σ-martingale density $Z^Q = \mathcal{E}(N^Q)$ such that N^Q has Jacod parameters $(0, f^Q, 0, 0)$ with respect to S^i. This uses that $X^{(0)}$ has no continuous local martingale part, so that the parameter β is not needed. By Lemma 2.3, $\sup\limits_{0<s\leq\cdot} f_s^Q(\Delta S_s^i)$ is locally Q-integrable, and so we get for a localising sequence $(\tau_k)_{k\in I\!N}$ that

$$\begin{aligned}
E_Q\Big[\int_0^{\tau_k}\int_{I\!R^d} f_t^Q(x) F_{0,t}^Q(dx)\,dA_t\Big] &= E_Q[(f^Q * \nu_0^Q)_{\tau_k}] \\
&= E_Q[(f^Q * \mu_0)_{\tau_k}] \\
&\leq E_Q\Big[\sup_{0<s\leq\tau_k} f_s^Q(\Delta S_s^i) I_{\{\Delta S_s^i \neq 0\}}\Big] < \infty.
\end{aligned}$$

Because $[\![0, \tau_k]\!]$ increases to $\Omega \times [0, \infty)$ as $k \to \infty$, we conclude that

$$G_{0,t}^Q(I\!R^d) := \int_{I\!R^d} f_t^Q(x) F_{0,t}^Q(dx) < \infty \qquad Q \otimes A\text{-a.e. on } \Omega \times [0, \infty). \tag{9.14}$$

Analogously, the measure $F_{0,t}^Q$ is $Q \otimes A$-a.e. finite since $X^{(0)}$ is a single-jump process; in fact,

$$E_Q\Big[\int_0^{\infty} F_{0,t}^Q(I\!R^d)\,dA_t\Big] = E_Q[(1 * \nu_0^Q)_\infty] = E_Q[(1 * \mu_0)_\infty] = Q[\tau < \infty, \Delta S_\tau^i \neq 0] < \infty. \tag{9.15}$$

The same computation as in (9.15) also shows that we can assume without loss of generality that $Q[\tau < \infty, \Delta S_\tau^i \neq 0] > 0$ and therefore that the set $\Gamma = \{F_0^Q(I\!R^d) > 0\} \in \mathcal{P}$ from (9.2) has $Q \otimes A$-measure > 0. Indeed, since $\Delta S_\tau^i = 0$ on Γ^c, we must by (9.1) also have $\widetilde{b} = 0$ on Γ^c because $Z^Q(I_{\Gamma^c} \cdot X^{(0)})$ is a Q-σ-martingale, and of course $F_0^Q = 0$ on Γ^c. So if $(Q \otimes A)(\Gamma) = 0$, we can take any positive constant for $\widetilde{f}^Q$ and then have (9.11), (9.12) $Q \otimes A$-a.e.; in fact, these hold trivially on Γ^c because F_0^Q and $\widetilde{b}$ vanish there, and Γ is by assumption a $Q \otimes A$-nullset. (Note that m_Q and m in (9.4), (9.5) then degenerate to the zero measure.) So that case is easily solved, and we therefore focus on the case where $(Q \otimes A)(\Gamma) > 0$. This implies that

$$C_Q^{-1} := E_Q\Big[\int_0^{\infty} F_{0,t}^Q(I\!R^d)\,dA_t\Big] > 0, \tag{9.16}$$

so that the constant C_Q defined by (9.16) is in $(0, \infty)$ by (9.15), and both m_Q and m from (9.4), (9.5) are probability measures. Moreover, we have

$$m_Q = m \otimes \bar{F}_0^Q \qquad \text{with } \bar{F}_0^Q := \frac{F_0^Q}{F_0^Q(I\!R^d)}, \text{ on } \Gamma, \tag{9.17}$$

and m_Q and m are equivalent to $Q \otimes A \otimes F_0^Q$ and $Q \otimes A$, respectively. Finally, by replacing A_t with $\tanh(A_t + t)$, we can also assume that A is bounded and strictly increasing.

Now $Z^Q X^{(0)}$ is a Q-σ-martingale, where $Z^Q = \mathcal{E}(N^Q)$ and N^Q has Jacod parameters $(0, f^Q, 0, 0)$ with respect to S^i. Therefore (9.1), Corollary 2.12 and the strict positivity of the predictable process φ imply that we have

$$\int_{\mathbb{R}^d} |x| f^Q(x) F_0^Q(dx) < \infty \qquad Q \otimes A\text{-a.e.,} \tag{9.18}$$

$$\widetilde{b} + \int_{\mathbb{R}^d} x f^Q(x) F_0^Q(dx) = 0 \qquad Q \otimes A\text{-a.e.} \tag{9.19}$$

This means that f^Q satisfies (9.12), (9.11); but we want a solution with additional properties.

Before we continue, we argue that (9.3) is well defined. First we note that (9.19) implies $\widetilde{b} = 0$ on the set $\Gamma^c = \{F_0^Q(\mathbb{R}^d) = 0\}$ so that $\{\widetilde{b} \neq 0\} \subseteq \Gamma$. This justifies the first inclusion in (9.3). Next, using $\psi(x) = \frac{x}{1+|x|}$, (9.19) can be rewritten as

$$-\widetilde{b} = \int_{\mathbb{R}^d} \psi(x) \bar{f}^Q(x) F_0^Q(dx) \qquad Q \otimes A\text{-a.e.}$$

with $\bar{f}^Q(x) := (1 + |x|) f^Q(x) > 0$ satisfying $\int_{\mathbb{R}^d} \bar{f}^Q(x) F_0^Q(dx) < \infty$ due to (9.14) and (9.18). Now on $\{\widetilde{b} \neq 0\} \subseteq \Gamma$, if the denominator in (9.3) becomes 0, we must have $(z^\top \psi(x))^- = 0$ $F_0^Q(dx)$-a.e., hence also

$$\int_{\mathbb{R}^d} \left(z^\top \psi(x) \bar{f}^Q(x)\right)^- F_0^Q(dx) = 0$$

and therefore

$$0 = \Big(\int_{\mathbb{R}^d} z^\top \psi(x) \bar{f}^Q(x) F_0^Q(dx) \Big)^- = (-z^\top \widetilde{b})^-.$$

This means that the numerator in (9.3) also vanishes, and so $\widetilde{\mathcal{R}}(Q)$ is well defined.

Step 2: To get from (9.19), (9.18) a better solution on $\{\widetilde{b} = 0\}$ to (9.12), (9.11), we use a DMW argument. On the probability space $(\Omega \times [0,\infty) \times \mathbb{R}^d, \widetilde{\mathcal{P}}, m_Q)$, consider the one-period model with filtration $\mathcal{G}_1 := \widetilde{\mathcal{P}}$, $\mathcal{G}_0 := \mathcal{P} \otimes \{\emptyset, \mathbb{R}^d\}$ and $X_1 := \psi$, $X_0 := -\bar{b}$ with

$$\bar{b} := \frac{\widetilde{b}}{G_0^Q(\mathbb{R}^d) + \int_{\mathbb{R}^d} |x| f^Q(x) F_0^Q(dx)} I_\Gamma = \frac{\widetilde{b}}{\int_{\mathbb{R}^d} (1 + |x|) f^Q(x) F_0^Q(dx)} I_\Gamma.$$

Note that $\bar{b}$ is predictable, hence $\mathcal{G}_0$-measurable, and well defined in view of (9.18) and (9.14). We claim that the above model is arbitrage-free in the sense that it satisfies NA. To see this, take some $H \in L^\infty(m, \mathcal{P}; \mathbb{R}^d)$ (which can be identified with a bounded $\mathbb{R}^d$-valued $\mathcal{G}_0$-measurable random variable) and suppose that $H^\top (X_1 - X_0) \geq 0$ m_Q-a.e. or, written out,

$$H^\top \left(\psi(x) + \bar{b}\right) \geq 0 \qquad m_Q\text{-a.e.}$$

We multiply this inequality by $(1+|x|)f^Q(x)>0$ and integrate with respect to F_0^Q, using $\psi(x)(1+|x|)=x$, to obtain

$$H^\top\Big(\int_{I\!R^d} x f^Q(x)F_0^Q(dx)+\bar b\int_{I\!R^d}(1+|x|)f^Q(x)F_0^Q(dx)\Big)\ge 0\qquad Q\otimes A\text{-a.e.}$$

But the second summand inside the brackets is $\widetilde{b}I_\Gamma=\widetilde{b}$ by the definition of $\bar b$, and so the sum in brackets is 0 by (9.19). As a consequence, the nonnegative original integrand of F_0^Q must vanish, which means that we must have

$$H^\top\big(\psi(x)+\bar b\big)=0\qquad m_Q\text{-a.e.}$$

or $H^\top(X_1-X_0)=0$ m_Q-a.e. This proves NA.

Because we have an arbitrage-free one-period model, the classic DMW theorem (see Theorem 2.4 in Dalang/Morton/Willinger (1990)) now implies the existence of a $\widetilde{\mathcal{P}}$-measurable function $\widehat{Y}>0$ which is bounded in (ω,t,x) and such that

$$X_0=E_{m_Q}[X_1\widehat{Y}\,|\,\mathcal{G}_0]\qquad m_Q\text{-a.e.}$$

In view of (9.17), this can be rewritten as $-\bar b=\int_{I\!R^d}\psi(x)\widehat{Y}(x)\bar F_0^Q(dx)$ $Q\otimes A$-a.e. or

$$\int_{I\!R^d} x\tfrac{\widehat{Y}(x)}{1+|x|}F_0^Q(dx)+\bar b F_0^Q(I\!R^d)=0\qquad Q\otimes A\text{-a.e.}\tag{9.20}$$

Note that since $\widehat{Y}$ is bounded and $F_0^Q(I\!R^d)<\infty$, we also have

$$\int_{I\!R^d}|x|\tfrac{\widehat{Y}(x)}{1+|x|}F_0^Q(dx)<\infty\qquad Q\otimes A\text{-a.e.}\tag{9.21}$$

We could multiply (9.20) with the denominator of $\bar b$ and divide by $F_0^Q(I\!R^d)$ to write (9.20) as

$$\int_{I\!R^d}\psi(x)\bar Y_0(x)F_0^Q(dx)+\widetilde{b}=0\qquad Q\otimes A\text{-a.e.}\tag{9.22}$$

with

$$\bar Y_0(x):=\tfrac{\widehat{Y}(x)}{F_0^Q(I\!R^d)}\int_{I\!R^d}(1+|x|)f^Q(x)F_0^Q(dx)I_\Gamma+I_{\Gamma^c},\tag{9.23}$$

but this is not really needed.

Step 3: We later want to view (9.22) as the statement that a certain mapping Ψ on functions Y has a zero in a certain function Y_0. To obtain good properties and a clear definition for that mapping, we need to deal with some further technical issues.

First of all, for proving Theorem 9.2, we can assume without loss of generality that $\{\widetilde{b} \neq 0\}$ is not evanescent. Indeed, on the set $\{\widetilde{b} = 0\}$, we have $\bar{b} = 0$ so that (9.20) yields

$$0 = I_{\{\tilde{b}=0\}}\Big(\widetilde{b} + \int_{I\!R^d} x \tfrac{\widehat{Y}(x)}{1+|x|} F_0^Q(dx)\Big) \qquad Q \otimes A\text{-a.e.} \tag{9.24}$$

So we can take

$$\widetilde{f}^Q(x) := \frac{\widehat{Y}(x)}{1+|x|} \qquad \text{on } \{\widetilde{b} = 0\}; \tag{9.25}$$

this is like $\widehat{Y}$ strictly positive and bounded uniformly in (ω, t, x), and it satisfies (9.12) and (9.11) on $\{\widetilde{b} = 0\}$ due to (9.20) and (9.21). So we assume that $\{\widetilde{b} \neq 0\}$ is not a $Q \otimes A$-nullset, and since $F_0^Q(I\!R^d) > 0$ on $\Gamma \supseteq \{\widetilde{b} \neq 0\}$ by Step 1, we then also have

$$E_Q\Big[\int_0^\infty I_{\{\tilde{b}_t \neq 0\}} F_{0,t}^Q(I\!R^d)\, dA_t\Big] > 0. \tag{9.26}$$

With

$$Y_0(x) := (1+|x|) f^Q(x) > 0, \tag{9.27}$$

we moreover obtain from (9.18) and (9.19) as well as $\psi(x) = \frac{x}{1+|x|}$ that

$$H := I_{\{\tilde{b} \neq 0\}} \int_{I\!R^d} |\psi(x)| Y_0(x) F_0^Q(dx) < \infty \qquad Q \otimes A\text{-a.e.}, \tag{9.28}$$

$$I_{\{\tilde{b} \neq 0\}}\Big(\widetilde{b} + \int_{I\!R^d} \psi(x) Y_0(x) F_0^Q(dx)\Big) = 0 \qquad Q \otimes A\text{-a.e.} \tag{9.29}$$

(We could also obtain this for $\bar{Y}_0$ from (9.23) instead of Y_0.)

9.3. An auxiliary mapping Ψ

Step 4: We now introduce the announced mapping Ψ; then (9.29) becomes the statement that Ψ has a zero in Y_0 given in (9.27). First of all, recalling the definition of H in (9.28), we introduce on the measurable spaces $(\Omega \times [0,\infty) \times I\!R^d, \widetilde{\mathcal{P}})$ and $(\Omega \times [0,\infty), \mathcal{P})$ the probability measures $\widehat{m}_Q$ and $\widehat{m}$ via

$$\begin{aligned} \widehat{m}_Q(d\omega, dt, dx) &:= \widehat{C_Q} I_{\{\tilde{b}_t(\omega) \neq 0\}} \frac{1}{1+H(\omega,t)} F_{0,t}^Q(\omega, dx)\, dA_t(\omega)\, Q(d\omega) \\ &= \widehat{C_Q} I_{\{\tilde{b}_t(\omega) \neq 0\}} \widehat{F}_{0,t}^Q(\omega, dx)\, dA_t(\omega)\, Q(d\omega), \end{aligned} \tag{9.30}$$

$$\begin{aligned} \widehat{m}(d\omega, dt) &:= \widehat{C_Q} I_{\{\tilde{b}_t(\omega) \neq 0\}} \frac{1}{1+H(\omega,t)} F_{0,t}^Q(\omega, I\!R^d)\, dA_t(\omega)\, Q(d\omega) \\ &= \widehat{C_Q} I_{\{\tilde{b}_t(\omega) \neq 0\}} \widehat{F}_{0,t}^Q(\omega, I\!R^d)\, dA_t(\omega)\, Q(d\omega), \end{aligned} \tag{9.31}$$

where

$$(9.32) \qquad \widehat{F}^Q_{0,t}(\omega, dx) := \frac{1}{1+H(\omega,t)} F^Q_{0,t}(\omega, dx)$$

and the normalising constant given by

$$(\widehat{C_Q})^{-1} = E_Q\Big[\int_0^\infty I_{\{\tilde{b}_t\neq 0\}} \tfrac{1}{1+H_t} F^Q_{0,t}(I\!R^d)\, dA_t\Big] = E_Q\Big[\int_0^\infty I_{\{\tilde{b}_t\neq 0\}} \widehat{F}^Q_{0,t}(I\!R^d)\, dA_t\Big]$$

is in $(0,\infty)$. To see the latter, note that $\widehat{C_Q} < \infty$ due to (9.26) and (9.28), and $\widehat{C_Q} > 0$ by (9.15). From the definitions, recalling that $\bar{F}^Q_0 = F^Q_0/F^Q_0(I\!R^d)$ on $\Gamma \supseteq \{\widetilde{b} \neq 0\}$, it is clear that $\widehat{m}_Q = \widehat{m} \otimes \bar{F}^Q_0$ and that $\widehat{m}$ is the marginal on $\Omega \times [0,\infty)$ of $\widehat{m}_Q$. Moreover, both m_Q from (9.4) and $\widehat{m}_Q$ are equivalent to $Q \otimes A \otimes F^Q_0$ on $\{\widetilde{b} \neq 0\}$, and we obviously have

$$(9.33) \qquad \widehat{C_Q}\, m_Q = C_Q(1+H)\widehat{m}_Q, \quad \widehat{C_Q}\, m = C_Q(1+H)\widehat{m} \qquad \text{on } \{\widetilde{b} \neq 0\}.$$

In particular, (9.33) implies that $d\widehat{m} = \text{const.} \frac{1}{1+H} I_{\{\widetilde{b}\neq 0\}}\, dm$ so that $\widehat{m} \ll m$ with a bounded density. In consequence, we have

$$(9.34) \qquad L^\Phi(m, \mathcal{P}) \subseteq L^\Phi(\widehat{m}, \mathcal{P}).$$

Recalling $X^{(0)}$ from (9.1) and $H \geq 0$ (which is predictable) from (9.28), we now define

$$(9.35) \qquad \widehat{X}^{(0)} := \frac{1}{1+H} \cdot X^{(0)} = \frac{1}{1+H} \cdot \widetilde{B} + \frac{\varphi_\tau \Delta S^i_\tau}{1+H_\tau} I_{[\![\tau,\infty[\![} = (\varphi\widehat{b}) \cdot A + \varphi \cdot (x * \widehat{\mu}_0),$$

where $\widehat{b} := \frac{\widetilde{b}}{1+H}$ and $\widehat{\mu}_0$ is the jump measure of the single-jump process $\frac{\Delta S^i_\tau}{1+H_\tau} I_{[\![\tau,\infty[\![}$. The Q-compensator of $\widehat{\mu}_0$ is called $\widehat{\nu}^Q_0$, and with $\widehat{F}^Q_0$ from (9.32), it is clearly given by

$$(9.36) \qquad \widehat{\nu}^Q_0(\omega, dt, dx) = \frac{1}{1+H(\omega,t)} \nu^Q_0(\omega, dt, dx) = \widehat{F}^Q_{0,t}(\omega, dx) dA_t(\omega).$$

Remark 9.5. Recall that the measure $M^Q_{\widehat{\mu}_0} = Q \otimes \widehat{\mu}_0$ on $\Omega \times [0,\infty) \times I\!R^d$ is given by

$$\int W\, dM^Q_{\widehat{\mu}_0} = E_Q\Big[\int_0^\infty \int_{I\!R^d} W(\omega,t,x) \widehat{\mu}_0(\omega, dt, dx)\Big]$$

for product-measurable functions $W \geq 0$. If $W \geq 0$ is even $\widetilde{\mathcal{P}}$-measurable, we can use the Q-compensator $\widehat{\nu}^Q_0$ of $\widehat{\mu}_0$ and (9.36), (9.30) to obtain from the above equality that

$$\int W I_{\{\tilde{b}\neq 0\}}\, dM^Q_{\widehat{\mu}_0} = E_Q\Big[\int_0^\infty \int_{I\!R^d} W(\omega,t,x) I_{\{\tilde{b}_t(\omega)\neq 0\}} \widehat{F}^Q_{0,t}(\omega, dx) dA_t(\omega)\Big] = (\widehat{C_Q})^{-1} \int W\, d\widehat{m}_Q.$$

We use this in Step 12 to relate integrabilities for $\widehat{m}_Q$ and $M^Q_{\tilde{\mu}_0}$ to each other. $\diamond$

We now introduce the spaces

$$U_0 := L^\Phi_+(\widehat{m}_Q, \widetilde{\mathcal{P}}), \qquad U_0^\circ := L^\Phi_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})$$

and recall from Lemma 3.3 that $L^\Phi \subseteq L^1$ since Φ satisfies (9.7). Because $|\psi|$ is bounded (by 1), the $I\!R^d$-valued function ψY on $\Omega \times [0,\infty) \times I\!R^d$ is therefore in $L^1(\widehat{m}_Q, \widetilde{\mathcal{P}}; I\!R^d)$ for every $Y \in U_0$. The next result shows that we also have this property for the function Y_0 constructed in Step 3.

Lemma 9.6. *For Y_0 from (9.27), the function ψY_0 on $\Omega \times [0,\infty) \times I\!R^d$ is in $L^1(\widehat{m}_Q, \widetilde{\mathcal{P}}; I\!R^d)$.*

Proof. We know that $Y_0 \geq 0$ is $\widetilde{\mathcal{P}}$-measurable, and since $z \mapsto \frac{z}{1+z} \leq 1$ on $[0,\infty)$ and A has been chosen bounded, using (9.30) and (9.28) yields

$$\begin{aligned} E_{\widehat{m}_Q}\big[|\psi| Y_0\big] &= \widehat{C_Q} E_Q\Big[\int_0^\infty I_{\{\tilde{b}\neq 0\}} \tfrac{1}{1+H} \int_{I\!R^d} |\psi(x)| Y_0(x) F_0^Q(dx)\, dA\Big] \\ &= \widehat{C_Q} E_Q\Big[\int_0^\infty I_{\{\tilde{b}\neq 0\}} \tfrac{H}{1+H}\, dA\Big] < \infty. \end{aligned}$$

This proves the result. **q.e.d.**

Continuing with Step 4, we now define the mapping Ψ on $\widetilde{\mathcal{P}}$-measurable functions Y with $\psi Y \in L^1(\widehat{m}_Q, \widetilde{\mathcal{P}}; I\!R^d)$ by

$$\Psi(Y) := I_{\{\tilde{b}\neq 0\}} \int_{I\!R^d} \psi(x)\big(Y(x) - Y_0(x)\big) \bar{F}_0^Q(dx). \tag{9.37}$$

Because $\widehat{m}_Q$ factorises as $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$, the quantity $\Psi(Y)$ can be viewed as (a nice version of) the conditional expectation $E_{\widehat{m}_Q}[\psi(\,\cdot\,)(Y - Y_0)\,|\,\mathcal{P} \otimes \{\emptyset, I\!R^d\}]$. For ease of notation, we write simply $E_{\widehat{m}_Q}[\,\cdot\,|\,\mathcal{P}\otimes\{\emptyset, I\!R^d\}] =: E_{\widehat{m}_Q}[\,\cdot\,|\,\mathcal{P}]$ in the sequel. Thanks to Lemma 9.6, Ψ is well defined by Fubini's theorem and maps U_0 into $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$, again because $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$. Moreover, dominated convergence for $\bar{F}_0^Q$ and $\widehat{m}_Q$ easily yields again by Lemma 9.6 that

$$\Psi(Y_0 \wedge n) \longrightarrow \Psi(Y_0) = 0 \qquad \text{both } \widehat{m}\text{-a.e. and in } L^1(\widehat{m}). \tag{9.38}$$

9.4. Getting the first part of Theorem 9.2 from a positive zero of Ψ

Step 5: We claim that the first assertion of Theorem 9.2 follows as soon as we show that

$$\Psi(U_0^\circ) = \Psi\big(L^\Phi_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})\big) \text{ contains } 0. \tag{9.39}$$

To see this, suppose $\Psi(\widetilde{Y}_0) = 0$ for a strictly positive $\widetilde{\mathcal{P}}$-measurable $\widetilde{Y}_0 \in L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$. Then

$$\widetilde{f}^Q(x) := I_{\Gamma^c} + I_\Gamma \frac{1}{1+|x|}\big(\widetilde{Y}_0(x) I_{\{\tilde{b}\neq 0\}} + \widehat{Y}(x) I_{\{\tilde{b}=0\}}\big)$$

is $\widetilde{\mathcal{P}}$-measurable, still in $L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$, and strictly positive $Q \otimes A \otimes F_0^Q$-a.e. Moreover, the definition of $\widetilde{f}^Q$ and (9.29) show that

$$0 = \Psi(\widetilde{Y}_0) = I_{\{\tilde{b}\neq 0\}} \int_{\mathbb{R}^d} \psi(x)\big(\widetilde{Y}_0(x) - Y_0(x)\big)\bar{F}_0^Q(dx) = \frac{I_{\{\tilde{b}\neq 0\}}}{F_0^Q(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d} x\widetilde{f}^Q(x)F_0^Q(dx) + \widetilde{b}\Big)$$

so that the above equality and the definition of $\widetilde{f}^Q$ together with (9.24), (9.25) yield

$$I_\Gamma\Big(\widetilde{b} + \int_{\mathbb{R}^d} x\widetilde{f}^Q(x)F_0^Q(dx)\Big) = I_{\Gamma\cap\{\tilde{b}\neq 0\}}0 + I_{\Gamma\cap\{\tilde{b}=0\}} \int_{\mathbb{R}^d} \psi(x)\widehat{Y}(x)F_0^Q(dx) = 0.$$

Moreover, we have $Q\otimes A$-a.e. that $x \mapsto \psi(x)\widetilde{Y}_0(x)$ is in $L^1(F_0^Q; \mathbb{R}^d)$ by Fubini's theorem and since $\widetilde{Y}_0 \in L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}}) \subseteq L^1(\widehat{m}_Q, \widetilde{\mathcal{P}})$. Combining this with (9.21) shows that

$$I_\Gamma \int_{\mathbb{R}^d} |x|\widetilde{f}^Q(x)F_0^Q(dx) < \infty \qquad Q\otimes A\text{-a.e.}$$

On the set Γ^c, we have $F_0^Q = 0$ as well as $\widetilde{b} = 0$, as already discussed in Step 1. So we also have (9.12) and (9.11) on Γ^c and hence in summary obtain the first part of the assertion of Theorem 9.2. Note that while we do have $Y_0 > 0$ and $\Psi(Y_0) = 0$, this is not yet enough to get (9.39) since we do not know whether Y_0 is in $L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$.

Step 6: Our goal is now to show the result (9.39) that $\Psi(L^\Phi_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})) = \Psi(U_0^\circ)$ contains 0. Conceptually, our proof follows the argument of Kabanov/Stricker (2001) for their (key) Lemma 4.1; so we assume that $0 \notin \Psi(U_0^\circ)$ and work towards a contradiction. To that end, we want to separate 0 from a set in the space $L^1(\widehat{m}, \mathcal{P}; \mathbb{R}^d)$ where Ψ takes its values. This is a very significant difference to Kabanov/Stricker (2001) who worked in $\mathbb{R}^m$ — since our space is infinite-dimensional, we must be careful to use appropriate notions of the "interior" of a set and a corresponding separation theorem. Our basic reference is Borwein/Lewis (1992).

Let X, later taken as $L^1(\widehat{m}, \mathcal{P}; \mathbb{R}^d)$, be a Banach space with (topological) dual X^*. For a convex subset $C \subseteq X$, the *quasi-relative interior* is defined as

$$\operatorname{qri} C := \big\{x \in C \,\big|\, \overline{\operatorname{cone}(C-x)} \text{ is a linear subspace of } X\big\} \subseteq C,$$

where $\overline{\operatorname{cone}(C-x)}$ is the closure in X of the cone generated by $C - x$. It is shown in Proposition 2.8 of Borwein/Lewis (1992) that $x \in C$ is in $\operatorname{qri} C$ if and only if the normal cone

$$N_C(x) := \{\xi \in X^* \,|\, \langle \xi, y - x\rangle \le 0 \text{ for all } y \in C\}$$

is a linear subspace of X^*. Moreover, we have the following separation result.

Theorem 9.7. [Daniele/Giuffrè/Idone/Maugeri (2007), Theorem 1] *Let $C \subseteq X$ be convex and $x_0 \in C \setminus \operatorname{qri} C$. Then there exists a nontrivial $\xi \in X^*$ such that $\langle \xi, x \rangle \le \langle \xi, x_0 \rangle$ for all $x \in C$.*

To apply this result, we take $X := L^1(\widehat{m}, \mathcal{P}; I\!R^d)$ so that $X^* = L^\infty(\widehat{m}, \mathcal{P}; I\!R^d)$ and set

$$C := \operatorname{conv}\big(\Psi(U_0^\circ) \cup \{0\}\big).$$

Then C is convex, a subset of $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$ and contains 0. Moreover, we claim that we have

Lemma 9.8. *With Ψ, U_0, U_0°, C defined as above, we have*

1) $\operatorname{qri}(\Psi(U_0)) = \Psi(U_0^\circ)$.

2) $\overline{\operatorname{cone}(C)} = \overline{\operatorname{cone}(\Psi(U_0))}$.

Proof. 1) This is proved at the end of this section in Steps 9 and 10 so as not to interrupt the overall flow of the argument; the inclusion "$\supseteq$" is easy, but "$\subseteq$" requires some work.

2) For the inclusion "$\subseteq$", note that $Y_n := Y_0 \wedge n$ is in U_0 and we have seen in (9.38) that $\Psi(Y_n) \to 0$ in $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$. So we have $0 \in \overline{\Psi(U_0)}$ and hence $\Psi(U_0^\circ) \cup \{0\} \subseteq \overline{\Psi(U_0)}$, giving

$$C \subseteq \operatorname{conv}\big(\overline{\Psi(U_0)}\big) \subseteq \operatorname{conv}\Big(\overline{\operatorname{cone}\big(\Psi(U_0)\big)}\Big) = \overline{\operatorname{cone}\big(\Psi(U_0)\big)}.$$

This implies "$\subseteq$". For the converse, start with $Y \in U_0$ and note that $Y_n := Y + \frac{1}{n}$ is a sequence in U_0° converging to Y. So $\Psi(Y_n) \in \Psi(U_0^\circ) \subseteq C \subseteq \operatorname{cone}(C)$, and since Ψ (which is up to a translation by a fixed vector just a conditional expectation) is continuous on $U_0 \subseteq L^1(\widehat{m}_Q, \widetilde{\mathcal{P}})$, we have $\Psi(Y_n) \to \Psi(Y)$ in $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$. But this implies that $\Psi(Y) \in \overline{\operatorname{cone}(C)}$, hence $\Psi(U_0) \subseteq \overline{\operatorname{cone}(C)}$, and so "$\supseteq$" follows. **q.e.d.**

Lemma 9.9. *With Ψ, U_0, U_0°, C defined as above, we have*

1) *$\Psi(U_0)$ contains 0. (This uses the assumption (9.13) that $\widetilde{\mathcal{R}}(Q) \in L^\Phi(m, \mathcal{P})$.)*

2) *If $0 \notin \Psi(U_0^\circ)$, then $0 \in C \setminus \operatorname{qri} C$.*

Proof. 1) Due to (9.13) and (9.34), we have $\widetilde{\mathcal{R}}(Q) \in L^\Phi(\widehat{m}, \mathcal{P})$. Theorem 10.2 below thus yields a $\widetilde{\mathcal{P}}$-measurable function $Y^* \ge 0$ on $\Omega \times [0, \infty) \times I\!R^d$ satisfying the Q-zero drift equations (10.2), (10.3), which are simply (9.28), (9.29) for Y^* instead of Y_0. So $\Psi(Y^*) = 0$ by (9.37) and (9.29), used for both Y^* and Y_0. Moreover, from Theorem 10.1 below, the construction of Y^* is such that $\|Y^*\|_{L^\infty(\bar{F}_0^Q)} = \operatorname*{ess\,sup}_{x \in I\!R^d} Y^*(\,\cdot\,, \,\cdot\,, x)$ is in $L^\Phi(\widehat{m}, \mathcal{P})$, due to (9.13), and we have $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$. So Y^* is in $L_+^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}}) = U_0$, and hence $\Psi(U_0)$ contains 0.

Proving Theorems 10.1 and 10.2 still needs substantial work; as outlined at the beginning of this section, we need to construct a solution to the Q-zero drift equations which is controlled by $\widetilde{\mathcal{R}}(Q)$. The detailed argument is given in Section 10 below.

2) By the definition of C, we have $0 \in C$. In addition, part 1) tells us that $0 \in \Psi(U_0)$. If $0 \notin \Psi(U_0^\circ)$, then part 1) of Lemma 9.8 yields that $0 \notin \mathrm{qri}\,(\Psi(U_0))$, which means by the definition and because $0 \in \Psi(U_0)$ that $\overline{\mathrm{cone}\,(\Psi(U_0))}$ is not a linear subspace of $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$. By part 2) of Lemma 9.8, also $\overline{\mathrm{cone}\,(C)}$ is then not a linear subspace, which means in turn, because $0 \in C$, that $0 \notin \mathrm{qri}\, C$. **q.e.d.**

As announced, suppose now that $0 \notin \Psi(U_0^\circ)$. Then $0 \in C \setminus \mathrm{qri}\, C$ by Lemma 9.9, and hence there exists by Theorem 9.7 a nontrivial $\xi \in X^* = L^\infty(\widehat{m}, \mathcal{P}; I\!R^d)$ such that

$$E_{\widehat{m}}[\xi^\top \Psi(Y)] = \langle \xi, \Psi(Y) \rangle \le 0 \qquad \text{for all } Y \in U_0^\circ, \tag{9.40}$$

since $C \supseteq \Psi(U_0^\circ)$. But (9.40) easily extends from U_0° to U_0 because as in the proof of Lemma 9.8, we have $\Psi(Y + \frac{1}{n}) \to \Psi(Y)$ in $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$, for any $Y \in U_0$. So we also get

$$E_{\widehat{m}}[\xi^\top \Psi(Y)] = \langle \xi, \Psi(Y) \rangle \le 0 \qquad \text{for all } Y \in U_0. \tag{9.41}$$

Step 7: Starting from (9.41), we now want to derive a contradiction to the assumption that $0 \notin \Psi(U_0^\circ)$. Define $Y_n := Y_0 \wedge n$ and, for appropriately chosen $D \in \widetilde{\mathcal{P}}$ and $\mathcal{P}$-measurable γ, $\widetilde{Y}_n := Y_n(1 + I_D \,\mathrm{sign}\,\gamma)$. Then both Y_n and $\widetilde{Y}_n$ are in $L^\infty_+(\widehat{m}_Q, \widetilde{\mathcal{P}}) \subseteq U_0$, and the definition of Ψ in (9.37) gives

$$\Psi(\widetilde{Y}_n) = \Psi(Y_n) + I_{\{\tilde{b} \neq 0\}} \mathrm{sign}\,\gamma \int_{I\!R^d} \psi(x) Y_n(x) I_D(x) \bar{F}_0^Q(dx) = \Psi(Y_n) + \mathrm{sign}\,\gamma E_{\widehat{m}_Q}[\psi Y_n I_D \,|\, \mathcal{P}].$$

We know from (9.38) that $\Psi(Y_n) \to 0$ in $L^1(\widehat{m}, \mathcal{P}; I\!R^d)$, and we also have $E_{\widehat{m}}[\xi^\top \Psi(\widetilde{Y}_n)] \le 0$ for all n by (9.41). Moreover, $0 \le Y_n \nearrow Y_0$ and hence $E_{\widehat{m}_Q}[\psi Y_n I_D \,|\, \mathcal{P}] \nearrow E_{\widehat{m}_Q}[\psi Y_0 I_D \,|\, \mathcal{P}]$ by monotone integration. Multiplying by $\xi^\top$ and taking the expectation under $\widehat{m}$ thus yields

$$0 \ge E_{\widehat{m}}\big[(\mathrm{sign}\,\gamma)\xi^\top E_{\widehat{m}_Q}[\psi Y_0 I_D \,|\, \mathcal{P}]\big] = E_{\widehat{m}}\big[\big|E_{\widehat{m}_Q}[\xi^\top \psi Y_0 I_D \,|\, \mathcal{P}]\big|\big],$$

if we choose $\gamma := \xi^\top E_{\widehat{m}_Q}[\psi(x) Y_0 I_D \,|\, \mathcal{P}]$ and note that ξ is $\mathcal{P}$-measurable and bounded. This means by the definition of $\widehat{m}$ and $\widehat{m}_Q$ in (9.30), (9.31) that $Q \otimes A$-a.e., we have

$$\int_{I\!R^d} \xi^\top \psi(x) Y_0(x) I_D(x) \bar{F}_0^Q(dx) = \int_{I\!R^d} \xi^\top x \tfrac{Y_0(x)}{1+|x|} I_D(x) \bar{F}_0^Q(dx) = 0 \qquad \text{on } \{\widetilde{b} \neq 0\}. \tag{9.42}$$

We choose the set $D := \{(\omega, t, x) \in \Omega \times [0, \infty) \times I\!R^d \,|\, x \in \mathrm{supp}\, \bar{F}_{0,t}^Q(\omega, \,\cdot\,)\}$ which is in $\widetilde{\mathcal{P}}$; see Delbaen/Schachermayer (2006), p.289, before Lemma 14.3.4. But then (9.42) implies that

$$\xi^\top x = 0 \qquad \text{for all } x \in \mathrm{supp}\, \bar{F}_{0,t}^Q(\omega, \,\cdot\,),\ Q \otimes A\text{-a.e. on } \{\widetilde{b} \neq 0\},$$

and because ξ is nontrivial, this is a contradiction — at least if $\bar{F}^Q_{0,t}(\omega,\,\cdot\,)$ has full support on $I\!R^d$ for $Q \otimes A$-almost all (ω, t).

Step 8: To deal with the fact that $\bar{F}^Q_0$ need not have full support, we proceed as follows. We consider for each $(\omega, t) \in \Omega \times [0,\infty)$ the linear subspace $L_{\omega,t} := \mathrm{lin}\,(\mathrm{supp}\,\bar{F}^Q_{0,t}(\omega,\,\cdot\,)) \subseteq I\!R^d$ generated by the support of the probability measure $\bar{F}^Q_{0,t}(\omega,\,\cdot\,)$ on $I\!R^d$ and define

$$\widetilde{X} := \big\{\widetilde{x} \in L^1(\widehat{m}, \mathcal{P}; I\!R^d) \,\big|\, \widetilde{x}(\omega,t) \in L_{\omega,t} \text{ for } Q \otimes A\text{-almost all } (\omega,t)\big\}.$$

Then $\widetilde{X}$ is a linear subspace of $L^1(\widehat{m}, \mathcal{P}; I\!R^d) = X$ and closed in X, since each $L_{\omega,t}$ is closed in $I\!R^d$. Moreover, the definition of Ψ in (9.37) implies that we actually have $\Psi(U_0) \subseteq \widetilde{X} \subseteq X$. So we can apply the separation argument in Step 6 with X replaced by $\widetilde{X}$ and get a nontrivial continuous linear functional $\widetilde{\xi} \in (\widetilde{X})^*$ with $\langle \widetilde{\xi}, \Psi(Y)\rangle \le 0$ for all $Y \in U^\circ_0$, like in (9.40). But by the Hahn–Banach theorem, $\widetilde{\xi}$ admits an extension to a continuous linear functional $\xi \in X^* = L^\infty(\widehat{m}, \mathcal{P}; I\!R^d)$ such that the restriction of ξ to $\widetilde{X}$ coincides with $\widetilde{\xi}$. Hence we get

$$E_{\widehat{m}}[\xi^\top \Psi(Y)] = \langle \xi, \Psi(Y)\rangle = \langle \widetilde{\xi}, \Psi(Y)\rangle \le 0 \qquad \text{for all } Y \in U^\circ_0,$$

and the argument in Step 7 then yields again

$$\xi^\top x = 0 \qquad \text{for all } x \in \mathrm{supp}\,\bar{F}^Q_{0,t}(\omega,\,\cdot\,),\ Q \otimes A\text{-a.e. on } \{\widetilde{b} \neq 0\}.$$

But this means that $\xi(\omega,t) \in L^\perp_{\omega,t}$ for $Q \otimes A$-almost all $(\omega,t) \in \{\widetilde{b} \neq 0\}$ and therefore, in view of the definition of $\widetilde{X}$, that

$$\big\langle \widetilde{\xi}, \widetilde{x}\big\rangle = \big\langle \xi, \widetilde{x}\big\rangle = E_{\widehat{m}}\big[\xi^\top \widetilde{x}\big] = 0 \qquad \text{for all } \widetilde{x} \in \widetilde{X}.$$

Because $\widetilde{\xi}$ is nontrivial on $\widetilde{X}$, this is the desired contradiction. So we have indeed the result (9.39) that $0 \in \Psi(U^\circ_0)$, and Step 5 produces the desired $\widetilde{f}^Q$.

9.5. Proof of part 1) in Lemma 9.8

Step 9: We now return to the assertion 1) in Lemma 9.8 that $\mathrm{qri}\,(\Psi(U_0)) = \Psi(U^\circ_0)$. We first prove an auxiliary result, recalling that $U_0 = L^\Phi_+(\widehat{m}_Q, \widetilde{\mathcal{P}})$ and $U^\circ_0 = L^\Phi_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})$. A very similar result is given as an example in Daniele/Giuffrè/Idone/Maugeri (2007). Recall also that $L^\Phi \subseteq L^1$ by Lemma 3.3 since Φ satisfies (9.7); this allows us below to work in L^1.

Lemma 9.10. *For a probability measure π on some measurable space, define for brevity $E_0 := L^\Phi_+(\pi)$ and $E^\circ_0 := L^\Phi_{++}(\pi)$. If we view E_0 and E°_0 as subsets of $L^1(\pi)$, we have*

$$\mathrm{qri}\,E_0 = E^\circ_0, \tag{9.43}$$

where the quasi-relative interior is with respect to $L^1(\pi)$.

Proof. We first claim that for any $y \in E_0$, we have

$$\text{(9.44)} \qquad \begin{aligned} N_{E_0}(y) &:= \{\varphi \in L^\infty(\pi) \,|\, \textstyle\int \varphi(y'-y)\,d\pi \le 0 \text{ for all } y' \in E_0\} \\ &= \big\{\varphi \in L^\infty_-(\pi) \,\big|\, \varphi = 0 \ \pi\text{-a.e. on } \{y>0\}\big\}. \end{aligned}$$

Indeed, the inclusion "$\supseteq$" holds since $\int \varphi(y'-y)\,d\pi = \int \varphi y'\,d\pi - \int \varphi y I_{\{y>0\}}\,d\pi = \int \varphi y'\,d\pi \le 0$ because $y \ge 0$, $\varphi \le 0$ by assumption and $y' \ge 0$. For "$\subseteq$", taking $y' := y + I_{\{\varphi\ge 0\}} \in E_0$ gives $0 \ge \int \varphi I_{\{\varphi\ge 0\}}\,d\pi$ because $\varphi \in N_{E_0}(y)$, so that we get $\varphi \le 0$ π-a.e., and choosing $y' := y(1 + I_{\{y>0\}}\operatorname{sign}\varphi) \in E_0$ gives $0 \ge \int |\varphi| y I_{\{y>0\}}\,d\pi$ so that also $\varphi = 0$ on $\{y>0\}$ π-a.e.

Having (9.44), the proof is now easy. If $y > 0$ π-a.e., then $N_{E_0}(y) = \{0\}$, by (9.44), is trivially a linear subspace of $L^\infty(\pi)$; and if $\pi(\{y=0\}) > 0$, then $-I_{\{y=0\}}$ is in $N_{E_0}(y)$, but $+I_{\{y=0\}}$ is not, again by (9.44), so that $N_{E_0}(y)$ is not a linear subspace. Therefore $y \in \operatorname{qri} E_0$ if and only if both $y \in E_0$ and $y > 0$ π-a.e., i.e. $y \in E_0^\circ$. This proves (9.43). **q.e.d.**

To prove part 1) of Lemma 9.8, we first note that writing for $Y \in L^1(\widehat{m}_Q, \widetilde{\mathcal{P}})$

$$\text{(9.45)} \qquad \Psi(Y) = E_{\widehat{m}_Q}[\psi(\,\cdot\,)(Y-Y_0)\,|\,\mathcal{P}] = E_{\widehat{m}_Q}[\psi Y\,|\,\mathcal{P}] - E_{\widehat{m}_Q}[\psi Y_0\,|\,\mathcal{P}] =: T(Y) - y_0$$

gives by Lemma 9.6 a continuous linear map $T: L^1(\widehat{m}_Q, \widetilde{\mathcal{P}}) \to L^1(\widehat{m}, \mathcal{P}; I\!R^d)$ and an element $y_0 \in L^1(\widehat{m}, \mathcal{P}; I\!R^d)$. Next, Lemma 9.10 and Theorem 3.4 of Borwein/Goebel (2003) yield

$$\text{(9.46)} \qquad T(U_0^\circ) = T(\operatorname{qri} U_0) \subseteq \operatorname{qri}\big(T(U_0)\big),$$

so that $C' := T(U_0)$ has $\operatorname{qri} C' \ne \emptyset$. Proposition 3 of Daniele/Giuffrè/Idone/Maugeri (2007) applied to C' therefore implies that $\operatorname{qri}\big(\Psi(U_0)\big) = \operatorname{qri}\big(T(U_0) - y_0\big) = \operatorname{qri}\big(T(U_0)\big) - y_0$. So if we show that

$$\text{(9.47)} \qquad \operatorname{qri}\big(T(U_0)\big) = T(U_0^\circ),$$

we get $\operatorname{qri}(\Psi(U_0)) = T(U_0^\circ) - y_0 = \Psi(U_0^\circ)$ and hence part 1) of Lemma 9.8. We have already shown in (9.46) the inclusion "$\supseteq$" for (9.47), and the converse is argued below in Step 10.

Step 10: To finish the proof of Lemma 9.8, we now show that

$$\text{(9.48)} \qquad \operatorname{qri}\big(T(U_0)\big) \subseteq T(U_0^\circ),$$

recalling from (9.45) and $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$ that

$$\text{(9.49)} \qquad T(Y) = E_{\widehat{m}_Q}[\psi Y\,|\,\mathcal{P}] = I_{\{\tilde{b}\ne 0\}} \int_{I\!R^d} \psi(x) Y(x) \bar{F}_0^Q(dx).$$

So take $q \in \operatorname{qri}(T(U_0)) \subseteq T(U_0)$ and some $Y \in U_0$ with $q = T(Y)$. Because $U_0 \subseteq L^1(\widehat{m}_Q, \widetilde{\mathcal{P}})$, Fubini's theorem implies that $x \mapsto Y(x)$ is in $L^1(\bar{F}_0^Q)$ for $\widehat{m}$-almost all (ω, t). Moreover, $x \mapsto \psi(x) = \frac{x}{1+|x|}$ is bounded, and the functions 1 and $\psi^i(x)$, $i = 1, \dots, d$, are linearly independent on $I\!R^d$. This allows us to use Theorem 2.9 of Borwein/Lewis (1991) and obtain for $\widehat{m}$-almost all (ω, t) a function $x \mapsto \widetilde{y}(\omega, t, x) \in L^\infty(\bar{F}_0^Q)$ with $\widetilde{y} \ge \delta(\omega, t) > 0$ such that

$$\int_{I\!R^d} \widetilde{y}(x) \bar{F}_0^Q(dx) = \int_{I\!R^d} Y(x) \bar{F}_0^Q(dx) \qquad \widehat{m}\text{-a.e.,} \tag{9.50}$$

$$\int_{I\!R^d} \psi(x) \widetilde{y}(x) \bar{F}_0^Q(dx) = \int_{I\!R^d} \psi(x) Y(x) \bar{F}_0^Q(dx) = q \qquad \widehat{m}\text{-a.e.,} \tag{9.51}$$

by (9.49). Note that $x \mapsto \widetilde{y}(\omega, t, x)$ is Borel-measurable for $\widehat{m}$-almost all (ω, t), but we have no information on joint measurability in (ω, t, x). However, this is fortunately not needed.

Now because $q = T(Y)$ is predictable, the set $\Delta := \{\widetilde{b} \neq 0\} \cap \{q \neq 0\}$ is in $\mathcal{P}$. On the probability space $(\Omega \times [0, \infty) \times I\!R^d, \widetilde{\mathcal{P}}, \widehat{m}_Q)$, consider the one-period model with $\mathcal{G}_1 := \widetilde{\mathcal{P}}$, $\mathcal{G}_0 := \mathcal{P} \otimes \{\emptyset, I\!R^d\}$, $X_1 := I_\Delta \psi$ and

$$X_0 := I_\Delta q' := \frac{I_\Delta q}{\int_{I\!R^d} \widetilde{y}(x) \bar{F}_0^Q(dx)}.$$

As in Step 2, we claim that this model is arbitrage-free. Indeed, if $H \in L^\infty(\widehat{m}, \mathcal{P}; I\!R^d)$ is such that $H^\top(X_1 - X_0) \ge 0$ $\widehat{m}_Q$-a.e., using $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$ implies that $\widehat{m}$-a.e., we have $I_\Delta H^\top(\psi(x) - q') \ge 0$ $\bar{F}_0^Q$-a.e. Multiplying with $\widetilde{y} > 0$, integrating and using (9.51) and the definition of q' gives $\widehat{m}$-a.e.

$$0 \le \int_{I\!R^d} I_\Delta H^\top \big(\psi(x) - q'\big) \widetilde{y}(x) \bar{F}_0^Q(dx) = I_\Delta H^\top \Big(\int_{I\!R^d} \psi(x) \widetilde{y}(x) \bar{F}_0^Q(dx) - q' \int_{I\!R^d} \widetilde{y}(x) \bar{F}_0^Q(dx) \Big) = 0.$$

Thus $\widetilde{y} > 0$ implies that $H^\top(X_1 - X_0) = I_\Delta H^\top(\psi(x) - q') = 0$ $\widehat{m}_Q$-a.e., and so we have indeed the no-arbitrage property NA. By the DMW theorem, we can therefore find a $\widetilde{\mathcal{P}}$-measurable bounded function $Y_1 > 0$ on $\Omega \times [0, \infty) \times I\!R^d$ such that $X_0 = E_{\widehat{m}_Q}[X_1 Y_1 \,|\, \mathcal{P}]$ $\widehat{m}$-a.e. or, using again the analogue $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$ of (9.17),

$$I_\Delta q = I_\Delta \int_{I\!R^d} \psi(x) Y_1(x) \Big(\int_{I\!R^d} \widetilde{y}(x) \bar{F}_0^Q(dx) \Big) \bar{F}_0^Q(dx) = I_\Delta T(Y_2) \qquad \widehat{m}\text{-a.e.,}$$

where, using (9.50),

$$Y_2(x) := Y_1(x) \int_{I\!R^d} \widetilde{y}(x) \bar{F}_0^Q(dx) = Y_1(x) \int_{I\!R^d} Y(x) \bar{F}_0^Q(dx) \qquad \widehat{m}\text{-a.e.}$$

Like Y_1 and $\widetilde{y}$, Y_2 is strictly positive $\widehat{m}_Q$-a.e., and the second representation also shows that Y_2 is $\widetilde{\mathcal{P}}$-measurable (even if we have no information on the joint measurability of $\widetilde{y}$).

Finally, Y_1 is bounded by a constant C, say, so that

$$Y_2(x) \leq C \int_{\mathbb{R}^d} Y(x)\bar{F}_0^Q(dx) \qquad \widehat{m}_Q\text{-a.e.} \tag{9.52}$$

Since $Y \in U_0 = L_+^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$ and $\widehat{m}_Q = \widehat{m} \otimes \bar{F}_0^Q$, we have $\int_{\mathbb{R}^d} |\Phi(\alpha Y(x))|\bar{F}_0^Q(dx) \in L^1(\widehat{m}, \mathcal{P})$ for some $\alpha > 0$. But due to (9.8), Φ is bounded below, say by $-b$ with a constant $b \geq 0$, so that $|\Phi(x)| \leq \Phi(x) + 2b$, and so convexity of Φ gives by Jensen's inequality

$$\begin{aligned}\left|\Phi\Big(\alpha \int_{\mathbb{R}^d} Y(x)\bar{F}_0^Q(dx)\Big)\right| &\leq 2b + \int_{\mathbb{R}^d} \Phi\big(\alpha Y(x)\big)\bar{F}_0^Q(dx) \\ &\leq 2b + \int_{\mathbb{R}^d} \big|\Phi\big(\alpha Y(x)\big)\big|\,\bar{F}_0^Q(dx) \in L^1(\widehat{m}, \mathcal{P}).\end{aligned}$$

So the right-hand side of (9.52) is in $L_+^\Phi(\widehat{m}, \mathcal{P})$ and constant in x; hence it is in $L_+^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$ as well, and so is then Y_2 because L_+^Φ is solid by Lemma 3.3. In summary, we have found $Y_2 \in U_0^\circ$ such that

$$I_\Delta T(Y) = I_\Delta q = I_\Delta T(Y_2) \qquad \widehat{m}\text{-a.e.}$$

Having dealt with $q \neq 0$, we now consider the predictable set $\Delta' := \{\widetilde{b} \neq 0\} \cap \{q = 0\}$. We look at $X_0 := I_{\Delta'} q' = 0$, $X_1 := I_{\Delta'}\psi$ with $\mathcal{G}_0, \mathcal{G}_1$ as before, and claim that this one-period model is also arbitrage-free. To argue this, we now have to exploit that $q \in \operatorname{qri}(T(U_0))$. We again take any $H \in L^\infty(\widehat{m}, \mathcal{P}; \mathbb{R}^d)$ with $H^\top(X_1 - X_0) \geq 0$ $\widehat{m}_Q$-a.e. and note that this implies

$$I_{\Delta'} H^\top \psi(x) \geq 0 \quad \text{and} \quad I_{\Delta'} H^\top q = 0 \qquad \widehat{m}_Q\text{-a.e.}$$

For any $y = T(g) = I_{\{\tilde{b} \neq 0\}} \int_{\mathbb{R}^d} \psi(x) g(x)\bar{F}_0^Q(dx)$ in $T(U_0)$ with $g \in U_0$, we thus obtain

$$-I_{\Delta'} H^\top (y - q) = -\int_{\mathbb{R}^d} I_{\Delta'} H^\top \psi(x) g(x)\bar{F}_0^Q(dx) + I_{\Delta'} H^\top q \leq 0 \qquad \widehat{m}\text{-a.e.}$$

and therefore $\langle -I_{\Delta'} H, y - q\rangle = E_{\widehat{m}}[-I_{\Delta'} H^\top (y - q)] \leq 0$ for all $y \in T(U_0)$, which means that $-I_{\Delta'} H$ is in $N_{T(U_0)}(q)$. But the latter is a linear subspace because $q \in \operatorname{qri}(T(U_0))$; so we also have $+I_{\Delta'} H \in N_{T(U_0)}(q)$, and this means that

$$E_{\widehat{m}_Q}[I_{\Delta'} H^\top \psi g] = E_{\widehat{m}}\Big[I_{\Delta'} \int_{\mathbb{R}^d} H^\top \psi(x) g(x)\bar{F}_0^Q(dx)\Big] = E_{\widehat{m}}[+I_{\Delta'} H^\top (y - q)] \leq 0$$

for all $y = T(g) \in T(U_0)$, i.e., for all $g \in U_0 \supseteq L_+^\infty(\widehat{m}_Q, \widetilde{\mathcal{P}})$. So we obtain that

$$I_{\Delta'} H^\top \psi(x) = 0 \quad \text{and still} \quad I_{\Delta'} H^\top q = 0 \qquad \widehat{m}_Q\text{-a.e.}$$

which implies that $H^\top(X_1 - X_0) = 0$ $\widehat{m}_Q$-a.e. So this model indeed satisfies NA, and hence the DMW theorem again yields the existence of a $\widetilde{\mathcal{P}}$-measurable bounded function $Y_3 > 0$ on $\Omega \times [0,\infty) \times I\!R^d$ with $X_0 = E_{\widehat{m}_Q}[X_1 Y_3 \,|\, \mathcal{P}]$ $\widehat{m}$-a.e. or, from the definitions of X_0 and X_1,

$$0 = I_{\Delta'} q = I_{\Delta'} \int\limits_{I\!R^d} \psi(x) Y_3(x) \bar{F}_0^Q(dx) = I_{\Delta'} T(Y_3).$$

So $Y_3 \in L^\infty_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}}) \subseteq U_0^\circ$ and thus $Y^* := I_\Delta Y_2 + I_{\Delta'} Y_3$ is in U_0° (because we already know that $Y_2 \in U_0^\circ$) and $T(Y^*) = I_\Delta q + I_{\Delta'} 0 = q$ $\widehat{m}$-a.e. But this means that $q \in T(U_0^\circ)$ and since $q \in \operatorname{qri}(T(U_0))$ was arbitrary, we have (9.48). This completes the proof of Lemma 9.8. **q.e.d.**

9.6. Finishing the proof of Theorem 9.2

Step 11: So far, we have worked under the global assumption that $\widetilde{\mathcal{R}}(Q)$ is in $L^\Phi(m, \mathcal{P})$. If we only have $\widetilde{\mathcal{R}}(Q) \in L^\Phi_{\mathrm{loc}}(m, \mathcal{P})$, there are stopping times $T_k \nearrow \infty$ Q-a.s. such that $\widetilde{\mathcal{R}}(Q) I_{[\![0,T_k]\!]} \in L^\Phi(m, \mathcal{P}) \subseteq L^\Phi(\widehat{m}, \mathcal{P})$ for each $k \in I\!N$, due to (9.34). Our arguments above then still give us via Theorem 10.2 a solution $Y^* \geq 0$ to (9.28), (9.29) on $\Omega \times [0,\infty) \times I\!R^d$, as in the proof of part 1) of Lemma 9.9. By Theorem 10.1, we also get $Y^* I_{[\![0,T_k]\!]} \in L^\Phi(\widehat{m}, \mathcal{P})$ for each k, as in the same proof. So we then argue for fixed k throughout on $[\![0, T_k]\!]$ only and obtain in Step 5 a function $\widetilde{f}^{Q,k} > 0$ defined on $[\![0, T_k]\!] \times I\!R^d$, simply by multiplying throughout by $I_{[\![0,T_k]\!]}$. Moreover, this yields $\widetilde{f}^{Q,k} \in L^\Phi(\widehat{m}, \mathcal{P})$. The desired $\widetilde{f}^Q$ is then obtained by piecing things together via $\widetilde{f}^Q := \widetilde{f}^{Q,k}$ on $]\!]T_{k-1}, T_k]\!] \times I\!R^d$; this function is clearly in $L^\Phi_{\mathrm{loc}}(\widehat{m}, \mathcal{P})$ by construction, it is > 0 $Q \otimes A \otimes F_0^Q$-a.e. since that measure has no mass on $[\![0]\!] \times I\!R^d$, and it satisfies (9.11), (9.12) since we have (9.28), (9.29) for $\widetilde{f}^{Q,k}$ on each $[\![0, T_k]\!]$.

Up to this point, with the exception of Theorem 10.1, we have proved the first part of Theorem 9.2. It only remains to construct the claimed Q-σ-martingale density $Z^{(0)}$ for $X^{(0)}$ with $Z^{(0)} \in L^\Phi_{\mathrm{loc}}(Q)$. This is done in

Step 12: Recall from (9.35) the process $\widehat{X}^{(0)} = \frac{1}{1+H} \cdot X^{(0)}$. Take $\widetilde{f}^Q$ constructed in the first part of Theorem 9.2, set $\widetilde{W}^Q := \widetilde{f}^Q - 1$ and then define

$$N^{(0)} := \widetilde{W}^Q * (\widehat{\mu}_0 - \widehat{\nu}_0^Q) = I_{]\!]0,\tau]\!]} \cdot N^{(0)}. \tag{9.53}$$

We first claim that $N^{(0)}$ is well defined and a local Q-martingale null at 0. For that, it is enough to argue that $\widetilde{W}^Q$ is in $\mathcal{G}^1_{\mathrm{loc}}(\widehat{\mu}_0)$ for Q; see the lines following (2.7). But $\widehat{\mu}_0$ is the jump measure of the single-jump process $\widehat{X}^{(0)}$ and $\widehat{\cdot}^{0,Q} \equiv 0$ by (6.2), or since $\widehat{X}^{(0)}$ is quasi-left-continuous like S^i and $X^{(0)}$. So $\widetilde{W}^Q \in \mathcal{G}^1_{\mathrm{loc}}(\widehat{\mu}_0)$ for Q reduces to showing that the single term $|\widetilde{W}^Q(\Delta \widehat{X}^{(0)})| I_{\{\Delta \widehat{X}^{(0)} \neq 0\}}$ is locally Q-integrable. Now $|\widetilde{W}^Q| \leq \widetilde{f}^Q + 1$ and $\widetilde{f}^Q$ is in $L^\Phi_{\mathrm{loc}}(\widehat{m}_Q, \widetilde{\mathcal{P}})$ by the first part of Theorem 9.2. By Remark 9.5, this is equivalent to saying that $\widetilde{f}^Q$ is $\widetilde{\mathcal{P}}$-measurable and in $L^\Phi_{\mathrm{loc}}(M^Q_{\hat{\mu}_0})$; so $\widetilde{f}^Q \in L^1_{\mathrm{loc}}(M^Q_{\hat{\mu}_0})$ due to (9.7) and Lemma 3.3, and hence $\widetilde{f}^Q \in \mathcal{G}^1_{\mathrm{loc}}(\widehat{\mu}_0)$ for Q, again in view of Remark 9.5. Thus $\widetilde{W}^Q \in \mathcal{G}^1_{\mathrm{loc}}(\widehat{\mu}_0)$ for Q and the claim follows.

Because the first part of Theorem 9.2 gives $\widetilde{f}^Q > 0$, we get $\Delta N^{(0)} > -1$ by Lemma 2.6. So $Z^{(0)} := \mathcal{E}(N^{(0)})$ is a local Q-martingale > 0 with $Z^{(0)}_0 = 1$, and $Z^{(0)}$ is in $L^\Phi_{\mathrm{loc}}(Q)$ by Lemma 3.7 because $\widetilde{f}^Q \in L^\Phi_{\mathrm{loc}}(M^Q_{\hat{\mu}_0})$. Finally, $\varphi\widetilde{f}^Q$ satisfies (9.11), (9.12), and Corollary 2.12 therefore implies in view of (9.1) that $Z^{(0)}X^{(0)}$ is a Q-σ-martingale. Thus $Z^{(0)}$ gives our desired Q-σ-martingale density for $X^{(0)}$, and the proof is complete. **q.e.d.**

9.7. The case where $\widetilde{\mathcal{R}}(Q)$ is locally bounded

Suppose in Theorem 9.2 that instead of (9.9), we assume that $\widetilde{\mathcal{R}}(Q)$ is locally bounded. We claim that we then can find $\widetilde{f}^Q > 0$ with (9.11), (9.12) which instead of satisfying (9.10) is even locally bounded, and that the resulting Q-σ-martingale density $Z^{(0)}$ for $X^{(0)}$ is also locally bounded. To see this, we argue as follows.

First of all, if $\widetilde{\mathcal{R}}(Q)$ is locally bounded, it also satisfies (9.9), and if $\widetilde{\mathcal{R}}(Q) \in L^\infty(m, \mathcal{P})$, then also $\widetilde{\mathcal{R}}(Q) \in L^\Phi(m, \mathcal{P})$. So we can again use all the arguments from the L^Φ-case, and we only need to examine where we can get extra results from the L^∞-condition.

We start with Lemma 9.9. If we replace the assumption (9.13) that $\widetilde{\mathcal{R}}(Q) \in L^\Phi(m, \mathcal{P})$ by $\widetilde{\mathcal{R}}(Q) \in L^\infty(m, \mathcal{P})$, the arguments from Theorems 10.1 and 10.2 give us a $Y^* \geq 0$ which is not only in $L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$, but even in $L^\infty(\widehat{m}_Q, \widetilde{\mathcal{P}})$. This ultimately rests on Theorem 10.2 where we construct an $h^* \geq 0$ which is controlled above by $\widetilde{\mathcal{R}}(Q)$. So in part 1) of Lemma 9.9, we even get that

$$\Psi(\mathcal{U}_0) \text{ contains } 0 \qquad \text{for } \mathcal{U}_0 := L^\infty(\widehat{m}_Q, \widetilde{\mathcal{P}}),$$

and we can and do continue our reasoning with $\mathcal{U}_0$ instead of $U_0 = L^\Phi(\widehat{m}_Q, \widetilde{\mathcal{P}})$.

In Steps 6–10, we prove the assertion (9.39) that $\Psi(U^\circ_0)$ contains 0. But the only difference between L^∞ and L^Φ is in part 1) of Lemma 9.9 which we have discussed and settled just before, and so the same proof also works to show that

$$\Psi(\mathcal{U}^\circ_0) \text{ contains } 0. \tag{9.39$'$}$$

In Step 5, we construct $\widetilde{f}^Q$ from the zero $\widetilde{Y}_0$ of Ψ. From Step 2, $\widehat{Y}$ is bounded; so if we have (9.39$'$) instead of (9.39), then $\widetilde{Y}_0 \in \mathcal{U}^\circ_0 = L^\infty_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})$, and then the explicitly constructed $\widetilde{f}^Q$ is also in $\mathcal{U}^\circ_0$. In particular, $\widetilde{f}^Q$ is bounded.

Steps 1–4 are not affected at all by the issue L^Φ or L^∞; so if we start with

$$\widetilde{\mathcal{R}}(Q) \in L^\infty(m, \mathcal{P}), \tag{9.13$'$}$$

we end up after Step 10 with $\widetilde{f}^Q \in L^\infty_{++}(\widehat{m}_Q, \widetilde{\mathcal{P}})$. If $\widetilde{\mathcal{R}}(Q)$ is locally bounded, the localisation in Step 11 thus yields an $\widetilde{f}^Q > 0$ which satisfies (9.11), (9.12) and is locally bounded.

Finally, in Step 12, if $\widetilde{f}^Q$ is locally bounded instead of in $L^\Phi_{\mathrm{loc}}(\widehat{m}_Q, \widetilde{\mathcal{P}})$, then $N^{(0)}$ is still well defined. Moreover, $N^{(0)}$ is then locally bounded since its jumps are all > -1 and controlled above by $\widetilde{f}^Q$. As a consequence, $Z^{(0)} = \mathcal{E}(N^{(0)})$ is also locally bounded, and this completes the argument for Remark 9.3.

10. Constructing nicely integrable σ-martingale densities

In this section, we present a novel approach to the construction of σ-martingale densities with extra (local) integrability properties. We do this for a single-jump process with continuous FV (drift) part; in view of Corollary 7.2 and Theorem 5.1, this is enough to handle the case of a general semimartingale as well. *We stay within the framework of Section 9 and in particular keep in force all the assumptions of Theorem 9.2.*

Since we want to work with the measures $\widehat{m}_Q, \widehat{m}$ from (9.30), (9.31), we start directly with the single-jump process

$$\widehat{X}^{(0)} = \frac{1}{1+H} \cdot X^{(0)} = (\varphi \widehat{b}) \cdot A + \varphi \cdot (x * \widehat{\mu}_0)$$

from (9.35), where the jump measure $\widehat{\mu}_0$ has the Q-compensator $\widehat{\nu}_0^Q(dt,dx) = \widehat{F}^Q_{0,t}(dx) dA_t$. We recall from below (9.35) and (9.32) that $\widetilde{b} = (1+H)\widehat{b}$ and $F^Q_0 = (1+H)\widehat{F}^Q_0$, and we introduce the function $h_0 := Y_0 \circ \psi^{-1}$ (note that $x \mapsto \psi(x) = \frac{x}{1+|x|}$ is a bijection) and the measure $\widehat{\varrho}$ defined by

$$\widehat{\varrho} := \widehat{F}^Q_0 \circ \psi^{-1}. \tag{10.1}$$

The key difference between $\widehat{\varrho}$ and $\widehat{F}^Q_0$ is that $\widehat{\varrho}$ has a compact support as all its mass lies in $U_1(0, I\!R^d)$. After division by $1+H$, we can then rewrite (9.28) and (9.29) as

$$\int_{I\!R^d} |\psi(x)| Y_0(x) \widehat{F}^Q_0(dx) = \int_{I\!R^d} |y| h_0(y) \widehat{\varrho}(dy) < \infty \qquad \widehat{m}\text{-a.e.}, \tag{10.2}$$

$$\widehat{b} + \int_{I\!R^d} \psi(x) Y_0(x) \widehat{F}^Q_0(dx) = \widehat{b} + \int_{I\!R^d} y h_0(y) \widehat{\varrho}(dy) = 0 \qquad \widehat{m}\text{-a.e.} \tag{10.3}$$

Note that $\widehat{m}$ and $Q \otimes A$ are equivalent on $\{\widetilde{b} \neq 0\}$ and that $\widehat{m}$ has all its mass on that set; so the indicator functions from (9.28), (9.29) are not needed here. Recalling $\widetilde{\mathcal{R}}(Q)$ from (9.3), we point out that this can be equivalently written as

$$\widetilde{\mathcal{R}}(Q) = \operatorname*{ess\,sup}_{z \in I\!R^d} \frac{(-z^\top \widehat{b})^-}{\int_{I\!R^d} (z^\top \psi(x))^- \, \widehat{F}^Q_0(dx)} = \operatorname*{ess\,sup}_{z \in I\!R^d} \frac{(-z^\top \widehat{b})^-}{\int_{I\!R^d} (z^\top y)^- \, \widehat{\varrho}(dy)} \qquad \widehat{m}\text{-a.e.} \tag{10.4}$$

As seen in Section 9 at the end of Step 3, we have at this point a (even strictly positive) solution Y_0 or h_0 to (10.2), (10.3). However, we want a solution which is in addition controlled by $\widetilde{\mathcal{R}}(Q)$. In this regard, we prove the following two results.

Theorem 10.1. *With the preceding notations, the following are equivalent:*

1) $\widetilde{\mathcal{R}}(Q) \in L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$.

2) *There exists a $\widetilde{\mathcal{P}}$-measurable function $Y^* \geq 0$ on $\Omega \times [0,\infty) \times I\!R^d$ which satisfies (10.2), (10.3) (viewed as a condition for Y and formulated via $\widehat{F}^Q_0$) and which is such that $\|Y^*\|_{L^\infty(\bar{F}^Q_0)} := \operatorname*{ess\,sup}_{x \in I\!R^d} Y^*(\,\cdot\,,\,\cdot\,,x) \in L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$; the ess sup is taken with respect to $\bar{F}^Q_0$.*

3) *There exists a $\widetilde{\mathcal{P}}$-measurable function $h^* \geq 0$ on $\Omega \times [0,\infty) \times I\!R^d$ which satisfies (10.2), (10.3) (viewed as a condition for h and formulated via $\widehat{\varrho}$) and which is such that $\|h^*\|_{L^\infty(\widehat{\varrho})} := \operatorname*{ess\,sup}_{y \in I\!R^d} h^*(\,\cdot\,,\,\cdot\,,y) \in L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$; the ess sup is taken with respect to $\widehat{\varrho}$.*

(The brackets around loc mean that we either put or omit loc in all three statements; both cases give a valid theorem.)

Theorem 10.2. *With the preceding notations, suppose that $\widetilde{\mathcal{R}}(Q) < \infty$ $\widehat{m}$-a.e. and take $\varepsilon > 0$. Then there exists an $I\!R^d$-valued predictable process $\bar{\lambda} = (\bar{\lambda}_t)_{t \geq 0}$ such that*

$$h^*(\omega,t,y) := \min\Big(\widetilde{\mathcal{R}}_t(Q)(\omega) + \varepsilon, \max\Big(0, (\Phi')^{-1}\big(-\bar{\lambda}_t^\top(\omega)y\big)\Big)\Big) \tag{10.5}$$

is a $\widetilde{\mathcal{P}}$-measurable function ≥ 0 on $\Omega \times [0,\infty) \times I\!R^d$ which satisfies

$$\int_{I\!R^d} |y| h^*(y)\widehat{\varrho}(dy) < \infty \qquad \widehat{m}\text{-a.e.}, \tag{10.6}$$

$$\widehat{b} + \int_{I\!R^d} y h^*(y)\widehat{\varrho}(dy) = 0 \qquad \widehat{m}\text{-a.e.} \tag{10.7}$$

The proof of Theorem 10.2 is more involved; so we first show how Theorem 10.2 quickly implies Theorem 10.1.

Proof of Theorem 10.1. Both (10.2), (10.3) only impose conditions on $\operatorname{supp}\widehat{m} \subseteq \{\widetilde{b} \neq 0\}$; so we can set h^* or Y^* to zero on $\{\widetilde{b} = 0\}$, and then the ess sup is the same for $\widehat{F}^Q_0$ and for $\bar{F}^Q_0$. Next, 2) and 3) are obviously equivalent via (10.1) and $Y^* := h^* \circ \psi$ and $h^* := Y^* \circ \psi^{-1}$, exploiting that ψ is a bijection. Moreover, Theorem 10.2 gives us a solution $h^* \geq 0$ to (10.2), (10.3) with $h^* \leq \widetilde{\mathcal{R}}(Q) + \varepsilon$, a bound which does not depend on y. So if $\widetilde{\mathcal{R}}(Q) \in L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$, then also $\|h^*\|_{L^\infty(\widehat{\varrho})} \leq \widetilde{\mathcal{R}}(Q) + \varepsilon$ is in $L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$. This shows that 1) implies 3). For the converse, note that (10.3) for h^* implies that

$$-z^\top \widehat{b} = \int_{I\!R^d} (z^\top y) h^*(y)\widehat{\varrho}(dy) \geq -\int_{I\!R^d} (z^\top y)^- \|h^*\|_{L^\infty(\widehat{\varrho})}\, \widehat{\varrho}(dy)$$

so that

$$(-z^\top \widehat{b})^- = \max\big(0, -(-z^\top \widehat{b})\big) \le \|h^*\|_{L^\infty(\hat{\varrho})} \int_{I\!R^d} (z^\top y)^- \,\widehat{\varrho}(dy).$$

In view of (10.4), this gives $\widetilde{\mathcal{R}}(Q) \le \|h^*\|_{L^\infty(\hat{\varrho})}$, and if the latter is in $L^\Phi_{(\mathrm{loc})}(\widehat{m}, \mathcal{P})$, so is then $\widetilde{\mathcal{R}}(Q)$. So 3) implies 1) and the proof is complete. **q.e.d.**

The proof of Theorem 10.2 is more involved; so we first give a short overview. We want to look at functions h on $I\!R^d$ that satisfy (10.7) and (10.6), omitting "$\widehat{m}$-a.e.". We introduce an optimisation criterion over functions h and view the "zero drift equation" (10.7) as a constraint on h. In a first step, we show the existence of a solution $\bar{h}$ to that constrained optimisation problem, exploiting that there exists by (10.3) a solution to (10.7). We then characterise the optimiser $\bar{h}$ via the first order conditions, using the Kuhn–Tucker theorem. This almost leads to the explicit representation in (10.5), except for one point. All the above is done for functions of y only, with fixed (ω, t); so in a last step, we show that the result in (10.5) can also be obtained in a measurable way, exploiting that the expression in (10.5) is sufficiently nice.

Remark 10.3. **1)** One subtlety of the above line of argument is that by working with fixed (ω, t), we get nice properties in y for the functions h as well as for the optimiser $\bar{h}$, e.g. boundedness or some integrability with respect to y. However, we have no control at all on how those properties depend on (ω, t), and hence we must take care to avoid making statements simultaneously with respect to (ω, t, y).

2) A positive aspect of part 1) is that our arguments for Theorem 10.2 do not yield, but also do not need global properties with respect to (ω, t, y). For proving Theorem 10.1, it therefore makes no difference whether we formulate that result with L^Φ_{loc} or with L^Φ. $\diamond$

So let us start with (10.7), where $\widehat{b}$ and $\widehat{\varrho}$ both depend on (ω, t). *From now on until further notice, we fix (ω, t) from a suitable set of full $\widehat{m}$-measure and write for brevity still $\widehat{b}$ for $\widehat{b}_t(\omega)$ and $\widehat{\varrho} = \widehat{\varrho}(\,\cdot\,)$ for $\widehat{\varrho}_t(\omega, \,\cdot\,)$.* This will make most quantities appearing in the sequel become implicitly dependent on (ω, t), even if we usually do not show this explicitly.

Our approach below to proving Theorem 10.2 is inspired from Cole/Goodrich (1993); so we use similar notations to facilitate references and comparisons. Define $\mathcal{X} := \operatorname{supp} \widehat{\varrho}$; this is compact because $\widehat{\varrho}$ by construction has no mass outside of $U_1(0, I\!R^d)$. For any Borel-measurable function $B : \mathcal{X} \to [0, \infty)$, we define

$$\begin{aligned}
(10.8)\quad \mathcal{H}_B := \mathcal{H}_B(\omega, t) := \{h : \mathcal{X} \to I\!R \,|\, h &\text{ is Borel-measurable and } 0 \le h(y) \le B(y) \\
&\text{for } \widehat{\varrho}_t(\omega, dy)\text{-almost all } y\}, \\
R_B := R_B(\omega, t) := \Big\{z \in I\!R^d \,\Big|\, z = \int_{\mathcal{X}} &y h(y) \widehat{\varrho}_t(\omega, dy) \text{ for some } h \in \mathcal{H}_B = \mathcal{H}_B(\omega, t)\Big\}.
\end{aligned}$$

Both $\mathcal{H}_B$ and R_B are clearly convex. For some fixed $\varepsilon > 0$ not depending on (ω, t), we set

$$B_\varepsilon := B_\varepsilon(\omega, t) := \widetilde{\mathcal{R}}_t(Q)(\omega) + \varepsilon$$

and view this as a constant function of y. Note that B_ε is finite by our assumption that $\widetilde{\mathcal{R}}(Q) < \infty$ $\widehat{m}$-a.e. We first collect some properties for later use.

Lemma 10.4. *For all $c \in I\!R$ and $w \in I\!R^d$, $w \neq 0$, we have $-w^\top \widehat{b} > c$ whenever one of the following two equivalent properties holds:*

$$\int_E w^\top y B_\varepsilon \,\widehat{\varrho}(dy) \geq c \qquad \textit{for all } E \in \mathcal{B}(\mathcal{X}). \tag{10.9}$$

$$w^\top \int_{\mathcal{X}} y h(y) \widehat{\varrho}(dy) \geq c \qquad \textit{for all } h \in \mathcal{H}_{B_\varepsilon}. \tag{10.10}$$

Proof. For the equivalence of (10.9) and (10.10), note on the one hand that (10.9) is just (10.10) for the particular choice $h := B_\varepsilon I_E$. Conversely, (10.10) follows from (10.9) by standard measure-theoretic induction.

Now suppose that (10.9) holds. Choosing first $E = \emptyset$ yields $c \leq 0$. Next, the definition of $\widetilde{\mathcal{R}}(Q)$ in (10.4) gives for $\widehat{\varrho}$-almost every $w \in I\!R^d$ that

$$\widetilde{\mathcal{R}}(Q) \geq \frac{(-w^\top \widehat{b})^-}{\int_{I\!R^d} (w^\top y)^- \,\widehat{\varrho}(dy)} = \frac{(-w^\top \widehat{b})^-}{\int_{\mathcal{X}} (w^\top y)^- \,\widehat{\varrho}(dy)}.$$

With $E_- := \{y \in \mathcal{X} \,|\, w^\top y < 0\}$, we therefore get

$$(-w^\top \widehat{b})^- \leq \widetilde{\mathcal{R}}(Q) \int_{\mathcal{X}} (w^\top y)^- \,\widehat{\varrho}(dy) \leq B_\varepsilon \int_{\mathcal{X}} (w^\top y)^- \,\widehat{\varrho}(dy) \leq - \int_{E_-} w^\top y B_\varepsilon \,\widehat{\varrho}(dy),$$

and because $B_\varepsilon = \widetilde{\mathcal{R}}(Q) + \varepsilon$, the middle inequality is strict if $\int_{\mathcal{X}} (w^\top y)^- \,\widehat{\varrho}(dy) \neq 0$, i.e. if $\widehat{\varrho}(E_-) > 0$. So if $\widehat{\varrho}(E_-) > 0$, (10.9) gives $-w^\top \widehat{b} \geq -(-w^\top \widehat{b})^- > + \int_{E_-} w^\top y B_\varepsilon \,\widehat{\varrho}(dy) \geq c$, as desired. On the other hand, if $\widehat{\varrho}(E_-) = 0$, then $y \mapsto w^\top y \geq 0$ $\widehat{\varrho}$-a.e. But we know from (10.3) that (for $\widehat{m}$-almost all (ω, t), to be accurate) $\widehat{b} + \int_{I\!R^d} y h_0(y) \widehat{\varrho}(dy) = 0$, and so the fact that $w^\top y \geq 0$ $\widehat{\varrho}$-a.e. and the definition of $\mathcal{X}$ imply that

$$-w^\top \widehat{b} = \int_{I\!R^d} w^\top y h_0(y) \widehat{\varrho}(dy) = \int_{\mathcal{X}} w^\top y h_0(y) I_{\{w^\top y > 0\}} \,\widehat{\varrho}(dy).$$

But $y \mapsto w^\top y$ cannot be identically 0 $\widehat{\varrho}$-a.e. on $\mathcal{X} = \operatorname{supp} \widehat{\varrho}$. So $\widehat{\varrho}(\{w^\top y > 0\}) > 0$, the last integral is > 0 since $h_0 > 0$ $\widehat{\varrho}$-a.e., and so $-w^\top \widehat{b} > 0 \geq c$. This completes the proof. **q.e.d.**

The next result follows from Lemma 2.6 of Cole/Goodrich (1993), but for completeness, we give a proof.

Lemma 10.5. *$-\widehat{b}$ is in the interior of R_{B_ε}, shortly $-\widehat{b} \in \mathrm{int}\,(R_{B_\varepsilon})$.*

Proof. We argue in a first step that $-\widehat{b}$ is in R_{B_ε}, using Lemma 10.4. To that end, we first show that R_{B_ε} is closed. Indeed, if (z_n) is a sequence in R_{B_ε} converging to some z, then $z_n = \int_{\mathcal{X}} y h_n(y) \widehat{\varrho}(dy)$ for a sequence (h_n) in $\mathcal{H}_{B_\varepsilon}$. Take $\bar{h}_n \in \mathrm{conv}(h_n, h_{n+1}, \ldots)$ with $(\bar{h}_n)$ converging to some $\bar{h}$ $\widehat{\varrho}$-a.e.; then also $\bar{h} \in \mathcal{H}_{B_\varepsilon}$, and dominated convergence due to compactness of $\mathcal{X}$ therefore implies that $\bar{z}_n = \int_{\mathcal{X}} y \bar{h}_n(y) \widehat{\varrho}(dy) \longrightarrow \int_{\mathcal{X}} y \bar{h}(y) \widehat{\varrho}(dy) =: \bar{z}$ which is in R_{B_ε}. But since $\bar{z}_n \in \mathrm{conv}(z_n, z_{n+1}, \ldots)$, the sequence $(\bar{z}_n)$ has the same limit as the convergent sequence (z_n), and so $z = \bar{z}$ is in R_{B_ε}, proving that R_{B_ε} is closed. Because R_{B_ε} is also convex, assuming that $-\widehat{b}$ is not in R_{B_ε} allows us to separate $-\widehat{b}$ strictly from R_{B_ε} by a hyperplane. So there exist then $c \in I\!R$ and $w \in I\!R^d$, $w \neq 0$, with $-w^\top \widehat{b} < c$ and $w^\top \int_{\mathcal{X}} y h(y) \widehat{\varrho}(dy) \geq c$ for all $h \in \mathcal{H}_{B_\varepsilon}$. But this contradicts Lemma 10.4, and so $-\widehat{b}$ is in R_{B_ε}.

Now consider again Lemma 10.4. This states that whenever R_{B_ε} lies on one side of a hyperplane determined by w and c (which is what (10.10) expresses), the point $-\widehat{b}$ lies strictly on the same side of that hyperplane. Because we already know that $-\widehat{b} \in R_{B_\varepsilon}$, this implies that we must even have $-\widehat{b} \in \mathrm{int}\,(R_{B_\varepsilon})$. **q.e.d.**

The following consequence of Lemma 10.5 will be used in our subsequent application of the Kuhn–Tucker theorem.

Lemma 10.6. *There exists a Borel-measurable $h : \mathcal{X} \to I\!R$ satisfying $\widehat{b} + \int_{I\!R^d} y h(y) \widehat{\varrho}(dy) = 0$ and $0 < \delta \leq h(y) \leq B_\varepsilon - \delta$ $\widehat{\varrho}$-a.e. for some $\delta > 0$. In other words, we have $-\widehat{b} = \int_{I\!R^d} y h(y) \widehat{\varrho}(dy)$ for some $h \in \mathrm{int}\,(\mathcal{H}_{B_\varepsilon})$, where the interior is with respect to the $L^\infty(\widehat{\varrho})$-norm.*

Proof. This is similar to the proof of Theorem 2.1 of Cole/Goodrich (1993), but we give a direct argument as our setting is slightly different. Because $\mathcal{H}_{B_\varepsilon} \subseteq L^\infty_+(\widehat{\varrho})$ is convex with $\mathrm{int}\,(\mathcal{H}_{B_\varepsilon}) \neq \emptyset$ due to $B_\varepsilon \geq \varepsilon > 0$, Corollary 2.14 in Borwein/Lewis (1992) implies that $\mathrm{int}\,(\mathcal{H}_{B_\varepsilon}) = \mathrm{qri}\,(\mathcal{H}_{B_\varepsilon})$. The mapping $A : \mathcal{H}_{B_\varepsilon} \to I\!R$ given by $A(h) := \int_{I\!R^d} y h(y)\, \widehat{\varrho}(dy)$ is well defined since $\widehat{\varrho}$ has compact support and $\mathcal{H}_{B_\varepsilon} \subseteq L^\infty_+(\widehat{\varrho})$, and is continuous and linear. Moreover, $R_{B_\varepsilon} = A(\mathcal{H}_{B_\varepsilon}) \subseteq I\!R$ is convex and finite-dimensional. Since $\mathrm{qri}\,(\mathcal{H}_{B_\varepsilon}) = \mathrm{int}\,(\mathcal{H}_{B_\varepsilon}) \neq \emptyset$ and $\mathrm{int}\,(R_{B_\varepsilon}) \neq \emptyset$ due to Lemma 10.5, Propositions 2.10, 2.4 and 2.14 in Borwein/Lewis (1992) imply that $A(\mathrm{qri}\,(\mathcal{H}_{B_\varepsilon})) = \mathrm{ri}\,(A(\mathcal{H}_{B_\varepsilon})) = \mathrm{qri}\,(A(\mathcal{H}_{B_\varepsilon})) = \mathrm{int}\,(A(\mathcal{H}_{B_\varepsilon})) = \mathrm{int}\,(R_{B_\varepsilon})$. By using again Lemma 10.5, the assertion follows. **q.e.d.**

To formulate our optimisation problem over functions, we recall $\mathcal{H}_B$ from (10.8). So

$$R_{B_\varepsilon} = \Big\{ z = \int_{I\!R^d} y h(y) \widehat{\varrho}(dy) \,\Big|\, h \in \mathcal{H}_{B_\varepsilon} \Big\} = \Big\{ z = \int_{\mathcal{X}} y h(y) \widehat{\varrho}(dy) \,\Big|\, h \in \mathcal{H}_{B_\varepsilon} \Big\},$$

and we now consider the problem to find

$$I := \inf \bigg\{ \int_{\mathcal{X}} \Phi\big(h(y)\big) \widehat{\varrho}(dy) \,\bigg|\, h \in \mathcal{H}_{B_\varepsilon} \text{ and } \int_{\mathcal{X}} y h(y) \widehat{\varrho}(dy) = -\widehat{b} \bigg\}. \tag{10.11}$$

Because $-\widehat{b} \in R_{B_\varepsilon}$, we are taking the infimum over a nonempty set; and as each $h \in \mathcal{H}_{B_\varepsilon}$ is bounded, we also have $I < \infty$ since Φ is continuous, hence bounded on $[0, B_\varepsilon]$. A standard argument shows that (10.11) has an optimiser $\bar{h}$, which is unique by strict convexity of Φ. Indeed, the set of functions $h \in \mathcal{H}_{B_\varepsilon}$ with $\int_{I\!R^d} y h(y) \widehat{\varrho}(dy) = -\widehat{b}$ is convex and each such h is nonnegative. Take a sequence (h_n) in that set with $I_n := \int_{\mathcal{X}} \Phi(h_n(y)) \widehat{\varrho}(dy)$ decreasing to I, and then take $\bar{h}_n \in \operatorname{conv}(h_n, h_{n+1}, \dots)$ converging $\widehat{\varrho}$-a.e. to some function $\bar{h}$. Then $\bar{h}$ is in $\mathcal{H}_{B_\varepsilon}$, and $|y\bar{h}_n| \le |y| B_\varepsilon$ plus compactness of $\mathcal{X}$ allows us to get via dominated convergence $\int_{\mathcal{X}} y \bar{h}(y) \widehat{\varrho}(dy) = \lim_{n\to\infty} \int_{\mathcal{X}} y \bar{h}_n(y) \widehat{\varrho}(dy) = -\widehat{b}$. Finally, Φ is continuous and bounded on $[0, B_\varepsilon]$ and $\bar{h}_n \le B_\varepsilon$ for all n; so dominated convergence, convexity of Φ and $I_n \searrow I$ yield

$$\int_{\mathcal{X}} \Phi\big(\bar{h}(y)\big) \widehat{\varrho}(dy) = \lim_{n\to\infty} \int_{\mathcal{X}} \Phi\big(\bar{h}_n(y)\big) \widehat{\varrho}(dy) \le \lim_{n\to\infty} \sup_{k \ge n} \int_{\mathcal{X}} \Phi\big(h_k(y)\big) \widehat{\varrho}(dy) = \lim_{n\to\infty} I_n = I.$$

This shows that $\bar{h}$ attains I. In particular, $\bar{h}$ satisfies the constraint (10.7) as well as (10.6), due to compactness of $\mathcal{X}$, and of course both without "$\widehat{m}$-a.e." since we have fixed (ω, t).

Our next result now describes $\bar{h}$ via the first order conditions for optimality.

Proposition 10.7. *The solution $\bar{h}$ to (10.11) can be written, for some $\lambda \in I\!R^d$, as*

$$\bar{h}(y) = \min\Big(B_\varepsilon, \max\big(0, (\Phi')^{-1}(-\lambda^\top y)\big)\Big). \tag{10.12}$$

Proof. We follow the arguments in the proof of Theorem 3.1 of Cole/Goodrich (1993), CG for short. This needs some changes because CG have the L^p-norm as their functional for optimisation, while we use a general convex Φ.

Almost as in CG [note their typo in the sign of G], define $G : L^\infty(\widehat{\varrho}) \to L^\infty(\widehat{\varrho}) \times L^\infty(\widehat{\varrho})$ by $h \mapsto G(h) := (h, B_\varepsilon - h)$. Then (10.11) is the problem to

$$\text{minimise} \int_{\mathcal{X}} \Phi\big(h(y)\big) \widehat{\varrho}(dy) \text{ over } h \in L^\infty(\widehat{\varrho}), \quad \text{subject to } G(h) \ge 0 \text{ and } \int_{\mathcal{X}} y h(y) \widehat{\varrho}(dy) = -\widehat{b}.$$

By the preceding argument, there exists a solution $\bar{h}$; and Lemmas 10.5 and 10.6 show that $-\widehat{b} \in \operatorname{int}(R_{B_\varepsilon})$ and there exists some $h \in L^\infty(\widehat{\varrho})$ with $G(h) \in \operatorname{int}\big(L^\infty_+(\widehat{\varrho}) \times L^\infty_+(\widehat{\varrho})\big)$ and satisfying the constraint $\int_{\mathcal{X}} yh(y)\widehat{\varrho}(dy) = -\widehat{b}$. As in CG, the Kuhn–Tucker theorem thus gives the existence of some nonzero $z_0^* = (z_1^*, z_2^*)$ in $\mathrm{ba}_+(\mathcal{X}) \times \mathrm{ba}_+(\mathcal{X})$, the dual of $L_+^\infty(\widehat{\varrho}) \times L_+^\infty(\widehat{\varrho})$, with $\langle G(\bar{h}), z_0^*\rangle = \langle \bar{h}, z_1^*\rangle + \langle B_\varepsilon - \bar{h}, z_2^*\rangle = 0$. Because $\bar{h} \geq 0$ and $B_\varepsilon - \bar{h} \geq 0$, this gives

$$\langle \bar{h}, z_1^*\rangle = 0 \qquad \text{and} \qquad \langle B_\varepsilon - \bar{h}, z_2^*\rangle = 0.$$

Next, $\bar{h}$ minimises $h \mapsto \int_{\mathcal{X}} \Phi(h(y))\widehat{\varrho}(dy) + \langle G(h), z_0^*\rangle$ over all h satisfying $\int_{\mathcal{X}} yh(y)\widehat{\varrho}(dy) = -\widehat{b}$. Using Lagrange multipliers therefore gives a vector $\lambda \in I\!R^d$ such that the mapping

$$h \mapsto J(h) := \int_{\mathcal{X}} \Phi\big(h(y)\big)\widehat{\varrho}(dy) + \langle G(h), z_0^*\rangle + \lambda^\top \Big(\widehat{b} + \int_{\mathcal{X}} yh(y)\widehat{\varrho}(dy)\Big)$$

is stationary at $\bar{h}$. So for any $h \in L^\infty(\widehat{\varrho})$ and any $\eta > 0$, we have

$$\begin{aligned} 0 &\leq \liminf_{\eta \searrow 0} \frac{\pm 1}{\pm\eta}\big(J(\bar{h} \pm \eta h) - J(\bar{h})\big) \\ &= \liminf_{\eta\searrow 0} \pm \int_{\mathcal{X}} \tfrac{1}{\pm\eta}\Big(\Phi\big(\bar{h}(y) \pm \eta h(y)\big) - \Phi\big(\bar{h}(y)\big)\Big)\widehat{\varrho}(dy) \pm \langle h, z_1^*\rangle \mp \langle h, z_2^*\rangle \pm \lambda^\top \int_{\mathcal{X}} yh(y)\widehat{\varrho}(dy). \end{aligned}$$

By Taylor's theorem, we have $\frac{1}{\pm\eta}(\Phi(\bar{h}(y) \pm \eta h(y)) - \Phi(\bar{h}(y))) = \Phi'(\tilde{h}(y))h(y)$ with $\tilde{h}(y)$ lying between $\bar{h}(y)$ and $\bar{h}(y) \pm \eta h(y)$. As $\eta \searrow 0$, we have $\tilde{h}(y) \to \bar{h}(y)$ and everything remains bounded by $C := \|\bar{h}\|_{L^\infty(\widehat{\varrho})} + \|h\|_{L^\infty(\widehat{\varrho})}$. So $\Phi'(\tilde{h}(y))$ tends to $\Phi'(\bar{h}(y))$ as $\eta \searrow 0$ and remains bounded by the maximum of the continuous function Φ' on the compact interval $[-C, C]$. Therefore we can use dominated convergence, the lim inf above is actually a limit, and since $\pm$ that limit is ≥ 0, we get for all $h \in L^\infty(\widehat{\varrho})$ that

$$0 = \int_{\mathcal{X}} \Big(\Phi'\big(\bar{h}(y)\big) + \lambda^\top y\Big)h(y)\widehat{\varrho}(dy) + \langle h, z_1^*\rangle - \langle h, z_2^*\rangle.$$

This is the generalisation to the Φ case of the equation (7) in CG.

From here on, we follow CG rather closely and hence give fewer details. We first obtain like there before (8) that

$$\Phi'\big(\bar{h}(y)\big) + \lambda^\top y = 0 \qquad \widehat{\varrho}\text{-a.e. on } \{0 < \bar{h} < B_\varepsilon\}$$

which gives

$$\bar{h}(y) = (\Phi')^{-1}(-\lambda^\top y) \qquad \widehat{\varrho}\text{-a.e. on } \{0 < \bar{h} < B_\varepsilon\}. \tag{10.13}$$

Next we argue as in CG before (9) [the signs in CG are correct here] that

$$\Phi'(0) + \lambda^\top y \geq 0 \qquad \widehat{\varrho}\text{-a.e. on } \{\bar{h} = 0\}; \tag{10.14}$$

note that the function Q in CG appears after (7) there and corresponds to our $\lambda^\top y$, and that we have an extra term $\Phi'(0)$ which does not appear in the CG case of $\Phi(y) = y^p$. Finally, again as in CG, before (10) [where the signs are also correct], we get

$$\Phi'\big(\bar{h}(y)\big) + \lambda^\top y \le 0 \qquad \widehat{\varrho}\text{-a.e. on } \{\bar{h} = B_\varepsilon\}. \tag{10.15}$$

But Φ' is increasing since Φ is convex, and so we can rewrite (10.13)–(10.15) as

$$\begin{aligned} \bar{h}(y) &= (\Phi')^{-1}(-\lambda^\top y) && \widehat{\varrho}\text{-a.e. on } \{0 < \bar{h} < B_\varepsilon\}, \\ 0 &\ge (\Phi')^{-1}(-\lambda^\top y) && \widehat{\varrho}\text{-a.e. on } \{\bar{h} = 0\}, \\ \bar{h}(y) &\le (\Phi')^{-1}(-\lambda^\top y) && \widehat{\varrho}\text{-a.e. on } \{\bar{h} = B_\varepsilon\}. \end{aligned}$$

By looking at the various cases and using that $0 \le \bar{h} \le B_\varepsilon$ and $B_\varepsilon > 0$, we finally obtain

$$\bar{h}(y) = \min\Big(B_\varepsilon, \max\big(0, (\Phi')^{-1}(-\lambda^\top y)\big)\Big) \qquad \widehat{\varrho}\text{-a.e.}$$

for some $\lambda \in I\!\!R^d$, which is (10.12). **q.e.d.**

Now we reinstate (ω, t). Up to here, we have proved that for $\widehat{m}$-a.e. (ω, t), there is a solution $\bar{h}$ to the Q-zero drift equation $\widehat{b} + \int_{I\!\!R^d} y h(y) \widehat{\varrho}(dy) = 0$ with $\bar{h}$ of the form (10.12) for some $\lambda \in I\!\!R^d$, which depends on (ω, t), as does $\bar{h}$. To finish the proof of Theorem 10.2, it remains to show that λ as a function of (ω, t) can be chosen in a $\mathcal{P}$-measurable way.

In order to apply a suitable measurable selection result, we introduce the mapping

$$\varphi(\omega, t, \lambda, y) := \min\Big(\widetilde{\mathcal{R}}_t(Q)(\omega) + \varepsilon, \max\big(0, (\Phi')^{-1}(-\lambda^\top y)\big)\Big) \tag{10.16}$$

on $\Omega \times [0, \infty) \times I\!\!R^d \times I\!\!R^d$, equipped with the σ-field $\mathcal{P} \otimes \mathcal{B}(I\!\!R^d) \otimes \mathcal{B}(I\!\!R^d) = \widetilde{\mathcal{P}} \otimes \mathcal{B}(I\!\!R^d)$. Since the right-hand side of (10.16) is clearly continuous in the three variables $\widetilde{\mathcal{R}}_t(Q)(\omega), \lambda, y$ and since $\widetilde{\mathcal{R}}(Q)$ is predictable, i.e. $\mathcal{P}$-measurable, we see that φ is product-measurable, with respect to $\widetilde{\mathcal{P}} \otimes \mathcal{B}(I\!\!R^d)$, and bounded in y by $\widetilde{\mathcal{R}}_t(Q)(\omega) + \varepsilon$. By Fubini's theorem and compactness of $\mathcal{X}(\omega, t) = \operatorname{supp} \widehat{\varrho}_t(\omega, \cdot)$, the function

$$\chi(\omega, t, \lambda) := \widehat{b}_t(\omega) + \int_{I\!\!R^d} y \varphi(\omega, t, \lambda, y) \widehat{\varrho}_t(\omega, dy)$$

is therefore well defined for $\widehat{m}$-a.a. (ω, t) and all λ, and $\widetilde{\mathcal{P}}$-measurable, i.e. product-measurable on $\Omega \times [0, \infty) \times I\!\!R^d$ with respect to $\mathcal{P} \otimes \mathcal{B}(I\!\!R^d)$; this uses that $\widehat{b}$ is also predictable.

Now define a (singleton-valued) correspondence F from $\Omega \times [0, \infty) \times I\!\!R^d$ to $I\!\!R^d$ by

$$F(\omega, t, \lambda) := \{\lambda + \chi(\omega, t, \lambda)\}$$

and note that $\lambda \in F(\omega,t,\lambda)$ means that $\chi(\omega,t,\lambda)=0$. For $\widehat{m}$-a.a. (ω,t), the set

$$\{\lambda \in I\!R^d \,|\, \lambda \in F(\omega,t,\lambda)\} = \Big\{\lambda \in I\!R^d \,\Big|\, \widehat{b}_t(\omega) + \int_{I\!R^d} y\varphi(\omega,t,\lambda,y)\widehat{\varrho}_t(\omega,dy) = 0\Big\} \tag{10.17}$$

is therefore nonempty due to Proposition 10.7 and the definition of φ in (10.16). Moreover, the mapping $(\omega,t,\lambda,z) \mapsto \lambda + \chi(\omega,t,\lambda) - z$ is clearly product-measurable with respect to $\widetilde{\mathcal{P}} \otimes \mathcal{B}(I\!R^d) = \mathcal{P} \otimes \mathcal{B}(I\!R^d) \otimes \mathcal{B}(I\!R^d) = \mathcal{P} \otimes \mathcal{B}(I\!R^d \times I\!R^d)$. So the graph of F,

$$\operatorname{Graph} F := \{(\omega,t,\lambda,z) \,|\, z \in F(\omega,t,\lambda)\} = \{(\omega,t,\lambda,z) \,|\, \lambda + \chi(\omega,t,\lambda) - z = 0\}$$

is in $\mathcal{P} \otimes \mathcal{B}(I\!R^d \times I\!R^d)$. By suitably defining χ and hence F on an $\widehat{m}$-nullset, we can achieve that the set in (10.17) is nonempty for all $(\omega,t) \in \Omega \times [0,\infty)$. This allows us to apply Theorem 7 in Tarafdar/Watson/Yuan (1997) and obtain a $\mathcal{P}$-measurable function $f : \Omega \times [0,\infty) \to I\!R^d$ such that $f(\omega,t) \in F(\omega,t,f(\omega,t))$ for $\widehat{m}$-a.a. (ω,t). Using the definitions of F, χ, φ and calling the function f now $\bar{\lambda}$, we thus have an $I\!R^d$-valued predictable process $\bar{\lambda} = (\bar{\lambda}_t)_{t \geq 0}$ such that

$$h^*(\omega,t,y) := \varphi\big(\omega,t,\bar{\lambda}_t(\omega),y\big) \tag{10.18}$$

is given by the right-hand side of (10.5) and satisfies, as in (10.17),

$$0 = \chi\big(\omega,t,\bar{\lambda}_t(\omega)\big) = \widehat{b}_t(\omega) + \int_{I\!R^d} y\varphi\big(\omega,t,\bar{\lambda}_t(\omega),y\big)\widehat{\varrho}_t(\omega,dy) \qquad \widehat{m}\text{-a.e.}$$

This is exactly (10.7) in view of (10.16), (10.18). Finally, (10.6) also holds since h^* is bounded in y by $\widetilde{\mathcal{R}}(Q) + \varepsilon$ and $\mathcal{X} = \operatorname{supp} \widehat{\varrho}$ is compact. The proof of Theorem 10.2 is complete. **q.e.d.**

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References

J. Borwein and R. Goebel (2003), "Notions of relative interior in Banach spaces", *Journal of Mathematical Sciences 115*, 2542–2553

J. M. Borwein and A. S. Lewis (1991), "Duality relationships for entropy-like minimization problems", *SIAM Journal on Control and Optimization 29*, 325–338

J. M. Borwein and A. S. Lewis (1992), "Partially finite convex programming, Part I: Quasi relative interiors and duality theory", *Mathematical Programming 57*, 15–48

A. S. Cherny and A. N. Shiryaev (2002), "Vector stochastic integrals and the fundamental theorems of asset pricing", *Proceedings of the Steklov Institute of Mathematics 237*, 6–49

T. Choulli and M. Schweizer (2015), "A result on integral functionals with infinitely many constraints", *Swiss Finance Institute Research Paper No. 15-38, available at* `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2662476`

T. Choulli and C. Stricker (2006), "More on minimal entropy-Hellinger martingale measure", *Mathematical Finance 16*, 1–19

T. Choulli, C. Stricker and J. Li (2007), "Minimal Hellinger martingale measures of order q", *Finance and Stochastics 11*, 399–427

R. E. Cole and R. K. Goodrich (1993), "L_p-spectral estimation with an L_∞-upper bound", *Journal of Optimization Theory and Applications 76*, 321–355

R. C. Dalang, A. Morton and W. Willinger (1990), "Equivalent martingale measures and no-arbitrage in stochastic securities market models", *Stochastics 29*, 185–201

P. Daniele, S. Giuffrè, G. Idone and A. Maugeri (2007), "Infinite dimensional duality and applications", *Mathematische Annalen 339*, 221–239

F. Delbaen and W. Schachermayer (1998), "The fundamental theorem of asset pricing for unbounded stochastic processes", *Mathematische Annalen 312*, 215–250

F. Delbaen and W. Schachermayer (2006), "The Mathematics of Arbitrage", *Springer*

C. Dellacherie and P.-A. Meyer (1982), "Probabilities and Potential B. Theory of Martingales", *North-Holland, Amsterdam*

I. Ekeland and R. Témam (1999), "Convex Analysis and Variational Problems", *SIAM Classics In Applied Mathematics*

B. Grigelionis (1973), "On nonlinear filtering theory and absolute continuity of measures, corresponding to stochastic processes", *in: G. Maruyama and Yu. V. Prokhorov (eds.),* *Proceedings of the Second Japan-USSR Symposium on Probability Theory, Lecture Notes in Mathematics 330, Springer*, 80–94

S.-W. He, J.-G. Wang and J.-A. Yan (1992), "Semimartingale Theory and Stochastic Calculus", *Science Press and CRC Press*

H. Hulley and M. Schweizer (2010), "M^6 — On minimal market models and minimal martingale measures", *in: C. Chiarella and A. Novikov (eds.), "Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen", Springer*, 35–51

S. D. Jacka (1992), "A martingale representation result and an application to incomplete financial markets", *Mathematical Finance 2*, 239–250

J. Jacod (1979), "Calcul Stochastique et Problèmes de Martingales", *Lecture Notes in Mathematics 714, Springer*

J. Jacod and A. N. Shiryaev (2003), "Limit Theorems for Stochastic Processes", *second edition, Springer*

Y. Kabanov and M. Safarian (2010), "Markets with Transaction Costs. Mathematical Theory", *Springer*

Y. Kabanov and C. Stricker (2001), "On equivalent martingale measures with bounded densities", *in: J. Azéma et al. (eds.), Séminaire de Probabilités XXXV, Lecture Notes in Mathematics 1755, Springer*, 139–148

J. Kallsen (2004), "σ-localization and σ-martingales", *Theory of Probability and Its Applications 48*, 152–163

C. Kardaras (2012), "Market viability via absence of arbitrage of the first kind", *Finance and Stochastics 16*, 651–667

D. O. Kramkov (1996), "On closure of a family of martingale measures and an optional decomposition of supermartingales", *Probability Theory and Its Applications 41*, 788–791

V. Prokaj and M. Rásonyi (2011), "Local and true martingales in discrete time", *Theory of Probability and Its Applications 55*, 325–332

M. Rásonyi (2001), "A note on equivalent martingale measures with bounded density", *in: M. Kohlmann and S. Tang (eds.), "Mathematical Finance: Workshop of the Mathematical Finance Research Project, Konstanz, Germany, October 5–7, 2000", Birkhäuser*, 302–306

D. B. Rokhlin (2010), "Lower bounds of martingale measure densities in the Dalang–Morton–Willinger theorem", *Theory of Probability and Its Applications 54*, 447–465

D. Rokhlin and W. Schachermayer (2006), "A note on lower bounds of martingale measure densities", *Illinois Journal of Mathematics 50*, 815–824

C. Stricker (1990), "Arbitrage et lois de martingale", *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 26*, 451–460

C. Stricker and J. A. Yan (1998), "Some remarks on the optional decomposition theorem", *in: J. Azéma et al. (eds.), Séminaire de Probabilités XXXII, Lecture Notes in Mathematics 1686, Springer*, 56–66

K. Takaoka and M. Schweizer (2014), "A note on the condition of no unbounded profit with bounded risk", *Finance and Stochastics 18*, 393–405

E. Tarafdar, P. Watson and X.-Z. Yuan (1997), "The measurability of Carathéodory set-valued mappings and random fixed point theorems", *Acta Mathematica Hungarica 74*, 309–319