Frontiers of Mathematical Finance
Vol. 1, No. 2, June 2022, pp. 249–286
doi:10.3934/fmf.2021010

MAKING NO-ARBITRAGE DISCOUNTING-INVARIANT: A NEW FTAP VERSION BEYOND NFLVR AND NUPBR

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(Communicated by Jakša Cvitanić)

Abstract. What is absence of arbitrage for non-discounted prices? How can one define this so that it does not change meaning if one decides to discount after all?

The answer to both questions is a new discounting-invariant no-arbitrage concept. As in earlier work, we define absence of arbitrage as the zero strategy or some basic strategies being *maximal*. The key novelty is that maximality of a strategy is defined in terms of *share* holdings instead of *value*. This allows us to generalise both NFLVR, by dynamic share efficiency, and NUPBR, by dynamic share viability. These new concepts are the same for discounted or undiscounted prices, and they can be used in general models under minimal assumptions on asset prices. We establish corresponding versions of the FTAP, i.e., dual characterisations in terms of martingale properties. As one expects, "properly anticipated prices fluctuate randomly", but with an *endogenous* discounting process which cannot be chosen a priori. An example with N geometric Brownian motions illustrates our results.

1. **Introduction.** We introduce and study an absence-of-arbitrage concept with three properties: (i) it is defined for original, non-discounted prices; (ii) it is discounting-invariant in full generality, i.e. under discounting with any positive semimartingale; (iii) it imposes minimal assumptions on the underlying model.

Consider a financial market on the right-open interval $[0,\infty)$ with $N \geq 2$ assets whose prices are modelled by an $\mathbb{R}^N$-valued semimartingale $S = (S_t)_{t\geq 0}$. We view this as a pure exchange economy; the unit of account in which prices are denominated is not tradable and can best be thought of as a perishable consumption good. We do not assume that there is an extra tradable riskless asset like cash or a money market account. In this setting, absence of arbitrage (AOA) should as usual capture the idea that one cannot get something out of nothing for free, and the denomination of prices should not matter — if a positive process $D = (D_t)_{t\geq 0}$

2020 *Mathematics Subject Classification.* 91G99, 91B02, 60G48.

Key words and phrases. Absence of arbitrage, maximal strategies, semimartingales, discounting, NFLVR, NUPBR, FTAP, σ-martingale discounter, share maximal, dynamic share viability, Black–Scholes model.

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describes how price units change over time, then S should satisfy AOA if and only if S/D does, for all D from a suitable class $\mathcal{D}$. Finding and characterising such an AOA concept is in our view a fundamental question in arbitrage theory.

Under the mild assumption that one asset price remains strictly positive, (i) and (ii) have been solved completely for models in *finite discrete time*; see the textbook Delbaen/Schachermayer [14, Chapter 2, in particular Section 2.5]. But this framework clearly does not qualify for (iii). For more general models, surprisingly, the existing literature does not provide a satisfactory solution. Despite the general view that all questions about arbitrage theory have been asked and answered in the works of Delbaen/Schachermayer (collected in [14]), nontrivial open questions remain. We illustrate this by a simple example.

Example 1.1. Let $N = 2$ so that there are only two assets available for trade and $S = (S^1, S^2)$. For concreteness, S^1 and S^2 could be two, possibly correlated, geometric Brownian motions. If we want to value a perpetual exchange option on these two assets, we should probably ensure that S is arbitrage-free. But what does this mean here?

As we discuss in detail in Section 5, neither the general theory from Delbaen/Schachermayer nor the recent work by Herdegen on numéraire-invariant AOA concepts give any good answer in Example 1.1. Delbaen/Schachermayer always start with *discounted* prices so that S has the form $S = (1, X)$ for some $\mathbb{R}^d$-valued semimartingale X (discounted prices of d risky assets). The very definition of AOA then changes in general (sometimes dramatically) if one discounts in a different way. In contrast, Herdegen does develop discounting-invariant concepts, but works on a *right-closed* interval $[0, T]$, which is crucial for his definitions and results alike. The generality of our setting needs something new.

Our approach borrows ideas and techniques from both Delbaen/Schachermayer and Herdegen, and complements them with a key new idea. As in Herdegen [17] and Herdegen/Schweizer [18], we define absence of arbitrage as the property that the zero strategy or a number of basic strategies are *maximal* in the sense that they cannot be "improved" by other strategies. In [17] as well as in earlier work of Delbaen/Schachermayer [10, 11, 12], improvements are measured in terms of *value* or wealth. Whenever a discounter D can go to 0 or $+\infty$, this approach breaks down for S/D, and handling prices on a right-open interval thus needs restrictive assumptions on D. We circumvent this difficulty by measuring "improvements" not in terms of value, but in terms of *shares* compared to a so-called *reference strategy* η, which intuitively represents a desirable investment. We prove as in Delbaen/Schachermayer [10] a key result which says that if one has an AOA property related to η, prices in units of the corresponding wealth process $V(\eta)$ must converge and hence can be defined on the *right-closed* interval $[0, \infty]$. This in turn allows us to exploit the results from Herdegen [17], after showing how his and our AOA conditions are related.

Our approach leads to genuinely discounting-invariant concepts in almost fully general frictionless semimartingale models of financial markets. We only assume $S \geq 0$ and the existence of a reference strategy, and the latter already holds for instance as soon as $\sum_{i=1}^N S^i > 0$ and $\sum_{i=1}^N S^i_- > 0$. Our main results are two new versions of the fundamental theorem of asset pricing (FTAP) — one for *dynamic share viability (DSV)*, our discounting-invariant counterpart of no unbounded profit with bounded risk (NUPBR), and one for *dynamic share efficiency (DSE)*, which extends no free lunch with vanishing risk (NFLVR). In contrast to the classic FTAP formulations of Delbaen/Schachermayer [10, 13] or Karatzas/Kardaras [27], the discounting process in our results cannot be chosen a priori, but is an endogenous part of the dual characterisation of absence of arbitrage.

Our discounting-invariant AOA framework lays the foundations for many possible future developments. One current project pursues a general treatment of the growth-optimal portfolio (GOP) and the benchmark approach; see Filipović/Platen [15]. We have already shown in Bálint/Schweizer [5] how ideas from the present paper can be used in the context of large financial markets. Stochastic portfolio theory (SPT) might benefit from our general perspective, given that its approach has some similarities. Finally, one can try to study utility maximisation, maybe in a discounting-invariant form similarly as in Kardaras [30], or under DSV instead of NUPBR; see e.g. [27] or Chau et al. [8].

The paper is structured as follows. Section 2 introduces basic concepts and presents our main results. Section 3 is the mathematical core; it shows how models on right-open intervals can be closed on the right under a weak AOA assumption, combines this with Herdegen [17] to prove dual characterisations of value maximality for a general setting, and connects our new concept of *share maximality* to the value maximality studied in [17]. This is used to prove the main results from Section 2. In Section 4, we relate our results to existing concepts, discuss the dependence on the choice of the reference strategy η appearing in share maximality, and connect to the classic setup. Section 5 provides a comparison to the literature and explains why our approach and the numéraire-independence in Delbaen/Schachermayer [11, 12] are conceptually quite different. Section 6 contains examples and counterexamples, including a detailed study of a market with N geometric Brownian motions, and the Appendix collects some technical proofs and auxiliary results.

2. **Concepts and main results.**

2.1. **Setup and new concepts.**

We always work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, assume that $\mathcal{F}_0$ is trivial and set $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. There are N basic traded assets whose prices are modelled by an $\mathbb{R}^N$-valued semimartingale S. If there is a bank account (we do not assume this in general), it must be one component of S. To have trading possible, we thus must have $N \geq 2$. All prices and values are expressed in some abstract, *non-tradable* accounting unit, which can be thought of as a perishable consumption good. Its only role is to make trading of asset shares possible.

We use general stochastic integration (in the sense of Jacod/Shiryaev [20, Chapter III.6] or Shiryaev/Cherny [37]). We call $L(S)$ the space of all $\mathbb{R}^N$-valued predictable S-integrable processes H and denote the (real-valued) stochastic integral of $H \in L(S)$ with respect to S by $H \bullet S := \int H \,\mathrm{d}S$. For any RCLL process Y, we set $Y_{0-} := Y_0$. The scalar product of $x, y \in \mathbb{R}^N$ is $x \cdot y := x^{\mathrm{tr}} y$, and the i-th unit vector in $\mathbb{R}^N$ is denoted by e^i.

Remark 2.1. We assume $S \geq 0$ is a semimartingale so that we can use general integrands for S. As in Delbaen/Schachermayer [10] or Kardaras/Platen [33], one could start with an adapted RCLL process S and impose an AOA property only with respect to *elementary* (piecewise constant) strategies. For the AOA concept we introduce below, this implies that $S/V(\vartheta)$ is a semimartingale for any self-financing elementary strategy ϑ whose wealth $V(\vartheta)$ and $V_-(\vartheta)$ are strictly positive. In particular, if $S = (1, X)$, then $X \geq 0$ must be a semimartingale. For precise formulations and results, we refer to Bálint/Schweizer [6].

Many of our results involve *discounting*, i.e. dividing prices by positive processes; this amounts to changing the unit of account, without any issues of tradability. Define $\mathcal{S}^m := \{\text{all } \mathbb{R}^m\text{-valued semimartingales}\}$ for any $m \in \mathbb{N}$ and set $\mathcal{S} := \mathcal{S}^1$, $\mathcal{S}_+ := \{D \in \mathcal{S} : D \geq 0\}$ and

$$\mathcal{S}_{++} := \{D \in \mathcal{S} : D > 0,\ D_- > 0\}.$$

Elements $D \in \mathcal{S}_{++}$ are called *discounters*, and we note that $1/D \in \mathcal{S}_{++}$ if $D \in \mathcal{S}_{++}$. Sometimes, we also need discounters from

$$\mathcal{S}_{++}^{\mathrm{unif}} := \Big\{ D \in \mathcal{S}_{++} : 0 < \inf_{t \geq 0} D_t \leq \sup_{t \geq 0} D_t < \infty \ P\text{-a.s.} \Big\}.$$

For $D \in \mathcal{S}_{++}$, we call S/D the *D-discounted prices*. The difference between discounters and deflators is discussed in Remark 2.11 below.

The following simple result will help later to provide more insight and intuition. We thank Kostas Kardaras for its statement and idea of proof.

Lemma 2.2. *Any discounter $D \in \mathcal{S}_{++}$ can be written as $D = N/L$, where $L \in \mathcal{S}_{++}$ is a local martingale > 0 with $L_0 = 1$ and $N \in \mathcal{S}_{++}$ is of finite variation with $N_0 = D_0$.*

Proof. We start with $D' := 1/D$ which is in $\mathcal{S}_{++}$ like D and set

$$C_t := \prod_{\substack{0 < s \leq t, \\ D'_s / D'_{s-} > 2}} \frac{D'_s}{D'_{s-}}, \qquad t \geq 0.$$

Then C is well defined with $C_0 = 1$, RCLL and increasing, hence of finite variation, because D' as an RCLL process has only finitely many large jumps on each compact interval. The ratio $Y := D'/C$ is then a semimartingale in $\mathcal{S}_{++}$ and $\frac{Y_s}{Y_{s-}} = \frac{D'_s/D'_{s-}}{C_s/C_{s-}} \leq 2$ as the denominator equals the numerator for all s where the latter is > 2, and is always ≥ 1 as C is increasing. So we get $Y \leq 2Y_-$, hence $\Delta Y \leq Y_-$, and also $\Delta Y = Y - Y_- \geq -Y_-$ because $Y \geq 0$. In consequence, $|\Delta Y| \leq Y_-$ is locally bounded, which implies that Y is a special semimartingale. But as Y is also in $\mathcal{S}_{++}$, it is well known that Y has a unique predictable multiplicative decomposition as $Y = LA$ with a local martingale $L > 0$ and a predictable process $A > 0$ of finite variation with $A_0 = Y_0$; see Jacod [19, Théorème (6.19)]. As $L_- > 0$ by the maximum principle for supermartingales, we also have $A_- > 0$ so that both L and A are in $\mathcal{S}_{++}$. Setting $N := \frac{1}{AC}$ completes the proof. □

Note that while the predictable multiplicative decomposition from [19] is unique, the optional multiplicative decomposition in Lemma 2.2 is not.

Self-financing strategies are integrands $\vartheta \in L(S)$ satisfying the self-financing condition

$$V(\vartheta) := \vartheta \cdot S = \vartheta_0 \cdot S_0 + \vartheta \bullet S.$$

We write $\vartheta \in \Theta^{\mathrm{sf}}$ and call $V(\vartheta)$ the *value process* of ϑ; this is in the same units as S because ϑ is in numbers of shares. For D-discounted prices $\tilde{S} = S/D$, we analogously have $V(\vartheta, \tilde{S}) := \vartheta \cdot \tilde{S} = V(\vartheta)/D$, the value process of ϑ in the units of $\tilde{S}$. Due to [17, Lemma 2.9], $\vartheta \in \Theta^{\mathrm{sf}}$ implies that both $\vartheta \in L(\tilde{S})$ and $V(\vartheta, \tilde{S}) = \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$

hold. Thus Θ^{sf} does not depend on units even if value processes do. We also need the spaces $\Theta^{\mathrm{sf}}_{+} := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \geq 0\} = \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_{+}\}$ and

$$\Theta^{\mathrm{sf}}_{++} := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_{++}\};$$

they do not depend on units either. For any $\eta \in \Theta^{\mathrm{sf}}_{++}$, the *$\eta$-discounted prices*

$$S^{\eta} := \frac{S}{V(\eta)} = \frac{S}{\eta \cdot S}$$

play an important role in the sequel. Note that we have $V(\eta, S^{\eta}) = \eta \cdot S^{\eta} \equiv 1$ and $(S^{\eta})^{\eta'} = S^{\eta'}$. Finally, a process Y is called *S-tradable* if it is the value process of some self-financing strategy, i.e., $Y = V(\vartheta)$ for some $\vartheta \in \Theta^{\mathrm{sf}}$.

The next concept is a crucial element in our approach.

Definition 2.3. A *reference strategy* is an $\eta \in \Theta^{\mathrm{sf}}_{++}$ with $\eta \geq 0$ (η is long-only).

A reference strategy is interpreted as a *desirable investment*; indeed, given that values are in terms of some perishable consumption good, $V(\eta) \in \mathcal{S}_{++}$ means that η keeps us forever from complete starvation. As η is expressed in numbers of shares, it does not depend in any way on the chosen unit of account — it is discounting-invariant.

In the sequel, *we usually assume that there exists a reference strategy η*, and some results impose the extra condition that η is bounded (uniformly in (ω, t)).

Remark 2.4. 1) The existence of a reference strategy η is a very weak condition on the process S. Indeed, let us consider the *market portfolio*, i.e. the strategy $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^N$ of holding one share of each asset. If we have nonnegative prices $S \geq 0$, then $\mathbb{1}$ is in Θ^{sf}_{+} and bounded, and all components of the $\mathbb{1}$-discounted price process $S^{\mathbb{1}} = S / \sum_{i=1}^{N} S^i$ have values between 0 and 1. Some authors call $S^{\mathbb{1}}$ the process of *relative market capitalisations*. (To be accurate, these terminologies are only appropriate if S^i describes not the share price of company i, but rather its market capitalisation.) If $S \geq 0$ and the sum $\sum_{i=1}^{N} S^i$ of all prices is strictly positive and has strictly positive left limits, we even have $\mathbb{1} \in \Theta^{\mathrm{sf}}_{++}$ so that the market portfolio is then a reference strategy. However, it is useful to work with a general reference strategy η because this gives a clearer view on a number of aspects.

2) A reference strategy is by definition long-only, which looks natural from an economic perspective. Mathematically, $\eta \geq 0$ is used in part 1) of the key Theorem 3.10 and therefore appears indirectly in many results throughout the paper.

Definition 2.5. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}$ is called an *η-buy-and-hold strategy* if it is of the form $\vartheta = c\eta$ (componentwise product) for some $c \in \mathbb{R}^N$.

A strategy ϑ is η-buy-and-hold if and only if it is a coordinatewise nonrandom multiple of η. If $\eta \equiv \mathbb{1}$ is the market portfolio and $c \geq 0$, this reduces to the classic concept of buying and holding a fixed number of shares of each asset; so the above concept is a natural generalisation. Note that η itself is always an η-buy-and-hold strategy.

To have simple notations and include all possible time horizons, we start from a stopping time ζ (with values in $[0, +\infty]$ as usual) and a model on the stochastic interval

$$[\![0, \zeta]\!] = \{(\omega, t) \in \Omega \times [0, \infty) : 0 \leq t \leq \zeta(\omega)\}.$$

Note that a stochastic interval is always a subset of $\Omega \times [0,\infty)$, even if one of its boundary points takes the value $+\infty$. Special cases are $\zeta \equiv T \in (0,\infty)$ or $\zeta \equiv \infty$, which yield models indexed by $[0,T]$ or $[0,\infty)$, respectively. We next extend (almost) all stochastic processes to $[\![0,\infty]\!]$ by keeping them constant on $[\![\zeta,\infty]\!]$. (There is one exception: to concatenate two strategies $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ at some stopping time τ, we sometimes define, for a mapping F, a new strategy of the form $I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}F(\vartheta^1,\vartheta^2)$. On the set $\{\tau = \zeta < \infty\}$, this is then constant for $t > \zeta(\omega)$, but maybe not for $t \geq \zeta(\omega)$.) In this way, we can and do assume that all processes are defined on $[\![0,\infty]\!] = \Omega \times [0,\infty) = [\![0,\infty[\![$.

With the above convention, we have

$$\inf_{t\geq 0} Y_t(\omega) = \begin{cases} \inf_{0\leq t<\infty} Y_t(\omega) & \text{for } \zeta(\omega) = \infty, \\ \inf_{0\leq t\leq \zeta(\omega)} Y_t(\omega) & \text{for } \zeta(\omega) < \infty, \end{cases}$$

$$\liminf_{t\to\infty} Y_t(\omega) = \begin{cases} \liminf_{t\to\infty} Y_t(\omega) & \text{for } \zeta(\omega) = \infty, \\ Y_\zeta(\omega), & \text{for } \zeta(\omega) < \infty, \end{cases}$$

etc. Of course, if we write $\lim_{t\to\infty} Y_t$, we must make sure that this limit exists on $\{\zeta = \infty\}$. For a model on $[0,T]$, inf and sup are then always over $0 \leq t \leq T$, and every lim, lim inf or lim sup is simply the value at T. (If we want a right-open interval $[0,T)$ with $T \in (0,\infty)$, we can map this bijectively to $[0,\infty)$ with a deterministic time-change.)

The next concept is fundamental for our paper.

Definition 2.6. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_+$ is called *share maximal (sm) for* η if there is no $[0,1]$-valued adapted process $\psi = (\psi_t)_{t\geq 0}$ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L^\infty_+ \setminus \{0\}$ and such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon$ and

$$\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\eta_t) \geq 0 \qquad P\text{-a.s.}$$

We mostly use this concept when η is a reference strategy. Then η is desirable, as explained after Definition 2.3, and $\psi\eta$ is a dynamic long-only portfolio whose factor ψ stabilises over time and which asymptotically achieves a significant part of η. Share maximality says that even with a little extra initial capital $\varepsilon > 0$, one cannot asymptotically improve ϑ via some $\hat{\vartheta}^\varepsilon$ in such a significant manner.

We also need the following concept inspired by Herdegen [17]; the difference to [17] is that we work here on a possibly right-open time interval. Note that we replace "strongly maximal" from [17] by the more explicit terminology "value maximal".

Definition 2.7. Fix an $\mathbb{R}^N$-valued semimartingale Y. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_+$ is called *value maximal (vm) for* Y if there is no $f \in L^0_+ \setminus \{0\}$ such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon, Y) \leq V_0(\vartheta, Y) + \varepsilon$ and

$$\liminf_{t\to\infty}\big(V_t(\hat{\vartheta}^\varepsilon, Y) - V_t(\vartheta, Y) - f\big) \geq 0 \qquad P\text{-a.s.}$$

Maximality of a strategy ϑ always means that ϑ cannot be improved. The key difference between Definitions 2.6 and 2.7 lies in how improvements are measured. For value maximality, the comparison is in terms of *value*, which makes the concept depend on the unit of account (for the chosen Y). In contrast, share maximality looks (via the reference strategy η) at *numbers of shares*, and this is independent of any unit for prices.

Given a maximality concept for strategies, we define viability and efficiency as in [17].

Definition 2.8. Fix $\eta \in \Theta^{\mathrm{sf}}$. We say that S satisfies *dynamic share viability (DSV) for* η if the zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}$ is share maximal for η, and *dynamic share efficiency (DSE) for* η if every η-buy-and-hold strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is share maximal for η.

It is a **key observation** that for fixed η, *share maximality for* η, *dynamic share viability for* η *and dynamic share efficiency for* η *are like* Θ^{sf} *all* discounting-invariant *with respect to* $\mathcal{S}_{++}$, in the sense that if one of these properties holds for S, it also holds for any D-discounted $\tilde{S} = S/D$ with *any discounter* $D \in \mathcal{S}_{++}$, and vice versa. In contrast, the strong (value) maximality for S from Herdegen [17] (and derived concepts like NINA there) is invariant under discounting by discounters $D \in \mathcal{S}^{\mathrm{unif}}_{++} \subsetneq \mathcal{S}_{++}$ (see Lemma 3.1 below), but not under discounting by general $D \in \mathcal{S}_{++}$ (see Example 3.2 below). In that sense, the value-related concepts and results from [17] are only numéraire- or discounting-invariant in a restricted manner. But for a general discounting-invariant framework, having invariance with respect to the full class $\mathcal{S}_{++}$ is crucial because the natural class of discounters on a right-open interval like $[0, \infty)$ is $\mathcal{S}_{++}$ and not only $\mathcal{S}^{\mathrm{unif}}_{++}$; see Example 1.1.

Remark 2.9. **1)** Theorems 2.14 and 4.1 below give equivalent characterisations for DSE for η, assuming among other things that η is a reference strategy and bounded (uniformly in (ω, t)). These results show that equivalent definitions of DSE for η are possible — one could as well stipulate that only η itself, or all bounded $\vartheta \in \Theta^{\mathrm{sf}}_{+}$, should be share maximal for η. We have opted for an intermediate definition to preserve the analogy to [17].

2) All our concepts depend on the choice of η. We discuss this in Section 4.2 and show there in particular that the dependence is fairly weak.

3) The idea of treating as central objects not value processes, but strategies/portfolios in numbers of shares has already been promoted by Yu. Kabanov in his geometric approach to markets with transaction costs; see the textbook by Kabanov/Safarian [22, in particular Section 3.1]. But as also stated in [22, Section 3.6.1], models with transaction costs are much less demanding in terms of stochastic calculus because strategies there are processes of finite variation. We cannot impose this in our frictionless market, and so the tools and techniques developed by Kabanov and co-authors cannot be used in our setup.

The preceding concepts are all about strategies and hence on the primal side. For a dual characterisation in terms of martingale properties, we need the following concept.

Definition 2.10. An $\mathcal{E}$*-discounter for an* $\mathbb{R}^N$*-valued semimartingale* Y, where $\mathcal{E} \in \{\sigma\text{-martingale, local martingale, martingale, UI martingale}\}$, is a $D \in \mathcal{S}_{++}$ such that Y/D is an $\mathcal{E}$.

Remark 2.11. **1)** In the literature, an $\mathcal{E}$*-deflator* for a class $\mathcal{Y}$ of processes is a strictly positive local martingale Z (often with $Z_0 = 1$) such that the product ZY is an $\mathcal{E}$ for all $Y \in \mathcal{Y}$. There are two differences to the notion of an $\mathcal{E}$-discounter. First, a deflator acts by multiplication (on $\mathcal{Y}$) while a discounter acts by division (on Y). More importantly, however, we impose no (local) martingale property on an $\mathcal{E}$-discounter D, nor on $1/D$. (Some definitions of an $\mathcal{E}$-deflator Z do not explicitly ask for Z to be a local martingale. But as $\mathcal{Y}$ invariably contains the process $Y \equiv 1$, this property follows from the definition and $Z > 0$.) An analogous comment applies to supermartingale deflators. In our setup, neither S nor the family $\{V(\vartheta) : \vartheta \in \Theta^{\mathrm{sf}}_{++}\}$ of value processes contains a constant process in general; so discounters are more natural and more general than deflators. In fact, using Lemma 2.2 to write $D = N/L$ shows that an $\mathcal{E}$-discounter for a process Y can be viewed as the combination of a discounter N of finite variation and an $\mathcal{E}$-deflator L for the N-discounted process Y/N.

2) The reciprocal of what we call a local martingale discounter is usually called a *stochastic discount factor (process)* in the financial economics literature; see for instance Back [4, Section 13.3].

2.2. **Main results: Two new versions of an FTAP.** After the preceding preliminaries, we can now state our first two main results.

Theorem 2.12. *Suppose $S \geq 0$ and there exists a reference strategy η. Then the following are equivalent:*
(a) *S satisfies dynamic share viability for η.*
(b) *There exists a σ-martingale discounter D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

Remark 2.13. As pointed out in the proof in Section 3.5, the implication (a) $\Rightarrow$ (b) in Theorem 2.12 does not need $S \geq 0$. The same applies to Theorem 2.14.

Theorem 2.14. *Suppose $S \geq 0$ and there exists a reference strategy η such that η and $S^\eta = S/(\eta \cdot S)$ are even bounded (uniformly in (ω, t)). Then the following are equivalent:*
(a) *S satisfies dynamic share efficiency for η.*
(b) *There exists a UI martingale discounter D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

The proofs of Theorems 2.12 and 2.14 need extra ideas and additional results. These are developed in Section 3 and used in Section 3.5 to prove the theorems.

Both Theorems 2.12 and 2.14 are modern formulations of the classic idea due to Samuelson [36] that "properly anticipated prices fluctuate randomly" or, in other words, suitably discounted prices form a martingale. The notion of "properly anticipated" or "suitably discounted" is in our paper captured by the *existence* of a process D which turns S via discounting to S/D into a "martingale". The *strength* of the martingale property of S/D (σ-martingale or UI martingale) depends on the strength of the initial no-arbitrage condition (viability or efficiency). In key contrast to the classic FTAP formulation of Delbaen/Schachermayer [10, 13], the discounting process *cannot be chosen a priori*, but is an endogenous part of the dual characterisation of absence of arbitrage. This idea already appears in Herdegen [17] (see also [18]) where the dual objects are *pairs* consisting of an S-tradable numéraire and an equivalent σ-martingale measure. Our σ-martingale discounter combines these compactly into a single process.

In Section 4, we give an extended result which contains both Theorems 2.12 and 2.14 as well as other equivalent properties. While this involves a minimal amount of repetition, it serves both to highlight the main contribution of the present paper and to explain its connection to the existing literature. Proving Theorems 2.12 and 2.14 involves the bulk of the work; the extra equivalences in Theorem 4.1 follow easily from known results.

The proofs of our main results involve several ideas and steps. We give here a short overview and implement this in Section 3. First, because share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$, we can work with any discounted price process S/D instead of the original S. We choose an S-tradable $D := V(\xi) = \xi \cdot S$ and show in Theorem 3.10 that if $\xi \geq 0$, then share maximality for ξ is equivalent to value maximality for $S^{\xi} = S/D$. This almost gives us access to the results from Herdegen [17] who derived dual characterisations for his strong (value) maximality, of 0 or of a fixed strategy, in terms of certain martingale properties for suitably discounted prices. (It is at this point that the endogenous discounter appears.) However, [17] crucially exploits that prices are defined on a *right-closed* time interval, and the numéraire-invariance in [17] is only with respect to the smaller, restrictive class $\mathcal{S}_{++}^{\mathrm{unif}}$ of discounters. Overcoming this needs an extra step. With a similar argument as in Delbaen/Schachermayer [10], we show that for tradably discounted prices S^{ξ} and under value maximality for S^{ξ} of 0, the value process $V(\vartheta, S^{\xi})$ of any self-financing strategy $\vartheta \in \Theta_{+}^{\mathrm{sf}}$ converges as $t \to \infty$. Therefore, all the $V(\vartheta, S^{\xi})$ are well defined on a right-closed interval (even if S or S^{ξ} is not), and this finally allows us to use the results from [17]. Combining everything yields our assertions.

3. **Theory and proofs.** This section is the mathematical core of the paper. Its first three subsections mirror the ideas and steps in the discussion at the end of Section 2. To have a clearer structure, we proceed in the reverse order than the above discussion. In addition, we need some classic concepts from the literature. The last subsection proves the two main results.

3.1. **Classic concepts.** To relate our work to the literature and present some important known results, we recall or rewrite some notions from the classic Delbaen/Schachermayer [10, 13] approach. For any $\mathbb{R}^k$-valued semimartingale Y and $a \geq 0$, we define

$$\begin{aligned} \Theta^{\mathrm{sf}}(Y) &:= \{\vartheta \in L(Y) : V(\vartheta, Y) := \vartheta \cdot Y = \vartheta_0 \cdot Y_0 + \vartheta \bullet Y\}, \\ \Theta_{+}^{\mathrm{sf}}(Y) &:= \{\vartheta \in \Theta^{\mathrm{sf}}(Y) : V(\vartheta, Y) \geq 0\}, \\ L_{\mathrm{adm}}^{a}(Y) &:= \{H \in L(Y) : H \bullet Y \geq -a\} \end{aligned}$$

as well as

$$\begin{aligned} \mathcal{G}_{\mathrm{adm}}^{a}(Y) := \Big\{ \lim_{t\to\infty} V_t(\vartheta, Y) - V_0(\vartheta, Y) : \vartheta \in \Theta_{+}^{\mathrm{sf}}(Y),\ V_0(\vartheta, Y) = a, \\ \lim_{t\to\infty} V_t(\vartheta, Y) \text{ exists} \Big\}. \end{aligned}$$

Each $g \in \mathcal{G}_{\mathrm{adm}}^{a}(Y)$ is the net outcome (final minus initial value) of a self-financing strategy ϑ (investing in Y) whose value is always $\geq -a$, with all quantities in the same units of account as Y. We then introduce the sets

$$\begin{aligned} \mathcal{G}_{\mathrm{adm}}(Y) &:= \bigcup_{a\geq 0} \mathcal{G}_{\mathrm{adm}}^{a}(Y) \\ &= \Big\{ \lim_{t\to\infty} V_t(\vartheta, Y) - V_0(\vartheta, Y) : \vartheta \in \Theta_{+}^{\mathrm{sf}}(Y),\ \lim_{t\to\infty} V_t(\vartheta, Y) \text{ exists} \Big\}, \\ \mathcal{C}_{\mathrm{adm}}(Y) &:= \mathcal{G}_{\mathrm{adm}}(Y) - L_{+}^{0}, \\ \overline{\mathcal{C}}_{\mathrm{adm}}^{\infty}(Y) &:= \overline{\mathcal{C}_{\mathrm{adm}}(Y) \cap L^{\infty}}^{\infty}, \end{aligned}$$

where the bar $^{-\infty}$ on the right-hand side denotes the norm closure in L^∞. Then we say that

- $\mathrm{NA}_\infty(Y)$ holds if $\mathcal{C}_{\mathrm{adm}}(Y) \cap L^\infty_+ = \{0\}$;
- $\mathrm{NUPBR}_\infty(Y)$ holds if $\mathcal{G}^1_{\mathrm{adm}}(Y)$ is bounded in L^0;
- $\mathrm{NFLVR}_\infty(Y)$ holds if $\overline{\mathcal{C}}^\infty_{\mathrm{adm}}(Y) \cap L^\infty_+ = \{0\}$.

It is very important to realise that whether or not one has in the basic model a riskless asset with a price of 1 makes a big difference. To see this, note first of all that $\Theta^{\mathrm{sf}}(Y) \subsetneq L(Y)$ in general; for instance, if $k = 1$ and $Y > 0$ is continuous, solving for $\vartheta = (\vartheta_0 Y_0 + \vartheta \bullet Y)/Y$ shows that any $\vartheta \in \Theta^{\mathrm{sf}}(Y)$ must be a continuous semimartingale. Next, any $\psi \in L(1, Y)$ has the form $\psi = (\psi^0, \vartheta)$, where ψ^0 is any real-valued predictable process and $\vartheta \in L(Y)$. Then clearly $\psi \bullet (1, Y) = \vartheta \bullet Y$, but $V(\psi, (1, Y)) = \psi \cdot (1, Y) = \psi^0 + \vartheta \cdot Y \neq \vartheta \cdot Y = V(\vartheta, Y)$ unless $\psi^0 \equiv 0$. Finally, choosing $\psi^0 := c + \vartheta_0 \cdot Y_0 + \vartheta \bullet Y - \vartheta \cdot Y$ implies that $\psi^0_0 = c$ and

$$V\big(\psi, (1, Y)\big) - V_0\big(\psi, (1, Y)\big) = \psi^0 + \vartheta \cdot Y - \psi^0_0 - \vartheta_0 \cdot Y_0 = \vartheta \bullet Y = \psi \bullet (1, Y). \tag{3.1}$$

Thus any pair $(c, \vartheta) \in L^0(\mathcal{F}_0) \times L(Y)$ can be identified with some $\psi \in \Theta^{\mathrm{sf}}(1, Y)$, and vice versa, with $c = \psi^0_0$; see Herdegen [17, Theorem 2.14] for a more general result. But as seen above, not any $\vartheta \in L(Y)$ is in $\Theta^{\mathrm{sf}}(Y)$.

Similar issues come up for absence-of-arbitrage considerations. Using the definition of $V(\vartheta, Y)$, then $\vartheta \bullet Y = \psi \bullet (1, Y)$ from above and finally (3.1), we can rewrite $\mathcal{G}^a_{\mathrm{adm}}(Y)$ as

$$\begin{aligned}
\mathcal{G}^a_{\mathrm{adm}}(Y) &= \Big\{ \lim_{t\to\infty} \vartheta \bullet Y_t : \vartheta \in \Theta^{\mathrm{sf}}(Y),\ \vartheta \bullet Y \geq -a,\ \lim_{t\to\infty} \vartheta \bullet Y_t \text{ exists} \Big\} \\
&\subsetneq \Big\{ \lim_{t\to\infty} \vartheta \bullet Y_t : \vartheta \in L(Y),\ \vartheta \bullet Y \geq -a,\ \lim_{t\to\infty} \vartheta \bullet Y_t \text{ exists} \Big\} \\
&= \Big\{ \lim_{t\to\infty} \psi \bullet (1, Y)_t : \psi \in L(1, Y),\ \psi \bullet (1, Y) \geq -a,\ \lim_{t\to\infty} \psi \bullet (1, Y)_t \text{ exists} \Big\} \\
&= \mathcal{G}^a_{\mathrm{adm}}(1, Y).
\end{aligned} \tag{3.2}$$

In view of the second line in (3.2), $\mathcal{G}_{\mathrm{adm}}(1, Y) = \bigcup_{a \geq 0} \mathcal{G}^a_{\mathrm{adm}}(1, Y)$ is precisely the set K_0 (or K) considered in Delbaen/Schachermayer [10, 13], and so

$$\mathrm{NX}_\infty(1, Y) = \text{classic NX for } Y, \qquad \text{with } \mathrm{X} \in \{\mathrm{A}, \mathrm{FLVR}, \mathrm{UPBR}\}.$$

We remark that the property $\mathrm{NUPBR}_\infty(1, Y)$ already appears without a name in [10, Corollary 3.4]; it was later called BK by Kabanov [23] and (classic) NUPBR (for Y) by Karatzas/Kardaras [27].

Classic NUPBR for Y means by definition, see [27], that $\mathcal{G}^1_{\mathrm{adm}}(1, Y)$ is bounded in L^0, whereas our notion $\mathrm{NUPBR}_\infty(Y)$ only imposes that the smaller set $\mathcal{G}^1_{\mathrm{adm}}(Y)$ is bounded in L^0. So if we have $\mathrm{NUPBR}_\infty(Y)$, we cannot apply in general the results from [27] because these rely on the stronger assumption $\mathrm{NUPBR}_\infty(1, Y)$. An analogous comment applies to classic NFLVR for Y, $\mathrm{NFLVR}_\infty(Y)$ and the results from [10, 13]. So one must be careful if one wants to apply results from the classic theory of mathematical finance in our setting.

If Y has a special form, things simplify. Start with an $\mathbb{R}^N$-valued semimartingale S, fix $\xi \in \Theta^{\mathrm{sf}}_{++}(S)$ and recall the ξ-discounted prices $S^\xi = S/(\xi \cdot S)$. Because $V(\xi, S^\xi) \equiv 1$, one can use Herdegen [17, Theorem 2.14] (which easily extends to $[\![0, \infty]\!]$) to show that

$$\{\vartheta \bullet S^\xi : \vartheta \in \Theta^{\mathrm{sf}}(S^\xi)\} = \{\psi \bullet (1, S^\xi) : \psi \in \Theta^{\mathrm{sf}}(1, S^\xi)\} = \{\vartheta \bullet S^\xi : \vartheta \in L(S^\xi)\}.$$

In words, the wealth processes one can generate from self-financing strategies are the same for the markets with S^ξ and with $(1, S^\xi)$. In view of (3.2), this implies

$$\mathcal{G}^a_{\mathrm{adm}}(S^\xi) = \Big\{ \lim_{t\to\infty} \vartheta \bullet S^\xi_t : \vartheta \in L(S^\xi),\, \vartheta \bullet S^\xi \geq -a,\, \lim_{t\to\infty} \vartheta \bullet S^\xi_t \text{ exists} \Big\}$$
$$= \mathcal{G}^a_{\mathrm{adm}}(1, S^\xi), \tag{3.3}$$

and in consequence that

$$\mathrm{NX}_\infty(S^\xi) = \text{classic NX for } S^\xi \qquad \text{for X} \in \{\mathrm{A}, \mathrm{FLVR}, \mathrm{UPBR}\}.$$

We can therefore use the classic theory and its results for *tradably* discounted prices S^ξ, but not for general prices Y.

3.2. **From a stochastic or right-open interval to a right-closed interval.** In this section, we use absence of arbitrage to pass from a model with a general time horizon (stochastic or not, finite or infinite) to a model effectively defined on $\Omega \times [0, \infty]$. This rests on a convergence result in the spirit of Delbaen/Schachermayer [10, Theorem 3.3] combined with ideas from Herdegen [17] to connect value maximality and NUPBR.

We begin with an auxiliary result.

Lemma 3.1. *Suppose $\vartheta \in \Theta^{\mathrm{sf}}_+$ is value maximal for S.*

1) *Value maximality is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$: If $D \in \mathcal{S}^{\mathrm{unif}}_{++}$, then ϑ is also value maximal for S/D. (The converse is clear as $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ implies $1/D \in \mathcal{S}^{\mathrm{unif}}_{++}$.)*

2) *Value maximal strategies form a cone: For any $\alpha \geq 0$, $\alpha\vartheta$ is also value maximal for S.*

Proof. See Appendix. □

The next example shows that $\mathcal{S}^{\mathrm{unif}}_{++}$ cannot be replaced by $\mathcal{S}_{++}$ in Lemma 3.1.

Example 3.2. *Value maximality is not discounting-invariant with respect to $\mathcal{S}_{++}$.* Consider the Black–Scholes model with $m = r = \sigma = 1$, so that $S^1_t = e^t$ and $S^2_t = e^{W_t + \frac{1}{2}t}$. Here, 0 is not value maximal for S because for any $\varepsilon > 0$, the strategy $\hat{\vartheta}^\varepsilon := \varepsilon \mathrm{e}^1 = (\varepsilon, 0)$ of buying and holding ε units of S^1 has $V_0(\hat{\vartheta}^\varepsilon) = \varepsilon$, but $\lim_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon) = +\infty$. But if we take $D := S^1 \in \mathcal{S}_{++} \setminus \mathcal{S}^{\mathrm{unif}}_{++}$, we obtain $S/D = S^{\mathrm{e}^1} = (1, e^{W_t - \frac{1}{2}t})$. This is a ($\sigma$-)martingale, and therefore 0 is value maximal for S^{e^1}; see Theorem 3.8 below (applied to $\xi \equiv \mathrm{e}^1$).

We first connect value maximality and NUPBR; this is similar to Herdegen [17, Proposition 3.24].

Proposition 3.3. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and recall the ξ-discounted price process given by $S^\xi = S/(\xi \cdot S)$. Then the following are equivalent:*

(a) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is value maximal for S^ξ.*

(b) *The set $\{\lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)$, H has bounded support on $[0, \infty)\}$ is bounded in L^0.*

(c) *The set $\{\liminf_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)\}$ is bounded in L^0.*

(d) *The set $\{\lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)$ and $\lim_{t\to\infty} H \bullet S^\xi_t$ exists$\}$ is bounded in L^0.*

(e) *$NUPBR_\infty(S^\xi)$ holds.*

Proof. (c) $\Rightarrow$ (d) $\Rightarrow$ (b) is clear; (b) $\Rightarrow$ (c) is from the proof of [10, Proposition 3.2]; and (d) $\Leftrightarrow$ (e) follows from the first equality in (3.3) and the definition of $\mathrm{NUPBR}_\infty(S^\xi)$.

We prove (c) $\Rightarrow$ (a) indirectly. If 0 is not vm for S^ξ, there are $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon = 1/n$ some $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^n, S^\xi) \leq 1/n$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^n, S^\xi) \geq f$ P-a.s. Then $\tilde{\vartheta}^n := n\hat{\vartheta}^n$ is in Θ^{sf}_+ with $V_0(\tilde{\vartheta}^n, S^\xi) \leq 1$, and $\tilde{\vartheta}^n$ is also in $L^1_{\mathrm{adm}}(S^\xi)$ because

$$0 \leq V(\tilde{\vartheta}^n, S^\xi) = V_0(\tilde{\vartheta}^n, S^\xi) + \tilde{\vartheta}^n \bullet S^\xi \leq 1 + \tilde{\vartheta}^n \bullet S^\xi.$$

Therefore, $\liminf_{t\to\infty} \tilde{\vartheta}^n \bullet S^\xi_t = \liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) - V_0(\tilde{\vartheta}^n, S^\xi) \geq nf - 1$ P-a.s. implies that (c) cannot hold as $f \in L^0_+ \setminus \{0\}$.

Finally, for (a) $\Rightarrow$ (b), suppose that (b) is not true. Then also the convex set

$$C := \Big\{ \lim_{t\to\infty} H \bullet S^\xi_t + 1 : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty) \Big\} \subseteq L^0_+$$

is not bounded in L^0. Lemma A.2 therefore yields a sequence $(H^n)_{n\in\mathbb{N}} \subseteq L^1_{\mathrm{adm}}(S^\xi)$, where each H^n has bounded support on $[0,\infty)$, and some $f \in L^0_+ \setminus \{0\}$ with $\lim_{t\to\infty} H^n \bullet S^\xi_t + 1 \geq nf$ P-a.s. for all $n \in \mathbb{N}$. Note that the limits exist because each H^n has bounded support. Consider the integrand $H^n \in L^1_{\mathrm{adm}}(S^\xi)$. By [17, Theorem 2.14] (and an easy extension to $[\![0,\infty]\!]$), there exists a corresponding $\vartheta^n \in \Theta^{\mathrm{sf}}_+$ with $V(\vartheta^n, S^\xi) - V_0(\vartheta^n, S^\xi) = H^n \bullet S^\xi$, where we can choose $V_0(\vartheta^n, S^\xi) = 1$. Defining $\tilde{\vartheta}^n := \vartheta^n/n \in \Theta^{\mathrm{sf}}_+$ yields

$$V(\tilde{\vartheta}^n, S^\xi) = V(\vartheta^n, S^\xi)/n = (H^n \bullet S^\xi + 1)/n,$$

hence $V_0(\tilde{\vartheta}^n, S^\xi) = 1/n$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) = \lim_{t\to\infty}(H^n \bullet S^\xi_t + 1)/n \geq f$ P-a.s. Thus 0 is not vm for S^ξ. $\square$

Our next result is of crucial importance. It is a variant of the key result in Delbaen/Schachermayer [10, Theorem 3.3] and shows that loosely speaking, *value processes expressed in good units of account converge under a weak no-arbitrage assumption.*

Proposition 3.4. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and suppose the zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is value maximal for S^ξ. Then for any $\vartheta \in \Theta^{\mathrm{sf}}_+$, $V_\infty(\vartheta, S^\xi) := \lim_{t\to\infty} V_t(\vartheta, S^\xi)$ exists and is finite, P-a.s.*

Proof. [1] Fix ξ as above and $H \in L^1_{\mathrm{adm}}(S^\xi)$. We first claim that $\lim_{t\to\infty} H \bullet S^\xi_t$ exists and is finite, P-a.s. This follows from upcrossing arguments as in Doob's martingale convergence theorem and is based on the proof of [10, Theorem 3.3]. Indeed, by Proposition 3.3, the value maximality for S^ξ of 0 implies that the set

$$\Big\{ \lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty) \Big\}$$

is bounded in L^0, so that the conclusion of [10, Proposition 3.1] holds (with S in [10] replaced by S^ξ here). A careful look at [10, Proposition 3.2 and Theorem 3.3] shows that all we need for the proofs of these two results is the conclusion of [10,

[1]As pointed out by an Associate Editor, one could also prove this result by using the FTAP from Karatzas/Kardaras [27]. We refrain from doing this because the argument does not become shorter and needs some extra explanations.

Proposition 3.1]. So we can repeat the proof of [10, Theorem 3.3] step by step [2] to obtain our auxiliary claim about the convergence of $H \bullet S^\xi$.

Now fix $\vartheta \in \Theta_+^{\mathrm{sf}}$, set $v_0 := V_0(\vartheta, S^\xi) \geq 0$ and define the strategy $\tilde{\vartheta} := \vartheta/(1+v_0)$. Then $\tilde{\vartheta}$ is in Θ_+^{sf} and $\tilde{\vartheta} \bullet S^\xi = V(\tilde{\vartheta}, S^\xi) - V_0(\tilde{\vartheta}, S^\xi) \geq -1$ as $V_0(\tilde{\vartheta}, S^\xi) = \frac{v_0}{1+v_0} \leq 1$, so that $\tilde{\vartheta}$ is in $L^1_{\mathrm{adm}}(S^\xi)$. By the first part,

$$V_t(\vartheta, S^\xi) = (1+v_0)V_t(\tilde{\vartheta}, S^\xi) = v_0 + (1+v_0)(\tilde{\vartheta}_t \bullet S^\xi_t)$$

therefore converges for $t \to \infty$ P-a.s. to a finite limit. □

Remark 3.5. Both Propositions 3.3 and 3.4 have ξ-discounted prices $S^\xi = S/(\xi \cdot S)$; so the discounter $\xi \cdot S = V(\xi)$ is S*-tradable.* One can ask if $V(\xi)$ could be replaced by an arbitrary $D \in \mathcal{S}_{++}$, and hence S^ξ by S/D, but this is not possible. Tradability of $V(\xi)$ is explicitly used in the proof of Proposition 3.3, (a) $\Rightarrow$ (b) when we use [17, Theorem 2.14]. For Proposition 3.4, we now give a counterexample with a nontradable D.

As in Proposition 3.4, suppose 0 is value maximal for S^ξ and define $D' \in \mathcal{S}_{++}^{\mathrm{unif}}$ by $D'_t = 2 + \sin t$. Then $D := D'V(\xi)$ is in $\mathcal{S}_{++}$ like $V(\xi)$, we have $S/D = S^\xi/D'$, and 0 is value maximal for S^ξ/D' by Lemma 3.1, 1), hence also for S/D. But for any $\vartheta \in \Theta_+^{\mathrm{sf}}$, we have $V(\vartheta, S/D) = V(\vartheta, S^\xi/D') = \frac{V(\vartheta, S^\xi)}{D'}$ which does not converge in general. For example, taking $\vartheta = \xi$ yields $V(\xi, S^\xi) \equiv 1$ so that $V(\vartheta, S/D) = 1/D'$.

The significance of Proposition 3.4 is that under its assumptions, the limit $V_\infty(\vartheta, S^\xi)$ exists P-a.s. for all $\vartheta \in \Theta_+^{\mathrm{sf}}$. Therefore $V(\vartheta, S^\xi)$ is well defined on the *right-closed* interval $[0, \infty]$, and as $V(\xi, S^\xi) \equiv 1$, the model S^ξ is on $[0, \infty]$ a numéraire market in the sense of Herdegen [17]. Hence *in the setting of Proposition 3.4, the situation is as if we had the market resulting from* S^ξ *defined up to* ∞, and so we can essentially use all results from [17] also for $[\![0, \infty]\!]$. More precisely, as long as we only use value processes of strategies in Θ_+^{sf}, we do not need S^ξ itself to be defined on $[0, \infty]$.

An important consequence is that the same weak AOA condition as above allows us to improve any self-financing strategy asymptotically by a value maximal strategy at no extra cost. This extends a result from [17, Theorem 4.1] to $[\![0, \infty]\!]$.

Lemma 3.6. *Fix* $\xi \in \Theta_{++}^{\mathrm{sf}}$ *and suppose the zero strategy* $0 \in \Theta_+^{\mathrm{sf}}$ *is value maximal for* S^ξ. *Then for any* $\vartheta \in \Theta_+^{\mathrm{sf}}$, *there exists a* $\hat{\vartheta} \in \Theta_+^{\mathrm{sf}}$ *which is value maximal for* S^ξ *and satisfies*

$$V_0(\hat{\vartheta}, S^\xi) = V_0(\vartheta, S^\xi) \qquad \text{and} \qquad \liminf_{t \to \infty} V_t(\hat{\vartheta} - \vartheta, S^\xi) \geq 0 \quad P\text{-a.s.}$$

Proof. Fix ξ as above. For any $\vartheta \in \Theta_+^{\mathrm{sf}}$, the limit $V_\infty(\vartheta, S^\xi)$ exists and is finite, P-a.s., by Proposition 3.4. In Definition 2.7 for S^ξ instead of S, we can thus replace the $\liminf$ by a limit, and so our value maximality for S^ξ is equivalent to the strong maximality of S^ξ on $[0, \infty]$ in the sense of [17]. In particular, having 0 vm for S^ξ is equivalent to having NINA on $[0, \infty]$ for S^ξ in the sense of [17]. Using [17, Theorem 4.1] on $[0, \infty]$ for S^ξ and rewriting $V_\infty(\hat{\vartheta}, S^\xi) \geq V_\infty(\vartheta, S^\xi)$ as $\liminf_{t \to \infty} V_t(\hat{\vartheta} - \vartheta, S^\xi) \geq 0$ then gives the result. □

[2] There are two minor unclear points or typos in the original proof in [10]. First, a set $A_2 \in \mathcal{F}_{t_2}$ such that $P[A_2 \Delta (B_1 \cap A)] > \alpha - \varepsilon_1 - \varepsilon_2$ is not a good approximation for $B_1 \cap A$; one should rather impose the requirement that $P[A_2 \Delta (B_1 \cap A)] < \varepsilon_2/2$. Second, it is not clear why $P[B_1 \cap A] > \alpha - \varepsilon_1$ should be true. However, it is clear that $P[B_1 \cap A] > \alpha - 2\varepsilon_1$, which is still sufficient to obtain the conclusion.

3.3. **Dual characterisation of value maximality.** In this section, we provide dual characterisations of value maximality for S^ξ, of the zero strategy 0 or of a given strategy ξ. This uses the results of Herdegen [17] and extends them to a general time horizon by exploiting Section 3.2.

Proposition 3.7. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. Then the following are equivalent:*

(a) *ξ is value maximal for S^ξ.*
(b) *Both $NA_\infty(S^\xi)$ and $NUPBR_\infty(S^\xi)$ hold.*
(c) *$NFLVR_\infty(S^\xi)$ holds.*

Proof. Both $\mathcal{C}_{\mathrm{adm}}(S^\xi)$ and $\mathcal{C}_{\mathrm{adm}}(S^\xi) \cap L^\infty$ are convex, and $\mathrm{NUPBR}_\infty(S^\xi)$ means that $\mathcal{G}^1_{\mathrm{adm}}(S^\xi)$ is bounded in L^0. Due to (3.3), (b) $\Leftrightarrow$ (c) can thus be proved like in Kabanov [23, Lemma 2.2].

Both (a) and (c) imply that $0 \in \Theta^{\mathrm{sf}}_+$ is vm for S^ξ; indeed, under (a), this follows by Lemma 3.1, 2), and under (c), we combine (c) $\Rightarrow$ (b) with Proposition 3.3. Proposition 3.4 and the subsequent discussion thus allow us to treat S^ξ as if it were defined on $[0,\infty]$, and then the proof of [17, Proposition 3.24, (c)], with T replaced by ∞, gives the result. $\square$

Recall that for $\mathcal{E} \in \{\sigma\text{-martingale, local martingale, martingale, UI martingale}\}$, an *$\mathcal{E}$-discounter* for an $\mathbb{R}^N$-valued semimartingale Y is a $D \in \mathcal{S}_{++}$ such that Y/D is an $\mathcal{E}$.

Theorem 3.8. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. Then the following are equivalent:*

(a) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is value maximal for S^ξ.*
(b) *There exists a strategy $\hat\vartheta \in \Theta^{\mathrm{sf}}_{++}$ which is value maximal for S^ξ and which has $V(\hat\vartheta, S^\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*
(c) *There exists a σ-martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S^ξ.*

Proof. (a) $\Rightarrow$ (b): By Lemma 3.6, we can find a $\hat\vartheta \in \Theta^{\mathrm{sf}}_+$ which is vm for S^ξ and satisfies $\liminf_{t\to\infty} V_t(\hat\vartheta - \xi, S^\xi) \ge 0$ P-a.s. Superadditivity of the lim inf plus $V(\xi, S^\xi) \equiv 1$ yields

$$\liminf_{t\to\infty} V_t(\hat\vartheta, S^\xi) \ge \liminf_{t\to\infty} V_t(\hat\vartheta - \xi, S^\xi) + \liminf_{t\to\infty} V_t(\xi, S^\xi) \ge 1 > 0 \qquad P\text{-a.s.}$$

But Proposition 3.4 and the subsequent discussion allow us to treat the market given by S^ξ as if it were defined up to ∞, and so $\inf_{t\ge 0} V_t(\hat\vartheta, S^\xi) > 0$ P-a.s. follows as in the proof of [17, Proposition 4.4], with T there replaced by ∞. On the other hand, $\limsup_{t\to\infty} V_t(\hat\vartheta, S^\xi) = \lim_{t\to\infty} V_t(\hat\vartheta, S^\xi) < \infty$ P-a.s. by Proposition 3.4, and because $V(\hat\vartheta, S^\xi) = V_0(\hat\vartheta, S^\xi) + \hat\vartheta \bullet S^\xi$ is RCLL, this implies $\sup_{t\ge 0} V_t(\hat\vartheta, S^\xi) < \infty$ P-a.s. Hence $V(\hat\vartheta, S^\xi)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$. We note for later use that $V(\hat\vartheta, S^\xi) = \hat\vartheta \cdot S^\xi = V(\hat\vartheta)/V(\xi)$.

(b) $\Rightarrow$ (c): Because $\hat\vartheta$ is vm for $S^\xi = S/(\xi \cdot S)$ and $V(\hat\vartheta, S^\xi) = (\hat\vartheta \cdot S)/(\xi \cdot S)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, $\hat\vartheta$ is by Lemma 3.1, 1) also vm for $S^\xi / V(\hat\vartheta, S^\xi) = S^{\hat\vartheta}$. Thus by Proposition 3.7, $\mathrm{NFLVR}_\infty(S^{\hat\vartheta})$ holds. Note that $V(\hat\vartheta, S^{\hat\vartheta}) \equiv 1$. By the discussion after [17, Definition 2.18] [3], we can apply [13, Theorem 1.1] to the price process $(1, X) := (V(\hat\vartheta, S^{\hat\vartheta}), S^{\hat\vartheta})$ of dimension $1 + N$, and so there exists a probability measure $Q \approx P$ (on $\mathcal{F} \supseteq \mathcal{F}_\infty$) such that $S^{\hat\vartheta}$ is a σ-martingale under Q. The density

[3] As has been pointed out to us by an Associate Editor, this amounts to saying that $\mathrm{NFLVR}_\infty(S^\xi) = \mathrm{NFLVR}_\infty(1, S^\xi)$ coincides with classic NFLVR for S^ξ, which allows to use [13, Theorem 1.1] for S^ξ. This again uses (3.3).

process Z of Q with respect to P is in $\mathcal{S}^{\mathrm{unif}}_{++}$ as it is a strictly positive P-martingale on the *right-closed* interval $[0,\infty]$. Thus also $D := V(\hat{\vartheta}, S^\xi)/Z$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, and $S^\xi/D = ZS^\xi/V(\hat{\vartheta}, S^\xi) = ZS^{\hat{\vartheta}}$ is a σ-martingale under P by the Bayes rule for stochastic calculus; see Kallsen [25, Proposition 5.1]. (In classic terminology, Z is a σ-martingale deflator for $S^{\hat{\vartheta}}$.)

(c) $\Rightarrow$ (a): Because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ and vm is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$ by Lemma 3.1, 1), we can equivalently prove vm of 0 for S^ξ or for S^ξ/D. Hence we can and do assume without loss of generality that S^ξ is a P-σ-martingale. If 0 is not vm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ such that $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \le \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi) \ge f$ P-a.s. Because we have $\hat{\vartheta}^\varepsilon \bullet S^\xi = V(\hat{\vartheta}^\varepsilon, S^\xi) - V_0(\hat{\vartheta}^\varepsilon, S^\xi) \ge -\varepsilon$ on $[0,\infty)$ P-a.s., the Ansel–Stricker lemma [3, Corollary 3.5] implies that $V(\hat{\vartheta}^\varepsilon, S^\xi)$ is a local P-martingale and a P-supermartingale. Combining this with Fatou's lemma and $f \in L^0_+ \setminus \{0\}$ yields

$$\varepsilon \ge V_0(\hat{\vartheta}^\varepsilon, S^\xi) \ge \liminf_{t\to\infty} E[V_t(\hat{\vartheta}^\varepsilon, S^\xi)] \ge E\Big[\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi)\Big] \ge E[f] > 0$$

for every $\varepsilon > 0$, which is a contradiction. □

Theorem 3.9. *Suppose that $\xi \in \Theta^{\mathrm{sf}}_{++}$ is such that both ξ and S^ξ are bounded (uniformly in (ω, t)). Then the following are equivalent:*

(a) *ξ is value maximal for S^ξ.*
(b) *There exists a UI martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S^ξ.*
(c) *Each bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is value maximal for S^ξ.*

Proof. (c) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (b): If ξ is vm for S^ξ, the same argument as in the proof of (b) $\Rightarrow$ (c) in Theorem 3.8 yields a $Q \approx P$ such that S^ξ is a σ-martingale under Q. Being uniformly bounded, S^ξ is even a UI martingale under Q, and so the same discounter $D := V(\xi, S^\xi)/Z = 1/Z$ as in the proof of Theorem 3.8 is now a UI martingale discounter for S^ξ and again in $\mathcal{S}^{\mathrm{unif}}_{++}$.

(b) $\Rightarrow$ (c): By Theorem 3.8, 0 is vm for S^ξ. Take any bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$. To show that ϑ is vm for S^ξ, as in the proof of (c) $\Rightarrow$ (a) in Theorem 3.8, we can assume that S^ξ is a UI martingale; so $S^\xi_\infty = \lim_{t\to\infty} S^\xi_t$ exists P-a.s. and in L^1, and then S^ξ is a martingale on $[0,\infty]$. Moreover, $V(\vartheta, S^\xi)$ is P-a.s. convergent as $t \to \infty$ by Proposition 3.4. For any stopping time τ, we have $|V_\tau(\vartheta, S^\xi)| \le \|\vartheta\|_\infty \sum_{i=1}^N |(S^\xi_\tau)^i|$, and the UI property of S^ξ on $[0,\infty]$ implies that $V(\vartheta, S^\xi)$ is of class (D). So $V(\vartheta, S^\xi)$ is even a UI martingale.

If ϑ is not vm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \le V_0(\vartheta, S^\xi) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S^\xi) \ge f$ P-a.s. As $\lim_{t\to\infty} V_t(\vartheta, S^\xi)$ exists, we even have $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi) \ge \lim_{t\to\infty} V_t(\vartheta, S^\xi) + f$ P-a.s., and $V(\hat{\vartheta}^\varepsilon, S^\xi)$ is a supermartingale by the same argument as for ϑ. Combining this with Fatou's lemma, the UI martingale property of $V(\vartheta, S^\xi)$ and $f \in L^0_+ \setminus \{0\}$ then gives a contradiction because

$$\begin{aligned} V_0(\vartheta, S^\xi) + \varepsilon \ge V_0(\hat{\vartheta}^\varepsilon, S^\xi) &\ge \liminf_{t\to\infty} E[V_t(\hat{\vartheta}^\varepsilon, S^\xi)] \\ &\ge E\Big[\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi)\Big] \ge E\Big[\lim_{t\to\infty} V_t(\vartheta, S^\xi)\Big] + E[f] \\ &= \lim_{t\to\infty} E[V_t(\vartheta, S^\xi)] + E[f] = V_0(\vartheta, S^\xi) + E[f] > V_0(\vartheta, S^\xi) \end{aligned}$$

for every $\varepsilon > 0$. □

Propositions 3.3 and 3.7 as well as Theorems 3.8 and 3.9 show a clear pattern: Thanks to the key result in Proposition 3.4, we can fairly easily extend the results from Herdegen [17] to a market with an infinite horizon, as long as we stick to ξ-discounted prices S^ξ. But what can be said if we want to start instead from the original prices S?

According to Lemma 3.1, 1), value maximality is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$, and $S^\xi = S/V(\xi)$. If we impose the extra condition that $V(\xi)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, all results still hold if we replace "value maximal for S^ξ" by "value maximal for S". Moreover, in Lemma 3.6, Theorem 3.8 and in (a)–(c) of Theorem 3.9, we can then also replace S^ξ by S.

In [17], the condition $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$ is automatically satisfied for any $\xi \in \Theta^{\mathrm{sf}}_{++}$ as the market there is defined on a *right-closed* time interval. In contrast, on a *right-open* interval like $[0,\infty)$ we consider here, the condition is very restrictive — just think of a non-discounted Black–Scholes model with an interest rate $r \neq 0$ and the market portfolio $\xi \equiv \mathbb{1}$. It is precisely the idea of replacing value maximality by share maximality which allows us to eliminate that restrictive condition and handle general models for S.

3.4. **Connecting share maximality and value maximality.** In this section, we show that under a very mild condition on the pair (S,ξ) of price process and strategy, share maximality for ξ and value maximality for S^ξ are equivalent. This is the key for proving our main results.

Theorem 3.10. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$.*

1) *If $\xi \geq 0$, any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is share maximal for ξ is value maximal for S^ξ.*

2) *If $S \geq 0$, any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is value maximal for S^ξ is also share maximal for ξ.*

The proof of Theorem 3.10 needs some preparation.

Lemma 3.11. *Suppose $S \geq 0$ and fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. If there is a strategy $\hat\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is value maximal for S^ξ, then $(\vartheta\cdot S)/(\xi\cdot S)$ is bounded in $t \geq 0$, P-a.s., for every $\vartheta \in \Theta^{\mathrm{sf}}_{+}$. In particular, S^ξ is bounded in $t \geq 0$, P-a.s.*

Proof. If $\hat\vartheta$ is vm for S^ξ, then 0 is vm for S^ξ by Lemma 3.1, 2). So Proposition 3.4 implies that for any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$, the process $(\vartheta\cdot S)/(\xi\cdot S) = \vartheta\cdot S^\xi = \vartheta_0\cdot S^\xi_0 + \vartheta \bullet S^\xi$ is P-a.s. convergent as $t \to \infty$ and hence bounded in $t \geq 0$, P-a.s. Choosing $\vartheta := \mathrm{e}^i$ for $i = 1,\dots,N$ gives the second assertion; note that $S \geq 0$ is used here to ensure that $\mathrm{e}^i \in \Theta^{\mathrm{sf}}_{+}$. □

In the proof of Theorem 3.10, we need to concatenate strategies which requires some notation. Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and a stopping time τ (as usual with values in $[0,\infty]$). The ξ-*concatenation* at time τ of $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ is defined by

$$\vartheta^1 \circledS^\xi_\tau \vartheta^2 := I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}\Big(I_\Gamma \vartheta^1 + I_{\Gamma^c}\big(\vartheta^2 + V_\tau(\vartheta^1-\vartheta^2, S^\xi)\xi\big)\Big)$$
$$\text{with } \Gamma := \{V_\tau(\vartheta^1) < V_\tau(\vartheta^2)\}. \tag{3.4}$$

The interpretation is clear: We start with ϑ^1 and follow this strategy until time τ where we compare its value to that of the competitor ϑ^2. If ϑ^1 is strictly cheaper, we stick to it. Otherwise, we liquidate ϑ^1_τ, start with ϑ^2 by buying ϑ^2_τ, and invest the nonnegative rest of the proceeds into ξ. Note that on $\{\tau = \infty\}$, we have

$\vartheta^1 \oslash^\xi_\tau \vartheta^2 = \vartheta^1$ so that the possibly undefined expressions ϑ^1_∞, ϑ^2_∞, S_∞ or S^ξ_∞ never appear.

Lemma 3.12. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and a stopping time τ. If ϑ^1, ϑ^2 are in Θ^{sf}, then so is $\vartheta^1 \oslash^\xi_\tau \vartheta^2$. If ϑ^1, ϑ^2 are in Θ^{sf}_+, then so is $\vartheta^1 \oslash^\xi_\tau \vartheta^2$.*

Proof. See Appendix. □

Proof of Theorem 3.10. 1) If ϑ is not vm for S^ξ, there are $f \in L^0_+ \setminus \{0\}$ and for any $\varepsilon = 1/n$ some $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $\hat{\vartheta}^n_0 \cdot S^\xi_0 = V_0(\hat{\vartheta}^n, S^\xi) \leq \vartheta_0 \cdot S^\xi_0 + 1/n$ and $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t) \geq f$ P-a.s. Choose $\delta > 0$ and $A \in \mathcal{F}$ with $P[A] > 0$ such that $f \geq 2\delta I_A$ P-a.s., and define

$$\begin{aligned}
\sigma'_n &:= \inf\{t \geq 0 : (\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t \geq \delta\},\\
\varphi_n &:= \inf\{t \geq 0 : P[\sigma'_n \leq t] \geq P[A](1 - 2^{-n+1})\},\\
\sigma_n &:= \sigma'_n \wedge \varphi_n \leq \varphi_n.
\end{aligned}$$

Then σ'_n is a stopping time, φ_n a bounded nonrandom time and σ_n a bounded stopping time. Moreover, $B_n := \{\sigma'_n \leq \varphi_n\} \in \mathcal{F}_{\varphi_n}$ satisfies $P[B_n] \geq P[A](1 - 2^{-n+1})$ and we have

$$(\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n}) \cdot S^\xi_{\sigma_n} = (\hat{\vartheta}^n_{\sigma'_n} - \vartheta_{\sigma'_n}) \cdot S^\xi_{\sigma'_n} \geq \delta \qquad \text{on } B_n,\ P\text{-a.s.} \tag{3.5}$$

by right-continuity. Due to $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t) \geq f \geq 0$ P-a.s.,

$$\tau_n := \inf\{t \geq \varphi_n : (\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t \geq -1/n\} \geq \varphi_n$$

is a P-a.s. finite-valued stopping time which satisfies $\tau_n \geq \sigma_n$.

We now consider the strategy

$$\tilde{\vartheta}^n := I_{[\![0,\tau_n]\!]}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta) + I_{]\!]\tau_n,\infty]\!]}\big(\vartheta + V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi)\xi\big) + \xi/n, \tag{3.6}$$

with $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta$ defined in (3.4). In words, we hold a $(1/n)$-multiple of ξ, switch at time σ_n from $\hat{\vartheta}^n$ to ϑ if the value of ϑ is at most the value of $\hat{\vartheta}^n$, and always switch to ϑ at time τ_n; in both cases, any difference in value is invested into ξ. Using $\xi \cdot S^\xi \equiv 1$, this gives

$$V_0(\tilde{\vartheta}^n, S^\xi) = \tilde{\vartheta}^n_0 \cdot S^\xi_0 = \hat{\vartheta}^n_0 \cdot S^\xi_0 + (\xi_0 \cdot S^\xi_0)/n \leq V_0(\vartheta, S^\xi) + 2/n.$$

Next, as $\hat{\vartheta}^n$ and ϑ are in Θ^{sf}_+, Lemma 3.12 yields $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta \in \Theta^{\mathrm{sf}}_+$, and therefore (3.6) gives $\tilde{\vartheta}^n \cdot S^\xi = V(\tilde{\vartheta}^n, S^\xi) \geq 0$ P-a.s. on $[\![0,\tau_n]\!]$. Using now $V(\xi, S^\xi) \equiv 1$ and the definition (3.4) allows us to compute, as in the proof of Lemma 3.12 in the Appendix, that

$$\begin{aligned}
V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) = I_{\{\tau_n=\sigma_n\}}&V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^\xi)\\
&+ I_{\{\tau_n>\sigma_n\}}\big(I_{\Gamma_n}V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^\xi)\\
&\qquad + I_{\Gamma_n^c}V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi)\big)
\end{aligned} \tag{3.7}$$

with $\Gamma_n := \{V_{\sigma_n}(\hat{\vartheta}^n) < V_{\sigma_n}(\vartheta)\}$. This shows that due to $\tau_n < \infty$ P-a.s., we always have

$$V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) \geq \min\big((\hat{\vartheta}^n_{\tau_n} - \vartheta_{\tau_n}) \cdot S^\xi_{\tau_n}, 0\big) \geq -1/n \qquad P\text{-a.s.} \tag{3.8}$$

Combining (3.6) and (3.8) and using $\xi \geq 0$ implies that on $]\!]\tau_n,\infty]\!]$, we have

$$\tilde{\vartheta}^n - \vartheta = V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi)\xi + \xi/n \geq 0, \tag{3.9}$$

hence $V(\tilde{\vartheta}^n, S^\xi) \geq V(\vartheta, S^\xi)$, and so $\tilde{\vartheta}^n$ is like ϑ in Θ^{sf}_+.

Now on the set B_n, we have $\sigma_n = \sigma'_n$, so $V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi) = (\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n}) \cdot S^\xi_{\sigma_n} \geq \delta$ P-a.s. as in (3.5) and therefore by (3.7) also

$$V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) = V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi) \geq \delta \qquad P\text{-a.s.}$$

Thus (3.9) and $\xi \geq 0$ yield $\tilde{\vartheta}^n - \vartheta \geq \delta\xi$ on B_n on $\rrbracket \tau_n, \infty \rrbracket$ and so, as $\tau_n < \infty$ P-a.s.,

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.} \tag{3.10}$$

Now define the $[0,1]$-valued adapted process $\psi^n = (\psi^n_t)_{t\geq 0}$ by $\psi^n_t := \delta E[I_{B_n}|\mathcal{F}_t]$. Then $\varphi_n < \infty$ and the fact that $B_n \in \mathcal{F}_{\varphi_n}$ yield $\psi^n_t = \delta I_{B_n}$ for $t \geq \varphi_n$ so that $\psi^n_\infty := \lim_{t\to\infty}\psi^n_t = \delta I_{B_n}$ P-a.s. Moreover, we also obtain via (3.10) that

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.}$$

Set $B := \bigcap_{n\in\mathbb{N}} B_n$ and $\psi_t := \delta E[I_B|\mathcal{F}_t]$ for $t \geq 0$. Then $\lim_{t\to\infty}\psi_t = \psi_\infty := \delta I_B$ P-a.s., and $B \subseteq B_n$ for all n implies $\psi \leq \psi^n$ for all n. Moreover, $\psi_\infty \in L^\infty_+ \setminus \{0\}$ because

$$\begin{aligned}P[B] \geq P[B\cap A] = P[A] - P\Bigg[A \cap \bigcup_{n\in\mathbb{N}} B^c_n\Bigg] &\geq P[A] - \sum_{n=1}^\infty P[A\cap B^c_n]\\ = P[A] - \sum_{n=1}^\infty (P[A] - P[A\cap B_n]) &\geq P[A]\Bigg(1 - \sum_{n=1}^\infty 2^{-n+1}\Bigg) = P[A]/2 > 0.\end{aligned}$$

So we have found ψ and for each $n \in \mathbb{N}$ a $\tilde{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $V_0(\tilde{\vartheta}^n, S^\xi) \leq V_0(\vartheta, S^\xi) + 2/n$ and

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) \geq 0 \qquad P\text{-a.s.,}$$

which contradicts the assumption that ϑ is sm for ξ.

2) If ϑ is not sm for ξ, there are a $[0,1]$-valued adapted $\psi = (\psi_t)_{t\geq 0}$ converging P-a.s. to $\psi_\infty := \lim_{t\to\infty}\psi_t \in L^\infty_+ \setminus \{0\}$ and for each $\varepsilon > 0$ a strategy $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon$, hence $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \leq V_0(\vartheta, S^\xi) + \varepsilon/V_0(\xi)$, and satisfying $\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t) \geq 0$ P-a.s. By Lemma 3.11, S^ξ is bounded in $t \geq 0$, P-a.s. Superadditivity of the lim inf, Lemma A.1, $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$ and $S^\xi \geq 0$ from $S \geq 0$ thus yield that P-a.s.,

$$\begin{aligned}\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S^\xi) &\geq \liminf_{t\to\infty}\big((\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t)\cdot S^\xi_t\big) + \liminf_{t\to\infty}\big((\psi_t\xi_t)\cdot S^\xi_t\big)\\ &\geq \Big(\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t)\Big)\cdot\Big(\liminf_{t\to\infty} S^\xi_t\Big) + \psi_\infty \geq \psi_\infty.\end{aligned}$$

So ϑ is not vm for S^ξ, and this completes the proof. $\square$

3.5. **Proofs of the main results.**

In this section, we prove the main results from Section 2.

Proof of Theorem 2.12. (a) $\Rightarrow$ (b): If S satisfies DSV for η, then $0 \in \Theta^{\mathrm{sf}}_+$ is sm for η and hence vm for S^η by Theorem 3.10, 1) for $\xi = \eta$. Theorem 3.8 for $\xi = \eta$ therefore yields a discounter $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a σ-martingale. Writing $S^\eta/D' = S/((\eta\cdot S)D')$ shows that $D := (\eta\cdot S)D' \in \mathcal{S}_{++}$ is a σ-martingale discounter for S. Moreover, $\eta\cdot(S/D) = 1/D'$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$ like D', and in particular, we have $\inf_{t\geq 0}(\eta_t\cdot(S_t/D_t)) > 0$ P-a.s. This does not need $S \geq 0$.

(b) $\Rightarrow$ (a): If D is a σ-martingale discounter for S, then $\tilde{S} := S/D$ is a σ-martingale. By [3, Corollary 3.5], $0 \leq V(\eta, \tilde{S}) = V_0(\eta, \tilde{S}) + \eta\bullet\tilde{S}$ is a P-supermartingale so that $\lim_{t\to\infty} V_t(\eta, \tilde{S})$ exists and is finite, P-a.s. (We cannot use Proposition 3.4 here because D need not be S-tradable; see Remark 3.5.) This yields $\sup_{t\geq 0}(\eta_t \cdot \tilde{S}_t) < \infty$ P-a.s., and as also $\inf_{t\geq 0}(\eta_t \cdot \tilde{S}_t) > 0$ P-a.s. by assumption, $V(\eta, \tilde{S}) = \eta \cdot (S/D)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$. Now $D' \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$ is a σ-martingale discounter for $\tilde{S}$, and so Theorem 3.8 applied to $\tilde{S}$ and $\xi = \eta$ implies that 0 is vm for $\tilde{S}^\eta$. By Theorem 3.10, 2) for $\tilde{S} \geq 0$ and $\xi = \eta$, 0 is then sm for η in the model $\tilde{S}$, and hence also in the model $S = \tilde{S}D$ because $D \in \mathcal{S}_{++}$ and share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$. So S satisfies DSV for η. □

Remark 3.13. The above proof shows the useful fact that for a σ-martingale discounter D for S, the properties $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$ and $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. are equivalent.

Proof of Theorem 2.14. This is very similar to the proof of Theorem 2.12, with the main difference that we use Theorem 3.9 instead of Theorem 3.8.

(a) ⇒ (b): If S satisfies DSE for η, every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ and in particular the reference strategy η is sm for η and hence vm for S^η by Theorem 3.10, 1). As η and S^η are bounded by assumption, Theorem 3.9 for $\xi = \eta$ yields the existence of some $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a UI martingale. As before, $D := (\eta \cdot S)D'$ is then a UI martingale discounter for S, and we also again get $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

(b) ⇒ (a): If $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ is a UI martingale deflator for S and we set $\tilde{S} := S/D$, we get $V(\eta, \tilde{S}) \in \mathcal{S}^{\mathrm{unif}}_{++}$ as before. Because η and S^η are bounded by assumption, Theorem 3.9 applied to $\tilde{S}$ and $\xi = \eta$ then yields that each bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is vm for $\tilde{S}^\eta$. But every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ is bounded like η itself, hence vm for $\tilde{S}^\eta$ and then sm for η as before. Thus S satisfies DSE for η. □

4. Additional results.

In this section, we first present a combined theorem which contains the main new results in Theorems 2.12 and 2.14 together with extra statements that connect our work to the literature. We then discuss to which extent our approach and results are robust towards the choice of a reference strategy, and finally present a number of results which clarify the relation of our AOA concepts to the classic theory and in the classic setup.

4.1. A more detailed version of our main results.

To connect our main results to existing concepts, we give the following combined and more detailed version of Theorems 2.12 and 2.14.

Theorem 4.1. *Suppose $S \geq 0$ and there exists a reference strategy η. Consider the following statements:*

- (e1) *S satisfies dynamic share efficiency for η.*
- (v1) *S satisfies dynamic share viability for η.*
- (e2) *Every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is value maximal for S^η.*
- (e2′) *The reference strategy η is bounded and value maximal for $S^\eta = S/(\eta \cdot S)$.*
- (v2) *The zero strategy 0 is value maximal for $S^\eta = S/(\eta \cdot S)$.*
- (e3) *There exists a UI martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*
- (v3) *There exists a σ-martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*
- (e4) *There exist a local martingale $L > 0$ with $L_0 = 1$ and a discounter N of finite variation such that the product $L(S/N)$ is a UI martingale and $\eta \cdot (LS/N)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$.*

$$
\begin{array}{ccc}
S \text{ satisfies DSE for } \eta & & S \text{ satisfies DSV for } \eta \\
(\text{every } \eta\text{-buy-and-hold } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is sm for } \eta) & & (0 \text{ is sm for } \eta) \\
\Updownarrow & & \Updownarrow \\
\text{every bounded } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is sm for } \eta & & 0 \in \Theta_+^{\mathrm{sf}} \text{ is sm for } \eta \\
\Updownarrow & & \Updownarrow \\
\eta \text{ is bounded and sm for } \eta & & 0 \text{ is sm for } \eta \\
\Updownarrow & & \Updownarrow \\
\eta \text{ is bounded and vm for } S^\eta & & 0 \text{ is vm for } S^\eta \\
\Updownarrow & & \Updownarrow \\
\text{every bounded } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is vm for } S^\eta & & \text{some } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is vm for } S^\eta \\
\Updownarrow & & \Updownarrow \\
\exists \text{ UI martingale discounter } D \text{ for } S & & \exists\ \sigma\text{-martingale discounter } D \text{ for } S \\
\text{with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}} & & \text{with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}} \\
\Updownarrow & \Longrightarrow & \Updownarrow \\
\exists \text{ FV discounter } N \text{ and} & & \exists \text{ FV discounter } N \text{ and} \\
\text{local martingale } L > 0 \text{ with} & & \text{local martingale } L > 0 \text{ with} \\
L(S/N) \text{ UI martingale and} & & L(S/N)\ \sigma\text{-martingale and} \\
L(\eta \cdot (S/N)) \in \mathcal{S}_{++}^{\mathrm{unif}} & & L(\eta \cdot (S/N)) \in \mathcal{S}_{++}^{\mathrm{unif}} \\
\Updownarrow & & \Updownarrow \\
\exists\ \vartheta \in \Theta_{++}^{\mathrm{sf}} \text{ and } Q \approx P \text{ with} & & \exists\ \vartheta \in \Theta_{++}^{\mathrm{sf}} \text{ and } Q \approx P \text{ with} \\
S/V(\vartheta) \text{ UI martingale and} & & S/V(\vartheta)\ \sigma\text{-martingale and} \\
V(\eta)/V(\vartheta) \in \mathcal{S}_{++}^{\mathrm{unif}} & & V(\eta)/V(\vartheta) \in \mathcal{S}_{++}^{\mathrm{unif}} \\
\Updownarrow & & \Updownarrow \\
S^\eta \text{ satisfies NFLVR}_\infty & & S^\eta \text{ satisfies NUPBR}_\infty \\
\Updownarrow & & \Updownarrow \\
S^\eta \text{ satisfies classic NFLVR} & & S^\eta \text{ satisfies classic NUPBR}
\end{array}
$$

FIGURE 1. Graphical summary of Theorem 4.1. Assumptions are $S \geq 0$ and that η is a reference strategy (which is assumed to exist). The shorthand sm stands for share maximal, vm stands for value maximal. The equivalences on the left side need in addition that η and S^η are bounded (uniformly in (ω, t)).

(v4) *There exist a local martingale $L > 0$ with $L_0 = 1$ and a discounter N of finite variation such that the product $L(S/N)$ is a σ-martingale and $\eta \cdot (LS/N)$ is in $\mathcal{S}_{++}^{\mathrm{unif}}$.*

(e5) *There exist a strategy $\vartheta \in \Theta_{++}^{\mathrm{sf}}$ and a probability measure $Q \approx P$ such that $S^\vartheta = S/V(\vartheta)$ is a UI martingale under Q and $V(\eta)/V(\vartheta) = (\eta \cdot S)/(\vartheta \cdot S)$ is in $\mathcal{S}_{++}^{\mathrm{unif}}$.*

(v5) *There exist a strategy $\vartheta \in \Theta_{++}^{\mathrm{sf}}$ and a probability measure $Q \approx P$ such that $S^\vartheta = S/V(\vartheta)$ is a σ-martingale under Q and $V(\eta)/V(\vartheta) = (\eta \cdot S)/(\vartheta \cdot S)$ is in $\mathcal{S}_{++}^{\mathrm{unif}}$.*

(e6) *$NFLVR_\infty(S^\eta)$ holds, i.e., $S^\eta = S/V(\eta)$ satisfies $NFLVR_\infty$.*

(v6) *$NUPBR_\infty(S^\eta)$ holds, i.e., $S^\eta = S/V(\eta)$ satisfies $NUPBR_\infty$.*

(e7) *$S^\eta = S/V(\eta)$ satisfies classic NFLVR.*

(v7) *$S^\eta = S/V(\eta)$ satisfies classic NUPBR.*

(e8) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is share maximal for η.*

(e8′) *The reference strategy* $\eta \in \Theta_+^{\mathrm{sf}}$ *is bounded and share maximal for* η.
(v8) *There exists some* $\vartheta \in \Theta_+^{\mathrm{sf}}$ *which is share maximal for* η.

Then we have (eK) $\Rightarrow$ (vK) *for* $\mathrm{K} = 1, \dots, 8$*, and the statements* (vK)*,* $\mathrm{K} = 1, \dots, 8$*, are equivalent among themselves. If in addition* η *and* S^η *are bounded (uniformly in* (ω, t)*), then also the statements* (eK)*,* $\mathrm{K} = 1, \dots, 8$*, are equivalent among themselves (including the prime* $'$ *versions).*

Figure 1 gives a graphical overview of this result.

Proof. First, (v1) $\Leftrightarrow$ (v2) is clear from the definition of DSV and Theorem 3.10 for $\xi = \eta$. Next, (v2) $\Leftrightarrow$ (v6) is (a) $\Leftrightarrow$ (e) from Proposition 3.3, and (v1) $\Leftrightarrow$ (v3) is (a) $\Leftrightarrow$ (b) from Theorem 2.12 in view of Remark 3.13. Moreover, (v2) $\Leftrightarrow$ (v8) is clear from Lemma 3.1, 2) together with Theorem 3.10 for $\xi = \eta$. The equivalence (v3) $\Leftrightarrow$ (v4) is immediate from Lemma 2.2. If we have (v1), the proof of Theorem 3.8, (a) $\Rightarrow$ (b) and then (b) $\Rightarrow$ (c), shows that (v5) holds. Finally, if we have (v5), the proof of Theorem 3.8, (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a), with $\xi = \eta$ shows that 0 is vm for S^η and hence sm for η by Theorem 3.10, 2).

The statements or definitions clearly yield (eK) $\Rightarrow$ (vK) for $\mathrm{K} = 1, \dots, 8$. Moreover, (v6) $\Leftrightarrow$ (v7) as well as (e6) $\Leftrightarrow$ (e7) follow from the discussion at the end of Section 3.1.

Now (e1) $\Leftrightarrow$ (e3) is (a) $\Leftrightarrow$ (b) in Theorem 2.14, again using Remark 3.13, and both (e2) $\Leftrightarrow$ (e8) and (e2′) $\Leftrightarrow$ (e8′) are due to Theorem 3.10. Next, (e2′) $\Leftrightarrow$ (e8) is (a) $\Leftrightarrow$ (c) from Theorem 3.9, and (e2′) $\Leftrightarrow$ (e6) is (a) $\Leftrightarrow$ (c) from Proposition 3.7, both for $\xi = \eta$. Moreover, the definition of DSE directly gives (e1) $\Rightarrow$ (e8′), and then also (e8) $\Rightarrow$ (e1) as η is bounded. Finally, the equivalences (e3) $\Leftrightarrow$ (e4) and (e1) $\Leftrightarrow$ (e5) are proved as above, with Theorem 3.9 replacing Theorem 3.8. $\square$

Remark 4.2. It is not difficult to prove the equivalence (v3) $\Leftrightarrow$ (v6) directly via the classic result from Karatzas/Kardaras [27, Theorem 4.12]. However, our main result in Theorem 2.12 is the equivalence of (v3) to the new AOA condition DSV for η in (v1). This is not available in the literature and has no easy proof, as one can see in Section 3.

4.2. **Robustness towards the choice of a reference strategy.** As already pointed out in Remark 2.9, 2), our concepts and main results depend on the choice of a reference strategy η. In this section, we show that this dependence is fairly weak, which means that our approach is quite robust towards the choice of η.

For two reference strategies η, η', consider the *ratio condition*

$$
\begin{aligned}
&(\eta' \cdot S)/(\eta \cdot S) = V(\eta')/V(\eta) \in \mathcal{S}_{++}^{\mathrm{unif}}, \text{ i.e.,} \\
&0 < \inf_{t \geq 0} \frac{V_t(\eta')}{V_t(\eta)} \leq \sup_{t \geq 0} \frac{V_t(\eta')}{V_t(\eta)} < \infty \qquad P\text{-a.s.}
\end{aligned}
\tag{4.1}
$$

As $\mathcal{S}_{++}^{\mathrm{unif}}$ is closed under taking reciprocals, (4.1) is symmetric in η and η'. Intuitively, (4.1) means that η and η' are comparable in a certain way.

Lemma 4.3. *Suppose* $S \geq 0$ *and there exist reference strategies* η, η'*. Fix* $\vartheta \in \Theta_+^{\mathrm{sf}}$*. If* (4.1) *holds,* ϑ *is share maximal for* η *if and only if it is share maximal for* η'*.*

Proof. If ϑ is sm for η, then it is vm for S^η by Theorem 3.10, 1). But $S^{\eta'} = S^\eta / D$ with $D := (\eta' \cdot S)/(\eta \cdot S) \in \mathcal{S}_{++}^{\mathrm{unif}}$ due to (4.1). Thus by Lemma 3.1, 1), ϑ is vm for $S^{\eta'}$ as well, and hence sm for η' by Theorem 3.10, 2). The converse is argued symmetrically. $\square$

Proposition 4.4. *Suppose $S \geq 0$ and there exist reference strategies η, η'.*

1) *If (4.1) holds, then DSV for η and DSV for η' are equivalent.*

2) *If η, η' as well as $S^\eta, S^{\eta'}$ are bounded (uniformly in (ω, t)), then DSE for η and DSE for η' are equivalent.*

Proof. 1) Apply Lemma 4.3 to $\vartheta \equiv 0$.

2) Because η and $S^{\eta'}$ are bounded, so is $(\eta \cdot S)/(\eta' \cdot S) = \eta \cdot S^{\eta'}$, and analogously, $(\eta' \cdot S)/(\eta \cdot S)$ is bounded. So (4.1) holds. If we have DSE for η, every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ and in particular η' is sm for η by Theorem 4.1, (e1) $\Rightarrow$ (e8). By Lemma 4.3, η' is thus also sm for η', and so Theorem 4.1, (e8′) $\Rightarrow$ (e1), gives DSE for η'. The converse argument is symmetric. $\square$

Note that the boundedness assumptions in Proposition 4.4, 2) are precisely those we impose in Theorem 2.14 to obtain a dual characterisation for DSE. So DSE is robust with respect to the choice of any reference strategy in that class.

By Proposition 4.4, 1), the ratio condition (4.1) is *sufficient* for DSV for η and η' to be equivalent. Using Theorem 3.10 and Lemma 3.11, one can show that it is also *necessary*.

Remark 4.5. Suppose $S \geq 0$ and $\sum_{i=1}^N S^i$ is strictly positive with strictly positive left limits. As seen in Remark 2.4, 1), the market portfolio $\mathbb{1}$ is then a reference strategy which has $\mathbb{1}$ and $S^{\mathbb{1}} = S/\sum_{i=1}^N S^i$ bounded (uniformly in (ω, t)). Any $\eta \in \Theta^{\mathrm{sf}}_+$ with $c\mathbb{1} \leq \eta \leq C\mathbb{1}$ for constants $0 < c \leq C < \infty$ is then also a reference strategy which has η and S^η bounded (uniformly in (ω, t)). In view of Proposition 4.4, 2), DSE for the market portfolio is thus the same as for any bounded reference strategy which always invests into all assets in a uniformly nondegenerate way. An "extreme" strategy like e^i, buy and hold a single fixed asset i, does not satisfy this.

4.3. **Connections to the classic results.** Theorem 4.1 shows that DSV is related to NUPBR, and DSE to NFLVR. We now study this in more detail in the classic setup $S = (1, X)$, for $X \geq 0$ as our results need $S \geq 0$.

If $S = (1, X)$ with $X \geq 0$, then $\mathbb{1}$ is a reference strategy (with $V(\mathbb{1}) \geq 1$) and $S^{\mathbb{1}} = S/V(\mathbb{1}) = S/\sum_{i=1}^N S^i$ is bounded (uniformly in (ω, t)). Both these properties hold for any $S \geq 0$ with $\sum_{i=1}^N S^i > 0$ and $\sum_{i=1}^N S^i_- > 0$. In contrast, e^i is a reference strategy for general $S \geq 0$ only if $S^i > 0$ and $S^i_- > 0$, and $S^{\mathrm{e}^i} = S/S^i$ has in general no boundedness properties. For $S = (1, X)$, e^1 is always a reference strategy and $S^{\mathrm{e}^1} = S$. But this relies crucially on the particular structure of $S = (1, X)$, and choosing e^1 as a reference strategy is therefore both more extreme and more delicate than choosing $\mathbb{1}$. The next result reflects this.

Proposition 4.6. *If $S = (1, X)$ for an $\mathbb{R}^d_+$-valued semimartingale $X \geq 0$, then classic NUPBR for X is equivalent to S satisfying DSV for e^1 and implies that S satisfies DSV for $\mathbb{1}$.*

Proof. Classic NUPBR for X is the same as $\mathrm{NUPBR}_\infty(1, X)$, and this is by Proposition 3.3 for $\xi \equiv \mathrm{e}^1$ equivalent to 0 being vm for $S^{\mathrm{e}^1} = S = (1, X)$. In turn, this is by Theorem 3.10 equivalent to 0 being sm for e^1, which is DSV for e^1 by definition. Next, DSV for $\mathbb{1}$ is the same as 0 being sm for $\mathbb{1}$, which is equivalent to 0 being vm for $S^{\mathbb{1}}$ by Theorem 3.10 again, now for $\xi \equiv \mathbb{1}$. As $S^{\mathbb{1}} = S/V(\mathbb{1}) = S^{\mathrm{e}^1}/V(\mathbb{1})$, vm for S^{e^1} is by Lemma 3.1, 1) the same as vm for $S^{\mathbb{1}}$ whenever $V(\mathbb{1}) \in \mathcal{S}^{\mathrm{unif}}_{++}$, and the point is now that this holds if X satisfies NUPBR. Indeed, $V(\mathbb{1}) \geq 1$ due to $X \geq 0$, and as NUPBR for X is equivalent to 0 being vm for $S^{\mathrm{e}^1} = S$, Proposition 3.4 for $\xi \equiv \mathrm{e}^1$ implies that $V(\mathbb{1})$ is convergent and hence bounded in $t \geq 0$, P-a.s. So $V(\mathbb{1}) \in \mathcal{S}_{++}^{\mathrm{unif}}$ and we are done. $\square$

The converse of the last implication in Proposition 4.6 is not true in general. A counterexample is given in Example 6.8. Thus our new concept of dynamic share viability, when used for the market portfolio $\mathbb{1}$, is more widely applicable than classic NUPBR. The same example also shows that DSV for $\mathbb{1}$ does not imply DSV for e^1 in general.

The situation with DSE versus NFLVR is more subtle. We first give a positive result.

Proposition 4.7. *If $S = (1, X)$ for an $\mathbb{R}_+^d$-valued semimartingale $X \geq 0$, then classic NFLVR for X is equivalent to S satisfying DSE for* e^1.

Proof. Classic NFLVR for X is the same as $\mathrm{NFLVR}_\infty(1, X)$, and this is by Proposition 3.3 for $\xi \equiv \mathrm{e}^1$ equivalent to e^1 being vm for $S^{\mathrm{e}^1} = S = (1, X)$. In turn, this is by Theorem 3.10 equivalent to e^1 being sm for e^1. But any e^1-buy-and-hold strategy ϑ is of the form $\vartheta = \lambda \mathrm{e}^1$ for some $\lambda \in \mathbb{R}$, because $\mathrm{e}^1 = (1, 0, \ldots, 0)$, and so ϑ is in Θ_+^{sf} if and only if $\lambda \geq 0$. Thanks to Lemma 3.1, 2), e^1 is therefore sm for e^1 if and only if every e^1-buy-and-hold $\vartheta \in \Theta_+^{\mathrm{sf}}$ is sm for e^1, which is DSE for e^1 by definition. $\square$

Remark 4.8. It looks tempting to use Theorem 4.1, (e1) $\Leftrightarrow$ (e8$'$), to shorten the above argument. But the proof of that equivalence uses that both η and S^η are bounded (uniformly in (ω, t)), which would place (for $\eta \equiv \mathrm{e}^1$) a massive restriction on $S^{\mathrm{e}^1} = S = (1, X)$.

If we want to use the reference strategy $\eta \equiv \mathbb{1}$, the situation for DSE versus NFLVR is different from DSV versus NUPBR. Neither of DSE for $\mathbb{1}$ and NFLVR for X implies the other in general in the classic case $S = (1, X)$. Example 6.10 shows that DSE for $\mathbb{1}$ does not imply NFLVR for X. Conversely, Example 6.7 shows that for $S = (1, X)$, we can have NFLVR for X while DSE for $\mathbb{1}$ fails. (We note that by Theorem 4.1 with $\eta \equiv \mathbb{1}$, S satisfying DSE for $\mathbb{1}$ is equivalent to $S^{\mathbb{1}}$ satisfying NFLVR_∞, and also to $S^{\mathbb{1}}$ satisfying DSE for $\mathbb{1}$, by discounting-invariance; but this only means that we have a result for $\mathbb{1}$-discounted prices $S^{\mathbb{1}}$, not for the original prices S.)

The background of this discrepancy is as follows. NFLVR for X is equivalent to e^1 being value maximal for $(1, X)$, whereas DSE for $\mathbb{1}$ is equivalent to $\mathbb{1}$ being value maximal for $S^{\mathbb{1}}$. Here, "e^1 value maximal" is weaker than "$\mathbb{1}$ value maximal" as $\mathrm{e}^1 \leq \mathbb{1}$, but "for $(1, X)$" is stronger than "for $S^{\mathbb{1}}$" as $(1, X) \geq S^{\mathbb{1}}$. Upon reflection, the discrepancy is actually not surprising; in fact, NFLVR is about how e^1 or $V(\mathrm{e}^1)$ fits into the market, whereas DSE for $\mathbb{1}$ looks at all the e^i, $i = 1, \ldots, N$. The next result makes this more precise.

Proposition 4.9. *Suppose that $S \geq 0$ and there exist reference strategies η, η'. Then $\mathrm{NFLVR}_\infty(S^\eta)$ plus $\inf_{t \geq 0}(\eta'_t \cdot S_t^\eta) > 0$ P-a.s. implies that η is share maximal for η'. In particular, if $S = (1, X)$ with $X \geq 0$, then classic NFLVR for X implies that* e^1 *is share maximal for* $\mathbb{1}$.

Proof. The second statement follows from the first for $\eta \equiv \mathrm{e}^1, \eta' \equiv \mathbb{1}$ by observing that $\mathbb{1} \cdot S^{\mathrm{e}^1} = 1 + \sum_{i=1}^d X^i \geq 1$. If we have $\mathrm{NFLVR}_\infty(S^\eta)$, then η and 0 are vm for S^η by Proposition 3.7 and Lemma 3.1, 2), and so $V(\eta', S^\eta)$ is convergent and hence bounded in $t \geq 0$, P-a.s., by Proposition 3.4. By assumption, $\inf_{t\geq 0} V_t(\eta', S^\eta) > 0$ P-a.s. so that $V(\eta', S^\eta) \in \mathcal{S}^{\mathrm{unif}}_{++}$. As η is vm for S^η, it is by Lemma 3.1, 1) also vm for $S^\eta/V(\eta', S^\eta) = S^{\eta'}$. By Theorem 3.10, 2), η is then sm for η'. □

To conclude, we briefly show how our approach yields new results even in the classic case. Note that the next result does not assume that $S \geq 0$.

Proposition 4.10. *Suppose that there exists an $\eta \in \Theta^{\mathrm{sf}}_{++}$. Then S^η satisfies NUPBR$_\infty$ if and only if there exists a σ-martingale discounter D for S^η with $D_\infty := \lim_{t\to\infty} D_t < \infty$ P-a.s.*

Proof. By Proposition 3.3 and Theorem 3.8 for $\xi = \eta$, S^η satisfies NUPBR$_\infty$ if and only if it admits a σ-martingale discounter $D' \in \mathcal{S}_{++}$ with the extra property $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$. Fix any σ-martingale discounter D for S^η. Because $V(\eta, S^\eta) \equiv 1$, writing

$$1/D = V(\eta, S^\eta)/D = V(\eta, S^\eta/D) = V_0(\eta, S^\eta/D) + \eta \bullet (S^\eta/D)$$

shows that $1/D$ is a σ-martingale like S^η/D, and in $\mathcal{S}_{++}$ like D. By Ansel/Stricker [3, Corollary 3.5], $1/D$ is thus a local martingale > 0, then a supermartingale > 0, and hence P-a.s. convergent to some finite limit. Then D itself is also P-a.s. convergent and $D_\infty > 0$ P-a.s., which implies $\inf_{t\geq 0} D_t > 0$ P-a.s. The extra property $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ thus holds if and only if $\sup_{t\geq 0} D_t < \infty$ P-a.s. or, equivalently by convergence, $D_\infty < \infty$ P-a.s. □

Corollary 4.11. *Suppose that X is an $\mathbb{R}^d$-valued semimartingale. Then X satisfies classic NUPBR if and only if there exists a local martingale $L > 0$ with $L_\infty := \lim_{t\to\infty} L_t > 0$ P-a.s. and such that the product LX is a σ-martingale.*

Proof. For $S = (1, X)$, $\eta \equiv \mathrm{e}^1$ is in $\Theta^{\mathrm{sf}}_{++}$ with $S^\eta = S^{\mathrm{e}^1} = S$. So we can apply Proposition 4.10 and take $L := 1/D$. The properties of L are all shown in the above proof. □

Corollary 4.11 sharpens the classic characterisation of NUPBR in Karatzas/Kardaras [27, Theorem 4.12] in two ways. First, X and X_- need not be strictly positive (i.e., we need not assume $X \in \mathcal{S}^d_{++}$); this is also pointed out in [27, Section 4.8]. More importantly, we get a σ-martingale deflator L for X itself, not only a supermartingale deflator $\tilde{L}$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$; see also Bálint/Schweizer [5, Lemma 2.13] for the connection between these two and other properties. Unlike $1/\tilde{L}$, however, $1/L$ cannot be chosen S-tradable in general; see Takaoka/Schweizer [39, Remark 2.8] for a counterexample and [5, Propositions 2.19 and 2.20] for related positive results. Corollary 4.11 also extends [39, Theorem 2.6] from a right-closed interval $[0, T]$ to a general setup.

5. **Comparison to the literature.**

This section compares our ideas and results to the existing literature. In contrast to Section 4.3, it is qualitative in the sense that it contains no results and proofs, but only discussion. We first consider absence-of-arbitrage (AOA) aspects and then discuss numéraire- or discounting-invariance.

5.1. **Absence of arbitrage.**

The two most used classic AOA notions in the literature are NFLVR (due to Delbaen/Schachermayer [10]) and the strictly weaker NUPBR (coined by Karatzas/Kardaras [27]). The latter condition was introduced under different names by different authors — BK in Kabanov [23], no cheap thrills in Loewenstein/Willard [34] or NA1 in Kardaras [29, 31]; see also Kabanov/Kramkov [24] for the notion of NAA1. By [29, Proposition 1] and Kabanov et al. [21, Lemma A.1], all these and NUPBR are equivalent.

It is very important to note that both NFLVR and NUPBR are classically only defined for *discounted* price processes of the form $S = (1, X)$. This entails a serious loss of generality — already in Example 1.1 with two GBMs S^1 and S^2, it can easily happen that the S^1-discounted model $S/S^1 = (1, S^2/S^1)$ satisfies classic NFLVR whereas the S^2-discounted model $S/S^2 = (S^1/S^2, 1)$ fails even the weaker property of classic NUPBR. In other words, the choice of discounting decides here whether we obtain a discounted model with or without arbitrage. The AOA properties of the original model (S^1, S^2) are thus not clarified at all. Similar concerns apply to general non-discounted models S. One might think that change-of-numéraire results could help to alleviate this issue, and we discuss this below in Section 5.2. But it turns out that classic results do not help.

Once one has a definition of AOA for $S = (1, X)$, one can ask about an equivalent alternative description. Dual characterisations, in terms of martingale properties for X, first focused on NFLVR, culminating in the classic FTAP due to Delbaen/Schachermayer [10, 13] that for a general $\mathbb{R}^d$-valued semimartingale X, NFLVR for $S = (1, X)$ is equivalent to the existence of an equivalent σ-martingale measure for the discounted prices X.

Even if NFLVR fails to hold, a market can still be nice enough to allow some AOA-type arguments. This has been exploited in several papers. Loewenstein/Willard [34] show in an Itô process setup that already no cheap thrills (NUPBR) is sufficient (and necessary) to solve utility maximization problems; see also Chau et al. [8]. In the benchmark approach presented in Platen/Heath [35], a market may violate NFLVR; but in units of the so-called numéraire portfolio, the theory works as if there was no arbitrage. An excellent discussion with more details can be found in Herdegen [17, Section 5.3]. For stochastic portfolio theory and the study of relative arbitrage (see Karatzas/Fernholz [26] for an overview), a market may have “arbitrage” in the sense of FLVR; but still portfolio choice can make sense, and hedging via superreplication can work. The comprehensive paper of Karatzas/Kardaras [27] shows that maximising growth rate, asymptotic growth or expected logarithmic utility from terminal wealth all make sense if and only if NUPBR holds. Another overview of these connections is given in the recent work of Choulli et al. [9].

In Bálint/Schweizer [5], we have recently studied the dependence of AOA conditions on the time horizon as part of an analysis of large financial markets; see [5, Section 5 and in particular Corollary 5.4]. We point out there in [5, Remark 5.6] that in contrast to common belief, NUPBR on $[0, \infty)$ is not stable under localisation. Put differently, NUPBR on $[0, \infty)$ (which is the original definition in [27]) is strictly stronger than having NUPBR on $[0, T]$ for all $T \in (0, \infty)$. For related work on the connection of NUPBR to the time horizon, we refer to Kardaras [32], Acciaio et al. [1] and Aksamit et al. [2]. See also Remark 5.1 below.

Like for NFLVR, the literature contains dual characterisations of NUPBR. Depending on the setting, they vary in the strength of the dual formulation; see Table 1 below for an overview. For $S = (1, X)$ on $[0, \infty)$ with $X \in \mathcal{S}^d_{++}$, Karatzas/Kardaras [27] show that NUPBR is equivalent to the existence of an S-tradable supermartingale discounter for all wealth processes of admissible self-financing strategies. On $[0, T]$, this is strengthened by Takaoka/Schweizer [39] to the existence of a σ-martingale discounter for X, where again $S = (1, X)$ but X can be in $\mathcal{S}^d$. Both Kardaras [31] and Kabanov et al. [21], inspired by the results and a counterexample in [39], work on $[0, T]$ with $S = (1, X)$ for $X \in \mathcal{S}^d$ and characterise NA1 (which is equivalent to NUPBR) by the existence of a local martingale discounter for all wealth processes of admissible self-financing strategies. In [31], this is done for $d = 1$ so that X is real-valued; the authors of [21] extend the result to $d \geq 1$ and in addition manage to find an *S-tradable* local martingale discounter under some $R \approx P$, for an R in any neighbourhood of P. An overview of the connections between different types of discounters is given in Bálint/Schweizer [5, Lemma 2.13].

	price process S	time	condition	dual condition
KK [27]	$(1, X) \in \mathcal{S}^{1+d}_{++}$	$[0, \infty)$	NUPBR	$\exists$ S-tradable SMD $D > 0$, $\forall$ $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, with $D_\infty > 0$
TS [39]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NUPBR	$\exists$ σMD $D > 0$ for X
K [31]	$(1, X) \in \mathcal{S}^{1+1}$	$[0, T]$	NA1	$\exists$ LMD $D > 0$, $\forall$ $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$
KKS [21]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NA1	$\exists$ S-tradable LMD $D > 0$, $\forall$ $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, in any neighbourhood of P
H [17]	in $\mathcal{S}^N$	$[0, T]$	NINA	$\exists$ (discounter, EσMM) pair for S
here	in $\mathcal{S}^N_+$	$[0, \infty)$	DSV for η	$\exists$ σMD $=$ LMD $D > 0$ for S with $\inf_{t \geq 0}(\eta_t \cdot (S_t/D_t)) > 0$

Table 1. Overview of existing FTAP-type results. Note that we have NA1 $=$ NUPBR on $[0, T]$.

Table 1 gives an overview of the dual characterisation results discussed above. We recall the space $\mathcal{S}^m$ of $\mathbb{R}^m$-valued semimartingales and use $\mathcal{S}^m_+$, $\mathcal{S}^m_{++}$ as in Section 2. The abbreviations SMD, σMD and LMD denote super-, σ- and local martingale discounters, respectively. The table compares Karatzas/Kardaras [27], Takaoka/Schweizer [39], Kardaras [31], Kabanov et al. [21], Herdegen [17] and the present article. Note that on a right-open interval, here called $[0, \infty)$, the dual characterisation always involves a condition at the right endpoint (here called ∞).

5.2. **Numéraire- or discounting-invariance.** As mentioned above, both classic NFLVR and NUPBR are only defined for discounted prices of the form $S = (1, X)$. It is natural to ask in general what happens to an AOA concept if one changes the numéraire, i.e., uses a different process for discounting. This can be done in two different directions, after initially fixing only a price process S:

(A) One can first fix a class $\mathcal{D}$ of discounting processes and then look for an AOA concept $\mathcal{A}$ which is invariant for the chosen class $\mathcal{D}$, in the sense that $\mathcal{A}$ holds simultaneously for all processes S/D with $D \in \mathcal{D}$.

(B) One can first fix an AOA concept $\mathcal{A}'$ and then look for a class $\mathcal{D}'$ of discounting processes which leaves the chosen $\mathcal{A}'$ invariant, in the same sense as above.

Both (A) and (B) are concerned with numéraire- or discounting-invariance; but their objectives and results are fundamentally different. In a nutshell, most of the classic results and in particular Delbaen/Schachermayer [11, 12] fall into category (B), whereas both Herdegen's and our approach here address (A). Put differently, we want to be liberal about the class $\mathcal{D}$ of allowed discounters and thus need to look for a suitable new AOA concept $\mathcal{A}$. In contrast, [11, 12] want to keep an established AOA concept $\mathcal{A}'$ and therefore look for restrictions on the class $\mathcal{D}'$ of discounters to achieve this.

Historically, probably the first to study questions of numéraire-invariance for AOA were Delbaen/Schachermayer [11] and Sin [38] (interestingly, these works do not cite each other). [38] studies problem (A) for the special case $\mathcal{D} = \{S^1, \dots, S^N\}$ and replaces for strategies $\vartheta \in \Theta^{\mathrm{sf}}$ the concept of admissibility used in [11] by that of *feasibility*, i.e., the value process should satisfy $V(\vartheta) \geq -V(c\mathbb{1}) = -\sum_{i=1}^N c^i S^i$, where $c^i \geq 0$ is the number of shares of asset i outstanding at time 0 and the product $c\mathbb{1}$ is componentwise. For $S = (1, X^{(1)})S^1$ with $S^1 > 0$ and $X^{(1)} \geq 0$ a semimartingale, the main result is then that $X^{(1)}$ satisfies NFLVR with *feasible* strategies if and only if $X^{(1)}$ admits an equivalent (true) martingale measure, and that this is also equivalent to NFLVR with feasible strategies for any $X^{(k)}$ with $S = (1, X^{(k)})S^k$ whenever $S^k > 0$ and $X^{(k)} \geq 0$ is a semimartingale. Thus one has indeed an answer to (A), and the new AOA concept $\mathcal{A}$ in [38] is NFLVR with feasible strategies. Essentially the same approach was redeveloped later in Yan [41] (who was apparently unaware of [38]).

In contrast, Delbaen/Schachermayer [11, 12] study problem (B) and answer the questions appearing there fairly exhaustively. Starting with $S = (1, X)$ and taking an S-tradable numéraire/discounter $D = V(\vartheta)$, they consider the two markets $S = (1, X)$ and $\tilde{S} = (\frac{1}{D}, \frac{X}{D})$ and show in [11] that if S satisfies classic NFLVR, then $\tilde{S}$ admits an equivalent σ-martingale measure if and only if $D_\infty - D_0$ is maximal in $\mathcal{G}_{\mathrm{adm}}(S)$. In the spirit of (B), this characterises those *S-tradable* discounters which preserve NFLVR. In [12], for such a D and under NFLVR for S, they derive an isometry between two spaces $\mathcal{G}(S)$ and $\mathcal{G}(\tilde{S})$ of (final values of) stochastic integrals. One key assumption for both results is $D_\infty > 0$; so in addition to being S-tradable, D must also be in $\mathcal{S}_{++}^{\mathrm{unif}}$. When we look at Example 1.1 from this perspective, we see that while both S^1, S^2 are S-tradable, none of them is in $\mathcal{S}_{++}^{\mathrm{unif}}$ — so both assets are not allowed as discounters in the Delbaen/Schachermayer framework, and the results from Delbaen/Schachermayer do not help to answer our question.

After Sin [38], problem (A) was taken up almost 20 years later (without citing [38]) by Herdegen [17] who worked on $[0, T]$ with a general $\mathbb{R}^N$-valued semimartingale S. He used the essentially largest possible class $\mathcal{D} = \mathcal{S}_{++}$ of discounters, which on $[0, T]$ coincides with $\mathcal{S}_{++}^{\mathrm{unif}}$ because all processes are defined up to and including the time horizon T, and introduced the discounting-invariant AOA condition NINA or dynamic (value) viability. This generalises NUPBR and is dually characterised by the existence of a (discounter, EσMM) pair (D, Q), meaning that $D \in \mathcal{S}_{++}$ and Q is an equivalent σ-martingale measure for S/D. In addition, [17] also presents a discounting-invariant alternative to NFLVR. It is called dynamic (value) efficiency and requires that not one particular asset, but each of the N basic assets (or, equivalently, the market portfolio $\mathbb{1}$) should satisfy (value) maximality. One key insight from Delbaen/Schachermayer [11] also reappears in [17] — NFLVR describes a maximality property of the discounting asset, but it does not say much about the market as a whole. (Our Proposition 4.9 extends that to the current framework.)

In the above terminology, the contribution of our paper can be succinctly described as follows. For an $\mathbb{R}^N_+$-valued semimartingale $S \geq 0$ on the right-open interval $[0, \infty)$, we consider the class $\mathcal{D} = \mathcal{S}_{++}$ of discounters and tackle problem (A). We introduce two new AOA concepts DSV and DSE which are discounting-invariant for $\mathcal{S}_{++}$ and provide dual characterisations.

An interesting related paper in discrete time is by Tehranchi [40]. The main result in Theorem 2.10 there is reminiscent of our Theorems 2.12 and 2.14, but has no dual condition at ∞. Moreover, the formulation in [40] hinges crucially on the discrete-time setup.

Remark 5.1. A recent new book by Karatzas/Kardaras [28] also introduces a discounting-invariant AOA concept under the name of *viability*. [28] consider a market with d risky assets $X = (X^1, \dots, X^d)$ and one riskless bank account $S^0 \equiv 1$, where each $X^i > 0$ is a continuous semimartingale. Viability in [28] turns out to be equivalent to $\mathrm{NUPBR}_{\mathrm{loc}}$ in the sense that X satisfies NUPBR on $[0, T]$ for all $T \in (0, \infty)$. It is easy to check that $\mathrm{NUPBR}_{\mathrm{loc}}$ is invariant under discounting with any strictly positive continuous semimartingale D because NUPBR on $[0, T]$ satisfies this. (This actually holds for general $\mathbb{R}^d$-valued semimartingales X and general semimartingales $D > 0$ with $D_- > 0$.) In our view, this viability concept has the drawback that it does not allow to consider problems that genuinely live on all of $[0, \infty)$ — the case of a perpetual exchange option as in Example 1.1 is a good illustration. Using instead directly (classic) NUPBR on $[0, \infty)$ is not a way out because this is discounting-invariant only for $D \in \mathcal{S}^{\mathrm{unif}}_{++} \subsetneq \mathcal{S}_{++} = \mathcal{D}$. (Continuity of X does not help for this latter point either.)

6. **Examples.**

This section illustrates our results by examples and counterexamples. Most are based on variants of one general example, and so we start with an analysis of that setup.

6.1. **Results for a model with N geometric Brownian motions.**

Example 6.1. For $i = 1, \dots, N$, let S^i be a geometric Brownian motion with parameters $m_i \in \mathbb{R}$, $\sigma_i \geq 0$; so $W = (W^1, \dots, W^N)$ is a vector of correlated Brownian motions and

$$\log S^i_t = \sigma_i W^i_t + \left(m_i - \frac{1}{2}\sigma_i^2\right)t, \qquad t \geq 0. \tag{6.1}$$

To construct W, we start with a vector $B = (B^1, \dots, B^N)$ of independent Brownian motions and a constant correlation matrix $\varrho \in \mathbb{R}^{N \times N}$; so ϱ is symmetric and positive semidefinite with $\varrho^{ii} = 1$ and $|\varrho^{ik}| \leq 1$ for $i, k = 1, \dots, N$. Using the Cholesky decomposition (see Golub/Van Loan [16, Theorem 4.2.7]), we write $\varrho = CC^\top$ for a lower triangular matrix $C \in \mathbb{R}^{N \times N}$ with $C^{ii} \geq 0$ for $i = 1, \dots, N$. (If ϱ is positive definite, as we assume in the sequel, we even have $C^{ii} > 0$ for $i = 1, \dots, N$, and then C is invertible.) Then

$$W := CB \tag{6.2}$$

defines Brownian motions $W^1, \dots, W^N$ with $\langle W^i, W^k \rangle_t = \varrho^{ik} t$.

To avoid degenerate situations, we assume ϱ is positive definite. We also assume that at most one of $\sigma_1, \dots, \sigma_N$ can be 0, and for definiteness, we suppose that $\sigma_N \geq 0$ and $\sigma_i > 0$ for $i = 1, \dots, N-1$. (If $\sigma_i = \sigma_k = 0$ for $i \neq k$, then $S^i_t = e^{m_i t}$, $S^k_t = e^{m_k t}$ and hence either $S^i \equiv S^k$ if $m_i = m_k$, or there is arbitrage if $m_i \neq m_k$.) The filtration $\mathbb{F}$ is generated by $S = (S^1, \dots, S^N)$, made right-continuous and complete. If $\sigma_N > 0$, then $\mathbb{F}$ is equivalently generated by W or (using that C is invertible) by B. If $\sigma_N = 0$, then $\mathbb{F}$ is generated by $W^1, \dots, W^{N-1}$. As $W^k = (CB)^k = \sum_{\ell=1}^N C^{k\ell} B^\ell$ and $C^{k\ell} = 0$ for $\ell > k$ because C is lower triangular, we see that each W^k can only depend on $B^1, \dots, B^k$. In particular, for $\sigma_N = 0$, $\mathbb{F}$ does not depend on B^N.

Example 6.2. If we take $N = 2$, then $C = \left(\begin{smallmatrix} 1 & 0 \\ \rho & 1 \end{smallmatrix}\right)$ with $\rho = \varrho^{12}$ and we obtain two basic cases. If $\sigma_2 > 0$, then S^1 and S^2 model two stocks given by *geometric Brownian motions (GBMs)* with correlation ρ. If $\sigma_2 = 0$ and we set $m := m_1$, $\sigma := \sigma_1$, $r := m_2$, we have the classic *Black–Scholes (BS) model* with bank account S^2 and stock S^1.

In the setting of Example 6.1, we want to know when S satisfies DSV or DSE for $\eta \equiv \mathbb{1}$. The key for that analysis is the following result.

Proposition 6.3. *In Example 6.1, there exists a unique S-tradable discounter $\bar{D}$ with the property that*

$$\bar{S}^i := S^i/\bar{D} = \mathcal{E}(\bar{\gamma}_i \cdot B), \qquad i = 1, \dots, N, \tag{6.3}$$

for vectors $\bar{\gamma}_1, \dots, \bar{\gamma}_N \in \mathbb{R}^N$. If $\sigma_N = 0$, then $\bar{\gamma}_i^N = 0$ for $i = 1, \dots, N$.

As the proof of Proposition 6.3 is rather lengthy, we postpone this for the moment and turn directly to consequences.

Theorem 6.4. *In Example 6.1,*

1) *S satisfies DSV for $\mathbb{1}$ if and only if one of the N processes S/S^i, $i = 1, \dots, N$, is a martingale;*

2) *S never satisfies DSE for $\mathbb{1}$.*

Proof. Because DSV and DSE are both discounting-invariant, we can argue for $\bar{S} = S/\bar{D}$ from Proposition 6.3 instead of S.

1) By Proposition 6.3, $\bar{S}^i = \mathcal{E}(\bar{\gamma}_i \cdot B)$ is a positive martingale for $i = 1, \dots, N$, and then so is $\mathbb{1} \cdot \bar{S} = \sum_{i=1}^N \bar{S}^i$. Because $B^1, \dots, B^N$ are independent, Yor's formula yields

$$\mathcal{E}(\bar{\gamma}_i \cdot B) = \mathcal{E}\left(\sum_{k=1}^N \bar{\gamma}_i^k B^k \right) = \prod_{k=1}^N \mathcal{E}(\bar{\gamma}_i^k B^k).$$

For every $\alpha \in \mathbb{R}$ and any Brownian motion $\tilde{B}$, the process $\mathcal{E}(\alpha \tilde{B})$ is a martingale, and it converges to 0 as $t \to \infty$ if $\alpha \neq 0$. Therefore

$$\lim_{t\to\infty} \bar{S}^i_t = \lim_{t\to\infty} \mathcal{E}(\bar{\gamma}_i \cdot B)_t = 0 \qquad P\text{-a.s. for } \bar{\gamma}_i \neq 0. \tag{6.4}$$

If there exists an i_0 with $\bar{\gamma}_{i_0} = 0$, then $\bar{S}^{i_0} \equiv 1$ and therefore $\mathbb{1} \cdot \bar{S}$ is a P-martingale with $\mathbb{1} \cdot \bar{S} \geq \bar{S}^{i_0} = 1$. So Theorem 2.12, (b) $\Rightarrow$ (a), with $\eta \equiv \mathbb{1}$ and $D \equiv 1$ implies that $\bar{S}$ (and also S) satisfies DSV for $\mathbb{1}$. On the other hand, if $\gamma_i \neq 0$ for all i, then $\lim_{t\to\infty} \mathbb{1} \cdot \bar{S}_t = 0$ P-a.s. due to (6.4). Because $\bar{D}$ is S-tradable, we have $\bar{D} = V(\bar{\vartheta}) = \bar{\vartheta} \cdot S$ for some $\bar{\vartheta} \in \Theta^{\mathrm{sf}}_{++}$ and therefore $\bar{\vartheta} \cdot \bar{S} = (\bar{\vartheta} \cdot S)/\bar{D} \equiv 1$. Thus $\lim_{t\to\infty} \frac{\bar{\vartheta}_t \cdot \bar{S}_t}{\mathbb{1} \cdot \bar{S}_t} = +\infty$ P-a.s., and Lemma 3.11 for $\xi \equiv \mathbb{1}$ implies for all strategies in

Θ_+^{sf} that they cannot be value maximal for $S^{\mathbb{1}}$ and then also not share maximal for $\mathbb{1}$ by Theorem 3.10, 1). Therefore DSV for $\mathbb{1}$ does not hold.

Up to here, we have shown that S satisfies DSV for $\mathbb{1}$ if and only if there exists an i_0 with $\bar{\gamma}_{i_0} = 0$. Then $\bar{S}^{i_0} \equiv 1$ which means that $S^{i_0} = \bar{D}$ and therefore $\bar{S} = S/S^{i_0}$. Because $\bar{S}$ is a P-martingale by Proposition 6.3, this proves the only if part. Conversely, if S/S^i is a P-martingale for some i, Theorem 2.12, (b) $\Rightarrow$ (a), with $\eta \equiv \mathbb{1}$ and $D = S^i$ implies that S satisfies DSV for $\mathbb{1}$; note that $\mathbb{1} \cdot (S_t/S_t^i) \geq 1$ P-a.s. This proves 1).

2) Because DSE implies DSV, the proof of 1) shows that we can only have DSE for $\mathbb{1}$ if $\bar{\gamma}_{i_0} = 0$ for some i_0. Then $\bar{D} = S^{i_0}$ and $S/S^{i_0} = \bar{S}$ is a martingale. In consequence, we must have $\bar{\gamma}_k \neq 0$ for all $k \neq i_0$; indeed, if $\bar{\gamma}_k = 0$, then also S/S^k is a martingale so that both S^k/S^{i_0} and S^{i_0}/S^k are martingales, which is impossible. We claim that e^k for $k \neq i_0$ is not value maximal for S/S^{i_0}. Indeed, because all the S^i start at 1, we have $V_0(\mathrm{e}^k, S/S^{i_0}) = 1 = V_0(\mathrm{e}^{i_0}, S/S^{i_0})$, and $S_t^k/S_t^{i_0} = \bar{S}_t^k = \mathcal{E}(\bar{\gamma}_k \cdot B)_t \to 0$ P-a.s. as $t \to \infty$ by (6.4) so that

$$\lim_{t\to\infty} V_t(\mathrm{e}^{i_0} - \mathrm{e}^k, S/S^{i_0}) = \lim_{t\to\infty} (1 - S_t^k/S_t^{i_0}) = 1 \in L_+^0 \setminus \{0\}.$$

But $S^{\mathbb{1}}/S^{\mathrm{e}^{i_0}} = \sum_{i=1}^N S^i/S^{i_0} \geq 1$ is in $\mathcal{S}_{++}^{\mathrm{unif}}$ as it is like S/S^{i_0} a martingale ≥ 0 and therefore convergent, hence bounded in $t \geq 0$, P-a.s. By Lemma 3.1, 1), e^k is then also not value maximal for $S^{\mathbb{1}}$ and thus also not share maximal for $\mathbb{1}$ by Theorem 3.10, 1) for $\xi \equiv \mathbb{1}$. As e^k is a $\mathbb{1}$-buy-and-hold strategy, this means that S does not satisfy DSE for $\mathbb{1}$. □

For Example 6.2 where $N = 2$, we can express the necessary and sufficient conditions for DSV for $\mathbb{1}$ directly in terms of the model coefficients. (One could also do this for general N by looking at the proof of Proposition 6.3, exploiting the expression for the $\bar{\gamma}_i$ there. However, the resulting conditions look much less concise, and so we omit them.)

Corollary 6.5. *In Example 6.2, S satisfies DSV for $\mathbb{1}$ if and only if*

$$m_i - \sigma_i^2 + \rho\sigma_1\sigma_2 = m_{3-i} \qquad \text{for } i = 1 \text{ or } i = 2. \tag{6.5}$$

If $\sigma_2 = 0$ so that we have the Black–Scholes model with parameters m, r, σ^2, condition (6.5) simplifies to

$$\frac{m-r}{\sigma^2} \in \{0, 1\}.$$

Proof. By Theorem 6.4, DSV for $\mathbb{1}$ holds if and only if either S^2/S^1 or S^1/S^2 is a martingale. Now we can plug in from (6.1) and compute. We omit the details. □

Finally, we turn to the

Proof of Proposition 6.3. 1) In a first step, we construct a discounter $\bar{D}$ such that (6.3) holds. For this, we need to find vectors $\bar{\gamma}_1, \dots, \bar{\gamma}_N \in \mathbb{R}^N$ such that

$$S^i/\mathcal{E}(\bar{\gamma}_i \cdot B) = \bar{D} \qquad \text{is independent of } i. \tag{6.6}$$

Now $\mathcal{E}(\bar{\gamma}_i \cdot B)_t = \exp(\gamma_i \cdot B_t - \frac{1}{2}|\bar{\gamma}_i|^2 t)$ and by (6.1) and (6.2), using $W^i = \mathrm{e}^i \cdot W$,

$$S_t^i = \exp\left(\sigma_i W_t^i + m_i t - \frac{1}{2}\sigma_i^2 t\right) = \exp\left(\sigma_i (\mathrm{e}^i)^\top C B_t + \left(m_i - \frac{1}{2}\sigma_i^2\right) t\right).$$

But $(\mathrm{e}^i)^\top C$ is the i-th row of the matrix C, and we write this as a column vector $c_i \in \mathbb{R}^N$. For the ratio $S^i/\mathcal{E}(\bar\gamma_i \cdot B)$ to be independent of i, we then must have

$$\sigma_i c_i - \bar\gamma_i = \sigma_1 c_1 - \bar\gamma_1 \qquad \text{for } i = 2, \dots, N \tag{6.7}$$

from the coefficients of B, and from the coefficients of t that

$$m_i - \frac{1}{2}\sigma_i^2 + \frac{1}{2}|\bar\gamma_i|^2 = m_1 - \frac{1}{2}\sigma_1^2 + \frac{1}{2}|\bar\gamma_1|^2 \qquad \text{for } i = 2, \dots, N. \tag{6.8}$$

Using (6.7) to compute $|\bar\gamma_i|^2$ and plugging this into (6.8) leads to

$$\begin{aligned}\bar\gamma_1 \cdot (\sigma_i c_i - \sigma_1 c_1) &= m_1 - \frac{1}{2}\sigma_1^2 - m_i + \frac{1}{2}\sigma_i^2 - \frac{1}{2}|\sigma_i c_i - \sigma_1 c_1|^2 \\ &=: \varphi_i \qquad \text{for } i = 2, \dots, N.\end{aligned} \tag{6.9}$$

We can already see from (6.7) that $\bar\gamma_2, \dots, \bar\gamma_N$ are affine functions of $\bar\gamma_1$, and we now claim that (6.9) determines the coordinates $\bar\gamma_1^i$ of $\bar\gamma_1$ as affine functions of $\bar\gamma_1^1$.

To argue the claim by an induction argument, we exploit the lower diagonal structure of C, which entails that $c_i^j = C^{ij} = 0$ for $j > i$. For $i = 2$, (6.9) therefore reduces to

$$\varphi_2 = \sigma_2 \bar\gamma_1 \cdot c_2 - \sigma_1 \bar\gamma_1 \cdot c_1 = (\sigma_2 c_2^1 - \sigma_1 c_1^1)\bar\gamma_1^1 + \sigma_2 c_2^2 \bar\gamma_1^2.$$

As $\sigma_2 > 0$ and $c_2^2 = C^{22} > 0$, this gives $\bar\gamma_1^2$ as an affine function of $\bar\gamma_1^1$. For $i > 2$, we write

$$\varphi_i - \varphi_{i-1} = \bar\gamma_1 \cdot (\sigma_i c_i - \sigma_{i-1} c_{i-1}) = \sigma_i c_i^i \bar\gamma_1^i + f(\bar\gamma_1^1, \dots, \bar\gamma_1^{i-1}),$$

where we use (6.9) and the lower triangular structure of C. Due to (6.9), f is an affine function of $\bar\gamma_1^1, \dots, \bar\gamma_1^{i-1}$, and by the induction hypothesis, this is in turn an affine function of $\bar\gamma_1^1$. As $c_i^i > 0$, also $\bar\gamma_1^i$ is an affine function of $\bar\gamma_1^1$ as long as $\sigma_i > 0$, which holds by assumption for $i < N$, i.e., $i \le N-1$. If $\sigma_N > 0$, we can also take $i = N$ and the claim is already proved. If $\sigma_N = 0$, we have seen before that $\mathbb{F}$ is generated by $W^1, \dots, W^{N-1}$ and contained in the filtration generated by $B^1, \dots, B^{N-1}$. Because $\bar D = S^i/\mathcal{E}(\bar\gamma_i \cdot B)$ must be adapted to $\mathbb{F}$, it cannot depend on B^N, and so we must have $\bar\gamma_i^N = 0$ for $i = 1, \dots, N$ if $\sigma_N = 0$. In particular, $\bar\gamma_1^N = 0$ is then trivially again an affine function of $\bar\gamma_1^1$.

Reversing the above steps shows that we can always construct vectors $\bar\gamma_1, \dots, \bar\gamma_N$ in $\mathbb{R}^N$ such that (6.6) holds; indeed, we use (6.9) to derive $\bar\gamma_1$ and then use (6.7) to get $\bar\gamma_2, \dots, \bar\gamma_N$. We also note that we have one degree of freedom because we can still choose $\bar\gamma_1^1$.

2) It remains to show that $\bar\gamma_1, \dots, \bar\gamma_N$ (or actually only $\bar\gamma_1^1$) can be chosen in such a way that $\bar D$ is S-tradable, meaning that $\bar D = V(\bar\vartheta) = \bar\vartheta \cdot S$ for some $\bar\vartheta \in \Theta_{++}^{\mathrm{sf}}$. Because the self-financing property is discounting-invariant, we write it for $\bar S$ instead of S as

$$V(\bar\vartheta, \bar S) = \bar\vartheta \cdot \bar S = V_0(\bar\vartheta, \bar S) + \int \bar\vartheta \, \mathrm{d}\bar S. \tag{6.10}$$

But $\bar\vartheta \cdot \bar S = (\bar\vartheta \cdot S)/\bar D \equiv 1$ if $\bar D$ is S-tradable via $\bar\vartheta$, and so (6.10) reduces to

$$0 = \bar\vartheta \, \mathrm{d}\bar S = \sum_{i=1}^N \bar\vartheta^i \, \mathrm{d}\bar S^i = \sum_{i=1}^N \bar\vartheta^i \bar S^i \, \mathrm{d}(\bar\gamma_i \cdot B), \tag{6.11}$$

where the last step uses (6.3). By switching from numbers of shares $\bar{\vartheta}^i$ to wealth amounts $\bar{\vartheta}^i \bar{S}^i =: \bar{\pi}^i$, we can rewrite (6.11) as

$$0 = \sum_{i=1}^N \bar{\pi}^i \,\mathrm{d}(\bar{\gamma}_i \cdot B) = \sum_{i=1}^N \bar{\pi}^i \sum_{k=1}^N \bar{\gamma}_i^k \,\mathrm{d}B^k = \sum_{k=1}^N \sum_{i=1}^N \bar{\pi}^i \bar{\gamma}_i^k \,\mathrm{d}B^k. \tag{6.12}$$

Note that $\sum_{i=1}^N \bar{\pi}^i = \sum_{i=1}^N \bar{\vartheta}^i \bar{S}^i = \bar{\vartheta} \cdot \bar{S} = 1$. Because $B^1, \dots, B^N$ are independent, (6.12) means that $\psi^k := \sum_{i=1}^N \bar{\pi}^i \bar{\gamma}_i^k$ must be zero for $k = 1, \dots, N$. But we know from (6.7) that $\bar{\gamma}_i = \bar{\gamma}_1 + \sigma_i c_i - \sigma_1 c_1$ for $i = 2, \dots, N$, and so the condition $\psi^k = 0$ becomes

$$0 = \sum_{i=1}^N \bar{\pi}^i \bar{\gamma}_i^k = \bar{\gamma}_1^k + \sum_{i=1}^N \bar{\pi}^i (\sigma_i c_i^k - \sigma_1 c_1^k) \qquad \text{for } k = 1, \dots, N. \tag{6.13}$$

Now we exploit again the lower triangular structure of C which tells us that we have $c_i^k = C^{ik} = 0$ for $k > i$, i.e., for $i < k$. This allows us to simplify (6.13) to

$$0 = \bar{\gamma}_1^k + \sum_{i=k}^N \bar{\pi}^i (\sigma_i c_i^k - \sigma_1 c_1^k) \qquad \text{for } k = 1, \dots, N. \tag{6.14}$$

Note that $c_1^k = 0$ for $k \geq 2$. Starting with $k = N$, we thus obtain

$$0 = \bar{\gamma}_1^N + \bar{\pi}^N \sigma_N c_N^N. \tag{6.15}$$

If $\sigma_N = 0$, we have seen in 1) that $\bar{\gamma}_1^N = 0$; so any choice of $\bar{\pi}^N$ satisfies (6.15). If $\sigma_N > 0$, using $c_N^N > 0$ shows that $\bar{\pi}^N$ is uniquely determined as a linear function of $\bar{\gamma}_1^N$ and hence by 1) as an affine function of $\bar{\gamma}_1^1$. For $k = N-1, \dots, 2$, we inductively write (6.14) as

$$0 = \bar{\gamma}_1^k + \bar{\pi}^k \sigma_k c_k^k + \sum_{i=k+1}^N \bar{\pi}^i \sigma_i c_i^k.$$

By the induction hypothesis and 1), respectively, the last sum and $\bar{\gamma}_1^k$ are both affine functions of $\bar{\gamma}_1^1$, and then so is $\bar{\pi}^k$ because $\sigma_k > 0$ and $c_k^k > 0$. This holds for $k = N, \dots, 2$. For $k = 1$, the summand for $i = 1$ vanishes and (6.14) becomes

$$0 = \bar{\gamma}_1^1 + \sum_{i=2}^N \bar{\pi}^i \sigma_i c_i^1. \tag{6.16}$$

But $\bar{\pi}^2, \dots, \bar{\pi}^N$ all are affine functions of $\bar{\gamma}_1^1$ as seen above, and so (6.16) is an affine equation for $\bar{\gamma}_1^1$ which obviously has a unique solution. This in turn determines $\bar{\pi}^2, \dots, \bar{\pi}^{N-1}$, and also $\bar{\pi}^N$ if $\sigma_N > 0$. Finally, $\bar{\pi}^1$ is given from the condition that $\sum_{i=1}^N \bar{\pi}^i = 1$. This completes the proof of 2).

The above proof shows that the vectors $\bar{\gamma}_i$ are unique, and so is then $\bar{D}$ by (6.6). □

6.2. **Explicit examples I.** This section gives explicit counterexamples for several wrong statements or implications. All these are based on the GBM setup from Section 6.1, and for concreteness and simplicity, we work with the BS model. So let $S_t^2 = e^{rt}$ and $S_t^1 = \exp(\sigma W_t + (m - \frac{1}{2}\sigma^2)t)$ with $m, r \in \mathbb{R}$ and $\sigma > 0$. We also need $X = S^1/S^2$ because $S/S^2 = (X, 1)$.

Example 6.6. *DSV for $\mathbb{1}$ does not imply DSE for $\mathbb{1}$.* If we take $m - r \in \{0, \sigma^2\}$, S satisfies DSV for $\mathbb{1}$ by Corollary 6.5. But S never satisfies DSE for $\mathbb{1}$, by Theorem 6.4, 2).

Example 6.7. *NFLVR for* $(X, 1)$ *does not imply DSE for* $\mathbb{1}$. Take $m = r$ so that X is a martingale; then clearly $S/S^2 = (X, 1)$ satisfies NFLVR_∞. But again by Theorem 6.4, 2), S never satisfies DSE for $\mathbb{1}$, and neither does S/S^2 because DSE is discounting-invariant.

Example 6.8. *DSV for* $\mathbb{1}$ *does not imply NUPBR for* $(X, 1)$. Now take $m - r = \sigma^2$ so that $X' = 1/X = S^2/S^1$ is a martingale. Then $(X, 1) = S/S^2$ satisfies DSV for $\mathbb{1}$ because S does by Corollary 6.5. However, $X'_t = \exp(-\sigma W_t - \frac{1}{2}\sigma^2 t) \to 0$ P-a.s. as $t \to \infty$; so $\lim_{t\to\infty} X_t = +\infty$ P-a.s. and $(X, 1)$ does not satisfy NUPBR_∞.

6.3. **Explicit examples II.** Some of our examples need models S which satisfy DSE, or UI martingales, and these requirements cannot be satisfied in the setup of Section 6.1. Theorem 6.4 shows that the GBM model never satisfies DSE for $\mathbb{1}$, and the appearing martingales are always stochastic exponentials $\mathcal{E}(\alpha\tilde{B})$ of some constant multiple of some Brownian motion $\tilde{B}$. Except for $\alpha = 0$ where $\mathcal{E}(\alpha\tilde{B}) \equiv 1$, such a martingale is never UI because it converges to 0 P-a.s. So we need to construct our examples in a different way.

For ease of exposition, we work in this section in (infinite) discrete time. Via piecewise constant interpolations of processes (LCRL for predictable, RCLL for optional) and piecewise constant filtrations, our models can be embedded in a continuous-time framework. Moreover, we use (only in this subsection) the notation $\Delta Y_n := Y_n - Y_{n-1}$ for the increment at time n of a generic discrete-time process $Y = (Y_n)_{n\in\mathbb{N}_0}$. Our examples have two building blocks.

A first basic ingredient is a martingale Y whose increments (or successor values) in each step only take two (different) values. The martingale condition then uniquely determines all one-step transition probabilities as a function of the Y-values, and so we can talk about "the" corresponding martingale. By choosing the increments or values in a suitable way, we can moreover ensure that Y is nonnegative and bounded, hence UI and P-a.s. convergent to some Y_∞ which closes Y on the right as a martingale (i.e., $Y = (Y_n)_{n\in\mathbb{N}_0\cup\{\infty\}}$ is a martingale). Finally, one can also ensure that Y_∞ only takes two values one of which is 0, and thus we obtain a UI martingale which converges to 0 with positive probability.

The second idea is more subtle. We want to work with a two-asset model and trade in such a way that our strategy involves the asymptotic behaviour of both assets in a specific nontrivial way. To this end, we construct $S = (S^1, S^2)$ such that in each step, exactly one of the assets has a price move, and these moves always alternate. This allows to predict which asset coordinate will move in the next step, which can be exploited to construct (switching) strategies with a desired behaviour; and as both coordinates move alternatingly, the resulting wealth process is influenced by each coordinate in turn.

Example 6.9. *DSV for* η *is not equivalent to the existence of a* σ*-martingale discounter* D *for* S*; the condition* $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P*-a.s. in Theorem 2.12 is indispensable.* To show this, we take $\eta \equiv \mathbb{1}$ and construct a bounded $\mathbb{R}^2$-valued martingale $S \geq 0$ satisfying $P[\lim_{t\to\infty} S_t = 0 \in \mathbb{R}^2] > 0$. Then $D \equiv 1$ is a UI martingale discounter for S and we have

$$P\Big[\inf_{t\geq 0}\big(\eta_t \cdot (S_t/D_t)\big) = 0\Big] \geq P\Big[\lim_{t\to\infty}(\mathbb{1} \cdot S_t) = 0\Big] > 0.$$

We then show that S does not satisfy DSV for $\mathbb{1}$.

To start the construction, let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the (unique) real-valued martingale with $Y_0 = 1$ which at any time $n \in \mathbb{N}$ only takes the two values $u_n = 2 - 2^{-n}$

or $d_n = 2^{-n}$. Then Y is P-a.s. strictly positive (but not bounded away from 0 uniformly in n) and bounded by 2. So (Y_n) converges to Y_∞ P-a.s., and clearly $P[Y_\infty = 2] = \frac{1}{2} = P[Y_\infty = 0]$.

Now let Y^1, Y^2 be independent copies of Y and define $S = (S^1, S^2)$ by $S^1_0 = 1$ and

$$S^1_{2n-1} = S^1_{2n} = Y^1_n \quad \text{for } n \in \mathbb{N}, \qquad S^2_{2n} = S^2_{2n+1} = Y^2_n \quad \text{for } n \in \mathbb{N}_0.$$

This gives for $n \in \mathbb{N}$ that $\Delta S^1_{2n-1} = \Delta Y^1_n$, $\Delta S^1_{2n} = 0$ and $\Delta S^2_{2n-1} = 0$, $\Delta S^2_{2n} = \Delta Y^2_n$ and in particular yields that the coordinates of S move alternatingly because

$$\Delta S^2_n I_{\{\Delta S^1_{n-1}=0\}} = 0 = \Delta S^1_n I_{\{\Delta S^2_{n-1}=0\}}. \tag{6.17}$$

Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the filtration generated by S. As S is like Y a bounded martingale, it converges to S_∞ P-a.s., and $B := \{\lim_{n\to\infty}(\mathbb{1}\cdot S_n) = 0\} = \{S_\infty = 0\}$ has $P[B] = \frac{1}{4} > 0$.

Because $(S_n)_{n\in\mathbb{N}_0}$ is strictly positive, $\eta \equiv \mathbb{1}$ is a reference strategy. If S satisfies DSV for $\mathbb{1}$, then 0 is share maximal for $\mathbb{1}$, hence value maximal for $S^{\mathbb{1}}$ by Theorem 3.10, 1) for $\xi \equiv \mathbb{1}$, and so Lemma 3.11 yields $\sup_{n\in\mathbb{N}_0}(\vartheta_n \cdot S_n)/(\mathbb{1}\cdot S_n) < \infty$ P-a.s. for all $\vartheta \in \Theta^{\mathrm{sf}}_+$. Write $(\vartheta\cdot S)/(\mathbb{1}\cdot S) = V(\vartheta)/(\mathbb{1}\cdot S)$. We exhibit below a strategy $\bar\vartheta \in \Theta^{\mathrm{sf}}_+$ with $V(\bar\vartheta) \equiv \varepsilon > 0$. This yields $\sup_{n\in\mathbb{N}_0}(\bar\vartheta_n \cdot S_n)/(\mathbb{1}\cdot S_n) = +\infty$ on B, and so S cannot satisfy DSV for $\mathbb{1}$.

To construct $\bar\vartheta$, we fix $\varepsilon > 0$ and consider the strategy which invests the amount ε at time 0 in asset 2 and subsequently reinvests at any time all its wealth into that asset which will not jump in the next period. More formally, we set $\bar\vartheta_0 := \bar\vartheta_1 := (0, \varepsilon)$ and

$$\bar\vartheta_{n+1} := I_{\{\Delta S^1_n=0\}}\left(0, \frac{\varepsilon}{S^2_n}\right) + I_{\{\Delta S^2_n=0\}}\left(\frac{\varepsilon}{S^1_n}, 0\right). \tag{6.18}$$

This is well defined because S^1, S^2 are both strictly positive, and $\mathcal{F}_n$-measurable (so that $\bar\vartheta$ is predictable) because S is adapted. Moreover, $S^2_0 = S^2_1 = 1$ yields $V_0(\bar\vartheta) = V_1(\bar\vartheta) = \varepsilon$, and

$$V_{n+1}(\bar\vartheta) = I_{\{\Delta S^1_n=0\}}\,\varepsilon\frac{S^2_{n+1}}{S^2_n} + I_{\{\Delta S^2_n=0\}}\,\varepsilon\frac{S^1_{n+1}}{S^1_n} = \varepsilon$$

as S^1, S^2 always jump alternatingly. So $V(\bar\vartheta) \equiv \varepsilon$, and $\bar\vartheta$ is also self-financing because

$$\Delta V_{n+1}(\bar\vartheta) - \bar\vartheta_{n+1}\cdot\Delta S_{n+1} = 0 - \bar\vartheta^1_{n+1}\Delta S^1_{n+1} - \bar\vartheta^2_{n+1}\Delta S^2_{n+1} \equiv 0$$

due to (6.18) and (6.17). So $\bar\vartheta$ has all the claimed properties, and this ends the example.

Example 6.10. *DSE for η need not imply NFLVR$_\infty$, not even for a classic model of the form $S = (1, X)$.* Similarly as in Example 6.9, let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the (unique) real-valued martingale valued in $(0,1)$ with $Y_0 = \frac{1}{2}$ and $Y_n \in \{\frac{1}{2}2^{-n}, 1 - \frac{1}{2}2^{-n}\}$. This converges P-a.s. to Y_∞ which takes the values 0 and 1 each with probability $\frac{1}{2}$. Set $Y' := 1 - Y$ and define

$$S := (1, X) := \left(1, \frac{Y'}{Y}\right) = \left(1, \frac{1-Y}{Y}\right).$$

Then $\mathbb{1}\cdot S = \frac{1}{Y}$ and so $S^{\mathbb{1}} = S/(\mathbb{1}\cdot S) = (Y, 1-Y)$ is a bounded P-martingale with $\mathbb{1}\cdot S^{\mathbb{1}} \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$. So S satisfies (e3) in Theorem 4.1 with $D = \mathbb{1}\cdot S$ and $\eta \equiv \mathbb{1}$, and this implies that S satisfies DSE for $\mathbb{1}$. However, we clearly have $X \geq 0$ and

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} \frac{1-Y_n}{Y_n} = +\infty \qquad \text{on } \{\lim_{n\to\infty} Y_n = 0\} = \{Y_\infty = 0\} =: B.$$

As $P[B] = \frac{1}{2} > 0$, $S = (1, X)$ does not satisfy NUPBR$_\infty$ and thus also not NFLVR$_\infty$.

Appendix. This section contains some technical proofs and auxiliary results.

For any function $z : [0,\infty) \to \mathbb{R}^N$, set $\underline{z}(\infty) := \liminf_{t\to\infty} z(t)$ (coordinatewise). If the limit exists, write $z(\infty) := \lim_{t\to\infty} z(t)$. In $\mathbb{R}_+$, the product of ∞ and 0 is 0.

Lemma A.1. *Suppose the functions $x, y : [0,\infty) \to \mathbb{R}^N$ satisfy*
(a) *$y \geq 0$ is bounded (uniformly in $t \geq 0$) by some $C < \infty$;*
(b) *$\underline{x}^i(\infty) \geq 0$ for $i = 1, \ldots, N$.*
Then

$$\underline{(x\cdot y)}(\infty) \geq \underline{x}(\infty) \cdot \underline{y}(\infty). \tag{A.1}$$

Proof. Fix $\varepsilon > 0$. Decompose $\{1, \ldots, N\}$ into indices ℓ with $\underline{x}^\ell(\infty) = \infty$ and indices m with $\underline{x}^m(\infty) < \infty$. For any ℓ and $t \geq T = T(\ell)$, we have $x^\ell(t) \geq 0$ and $y^\ell(t) \geq \frac{1}{2}\underline{y}^\ell(\infty)$, and for any m, we get $x^m(t) \geq \underline{x}^m(\infty) - \varepsilon$ for $t \geq T = T(m,\varepsilon)$ and $0 \leq y^m(t) \leq C$ for all t. This implies

$$x^m(t)y^m(t) \geq (\underline{x}^m(\infty) - \varepsilon)y^m(t) \geq \underline{x}^m(\infty)y^m(t) - \varepsilon C$$

and therefore

$$\begin{aligned}(x\cdot y)(t) &= \sum_\ell x^\ell(t)y^\ell(t) + \sum_m x^m(t)y^m(t)\\ &\geq \frac{1}{2}\sum_\ell x^\ell(t)\underline{y}^\ell(\infty) + \sum_m \big(\underline{x}^m(\infty)y^m(t) - \varepsilon C\big).\end{aligned}$$

Let $t \to \infty$ and use on the right-hand side the superadditivity of lim inf, $y \geq 0$ and the fact that $\underline{x}^m(\infty) \in [0,\infty)$ for all m, to obtain

$$\underline{(x\cdot y)}(\infty) \geq \frac{1}{2}\sum_\ell \underline{x}^\ell(\infty)\underline{y}^\ell(\infty) + \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C.$$

If there is an ℓ with $\underline{y}^\ell(\infty) > 0$, the right-hand side is $+\infty$ and (A.1) holds trivially. So we can assume for the rest of the proof that $\underline{y}^\ell(\infty) = 0$ for all ℓ; then $\underline{x}^\ell(\infty)\underline{y}^\ell(\infty) = 0$ for all ℓ by our convention, and we end up with

$$\underline{(x\cdot y)}(\infty) \geq \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C = \sum_{i=1}^N \underline{x}^i(\infty)\underline{y}^i(\infty) - N\varepsilon C.$$

Letting $\varepsilon \searrow 0$ then again gives (A.1) and completes the proof. ☐

Proof of Lemma 3.1. If ϑ is not vm for S/D, there are $f \in L^0_+\setminus\{0\}$ and for any $\varepsilon > 0$ some $\hat\vartheta^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat\vartheta^\varepsilon, S/D) \leq V_0(\vartheta, S/D) + \varepsilon$, hence $V_0(\hat\vartheta^\varepsilon) \leq V_0(\vartheta) + \varepsilon D_0$, and

$$\liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \vartheta, S/D) \geq f \geq 0 \qquad P\text{-a.s.} \tag{A.2}$$

As $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ has $\inf_{t\geq 0} D_t > 0$ P-a.s., $f' := f \liminf_{t\to\infty} D_t$ is in $L^0_+\setminus\{0\}$. Because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ also has $\sup_{t\geq 0} D_t < \infty$ P-a.s., (A.2) implies by Lemma A.1 that

$$\begin{aligned}\liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \vartheta) &= \liminf_{t\to\infty} \big(V_t(\hat\vartheta^\varepsilon - \vartheta, S/D)D_t\big)\\ &\geq \liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \vartheta, S/D) \liminf_{t\to\infty} D_t \geq f' \qquad P\text{-a.s.}\end{aligned}$$

This shows that ϑ is not vm for S either.

For the second part, if $\alpha\vartheta$ is not vm for S, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat\vartheta^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat\vartheta^\varepsilon) \le V_0(\alpha\vartheta) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \alpha\vartheta) \ge f$ P-a.s. Now we distinguish two cases. If $\alpha > 0$, then $\tilde\vartheta := \hat\vartheta^\varepsilon/\alpha \in \Theta^{\mathrm{sf}}_+$ satisfies

$$V_0(\tilde\vartheta) = V_0(\hat\vartheta^\varepsilon)/\alpha \le V_0(\vartheta) + \varepsilon/\alpha,$$
$$\liminf_{t\to\infty} V_t(\tilde\vartheta - \vartheta) = \liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \alpha\vartheta)/\alpha \ge f/\alpha \qquad P\text{-a.s.}$$

So ϑ is not vm for S as f/α is in $L^0_+ \setminus \{0\}$. If $\alpha = 0$, then $\tilde\vartheta := \vartheta + \hat\vartheta^\varepsilon \in \Theta^{\mathrm{sf}}_+$ has

$$V_0(\tilde\vartheta) \le V_0(\vartheta) + V_0(\alpha\vartheta) + \varepsilon = V_0(\vartheta) + \varepsilon,$$
$$\liminf_{t\to\infty} V_t(\tilde\vartheta - \vartheta) = \liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon - \alpha\vartheta) \ge f \qquad P\text{-a.s.;}$$

so again ϑ is not vm for S. ☐

Proof of Lemma 3.12. For brevity, we define the set $\Gamma := \{V_\tau(\vartheta^1) < V_\tau(\vartheta^2)\} \in \mathcal{F}_\tau$ and set $\varphi := \vartheta^1 \oslash^\xi_\tau \vartheta^2$. We use $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$, which also gives $\xi \bullet S^\xi \equiv 0$. Then using the definition of φ, the general fact that $XI_{[\![0,\tau]\!]} = X^\tau - X_\tau I_{]\!]\tau,\infty[\![}$, the fact that ϑ^1, ϑ^2 are self-financing and again the definition of φ yields

$$\begin{aligned}
&V(\varphi, S^\xi)\\
&= I_{[\![0,\tau]\!]} V(\vartheta^1, S^\xi) + I_{]\!]\tau,\infty[\![}\big(I_\Gamma V(\vartheta^1, S^\xi) + I_{\Gamma^c} V(\vartheta^2, S^\xi) + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\big)\\
&= \big(V(\vartheta^1, S^\xi)\big)^\tau + I_{]\!]\tau,\infty[\![}\Big(I_\Gamma\big(V(\vartheta^1, S^\xi) - V_\tau(\vartheta^1, S^\xi)\big) + I_{\Gamma^c}\big(V(\vartheta^2, S^\xi) - V_\tau(\vartheta^2, S^\xi)\big)\Big)\\
&= V_0(\vartheta^1, S^\xi) + (\vartheta^1 I_{[\![0,\tau]\!]})\bullet S^\xi + \Big(I_{]\!]\tau,\infty[\![}\big(I_\Gamma \vartheta^1 + I_{\Gamma^c}\vartheta^2 + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\xi\big)\Big)\bullet S^\xi\\
&= V_0(\varphi, S^\xi) + \varphi\bullet S^\xi.
\end{aligned}$$

This shows that φ is self-financing. If both ϑ^1, ϑ^2 are in Θ^{sf}_+, the second line above is nonnegative so that also φ is in Θ^{sf}_+. ☐

The next auxiliary result is extracted from the proof of [29, Proposition 1].

Lemma A.2. *A convex set $C \subseteq L^0_+$ is bounded in L^0 if and only if C contains no sequence $(V^n)_{n\in\mathbb{N}}$ satisfying $V^n \ge n\xi$ P-a.s. for all $n \in \mathbb{N}$ and for some $\xi \in L^0_+ \setminus \{0\}$.*

Proof. The only if part is clear. For the if part, suppose C is not bounded in L^0 and let $\Omega_u \in \mathcal{F}$ be as in [7, Lemma 2.3]. (In the terminology of [7], C is hereditarily unbounded in probability on Ω_u.) Note that $P[\Omega_u] > 0$ because $P[\Omega_u] = 0$ would imply that C is bounded in L^0. Then [7, Lemma 2.3, part 4)] implies with $\varepsilon := 2^{-n}$ that for each $n \in \mathbb{N}$, there is some $V^n \in C$ such that

$$P[\{V^n \le n\} \cap \Omega_u] \le P[\{V^n \le 2^n\} \cap \Omega_u] \le 2^{-n}.$$

Take $N \in \mathbb{N}$ with $\sum_{n=N}^\infty 2^{-n} \le P[\Omega_u]/2$. For $n \ge N$, set $A_n := \{V^n > n\} \cap \Omega_u \in \mathcal{F}$ and define $A := \bigcap_{n\ge N} A_n \in \mathcal{F}$ so that $V^n \ge nI_{A_n} \ge nI_A$ due to $V^n \in C \subseteq L^0_+$. Then

$$P[A] \ge P[\Omega_u] - \sum_{n=N}^\infty P[A_n^c \cap \Omega_u] \ge P[\Omega_u]/2 > 0$$

shows that $\xi := I_A \in L^0_+ \setminus \{0\}$, and we have $V^n \ge n\xi$ P-a.s. for all $n \in \mathbb{N}$. But this contradicts the assumption for the if part, and so we are done. ☐

Acknowledgments. We gratefully acknowledge financial support by the ETH Foundation via the Stochastic Finance Group at ETH Zurich and by the Swiss Finance Institute (SFI).

This paper has been presented on many different occasions, and we thank everyone, including two anonymous referees and a thorough Associate Editor, who provided questions, criticisms and general feedback. It is a pleasure to thank Michael Obergfell who worked out a substantial part of the example in Section 6.1.

REFERENCES

[1] B. Acciaio, C. Fontana and C. Kardaras, Arbitrage of the first kind and filtration enlargements in semimartingale financial models, *Stochastic Process. Appl.*, **126** (2016), 1761–1784.

[2] A. Aksamit, T. Choulli, J. Deng and M. Jeanblanc, No-arbitrage up to random horizon for quasi-left-continuous models, *Finance Stoch.*, **21** (2017), 1103–1139.

[3] J.-P. Ansel and C. Stricker, Couverture des actifs contingents et prix maximum, *Ann. Inst. H. Poincaré Probab. Statist.*, **30** (1994), 303–315.

[4] K. E. Back, *Asset Pricing and Portfolio Choice Theory*, Oxford University Press, 2010.

[5] D. Á. Bálint and M. Schweizer, Large financial markets, discounting, and no asymptotic arbitrage, *Theory Probab. Appl.*, **65** (2020), 191–223.

[6] D. Á. Bálint and M. Schweizer, Properly discounted asset prices are semimartingales, *Math. Financ. Econ.*, **14** (2020), 661–674.

[7] W. Brannath and W. Schachermayer, A bipolar theorem for $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$, In: *J. Azéma et al. (eds.), Séminaire de Probabilités XXXIII*, Lecture Notes in Mathematics, vol. 1709, Springer, Berlin, 1999, 349–354.

[8] H. N. Chau, A. Cosso, C. Fontana and O. Mostovyi, Optimal investment with intermediate consumption under no unbounded profit with bounded risk, *J. Appl. Probab.*, **54** (2017), 710–719.

[9] T. Choulli, J. Deng and J. Ma, How non-arbitrage, viability and numéraire portfolio are related, *Finance Stoch.*, **19** (2015), 719–741.

[10] F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing, *Math. Ann.*, **300** (1994), 463–520.

[11] F. Delbaen and W. Schachermayer, The no-arbitrage property under a change of numéraire, *Stochastics Stochastics Rep.*, **53** (1995), 213–226.

[12] F. Delbaen and W. Schachermayer, The Banach space of workable contingent claims in arbitrage theory, *Ann. Inst. H. Poincaré Probab. Statist.*, **33** (1997), 113–144.

[13] F. Delbaen and W. Schachermayer, The fundamental theorem of asset pricing for unbounded stochastic processes, *Math. Ann.*, **312** (1998), 215–250.

[14] ———, *The Mathematics of Arbitrage*, Springer, Berlin, 2006.

[15] D. Filipović and E. Platen, Consistent market extensions under the benchmark approach, *Math. Finance*, **19** (2009), 41–52.

[16] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed., Johns Hopkins University Press, Baltimore, 2013.

[17] M. Herdegen, No-arbitrage in a numéraire-independent modeling framework, *Math. Finance*, **27** (2017), 568–603.

[18] M. Herdegen and M. Schweizer, Strong bubbles and strict local martingales, *Int. J. Theor. Appl. Finance*, **19** (2016), 1650022, 44 pp.

[19] J. Jacod, *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, vol. 714, Springer, Berlin, 1979.

[20] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Second ed., Springer-Verlag, Berlin, 2003.

[21] Y. Kabanov, C. Kardaras and S. Song, No arbitrage of the first kind and local martingale numéraires, *Finance Stoch.*, **20** (2016), 1097–1108.

[22] Y. Kabanov and M. Safarian, *Markets with Transaction Costs. Mathematical Theory*, Springer Finance. Springer-Verlag, Berlin, 2009.

[23] Yu. M. Kabanov, On the FTAP of Kreps–Delbaen–Schachermayer, In: *Yu. M. Kabanov et al. (eds.), Statistics and Control of Stochastic Processes: The Liptser Festschrift*, World Scientific, River Edge, NJ, 1997, 191–203.

[24] Yu. M. Kabanov and D. O. Kramkov, Large financial markets: Asymptotic arbitrage and contiguity, *Theory Probab. Appl.*, **39** (1995), 182–187.

[25] J. Kallsen, σ-localization and σ-martingales, *Teor. Veroyatnost. i Primenen.*, **48** (2003), 152–163.

[26] I. Karatzas and R. Fernholz, Stochastic portfolio theory: An overview, In: *A. Bensoussan and Q. Zhang (eds.), Handbook of Numerical Analysis, Special Volume: Mathematical Modeling and Numerical Methods in Finance*, Elsevier, 2009, 89–167.

[27] I. Karatzas and C. Kardaras, The numéraire portfolio in semimartingale financial models, *Finance Stoch.*, **11** (2007), 447–493.

[28] ———, *Portfolio Theory and Arbitrage: A Course in Mathematical Finance*, American Mathematical Society, 2021.

[29] C. Kardaras, Finitely additive probabilities and the fundamental theorem of asset pricing, In: *C. Chiarella and A. Novikov (eds.), Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen*, Springer, Berlin, 2010, 19–34.

[30] C. Kardaras, Numéraire-invariant preferences in financial modeling, *Ann. Appl. Probab.*, **20** (2010), 1697–1728.

[31] C. Kardaras, Market viability via absence of arbitrage of the first kind, *Finance Stoch.*, **16** (2012), 651–667.

[32] ———, A time before which insiders would not undertake risk, In: *Y. Kabanov et al. (eds.), Inspired by Finance. The Musiela Festschrift*, Springer, 2014, 349–362.

[33] C. Kardaras and E. Platen, On the semimartingale property of discounted asset price-processes, *Stochastic Process. Appl.*, **121** (2011), 2678–2691.

[34] M. Loewenstein and G. A. Willard, Local martingales, arbitrage, and viability. Free snacks and cheap thrills, *Economic Theory*, **16** (2000), 135–161.

[35] E. Platen and D. Heath, *A Benchmark Approach to Quantitative Finance*, Springer Finance. Springer-Verlag, Berlin, 2006.

[36] P. A. Samuelson, Proof that properly anticipated prices fluctuate randomly, *Industrial Management Review*, **6**(2) (1965), 41–49.

[37] A. N. Shiryaev and A. S. Chernyi, Vector stochastic integrals and the fundamental theorems of asset pricing, *Proc. Steklov Inst. Math.*, **237** (2002), 6–49.

[38] C. A. Sin, *Strictly Local Martingales and Hedge Ratios on Stochastic Volatility Models*, PhD thesis, Cornell University (1996), available online at https://ecommons.cornell.edu/bitstream/handle/1813/9055/TR001171.pdf.

[39] K. Takaoka and M. Schweizer, A note on the condition of no unbounded profit with bounded risk, *Finance Stoch.*, **18** (2014), 393–405.

[40] M. R. Tehranchi, Arbitrage theory without a numéraire, preprint, arXiv:1410.2976v2, 2015.

[41] J.-A. Yan, A new look at the fundamental theorem of asset pricing, *J. Korean Math. Soc.*, **35** (1998), 659–673.

Received July 2021; revised December 2021; early access May 2022.

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