Dynamic indifference valuation via convex risk measures

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Abstract: The (subjective) indifference value of a payoff in an incomplete financial market is that monetary amount which leaves an agent indifferent between buying or not buying the payoff when she always optimally exploits her trading opportunities. We study these values over time when they are defined with respect to a dynamic monetary concave utility functional, i.e., minus a dynamic convex risk measure. For that purpose, we prove some new results about families of conditional convex risk measures. We study the convolution of abstract conditional convex risk measures and show that it preserves the dynamic property of time-consistency. Moreover, we construct a dynamic risk measure (or utility functional) associated to superreplication in a market with trading constraints and prove that it is time-consistent. By combining these results, we deduce that the corresponding indifference valuation functional is again time-consistent. As an auxiliary tool, we establish a variant of the representation theorem for conditional convex risk measures in terms of equivalent probability measures.

Key words: utility indifference valuation, monetary concave utility functionals, time-consistency, convolution, representation of risk measures, convex risk measures, incomplete markets

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0. Introduction

This paper introduces and studies valuation by indifference in an incomplete financial market for an agent whose time t preferences over payoffs are given by a *monetary concave utility functional (MCUF)* Φ_t. The corresponding indifference valuation p_t is determined by requiring that it leaves the agent indifferent, when she optimally exploits her trading opportunities, between buying or not buying the payoff X to be valued. Formally, p_t is defined via

$$\operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \Phi_t\big(g - p_t(X) + X\big) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \Phi_t(g),$$

where the set $\mathcal{C}_t$ of payoffs superreplicable from time t with zero wealth encodes the trades available in the market. Both $\Phi_t(X)$ and $p_t(X)$ are $\mathcal{F}_t$-measurable.

While this approach has been extensively studied for the case where Φ_t comes from a von Neumann-Morgenstern expected utility, we work here with *monetary* utility functionals. These are *translation invariant* in the sense that $\Phi_t(X + a_t) = \Phi_t(X) + a_t$ if a_t is $\mathcal{F}_t$-measurable. Hence $-\Phi_t$ is simply a (conditional) convex risk measure, and we are led to study general questions about conditional risk measures. It turns out that up to normalization, p_t is as in Barrieu/El Karoui (2005) the *convolution* of Φ_t with the *market functional*; the latter is associated to $\mathcal{C}_t$ and constructed like in Föllmer/Schied (2002) from the underlying financial market with the help of the optional decomposition under constraints. Thus we need results about the convolution of two abstract conditional convex risk measures. Because pricing in financial markets is done with the help of equivalent martingale measures, we also want a *representation* for conditional convex risk measures in terms of their convex conjugate functionals via *equivalent* probability measures. We obtain a result which is sharper than those in the existing literature. Finally, a key issue is to ensure *time-consistency* for the dynamic behaviour of $p = (p_t)$.

Although various aspects of our approach have been studied before, the combined treatment of all ideas at the general and conditional level seems to be new. Most previous results are only given unconditionally for $t = 0$; this applies to the indifference valuation via risk measures in Xu (2006) or (briefly) in Barrieu/El Karoui (2005), to the construction of the market functional in Föllmer/Schied (2002), or to the convolution in Barrieu/El Karoui (2005). Some conditional results are available; Detlefsen/Scandolo (2005) and Cheridito/Delbaen/Kupper (2006) provide representations for convex risk measures, Barrieu/El Karoui (2004) discuss the convolution for dynamic MCUFs which are given by backward stochastic differential equations (BSDEs), and Larsen/Pirvu/Shreve/Tütüncü (2005) treat indifference valuation for a special Φ_t. Jobert/Rogers (2006) study several of the above issues in finite discrete time over a finite probability space. Our general results that convolution preserves time-consistency and that the market functional in an incomplete market with trading constraints is time-consistent seem to be new.

The paper is structured as follows. After briefly summarizing notation in Section 1, we define and study dynamic MCUFs $\Phi = (\Phi_t)$ in Section 2. We prove in Theorem 4 a representation of Φ_t in terms of its concave conjugate functional α_t via equivalent probability measures, and we also define and characterize (strong) time-consistency. Section 3 studies the convolution of two abstract MCUFs and proves in Theorem 7 that this operation preserves (strong) time-consistency. The arguments rely on the representation result from Section 2. In Section 4, we first present indifference valuation for a general setting before we construct in Theorem 11 the market DMCUF associated to a family of sets $\mathcal{C}_t$ describing a financial market with trading constraints. In particular, we show that this DMCUF is strongly time-consistent. Combining this with the convolution results from Section 3 immediately gives the valuation functional p and its desired properties. Finally, we discuss some connections between this valuation approach and arbitrage opportunities.

This paper is a shortened and condensed version of Klöppel/Schweizer (2005), henceforth abbreviated as KS. In addition to giving much more details and discussions, KS also contains a section devoted to the case where Φ, and then also p, can be described by BSDEs, and some results on connections to good-deal bounds.

1. Notations

We work on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $I\!F = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying under P the usual conditions. $T \in (0, \infty)$ is a fixed time horizon and we assume that $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_0$ is P-trivial. We denote by $L^\infty(\mathcal{F}_t)$ the space of all (equivalence classes of) $\mathcal{F}_t$-measurable random variables in $L^\infty = L^\infty(P)$ and by $L^0(\mathcal{F}_t; Y)$ the set of all (equivalence classes of) $\mathcal{F}_t$-measurable mappings $\Omega \to Y$. An *$\mathcal{F}_t$-partition* is a family of pairwise disjoint sets $(A_n)_{n \in I\!N}$ in $\mathcal{F}_t$ whose union is Ω. For a subset $A \subseteq L^\infty$, we write $\overline{A}$ for the closure of A in $\sigma(L^\infty, L^1)$.

As in Föllmer/Schied (2004), $\mathcal{M}_1$ denotes the set of all probability measures Q on $(\Omega, \mathcal{F})$ and $\mathcal{M}_1(P)$ the set of all $Q \in \mathcal{M}_1$ with $Q \ll P$. We identify $\mathcal{M}_1(P)$ with a subset of $L^1 = L^1(P)$ via the density $\frac{dQ}{dP}$ and call the RCLL P-martingale $Z^Q_\centerdot := E_P\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_\centerdot\right]$ the density process of Q with respect to P. For any subset $\mathcal{Q} \subseteq \mathcal{M}_1$, we set $\mathcal{Q}^e := \{Q \in \mathcal{Q} \,|\, Q \approx P\}$. All (in)equalities are assumed to hold P-a.s.

2. Monetary concave utility functionals

In this section, we study dynamic monetary concave utility functionals $\Phi = (\Phi_t)$; these are up to a sign families of conditional convex risk measures. Theorem 4 provides a representation of Φ_t in terms of its concave conjugate functional α_t via equivalent probability measures. This is a variant of recent results by Detlefsen/Scandolo (2005) and Cheridito/Delbaen/Kupper (2006). We also define and characterize time-consistency.

A textbook account on (static) risk measures can be found in Föllmer/Schied (2004), and we frequently refer to the papers by Barrieu/El Karoui (2005), Cheridito/Delbaen/Kupper (2006) and Detlefsen/Scandolo (2005). These are abbreviated FS, BEK, CDK and DS.

Definition. Fix $t \in [0,T]$. A *monetary concave utility functional (MCUF) at time t* is a mapping $\Phi_t : L^\infty \to L^\infty(\mathcal{F}_t)$ satisfying

A) *monotonicity:* $X_1 \le X_2$ implies $\Phi_t(X_1) \le \Phi_t(X_2)$,

B) *($\mathcal{F}_t$-)translation invariance:* $\Phi_t(X + a_t) = \Phi_t(X) + a_t \quad$ for $a_t \in L^\infty(\mathcal{F}_t)$,

C) *concavity:* $\Phi_t\big(\beta X_1 + (1-\beta)X_2\big) \ge \beta\Phi_t(X_1) + (1-\beta)\Phi_t(X_2) \quad$ for $\beta \in [0,1]$.

We say that an MCUF Φ_t is *normalized* if $\Phi_t(0) = 0$. A *monetary coherent utility functional (MCohUF) at time t* is an MCUF satisfying in addition

D) *positive homogeneity:* $\Phi_t(\lambda X) = \lambda \Phi_t(X)$ for $\lambda \ge 0$.

A *dynamic MCUF* (or *dynamic MCohUF*), shortly DMCUF (or DMCohUF), is a family $\Phi = (\Phi_t)_{0 \le t \le T}$ such that each Φ_t is an MCUF (or MCohUF) at time t.

For an interpretation of Φ_t, view X as a discounted payoff at time T expressed in monetary units of some numeraire that can be freely transferred over time. All payoffs occur at time T only, and in particular we do not consider payoff streams. We interpret $\Phi_t(X)$ as the subjective usefulness (or utility), in monetary units, that some agent assigns to X at time t; this explains and motivates the translation invariance B), sometimes also called monetary property. However, it need not be possible to sell X at time t for the amount $\Phi_t(X)$, since this requires an agent willing to pay $\Phi_t(X)$ for X and such an agent need not exist. Thus $\Phi_t(X)$ has more the character of a subjective value than of a price quoted in a market.

An additional property one might require of Φ_t is

E) *$\mathcal{F}_t$-regularity:* $\Phi_t(\mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2) = \mathbf{1}_A \Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2) \quad$ for $A \in \mathcal{F}_t$.

But M. Kupper has pointed out to us that monotonicity and translation invariance already imply E) as follows; see also Proposition 3.3 of CDK. First of all, we have $\mathbf{1}_A \Phi_t(X\mathbf{1}_A) = \mathbf{1}_A \Phi_t(X)$ for $X \in L^\infty$ and $A \in \mathcal{F}_t$, because A) and B) yield

$$\mathbf{1}_A \Phi_t(X) \mathrel{\substack{\le \\ \ge}} \mathbf{1}_A \Phi_t\big(X\mathbf{1}_A \pm \|X\|_{L^\infty}\mathbf{1}_{A^c}\big) = \mathbf{1}_A \Phi_t(X\mathbf{1}_A).$$

Applying this to $X = \mathbf{1}_A X_1 + \mathbf{1}_{A^c} X_2$ gives

$$\Phi_t(X) = \mathbf{1}_A \Phi_t(X\mathbf{1}_A) + \mathbf{1}_{A^c}\Phi_t(X\mathbf{1}_{A^c}) = \mathbf{1}_A \Phi_t(X_1) + \mathbf{1}_{A^c}\Phi_t(X_2).$$

Remarks. 1) For later use, we note that properties A) and B) imply that Φ_t is Lipschitz-continuous on L^∞. This is well known; see Lemma 4.3 in FS.

2) An MCUF Φ_t at time t automatically satisfies not only C), but even the stronger property of $\mathcal{F}_t$-concavity, where $\beta \in L^0(\mathcal{F}_t;[0,1])$. This can be proved by the standard measure-theoretic induction, using the $\mathcal{F}_t$-regularity and Lipschitz-continuity of Φ_t. So $-\Phi_t$ is almost an $\mathcal{F}_t$-conditional convex risk measure in the sense of DS; the only difference is that DS insist on having Φ_t normalized.

3) Since $\mathcal{F}_0$ is trivial, $-\Phi_0$ is simply a convex risk measure in the usual sense; see Chapter 4 in FS. We call $t = 0$ the *static* or *unconditional* case. $\diamond$

In contrast to most authors, we deliberately do not assume that an MCUF Φ_t is normalized. Normalization seems reasonable if X models a change in wealth, but it can be inappropriate if X is some payoff whose utility we want to measure. In fact, if our agent can trade in a financial market, she might obtain with zero initial capital a position she personally strictly prefers to the payoff 0. (This has nothing to do with arbitrage, but only with preferences.) In such a situation, she probably assigns non-zero utility to 0. Note that this again uses the idea that $\Phi_t(X)$ should in general be viewed as a subjective value/usefulness rather than as a (market) price, which must be normalized to avoid arbitrage.

MCohUFs are always normalized, and MCUFs can be normalized by subtracting $\Phi_t(0)$ from the original functional. This changes levels of utility, but not the ordering induced by Φ_t. However, insisting on normalization can lead to problems; see Example 1 in Section 4.

Definition. For an MCUF Φ_t at time t, the *acceptance set* is $\mathcal{A}_t := \{X \in L^\infty \,|\, \Phi_t(X) \geq 0\}$, and elements of $\mathcal{A}_t$ are called *acceptable* (with respect to Φ_t, to be precise).

It is well known from the theory of static risk measures that an MCUF Φ_0 at time 0 can be equivalently described by its acceptance set; see Propositions 4.6 and 4.7 in FS. This also holds true for the conditional case if we use (like in CDK) a conditional form of the L^∞-norm as follows. For $X \in L^\infty$ and $t \in [0,T]$, set $\|X\|_t := \operatorname{ess\,inf}\big\{m_t \in L^\infty(\mathcal{F}_t)\,\big|\,|X| \leq m_t\big\}$ and call $\mathcal{B} \subseteq L^\infty$ *closed with respect to* $\|\cdot\|_t$ if for any sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ such that $\lim\limits_{n\to\infty} \|X_n - X\|_t = 0$ for some $X \in L^\infty$, we also have $X \in \mathcal{B}$. This holds for instance if $\mathcal{B}$ is closed in $\sigma(L^\infty, L^1)$.

Lemma 1. *The acceptance set $\mathcal{A}_t$ of an MCUF Φ_t at time t has the following properties:*

i) *$\mathcal{A}_t$ is non-empty and convex.*

ii) $\operatorname{ess\,sup}\{m_t \in L^\infty(\mathcal{F}_t) \,|\, -m_t \in \mathcal{A}_t\} = \operatorname{ess\,sup}\big(-\mathcal{A}_t \cap L^\infty(\mathcal{F}_t)\big) \in L^\infty$.

iii) *$-\mathcal{A}_t$ is solid, i.e., $X \in \mathcal{A}_t$, $Y \in L^\infty$ and $Y \geq X$ imply that $Y \in \mathcal{A}_t$.*

iv) *$\mathcal{A}_t$ is $\mathcal{F}_t$-regular, i.e., $X, Y \in \mathcal{A}_t$ and $A \in \mathcal{F}_t$ implies that $\mathbf{1}_A X + \mathbf{1}_{A^c} Y \in \mathcal{A}_t$.*

Moreover, $\mathcal{A}_t$ is closed with respect to $\|\cdot\|_t$. Finally, if Φ_t is an MCohUF, then $\mathcal{A}_t$ is a cone containing 0.

Proof. For the closedness property, see Proposition 3.6 in CDK and the Appendix. The rest follows from the definition as in the static case; see Proposition 4.6 in FS. **q.e.d.**

Definition. A subset $\mathcal{B}$ of L^∞ with the properties i) – iv) in Lemma 1 is called a *pre-acceptance set at time* t.

Lemma 2. *Let $\mathcal{B}$ be a pre-acceptance set at time t and define a mapping $\Phi_t^{\mathcal{B}}$ on L^∞ by*

$$\Phi_t^{\mathcal{B}}(X) := \operatorname{ess\,sup}\{m_t \in L^\infty(\mathcal{F}_t) \,|\, X - m_t \in \mathcal{B}\} = \operatorname{ess\,sup}\big((X - \mathcal{B}) \cap L^\infty(\mathcal{F}_t)\big). \tag{2.1}$$

Then:

1) *$\Phi_t^{\mathcal{B}}$ is an MCUF at time t.*
2) *If $\mathcal{B}$ is in addition closed with respect to $\|\cdot\|_t$, then $\mathcal{B}$ is the acceptance set of $\Phi_t^{\mathcal{B}}$.*
3) *If $\mathcal{B}$ is the acceptance set $\mathcal{A}_t$ of some MCUF Φ_t at time t, then $\Phi_t = \Phi_t^{\mathcal{B}}$, i.e., we can recover Φ_t from its acceptance set $\mathcal{A}_t$ as $\Phi_t = \Phi_t^{\mathcal{A}_t}$.*
4) *If $\mathcal{B}$ is a cone containing 0, then $\Phi_t^{\mathcal{B}}$ is an MCohUF at time t.*

Proof. This follows from Proposition 3.10 of CDK; see Appendix. **q.e.d.**

Our next goal is now to provide a representation for an MCUF Φ_t via its concave conjugate functional, which is defined as follows.

Definition. The *concave conjugate functional* of an MCUF Φ_t at time t is the mapping $\alpha_t : \mathcal{P}_t^{\approx} \to L^0\big(\mathcal{F}_t; [-\infty, +\infty)\big)$,

$$Q \mapsto \alpha_t(Q) := \operatorname*{ess\,inf}_{X \in L^\infty} \big\{E_Q[X|\mathcal{F}_t] - \Phi_t(X)\big\}, \tag{2.2}$$

where $\mathcal{P}_t^{\approx} := \{Q \in \mathcal{M}_1 \,|\, Q \approx P \text{ on } \mathcal{F}_t\} \supseteq \mathcal{M}_1^e(P)$ is the largest subset of $\mathcal{M}_1(P)$ on which the essential infimum is well-defined.

Lemma 3. *The concave conjugate α_t of an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$ can be written as*

$$\alpha_t(Q) = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t} E_Q[X|\mathcal{F}_t] \qquad \text{for } Q \in \mathcal{P}_t^{\approx}, \tag{2.3}$$

and it has the following σ-pasting property: If Q^n, $n \in I\!N$, are in $\mathcal{M}_1^e(P)$ with density processes Z^n, if $(A_n)_{n \in I\!N}$ is an $\mathcal{F}_t$-partition of Ω, and if $\bar{Q} \in \mathcal{M}_1^e(P)$ is defined by $\frac{d\bar{Q}}{dP} := \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \frac{Z_T^n}{Z_t^n}$, then $\alpha_t(\bar{Q}) = \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \alpha_t(Q^n)$.

Proof. The second assertion is straightforward, and (2.3) is already known; see Remark 4.16 in FS or Remark 9 in DS. **q.e.d.**

Definition. An MCUF Φ_t at time t is called *continuous from above (below)* if $\lim_{n\to\infty} \Phi_t(X_n) = \Phi_t(X)$ for any sequence $(X_n)_{n\in\mathbb{N}}$ in L^∞ decreasing (increasing) to some $X \in L^\infty$.

One can easily show that like in the static case, continuity from below implies continuity from above; see Lemma 3.14 in KS. And also in analogy to the static case (see Theorem 4.31 in FS), the next result shows that for an MCUF, existence of a representation via the concave conjugate, continuity from above, and $\sigma(L^\infty, L^1)$-closedness of the acceptance set are all equivalent. A detailed comparison with other related results is given after Lemma 5.

Theorem 4. *For an MCUF Φ_t at time t with acceptance set $\mathcal{A}_t$, the following are equivalent:*

I) *Φ_t is continuous from above and satisfies*

$$\inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty \qquad \text{for some } \tilde{Q} \in \mathcal{M}_1^e(P). \tag{2.4}$$

II) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_1^e(P)} \big\{ E_Q[X|\mathcal{F}_t] - \alpha_t^0(Q) \big\} \tag{2.5}$$

for a mapping $\alpha_t^0 : \mathcal{M}_1^e(P) \to L^0\big(\mathcal{F}_t; [-\infty, +\infty)\big)$ which has the σ-pasting property.

III) *Φ_t can be represented as*

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_1^e(P)} \big\{ E_Q[X|\mathcal{F}_t] - \alpha_t(Q) \big\}, \tag{2.6}$$

where α_t is the concave conjugate of Φ_t.

IV) *$\mathcal{A}_t$ is closed in $\sigma(L^\infty, L^1)$ and satisfies (2.4).*

If Φ_t satisfies one of the above properties and is in addition positively homogeneous, hence an MCohUF, it can be represented as

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}^e} E_Q[X|\mathcal{F}_t] \tag{2.7}$$

for some set $\mathcal{Q} \subset \mathcal{M}_1(P)$ with $\mathcal{Q} \cap \mathcal{M}_1^e(P) \neq \emptyset$. $\mathcal{Q}$ can be chosen convex and closed in L^1.

Proof. See Appendix.

Definition. If one of the properties I) – IV) is satisfied, we call Φ_t *well-representable*.

Remark. Theorem 4 also allows us to *define* Φ_t by (2.5) from some mapping α_t^0 from $\mathcal{M}_1^e(P)$ into $L^0\big(\mathcal{F}_t; [-\infty, \infty)\big)$. The resulting Φ_t is an MCUF at time t and continuous from above, even if α_t^0 does not have the σ-pasting property; this follows from Theorem 1 in DS as can be seen from our proof of Theorem 4. Defining Φ_t via (2.5) is particularly useful in the coherent case since one can specify an entire DMCohUF via (2.7) by a single set $\mathcal{Q}$. For economic interpretations and examples, see Section 4.3 of FS. $\diamond$

One difference to other related representation results is our use of the condition (2.4) that $\inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ for some $\tilde{Q} \in \mathcal{M}_1^e(P)$. Before discussing this difference in more detail, let us first show how (2.4) can be ensured from a relevance condition on Φ_t; see Definition 4.32 and Corollaries 4.34 and 9.30 in FS for this economically very natural concept.

Definition. An MCUF Φ_t at time t is called *relevant* or *sensitive* if $P\big[\Phi_t(-\mathbf{1}_B) < \Phi_t(0)\big] > 0$ for any $B \in \mathcal{F}$ with $P[B] > 0$.

Lemma 5. *Let Φ_t be an MCUF at time t.*

a) *If Φ_t is continuous from above and relevant, there exists some $\tilde{Q} \in \mathcal{M}_1^e(P)$ such that $\inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$. Hence (2.4) holds, and in particular, Φ_t is well-representable.*

b) *If Φ_t is well-representable and an MCohUF at time t, then Φ_t is relevant.*

Proof. See Appendix.

Theorem 4 is very similar to Theorem 1 in DS and Theorems 3.16, 3.18 and 3.23 in CDK. Obvious differences are changes of signs in DS and that CDK work with MCUFs on processes instead of only random variables like here. In the Appendix, we briefly sketch how their notation can be translated to our setting. But the main difference is that DS and CDK assume in I) only that Φ_t is continuous from above. They then obtain representations like in (2.6), but the set of measures appearing in their results is

$$\mathcal{P}_t^{=} := \{Q \ll P \,|\, Q = P \text{ on } \mathcal{F}_t\},$$

which explicitly depends on $\mathcal{F}_t$. As in DS, we call an MCUF with that structure *representable*. By imposing the additional condition (2.4) on Φ_t, we have in contrast a representation with one set $\mathcal{M}_1^e(P)$ for all t and, more importantly, a representation in terms of measures which are equivalent to P. The term well-representable is meant to highlight this difference.

To be accurate, things are even more subtle. In their Theorem 3.23, CDK also provide a representation like (2.6) in terms of $\mathcal{M}_1^e(P)$. However, they assume for this that Φ_t is relevant, which by Lemma 5 is sufficient (but not necessary) for (2.4). In contrast, we show that the weaker condition (2.4) is already sufficient for the representation in (2.6), and that (together with continuity from above) it is actually also necessary.

Other related conditional representation results for convex risk measures can be found in Rosazza Gianin (2004) in the context of BSDEs. In the coherent case, things become simpler;

see for instance Riedel (2004), Roorda/Schumacher/Engwerda (2005) or Artzner/Delbaen/Eber/Heath/Ku (2004). The recent work of Weber (2006) is less relevant for our goals, because law-invariance does not fit well with the notion of hedging.

To study relations between MCUFs at different points in time, we now introduce a notion of time-consistency.

Definition. A DMCUF $\Phi := (\Phi_t)_{0\le t\le T}$ is called *time-consistent* if for $X, Y \in L^\infty$ and $s \le t$,

$$\Phi_t(X) = \Phi_t(Y) \quad \text{implies that} \quad \Phi_s(X) = \Phi_s(Y). \tag{2.8}$$

Φ is called *strongly time-consistent* if in addition $\mathcal{A}_t \subseteq \mathcal{A}_s$ for $t \ge s$.

In the literature, one can find several differing definitions of time-consistency; see for instance Peng (2004), Weber (2006), or Artzner/Delbaen/Eber/Heath/Ku (2004) for an overview. For our purposes, (2.8) means that indifference at time t between two payoffs X and Y carries over to any earlier time $s < t$, i.e., when less information is available. As the "=" could be replaced by "$\ge$" signs in (2.8), time-consistency preserves the ordering between payoffs over time, but does not fix the level at which this occurs. Unless all Φ_t are normalized, (2.8) hence does not guarantee that an X acceptable in t is also acceptable at time $s < t$; this requires strong time-consistency. We do not impose normalization here since we later consider operations on DMCUFs which preserve (even strong) time-consistency, but may change the initial utility level; see the remark after Theorem 7 and Example 1 in Section 4.

For a DMCUF $(\Phi_t)_{0\le t\le T}$ with acceptance sets $(\mathcal{A}_t)_{0\le t\le T}$ and for $s \le t$, we use the notation $\mathcal{A}_s(\mathcal{F}_t) := \mathcal{A}_s \cap L^\infty(\mathcal{F}_t)$. We note that $\Phi_{sot} := \Phi_s \circ \Phi_t$ is an MCUF at time s and denote by $\mathcal{A}_{sot}$ its acceptance set. Similarly as in Theorems 6.2 and 7.9 in Delbaen (2006b), time-consistency can then be characterized as follows; see also Proposition 8 of DS.

Lemma 6. *For a DMCUF $\Phi = (\Phi_t)_{0\le t\le T}$, the properties*

a) $\Phi_{sot} = \Phi_s$ *for all $s \le t$,*

b) $\mathcal{A}_s = \mathcal{A}_{sot}$ *for all $s \le t$,*

c) $\mathcal{A}_s = \mathcal{A}_s(\mathcal{F}_t) + \mathcal{A}_t$ *for all $s \le t$,*

are all equivalent and imply

d) Φ *is time-consistent.*

If Φ is normalized, i.e., $\Phi_t(0) \equiv 0$ for all $t \in [0, T]$, then d) is equivalent to a) – c).

Proof. See Appendix.

For a normalized DMCUF, time-consistency and strong time-consistency are the same. In fact, $\Phi_s(0) = 0$ implies $0 \in \mathcal{A}_s(\mathcal{F}_t)$ and thus $\mathcal{A}_t \subseteq \mathcal{A}_s$ by c) of Lemma 6. However, a DMCUF can be strongly time-consistent without being normalized; see Example 1 in Section 4. Note also that each of the equivalent properties a) – c) in Lemma 6 implies that Φ is normalized. To see this for a), simply write $\Phi_t(0) = \Phi_{t \circ t}(0) = \Phi_t\big(0 + \Phi_t(0)\big) = \Phi_t(0) + \Phi_t(0) = 2\Phi_t(0)$.

Suppose that a DMCUF Φ satisfies $\mathcal{A}_t \subseteq \mathcal{A}_s$ for $t \geq s$. Then $t \mapsto \inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X]$ is increasing and thus $\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ holds for all t as soon as we have this for $t = 0$, i.e., if $\alpha_0(\tilde{Q}) = \inf_{X \in \mathcal{A}_0} E_{\tilde{Q}}[X] > -\infty$. Hence the condition (2.4) in Theorem 4 simplifies in this case. Similarly, a time-consistent DMCUF Φ with Φ_0 relevant has Φ_t relevant for all t.

Remarks. 1) Although time-consistency is desirable in many situations, it is also quite restrictive. Section 7.2 of KS gives an example of an MCohUF at time 0 which cannot be extended to a time-consistent DMCUF.

2) A proper treatment of time-consistency ought to take into account the influence of the final time horizon T. This is discussed in KS, but omitted here for reasons of space. $\diamond$

3. Convolutions of monetary concave utility functionals

In this section, we study the convolution of two MCUFs. We extend earlier work by Delbaen (2000) and BEK from the static to the abstract conditional case and prove that the dynamic property of (strong) time-consistency is preserved by convolution. Convoluting an MCUF with a pre-acceptance set is also a key step for the construction of a dynamic indifference valuation in the next section. The proofs rely on the representation result in Theorem 4.

Definition. If Φ_t^1 and Φ_t^2 are MCUFs at time t, their *convolution* is defined as

$$\Phi_t^1 \Box \Phi_t^2(X) := \operatorname*{ess\,sup}_{Y \in L^\infty} \big\{\Phi_t^1(X+Y) + \Phi_t^2(-Y)\big\} \qquad \text{for } X \in L^\infty. \tag{3.1}$$

If $\mathcal{B} \subseteq L^\infty$ is non-empty, convex and $\mathcal{F}_t$-regular, the *convolution* of Φ_t^1 and $\mathcal{B}$ is defined as

$$\Phi_t^1 \Box \mathcal{B}(X) := \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) \qquad \text{for } X \in L^\infty. \tag{3.2}$$

As a purely mathematical concept, the above convolution on risk measures was introduced and studied by Delbaen (2000) in the static and coherent case; see also Delbaen (2006a) for an economic interpretation. One motivation for studying $\Phi_t^1 \Box \Phi_t^2$ comes from a problem of risk transfer between two agents with preferences given by Φ_t^1 and Φ_t^2; see Barrieu/El Karoui (2004, 2005). Convoluting Φ_0^1 and Φ_0^2 also corresponds to finding a Pareto-efficient exchange between two individuals with preferences Φ_0^1 and Φ_0^2. This has been pointed out to us by N. Touzi and is discussed in more detail in Jouini/Schachermayer/Touzi (2005) or KS.

The main result of this section is an extension of Theorem 3.6 in BEK in several directions. We show that the convolution operation produces a new MCUF and also preserves (strong) time-consistency. All this is done in a conditional and abstract setting, in contrast to BEK who only treat the static abstract case, and also to Barrieu/El Karoui (2004) who study in the dynamic case a class of DMCUFs defined via BSDEs. Moreover, the question of time-consistency for convolutions of DMCUFs seems not to have been addressed so far in a general setting. In technical terms, the main difficulty in this section is related to closedness properties of acceptance sets; this comes up when we need to identify the acceptance set of the convolution $\Phi_t^1 \Box \Phi_t^2$.

Theorem 7. *For $i = 1, 2$, let Φ_t^i be an MCUF at time t with acceptance set $\mathcal{A}_t^i$ and concave conjugate α_t^i. Assume that $\Phi_t^1 \Box \Phi_t^2(0) \in L^\infty$. Then:*

a) *$\Phi_t^1 \Box \Phi_t^2$ is an MCUF at time t, and*

$$\Phi_t^1 \Box \Phi_t^2(X) = \Phi_t^1 \Box \mathcal{A}_t^2(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \left\{ \Phi_t^1(X+Y) + \Phi_t^2(-Y) \right\} \qquad \text{for } X \in L^\infty, \tag{3.3}$$

where $\mathcal{B}$ is an arbitrary subset of L^∞ containing $\mathcal{A}_t^2$.

b) *If Φ_t^1 and Φ_t^2 are both coherent, so is $\Phi_t^1 \Box \Phi_t^2$.*

c) *If Φ_t^1 or Φ_t^2 is continuous from below, so is $\Phi_t^1 \Box \Phi_t^2$. Moreover, $\Phi_t^1 \Box \Phi_t^2$ is then representable, its concave conjugate $\alpha_t^{1\Box 2}$ is given by*

$$\alpha_t^{1\Box 2}(Q) = \alpha_t^1(Q) + \alpha_t^2(Q) \qquad \text{for } Q \in \mathcal{P}_t^{\approx}, \tag{3.4}$$

and its acceptance set $\mathcal{A}_t^{1\Box 2}$ is given by

$$\mathcal{A}_t^{1\Box 2} = \overline{\mathcal{A}_t^1 + \mathcal{A}_t^2}, \tag{3.5}$$

where the closure is taken in $\sigma(L^\infty, L^1)$. If in addition we have

$$\inf_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E_{\tilde{Q}}[X] > -\infty \qquad \text{for some } \tilde{Q} \in \mathcal{M}_1^e(P), \tag{3.6}$$

then $\Phi_t^1 \Box \Phi_t^2$ is also well-representable.

d) *Suppose that $\Phi^i = (\Phi_t^i)_{0 \le t \le T}$ for $i = 1, 2$ are (strongly) time-consistent DMCUFs such that for each $t \in [0, T]$, Φ_t^1 or Φ_t^2 is continuous from below and $\Phi_t^1 \Box \Phi_t^2(0) \in L^\infty$. Then $\Phi^1 \Box \Phi^2 = (\Phi_t^1 \Box \Phi_t^2)_{0 \le t \le T}$ is also a (strongly) time-consistent DMCUF.*

Remarks. **1)** Like in Section 2, condition (3.6) simplifies if $\Phi^1 \Box \Phi^2$ is strongly time-consistent; it is then enough if $\inf_{X \in \mathcal{A}_0^1 + \mathcal{A}_0^2} E_{\tilde{Q}}[X] = \alpha_0^1(\tilde{Q}) + \alpha_0^2(\tilde{Q}) > -\infty$ for some $\tilde{Q} \in \mathcal{M}_1^e(P)$.

2) $\Phi_t^1 \Box \Phi_t^2$ need not be normalized even if Φ_t^1 and Φ_t^2 both are. This is the main reason why we abandon the requirement of normalization. An example is given in Section 4. $\diamond$

In the proof of Theorem 7, we use the following auxiliary result.

Lemma 8. *Take an MCUF Φ_t^1 at time t and a non-empty, convex and $\mathcal{F}_t$-regular set $\mathcal{B} \subseteq L^\infty$. If $\Phi_t^1 \Box \mathcal{B}(0) \in L^\infty$, then:*

a) *$\Phi_t^1 \Box \mathcal{B}$ is an MCUF at time t.*

b) *If Φ_t^1 is coherent and $\mathcal{B}$ a convex cone containing 0, then $\Phi_t^1 \Box \mathcal{B}$ is an MCohUF at time t.*

c) *If Φ_t^1 is continuous from below, so is $\Phi_t^1 \Box \mathcal{B}$.*

Proof. This is very similar to the arguments in BEK in the proofs of Corollary 3.7 and Theorem 3.6 there. Hence we omit the details. **q.e.d.**

Proof of Theorem 7. To shorten notation, we write $\Phi_t := \Phi_t^1 \Box \Phi_t^2$ and $\alpha_t := \alpha_t^{1\Box 2}$ for $t \in [0,T]$. The argument for $\Phi_t = \Phi_t^1 \Box \mathcal{A}_t^2$ is straightforward, and the second equality in (3.3) follows since $\Phi_t^2(-Y) \geq 0$ iff $-Y \in \mathcal{A}_t^2$. Using Lemma 8 yields a) and the continuity assertion in c), and together with Lemma 1 also b). Once we know that Φ_t is continuous from below, hence also from above, it is representable by Theorem 1 of DS. The representation (3.4) of $\alpha_t^{1\Box 2}$ can be shown like in Theorem 3.6 of BEK, and so it only remains to prove (3.5) and d).

To show d), suppose first that Φ^1 and Φ^2 are time-consistent. We may also assume that they are normalized, because the MCUFs $\hat{\Phi}_u^i(X) := \Phi_u^i(X) - \Phi_u^i(0)$ for $i = 1,2$ are, we have $\Phi_u(X) = \hat{\Phi}_u^1 \Box \hat{\Phi}_u^2(X) + \big(\Phi_u^1(0) + \Phi_u^2(0)\big)$, and time-consistency is not affected by translation. So let $s \leq t$ and X_1, X_2 be such that

$$\Phi_t(X_1) = \Phi_t(X_2) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^2} \Phi_t^1(X_2 + Y). \tag{3.7}$$

By (3.3) it suffices to show that we then have

$$\Phi_s^1 \Box \mathcal{A}_s^2(X_1) = \operatorname*{ess\,sup}_{Y' \in -\mathcal{A}_s^2} \Phi_s^1(X_1 + Y') = \operatorname*{ess\,sup}_{Y' \in -\mathcal{A}_s^2} \Phi_s^1(X_2 + Y').$$

Now Lemma 6 implies that

$$\Phi_s^1(X) = \Phi_s^1\big(\Phi_t^1(X)\big) \qquad \text{for } X \in L^\infty, \tag{3.8}$$

$$\mathcal{A}_s^2 = \mathcal{A}_s^2(\mathcal{F}_t) + \mathcal{A}_t^2, \tag{3.9}$$

and Lemma 1 applied to $\mathcal{A}_t^2$ and $\mathcal{F}_t$-regularity of Φ_t^1 yield that $\big\{\Phi_t^1(X+Y) \,\big|\, Y \in -\mathcal{A}_t^2\big\}$ is a lattice for any $X \in L^\infty$. Hence there is a sequence (Y_n) in $-\mathcal{A}_t^2$ such that $\operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^2} \Phi_t^1(X+Y) = \nearrow\text{-}\lim_{n\to\infty} \Phi_t^1(X+Y_n)$. Moreover, $\big(\Phi_t^1(X+Y_n)\big)_{n \in \mathbb{N}}$ is uniformly bounded due to (3.3) because

$$-\|X+Y_1\|_{L^\infty} \leq \Phi_t^1(X+Y_1) \leq \Phi_t^1(X+Y_n) \leq \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^2} \Phi_t^1(X+Y) = \Phi_t(X) \in L^\infty.$$

Hence translation invariance and continuity from below of Φ^1_s imply for any $\hat{Y} \in \mathcal{A}^2_s(\mathcal{F}_t)$ that

$$\Phi^1_s\Big(\operatorname*{ess\,sup}_{Y\in -\mathcal{A}^2_t} \Phi^1_t(X+Y+\hat{Y})\Big) = \nearrow\text{-}\lim_{n\to\infty} \Phi^1_s\big(\Phi^1_t(X+Y_n)+\hat{Y}\big) \le \operatorname*{ess\,sup}_{Y\in -\mathcal{A}^2_t} \Phi^1_s\big(\Phi^1_t(X+Y)+\hat{Y}\big),$$

and by monotonicity of Φ^1_s, we even must have equality. Combining this with (3.8), (3.9) and using (3.7) to exchange X_1 for X_2, we get

$$\begin{aligned}
\operatorname*{ess\,sup}_{Y'\in -\mathcal{A}^2_s} \Phi^1_s(X_1+Y') &= \operatorname*{ess\,sup}_{\hat{Y}\in -\mathcal{A}^2_s(\mathcal{F}_t)} \operatorname*{ess\,sup}_{Y\in -\mathcal{A}^2_t} \Phi^1_s\big(\Phi^1_t(X_1+Y+\hat{Y})\big)\\
&= \operatorname*{ess\,sup}_{\hat{Y}\in -\mathcal{A}^2_s(\mathcal{F}_t)} \Phi^1_s\Big(\operatorname*{ess\,sup}_{Y\in -\mathcal{A}^2_t} \Phi^1_t(X_2+Y)+\hat{Y}\Big)\\
&= \operatorname*{ess\,sup}_{Y'\in -\mathcal{A}^2_s} \Phi^1_s(X_2+Y'),
\end{aligned}$$

where the last equality is obtained by doing the same steps in reverse order with X_1 replaced by X_2. This shows that Φ is time-consistent. If Φ^1, Φ^2 are strongly time-consistent, we have in addition $\mathcal{A}^i_t \subseteq \mathcal{A}^i_s$ for $t \ge s$ and $i = 1, 2$, and thus also $\overline{\mathcal{A}^1_t + \mathcal{A}^2_t} \subseteq \overline{\mathcal{A}^1_s + \mathcal{A}^2_s}$. Hence (3.5) will imply that Φ is strongly time-consistent as well, and so d) is proved.

Finally we turn to (3.5). If $X_i \in \mathcal{A}^i_t$ for $i = 1, 2$, then $\Phi_t(X_1+X_2) \ge \Phi^1_t(X_1)+\Phi^2_t(X_2) \ge 0$ shows that $X_1+X_2 \in \mathcal{A}^{1\square 2}_t$, and because $\mathcal{A}^{1\square 2}_t$ is closed in $\sigma(L^\infty, L^1)$ by Theorem 4, we obtain $\overline{\mathcal{A}^1_t + \mathcal{A}^2_t} \subseteq \mathcal{A}^{1\square 2}_t$. For the converse inclusion, we claim that

$$\text{(3.10)} \qquad \inf_{X\in\mathcal{A}^{1\square 2}_t} E[ZX] = \inf_{X\in\mathcal{A}^1_t+\mathcal{A}^2_t} E[ZX] = \inf_{X\in\overline{\mathcal{A}^1_t+\mathcal{A}^2_t}} E[ZX] \qquad \text{for all } Z \in L^1_+;$$

observe that the second equality follows from the first one because we already know that $\mathcal{A}^1_t + \mathcal{A}^2_t \subseteq \overline{\mathcal{A}^1_t + \mathcal{A}^2_t} \subseteq \mathcal{A}^{1\square 2}_t$. Then if the inclusion "$\subseteq$" in (3.5) is not true, there exists some $X' \in \mathcal{A}^{1\square 2}_t \setminus \overline{\mathcal{A}^1_t + \mathcal{A}^2_t}$, and the Hahn-Banach theorem yields some $Z' \in L^1$ with

$$\text{(3.11)} \qquad \inf_{X\in\overline{\mathcal{A}^1_t+\mathcal{A}^2_t}} E[XZ'] > E[X'Z'] > -\infty.$$

But since $-\big(\overline{\mathcal{A}^1_t + \mathcal{A}^2_t}\big)$ is solid, we must have $Z' \ge 0$, and so (3.11) contradicts (3.10).

To complete the proof, it remains to establish (3.10). To that end, we first use Lemma 3, (3.4) and again Lemma 3 to obtain

$$\begin{aligned}
\text{(3.12)} \qquad \operatorname*{ess\,inf}_{X\in\mathcal{A}^{1\square 2}_t} E_Q[X|\mathcal{F}_t] &= \alpha^{1\square 2}_t(Q)\\
&= \operatorname*{ess\,inf}_{X_1\in\mathcal{A}^1_t} E_Q[X_1|\mathcal{F}_t] + \operatorname*{ess\,inf}_{X_2\in\mathcal{A}^2_t} E_Q[X_2|\mathcal{F}_t]\\
&= \operatorname*{ess\,inf}_{X\in\mathcal{A}^1_t+\mathcal{A}^2_t} E_Q[X|\mathcal{F}_t] \qquad \text{for all } Q \in \mathcal{P}^{\approx}_t.
\end{aligned}$$

Now up to normalization, $\mathcal{P}_t^{\approx}$ can be identified with

$$\mathcal{Z}_t := \left\{ Z \in L^1_+ \,\middle|\, \text{for all } A \in \mathcal{F}_t,\ P[A] = 0 \text{ iff } Z\mathbf{1}_A = 0 \right\}.$$

Hence (3.12) implies that

$$\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\square 2}} E[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX|\mathcal{F}_t] \qquad \text{for all } Z \in \mathcal{Z}_t. \tag{3.13}$$

To extend this to all $Z \in L^1_+$, fix $Z \in L^1_+$ and define $B \in \mathcal{F}_t$ up to nullsets by $\mathbf{1}_B := \operatorname{ess\,sup}\{\mathbf{1}_A \,|\, A \in \mathcal{F}_t \text{ and } Z\mathbf{1}_A = 0\}$ so that $Z\mathbf{1}_{B^c} = Z$. Because Φ_t is representable, we have

$$L^\infty \ni -\Phi_t(0) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t^{=}} \alpha_t(Q) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t^{=}} \Big(\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\square 2}} E_Q[X|\mathcal{F}_t] \Big)$$

and so there exists some $Q' \in \mathcal{P}_t^{=}$ with density Z'_T such that $\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\square 2}} E_{Q'}[X|\mathcal{F}_t] \in L^\infty$. Then $\widehat{Z} := Z'_T\mathbf{1}_B + Z\mathbf{1}_{B^c}$ is in $\mathcal{Z}_t$ and

$$\mathbf{1}_{B^c} E[ZX|\mathcal{F}_t] = \mathbf{1}_{B^c} E[\widehat{Z}X|\mathcal{F}_t]. \tag{3.14}$$

Using $Z = Z\mathbf{1}_{B^c}$, (3.14), (3.13) for $\widehat{Z}$ and then reversing the steps again yields

$$\operatorname*{ess\,inf}_{X \in \mathcal{A}_t^{1\square 2}} E[ZX|\mathcal{F}_t] = \operatorname*{ess\,inf}_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX|\mathcal{F}_t]$$

as desired. Because $\left\{ E[ZX|\mathcal{F}_t] \,\middle|\, X \in \mathcal{B} \right\}$ is a lattice for $\mathcal{B} \in \left\{ \mathcal{A}_t^{1\square 2}, \mathcal{A}_t^1 + \mathcal{A}_t^2 \right\}$ by $\mathcal{F}_t$-regularity, we can interchange infimum and expectation to obtain

$$\inf_{X \in \mathcal{A}_t^{1\square 2}} E[ZX] = \inf_{X \in \mathcal{A}_t^1 + \mathcal{A}_t^2} E[ZX]$$

for every $Z \in L^1_+$. This establishes (3.10) and completes the proof. **q.e.d.**

If Φ_t^1 is an MCUF and $\mathcal{B}$ a pre-acceptance set at time t, Lemma 8 implies that $\Phi_t := \Phi_t^1 \square \mathcal{B}$ is again an MCUF, provided that $\Phi_t(0) \in L^\infty$. In the sequel, we want to have a maximum of good properties for that Φ_t with a minimum of assumptions on $\mathcal{B}$. To make this more precise, recall from Lemma 2 the MCUF $\Phi_t^{\mathcal{B}}$ associated to $\mathcal{B}$. From (2.1) and (3.3), it seems natural to expect that $\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\mathcal{B}}$ and that the acceptance set of Φ_t should be $\overline{\mathcal{A}_t^1 + \mathcal{B}}$ in view of (3.5). However, this can be deduced from the preceding results only if Φ_t^1 is continuous from below and $\mathcal{B}$ is the acceptance set of $\Phi_t^{\mathcal{B}}$, e.g., if $\mathcal{B}$ is closed in $\sigma(L^\infty, L^1)$. Because the latter is often hard to check, we do not want to make that assumption. So we first work with the $\sigma(L^\infty, L^1)$-closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ since we have precise results for $\Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}$, and then show that the latter coincides with $\Phi_t^1 \square \mathcal{B}$.

Proposition 9. *Let $\mathcal{B}$ be a pre-acceptance set and Φ_t^1 an MCUF at time t with acceptance set $\mathcal{A}_t^1$ and concave conjugate α_t^1. Denote by $\overline{\mathcal{B}}$ the closure of $\mathcal{B}$ in $\sigma(L^\infty, L^1)$. If $\Phi_t^1 \square \mathcal{B}(0) = \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(Y) \in L^\infty$, then*

$$\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\mathcal{B}}. \tag{3.15}$$

If in addition Φ_t^1 is continuous from below and

$$\operatorname*{ess\,sup} \big(- \overline{\mathcal{B}} \cap L^\infty(\mathcal{F}_t) \big) \in L^\infty, \tag{3.16}$$

then

$$\Phi_t^1 \square \mathcal{B} = \Phi_t^1 \square \Phi_t^{\overline{\mathcal{B}}}. \tag{3.17}$$

In particular, $\Phi_t := \Phi_t^1 \square \mathcal{B}$ is then continuous from below, with concave conjugate

$$\alpha_t(Q) = \alpha_t^1(Q) + \alpha_t^{\overline{\mathcal{B}}}(Q) := \alpha_t^1(Q) + \operatorname*{ess\,inf}_{Y \in \overline{\mathcal{B}}} E_Q[Y|\mathcal{F}_t] \tag{3.18}$$

and acceptance set

$$\mathcal{A}_t = \overline{\mathcal{A}_t^1 + \overline{\mathcal{B}}} = \overline{\mathcal{A}_t^1 + \mathcal{B}}.$$

Proof. If $\mathcal{A}_t^{\mathcal{B}}$ denotes the acceptance set of $\Phi_t^{\mathcal{B}}$, then $\mathcal{B} \subseteq \mathcal{A}_t^{\mathcal{B}}$ so that (3.3) implies

$$\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \Phi_t^1(X+Y) \geq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y) = \Phi_t^1 \square \mathcal{B}(X). \tag{3.19}$$

Since $\Phi_t^{\mathcal{B}}$ is non-negative on $\mathcal{A}_t^{\mathcal{B}}$, (3.3) also yields

$$\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) \leq \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big) \leq \operatorname*{ess\,sup}_{Y \in L^\infty} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big) = \Phi_t^1 \square \Phi_t^{\mathcal{B}}(X)$$

so that $\Phi_t^1 \square \Phi_t^{\mathcal{B}}(X) = \operatorname*{ess\,sup}_{Y \in -\mathcal{A}_t^{\mathcal{B}}} \big(\Phi_t^1(X+Y) + \Phi_t^{\mathcal{B}}(-Y) \big)$. In view of (3.19), it thus suffices to show that for each $Y' \in -\mathcal{A}_t^{\mathcal{B}}$,

$$\Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') \leq \operatorname*{ess\,sup}_{Y \in -\mathcal{B}} \Phi_t^1(X+Y). \tag{3.20}$$

Pick a sequence (m_t^n) in $L^\infty(\mathcal{F}_t)$ and an $\mathcal{F}_t$-partition (A_n) with $-Y' - m_t^n \in \mathcal{B}$ for all n and

$$\Phi_t^{\mathcal{B}}(-Y') \leq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n + \varepsilon,$$

for a fixed $\varepsilon > 0$. Then translation invariance of Φ_t^1 implies that

$$\begin{aligned}\operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X+Y) &= \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \operatorname*{ess\,sup}_{Y\in-\mathcal{B}} \Phi_t^1(X+Y) \\ &\geq \sum_{n=1}^{\infty} \mathbf{1}_{A_n} \Phi_t^1(X+Y'+m_t^n) \\ &= \Phi_t^1(X+Y') + \sum_{n=1}^{\infty} \mathbf{1}_{A_n} m_t^n \\ &\geq \Phi_t^1(X+Y') + \Phi_t^{\mathcal{B}}(-Y') - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves (3.20) and hence (3.15).

If we now assume (3.16), $\overline{\mathcal{B}}$ is like $\mathcal{B}$ a pre-acceptance set at time t and thus by Lemma 2 the acceptance set of the MCUF $\Phi_t^{\overline{\mathcal{B}}}$. So it is enough to prove (3.17) because all claimed properties then follow from Theorem 7 and Lemma 3, and as $\Phi_t := \Phi_t^1 \Box \mathcal{B}$ and $\Phi_t^1 \Box \Phi_t^{\overline{\mathcal{B}}}$ both are MCUFs at time t, they coincide if their acceptance sets $\mathcal{A}_t$ and $\overline{\mathcal{A}_t^1 + \overline{\mathcal{B}}} = \overline{\mathcal{A}_t^1 + \mathcal{B}}$ agree. By the assumptions and Lemma 8, Φ_t is continuous from below, hence also from above; so $\mathcal{A}_t$ is closed in $\sigma(L^\infty, L^1)$ due to the implication "I) $\Longrightarrow$ IV)" in Theorem 4. Because the definition of Φ_t gives $\mathcal{A}_t^1 + \mathcal{B} \subseteq \mathcal{A}_t$, we obtain $\overline{\mathcal{A}_t^1 + \mathcal{B}} \subseteq \mathcal{A}_t$; the converse inclusion is trivial since (3.2) and (3.3) with $\mathcal{A}_t^2 = \overline{\mathcal{B}}$ give

$$\Phi_t(X) \leq \operatorname*{ess\,sup}_{Y\in-\overline{\mathcal{B}}} \Phi_t^1(X+Y) = \Phi_t^1 \Box \Phi_t^{\overline{\mathcal{B}}}(X) \qquad \text{for } X \in L^\infty.$$

This completes the proof. **q.e.d.**

4. Indifference valuation via monetary concave utility functionals

In this section, we define and study a valuation by indifference with respect to an agent's subjective DMCUF Φ in a financial market with trading constraints. The main work involved is the construction of the *market DMCUF* whose acceptance sets consist exactly of (minus) those payoffs that can be superreplicated at zero cost. We extend an idea of Föllmer/Schied (2002) by using the optional decomposition under constraints dynamically over time, and notably show that the resulting market DMCUF is strongly time-consistent. The indifference valuation functional is obtained by normalizing the convolution of Φ with the market DMCUF. Finally, we discuss the connections between this valuation approach and arbitrage opportunities.

Valuation by indifference with respect to a von Neumann-Morgenstern expected utility is an old theme and has been much studied again in the last years. An early reference is

Hodges/Neuberger (1989); Frittelli (2000) and Rouge/El Karoui (2000) are at the start of the recent resurgence of activity, and Becherer (2003) and Henderson/Hobson (2004) contain overviews and many more references. But explicit results are hard to obtain because except for the exponential case, the utility-based certainty equivalent is not translation invariant.

The idea of replacing expected utility by a monetary (hence translation invariant) utility functional and the naturally ensuing link to the convolution with the market functional have only emerged rather recently. Perhaps the earliest reference where this can be found in a general abstract (but static) form is Jaschke/Küchler (2001), even though the formulation there is for coherent risk measures and cast in terms of good-deal bounds. Subsequent developments in that latter direction include Černý/Hodges (2002) and Staum (2004), among others; see also Section 6 of KS and our forthcoming work Klöppel/Schweizer (2006). Indifference valuation proper is briefly mentioned in BEK and discussed in more detail in Xu (2006) which also contains a number of worked examples. However, both deal only with the static case, and Xu (2006) has no constraints in the market. Larsen/Pirvu/Shreve/Tütüncü (2005) contains a dynamic treatment for a particular class of examples where Φ is given via a finite set of scenario and stress measures, generalizing an idea from Carr/Geman/Madan (2001). None of these works study the issue of time-consistency.

To introduce the basic idea, consider a mapping $U_t : L^0 \to L^0(\mathcal{F}_t)$ and a set $\mathcal{C}_t \subseteq L^0$. Fix the time horizon T and recall from Section 2 that all monetary quantities are in units of a fixed numeraire, eliminating the need for a riskless interest rate. We think of $\mathcal{C}_t$ as all payoffs in T that can be superreplicated when starting from time t with zero capital, and of $U_t(X)$ as the subjective monetary utility or value assigned by some agent at time t to a payoff X in T. Thus U_t should be $\mathcal{F}_t$-translation invariant in the sense that $U_t(X + a_t) = U_t(X) + a_t$ for $a_t \in L^0(\mathcal{F}_t)$. For a given wealth $x_t \in L^0(\mathcal{F}_t)$, the *time t utility indifference (buyer) value* $p_t(X)$ for X is implicitly defined by

$$\operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(x_t + g) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t\big(x_t - p_t(X) + g + X\big). \tag{4.1}$$

This value makes our agent indifferent (according to U_t) between buying the asset X or not, provided that she always optimally exploits her trading opportunities. Translation invariance allows us to solve (4.1) for $p_t(X)$ and write (assuming that all is well-defined)

$$p_t(X) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(X + g) - \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(g) = U_t^{\mathrm{opt}}(X) - U_t^{\mathrm{opt}}(0), \tag{4.2}$$

where

$$U_t^{\mathrm{opt}}(X) := \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} U_t(X + g) \tag{4.3}$$

is the maximal utility the agent can achieve from X by trading optimally in the market.

Since U_t is translation invariant, $p_t(X)$ has the pleasant feature that it does not depend on the wealth x_t which is known at time t. However, a valuation should take into account possible *future* commitments our agent may have made. This can be captured by generalizing (4.1) and defining $p_t(X;Y)$, the value of X subject to a later income Y, via

$$\operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t+g+Y) = \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t\big(x_t - p_t(X;Y)+g+Y+X\big).$$

Like before, this yields $p_t(X;Y) = U_t^{\mathrm{opt}}(Y+X) - U_t^{\mathrm{opt}}(Y)$, and together with (4.2) gives

$$p_t(X+Y;0) = p_t(X+Y) = U_t^{\mathrm{opt}}(X+Y) - U_t^{\mathrm{opt}}(0) = p_t(Y)+p_t(X;Y) = p_t(Y;0)+p_t(X;Y).$$

Hence this valuation is *iterative*: It does not matter whether the agent values the assets in bulk or one after another, if she takes into account what she has already committed to.

Remark. We can analogously define a *time t utility indifference seller value* $p_t^s(X)$ by

$$\operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t(x_t+g) = \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} U_t\big(x_t + p_t^s(X)+g-X\big).$$

Because this gives $p_t^s(X) = -p_t(-X)$, it is enough to study $p_t(X)$ only. ◇

To link the above formal discussion to previous concepts, we must be more precise about U_t and $\mathcal{C}_t$. So we assume from now on that $-\mathcal{C}_t \subseteq L^\infty$ is a pre-acceptance set at time t. Then

$$\Phi_t^{-\mathcal{C}_t}(X) = \operatorname{ess\,sup}\big((X-\mathcal{C}_t)\cap L^\infty(\mathcal{F}_t)\big) \qquad \text{for } X \in L^\infty$$

is by Lemma 2 an MCUF at time t, and the above interpretation of $\mathcal{C}_t$ shows that

$$-\Phi_t^{-\mathcal{C}_t}(-X) = \operatorname{ess\,inf}\{m_t \in L^\infty(\mathcal{F}_t) \,|\, X \in m_t + \mathcal{C}_t\}$$

can be viewed as the time t superreplication price for X. Hence it is natural to call $\Phi_t^{-\mathcal{C}_t}$ the *market MCUF* induced by $\mathcal{C}_t$. We also assume from now on that $U_t = \Phi_t$ is an MCUF at time t. The analogue of U_t^{opt} from (4.3) is then

$$\Phi_t^{\mathrm{opt}}(X) := \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} \Phi_t(X+g) = \Phi_t \square (-\mathcal{C}_t)(X),$$

and if

$$\Phi_t^{\mathrm{opt}}(0) = \operatorname*{ess\,sup}_{g\in\mathcal{C}_t} \Phi_t(g) \in L^\infty, \tag{4.4}$$

we know from Proposition 9 that $\Phi_t^{\mathrm{opt}} = \Phi_t \square \Phi_t^{-\mathcal{C}_t}$ is the convolution of the subjective MCUF Φ_t with the market MCUF. Moreover, Lemma 8 implies that Φ_t^{opt} is an MCUF at time t, and continuous from below if Φ_t is. The corresponding utility indifference valuation functional

$$p_t(X) = \Phi_t^{\mathrm{opt}}(X) - \Phi_t^{\mathrm{opt}}(0) = \Phi_t \square \Phi_t^{-\mathcal{C}_t}(X) - \Phi_t \square \Phi_t^{-\mathcal{C}_t}(0) \tag{4.5}$$

has then the same properties and is in addition normalized. Note in particular that $p_t(X)$ is now well-defined for any $X \in L^\infty$.

One natural requirement for the indifference valuation DMCUF $p = (p_t)$ is that it should be time-consistent. This will follow from Theorem 7 if both the a priori DMCUF Φ and the market DMCUF are. The main contribution of this section is to show that the market DMCUF is indeed well-defined and strongly time-consistent if the family $(\mathcal{C}_t)_{0 \le t \le T}$ corresponds in a natural way to trading under constraints in an arbitrage-free model.

The idea for using the optional decomposition under constraints to construct an MCUF describing a financial market is due to Föllmer/Schied (2002) in the static case; see also Section 4.8 in FS. The conditional case for fixed t is to a large extent analogous, but we still need to give some details because we additionally want to show time-consistency. It is quite intuitive that this should hold, but it has apparently not been studied or proved so far.

Our setup is almost the same as in Föllmer/Kramkov (1997) (abbreviated FK), and so we keep the exposition deliberately short. We model the (discounted) prices of our primary traded assets by a locally bounded $I\!R^d$-valued P-semimartingale $S = (S_t)_{0 \le t \le T}$ and assume the no-arbitrage condition (NFLVR) from Delbaen/Schachermayer (1994) so that the set $I\!P^e$ of equivalent local martingale measures for S is non-empty, i.e., there exists a $Q^* \in \mathcal{M}_1^e(P)$ such that S is a local Q^*-martingale. $L^a_{\mathrm{loc}}(S)$ is the set of all $I\!R^d$-valued predictable S-integrable processes H such that the stochastic integral process $H \raisebox{0.2ex}{$\scriptstyle\bullet$} S = \int H \, dS$ is locally bounded from below. A *portfolio* is a triple $\Pi = (x, H, K)$ with $x \in I\!R$, $H \in L^a_{\mathrm{loc}}(S)$ and K an increasing adapted RCLL process null at 0, and its *value process* is $V^\Pi = V^{x,H,K} = x + H \raisebox{0.2ex}{$\scriptstyle\bullet$} S - K$. The interpretation of Π is as usual: x is the time 0 initial capital, H specifies the number of units of each asset held, and K describes cumulative consumption.

Definition. An *admissible hedging set* is a subset $\mathcal{H} \subseteq L^a_{\mathrm{loc}}(S)$ containing 0 and such that

i) $\mathcal{H}$ is predictably convex: for $H^1, H^2 \in \mathcal{H}$ and any $[0,1]$-valued predictable process h, the process $hH^1 + (1-h)H^2$ again belongs to $\mathcal{H}$.

ii) $\mathcal{H}$ is closed in $L^a_{\mathrm{loc}}(S)$ with respect to the metric $d_S(H^1, H^2) := D_E(H^1 \raisebox{0.2ex}{$\scriptstyle\bullet$} S, H^2 \raisebox{0.2ex}{$\scriptstyle\bullet$} S)$, where D_E is the Émery distance on the space of real semimartingales.

A portfolio $\Pi = (x, H, K)$ is called *$\mathcal{H}$-constrained* if $H \in \mathcal{H}$; it is an *$\mathcal{H}$-constrained hedging portfolio for $X \in L^\infty$* if in addition V^Π is uniformly bounded from below and $V^\Pi_T \ge X$. Finally, such a Π is *minimal* if $V^\Pi \le V^{\Pi'}$ for any $\mathcal{H}$-constrained hedging portfolio Π' for X.

The idea behind the above concepts is that $\mathcal{H}$ describes the constraints imposed on trading since we may only use strategies H from $\mathcal{H}$; the simplest case, namely unconstrained trading, is given by $\mathcal{H} = L^a_{\mathrm{loc}}(S)$. Each admissible hedging set $\mathcal{H}$ induces a corresponding non-empty set $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{M}_1^e(P)$ of probability measures Q and a family of increasing predictable processes $A^{\mathcal{H}}(Q)$ indexed by $Q \in \mathcal{P}(\mathcal{H})$. In FK, with $\mathcal{S} = \{H \cdot S \,|\, H \in \mathcal{H}\}$, these are called

$$\mathcal{P}(\mathcal{S}) := \big\{Q \in \mathcal{M}_1^e(P) \,\big|\, \text{there exists an increasing predictable process } A = A(Q, \mathcal{S}) \text{ such that } Y - A \text{ is a local } Q\text{-supermartingale for every } Y \in \mathcal{S}\big\}$$

and the *upper variation* process

$$A^{\mathcal{S}}(Q) := \text{smallest } A \text{ (with respect to the strong order) satisfying the above conditions;}$$

we prefer here to emphasize the dependence on $\mathcal{H}$. We do not really need here the definitions of $\mathcal{P}(\mathcal{H})$ or $A^{\mathcal{H}}(Q)$, but we mention (and use) two facts. For $\mathcal{H} := L^a_{\mathrm{loc}}(S)$, we simply have $\mathcal{P}(\mathcal{H}) = I\!P^e$; and $A^{\mathcal{H}}(Q) \equiv 0$ for any $Q \in I\!P^e$ and any $\mathcal{H}$.

Our framework here is slightly different from the one used in FK, but their arguments can be adapted to yield the following result. It is a slight modification of Proposition 4.1 in FK and we refer to KS for both a discussion of differences and the details of the proof.

Proposition 10. *Fix an admissible hedging set $\mathcal{H}$. For any $X \in L^\infty$, there exists a minimal $\mathcal{H}$-constrained hedging portfolio $\hat{\Pi} = (\hat{x}, \hat{H}, \hat{K})$. Its value process equals*

$$\hat{V}_t(X) = \hat{x} + \int_0^t \hat{H}_s \, dS_s - \hat{K}_t = \operatorname*{ess\,sup}_{Q \in \mathcal{P}(\mathcal{H})} \big\{E_Q[X|\mathcal{F}_t] - E_Q[A^{\mathcal{H}}(Q)_T - A^{\mathcal{H}}(Q)_t \,|\, \mathcal{F}_t]\big\} \tag{4.6}$$

and is in particular uniformly bounded.

Note that in the unconstrained case $\mathcal{H} = L^a_{\mathrm{loc}}(S)$, (4.6) reduces to the well-known representation $\operatorname*{ess\,sup}_{Q \in I\!P^e} E_Q[X|\mathcal{F}_\cdot]$ of the superreplication price process. For a general admissible hedging set $\mathcal{H}$, the mapping $X \mapsto -\hat{V}_t(-X)$ on L^∞ is always an MCUF at time t; this follows from (4.6) and the remark after Theorem 4 by setting

$$\alpha_t^0(Q) := \begin{cases} -E_Q[A^{\mathcal{H}}(Q)_T - A^{\mathcal{H}}(Q)_t \,|\, \mathcal{F}_t] & \text{if } Q \in \mathcal{P}(\mathcal{H}) \\ -\infty & \text{else.} \end{cases} \tag{4.7}$$

With these preliminaries, we are now well armed to construct a good family $(\mathcal{C}_t)_{0 \le t \le T}$. We fix an admissible hedging set $\mathcal{H}$, define for each $t \in [0, T]$ the set

$$\mathcal{H}_t := \left\{H \in \mathcal{H} \,\middle|\, \int_t^{\cdot} H_s \, dS_s \text{ is uniformly bounded from below on } [t, T]\right\}$$

and set

$$\mathcal{C}_t := \left(\left\{\int_t^T H_s \, dS_s \,\middle|\, H \in \mathcal{H}_t\right\} - L^0_+\right) \cap L^\infty. \tag{4.8}$$

Then we can apply Proposition 10 to prove that the above sets $\mathcal{C}_t$ yield a strongly time-consistent market DMCUF $(\Phi_t^{-\mathcal{C}_t})_{0\le t\le T}$.

Theorem 11. *Define $\mathcal{C}_t$ by (4.8). For $X \in L^\infty$ and $t \in [0,T]$, set*

$$\hat{\Phi}_t(X) := -\hat{V}_t(-X), \tag{4.9}$$

where $\hat{V}(-X)$ is the value process of the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$ from Proposition 10. Then $\hat{\Phi} = (\hat{\Phi}_t)_{0\le t\le T}$ is a well-representable strongly time-consistent DMCUF, and the acceptance set of $\hat{\Phi}_t$ is $-\mathcal{C}_t$ so that $\hat{\Phi}_t = \Phi_t^{-\mathcal{C}_t}$ on L^∞. In particular, each $\mathcal{C}_t$ is closed in $\sigma(L^\infty, L^1)$.

Proof. It is clear from Proposition 10 and the subsequent remark that $\hat{\Phi}$ is a DMCUF. One can check from the properties of $A^{\mathcal{H}}(Q)$ that α_t^0 from (4.7) has the σ-pasting property on the set of those $Q \in \mathcal{M}_1^e(P)$ where $\alpha_t^0(Q) \not\equiv -\infty$; see Lemma 5.10 in KS. A closer look at the proof of Theorem 4 shows that this is enough to prove that each $\hat{\Phi}_t$ is well-representable so that its acceptance set $\hat{\mathcal{A}}_t$ is closed in $\sigma(L^\infty, L^1)$. Since clearly $\mathcal{C}_t \subseteq \mathcal{C}_s$ for $t \ge s$, it remains to prove time-consistency and $-\mathcal{C}_t = \hat{\mathcal{A}}_t$.

For any $H \in \mathcal{H}_t$ and $Y \in L^0_+$ with $g = \int_t^T H_s\, dS_s - Y \in \mathcal{C}_t$, we construct an $\mathcal{H}$-constrained hedging portfolio $\Pi' = (0, H', K')$ for g by choosing $H' := H\mathbf{1}_{]\!]t,T]\!]}$ and $K' := Y\mathbf{1}_{[\![T]\!]}$; note that $H' \in \mathcal{H}$ by predictable convexity and that $V^{\Pi'}$ is zero up to time t and uniformly bounded from below since $H \in \mathcal{H}_t$ and $g \in L^\infty$. Hence $\hat{V}_t(g) \le V_t^{\Pi'} = 0$ implies $\hat{\Phi}_t(-g) = -\hat{V}_t(g) \ge 0$ and thus $-g \in \hat{\mathcal{A}}_t$ so that $-\mathcal{C}_t \subseteq \hat{\mathcal{A}}_t$. For the converse inclusion, fix $X \in \hat{\mathcal{A}}_t$ and denote by $\hat{\Pi} = (\hat{x}, \hat{H}, \hat{K})$ the minimal $\mathcal{H}$-constrained hedging portfolio for $-X$. Since $\hat{K}$ is increasing and $V^{\hat{\Pi}}$ is uniformly bounded,

$$V_u^{\hat{\Pi}} = V_t^{\hat{\Pi}} + \int_t^u \hat{H}_s\, dS_s - (\hat{K}_u - \hat{K}_t) \quad , \qquad t \le u \le T \tag{4.10}$$

shows that $\hat{H} \in \mathcal{H}_t$. Moreover, if we take $u = T$ in (4.10) and recall that $V_t^{\hat{\Pi}} \le 0$ (since $X \in \hat{\mathcal{A}}_t$) and $V_T^{\hat{\Pi}} \ge -X$, this also yields $-X \in \mathcal{C}_t$. Hence $-\mathcal{C}_t = \hat{\mathcal{A}}_t$.

To show time-consistency, let $s < t$ and suppose that for some $X, Y \in L^\infty$, we have $\hat{\Phi}_t(X) = \hat{\Phi}_t(Y)$, but $P\big[\hat{\Phi}_s(X) > \hat{\Phi}_s(Y)\big] > 0$. Denote by Π^X, Π^Y the minimal $\mathcal{H}$-constrained hedging portfolios for $-X$ and $-Y$ and define another $\mathcal{H}$-constrained hedging portfolio Π' for $-Y$ by switching from Π^X to Π^Y at time t, or more precisely via

$$x' := x^X, \qquad H' := H^X\mathbf{1}_{[\![0,t]\!]} + H^Y\mathbf{1}_{]\!]t,T]\!]}, \qquad K' := K^X\mathbf{1}_{[\![0,t]\!]} + (K^Y - K_t^Y + K_t^X)\mathbf{1}_{]\!]t,T]\!]}.$$

Note that $H' \in \mathcal{H}$ by predictable convexity and that $V^{\Pi'} = V^{\Pi^X}\mathbf{1}_{[\![0,t]\!]} + V^{\Pi^Y}\mathbf{1}_{]\!]t,T]\!]}$ because $V_t^{\Pi^X} = V_t^{\Pi^Y}$. Therefore $V^{\Pi'}$ is in particular uniformly bounded so that Π' is an $\mathcal{H}$-constrained hedging portfolio for $-Y$. But now

$$P\big[V_s^{\Pi'} < V_s^{\Pi^Y}\big] = P\big[V_s^{\Pi^X} < V_s^{\Pi^Y}\big] = P\big[-\hat{\Phi}_s(X) < -\hat{\Phi}_s(Y)\big] > 0$$

contradicts the minimality of Π^Y. Hence $\hat{\Phi}_s(X) = \hat{\Phi}_s(Y)$ and $\hat{\Phi}$ is time-consistent. **q.e.d.**

After these preparations, we easily construct the utility indifference valuation functional and conclude that it inherits all the above nice properties. This achieves our main goal.

Theorem 12. *Let Φ be a time-consistent DMCUF such that each Φ_t is continuous from below. Fix an admissible hedging set $\mathcal{H}$ and define $\mathcal{C}_t$ by (4.8). If (4.4) is satisfied for each t, the corresponding utility indifference valuation DMCUF p from (4.5) is strongly time-consistent and each p_t is continuous from below.*

Proof. Due to (4.4), Theorem 11 and Theorem 7 imply that

$$\Phi_t^{\mathrm{opt}} := \Phi_t \Box (-\mathcal{C}_t) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t} = \Phi_t \Box \hat{\Phi}_t \qquad \text{for } t \in [0,T]$$

gives a time-consistent DMCUF and that each Φ_t^{opt} is continuous from below. Since p_t is obtained from Φ_t^{opt} simply by normalization, p is even strongly time-consistent. **q.e.d.**

Remarks. 1) We have two reasons for using the result in Proposition 10 about superreplication under constraints. For one thing, this is the easiest way to obtain closedness of $\mathcal{C}_t$; the alternative would be to repeat (the difficult) part of the proofs in FK. More importantly, however, Proposition 10 gives us for the integrals $H \cdot S$ lower bounds which are uniform over varying time intervals, which is crucial when proving time-consistency. For a more detailed discussion, see KS.

2) The sets $\mathcal{C}_t$ in (4.8) satisfy the natural no-arbitrage requirement

$$\Phi_t^{-\mathcal{C}_t}(0) = \operatorname{ess\,sup}\big(\mathcal{C}_t \cap L^\infty(\mathcal{F}_t)\big) \le 0, \tag{4.11}$$

saying that one cannot superreplicate from t on at zero cost something known at time t and positive. In fact, if $g = \int_t^T H_s\, dS_s - Y$ with $H \in \mathcal{H}_t$ and $Y \in L^0_+$, the process $\int_t^{\bullet} H\, dS$ on $[t,T]$ is under $Q^* \in I\!P^e$ a local martingale and a supermartingale, both because it is uniformly bounded from below. This implies $E_{Q^*}[g|\mathcal{F}_t] \le 0$, and so any $g \in \mathcal{C}_t \cap L^\infty(\mathcal{F}_t)$ must be ≤ 0.

3) During the revision of this paper, we have learnt that Cheridito/Kupper (2006) have obtained a sort of converse to Theorem 12. More precisely, let $U = (U_t)_{0\le t\le T}$ be a dynamic utility functional satisfying A) and E), but not necessarily the translation invariance B). For a given financial market and some random endowment $V \in L^\infty(\mathcal{F}_T)$, define the conditional *certainty equivalent* functionals $c_t^V : L^\infty \to L^\infty(\mathcal{F}_t)$ by

$$U_t^{\mathrm{opt}}\big(V + c_t^V(X)\big) = U_t^{\mathrm{opt}}(V + X) \qquad \text{for } t \in [0,T]$$

and the indifference price functionals $p_t^V : L^\infty \to L^\infty(\mathcal{F}_t)$ by

$$U_t^{\rm opt}\big(V + X - p_t^V(X)\big) = U_t^{\rm opt}(V) \qquad \text{for } t \in [0,T].$$

If U is translation invariant, then each U_t is Lipschitz-continuous, and p^V and c^V coincide. Taking $V \equiv 0$ then yields our dynamic indifference valuation functional

$$p = p^0 = c^0 = U^{\rm opt} - U^{\rm opt}(0).$$

(All this is formal; we ignore existence and uniqueness issues for this explanation.)

Now assume for U that A), C) and E) hold and that U is time-consistent. Then (omitting technical conditions) Theorem 12 says that translation invariance of U implies that p is again time-consistent. The converse result obtained by Cheridito/Kupper (2006) (with a degenerate financial market and again omitting technical conditions) reads as follows. Assume that U satisfies A) (essentially), continuity on L^∞ and E), and that U is time-consistent. Then time-consistency of the indifference prices p^V for all V implies that the certainty equivalent c^V is translation invariant for all V. ◇

For unconstrained trading, the structure of $\Phi^{\rm opt}$ becomes more explicit. In the static case, this has for instance been studied in Chapter 4.8 in FS or Corollary 3.7 of BEK. Starting from Φ_0 with concave conjugate α_0, they show that the convolution $\Phi_0^{\rm opt}$ of Φ_0 with the market functional $\Phi_0^{-\mathcal{C}_0}$ is obtained by replacing in the representation (2.6) of Φ_0 the infimum over all of $\mathcal{M}_1(P)$ with that over the smaller set $I\!P^a$ of absolutely continuous local martingale measures for S. In other words, we have

$$\Phi_0^{\rm opt}(X) = \inf_{Q \in I\!P^a} \big(E_Q[X] - \alpha_0(Q)\big). \tag{4.12}$$

The next result is a dynamic analogue. Its proof is a bit lengthy, but contains no major new ideas; hence we omit it and refer interested readers to Section 6 of KS for more details.

Proposition 13. *Let Φ_t be an MCUF at time t with acceptance set $\mathcal{A}_t$ and concave conjugate α_t. Assume that Φ_t is continuous from below and that $\inf_{X \in \mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty$ for some $\tilde{Q} \in I\!P^e$. Define $\mathcal{C}_t$ by (4.8) with $\mathcal{H} := L^a_{\rm loc}(S)$ and assume that (4.4) holds. Then we have the representation*

$$\Phi_t^{\rm opt}(X) = \Phi_t \Box \Phi_t^{-\mathcal{C}_t}(X) = \operatorname*{ess\,inf}_{Q \in I\!P^e} \big\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\big\} \qquad \textit{for } X \in L^\infty. \tag{4.13}$$

Remarks. 1) The direct analogue of (4.12) would be to take the essential infimum over all local martingale measures for the process $(S_u)_{t \le u \le T}$ on $[t,T]$. However, (4.13) shows that we even have a representation where the set of Q's does not depend on t and consists of equivalent measures.

2) Both (4.4) and the assumption that $\inf_{X \in \mathcal{A}_t} E_Q[X] > -\infty$ for some $Q \in I\!P^e$ formalize the intuitive requirement that the a priori preferences Φ_t should fit together with the financial market. Like in the comment after Lemma 6, the second condition (involving Q) need only hold for $t = 0$ if Φ is strongly time-consistent.

3) To apply Theorem 7 to Φ_t and $\Phi_t^{-\mathcal{C}_t}$, it is enough that one of these MCUFs is continuous from below. We have good reasons for imposing this on Φ_t and not on the market MCUF $\Phi_t^{-\mathcal{C}_t}$. In the unconstrained case, we obtain in analogy to (4.12) that $\Phi_0^{-\mathcal{C}_0}(X) = \inf_{Q \in I\!P^a} E_Q[X]$. But if $\Phi_0^{-\mathcal{C}_0}$ is continuous from below, Corollary 4.35 of FS implies that $I\!P^a$ is weakly compact, and if then in addition S is continuous and $I\!F$ is quasi-left-continuous, Corollary 7.2 of Delbaen (1992) implies that $I\!P^a$ is a singleton which means that the market is complete. Assuming continuity for the market MCUF may thus be rather restrictive. ◇

Before turning to the relation between arbitrage and the above valuation, let us give an example that illustrates several points we have discussed so far.

Example 1. Let $u(x) = -e^{-x}$ be the exponential utility function and

$$\Phi_t(X) := u^{-1}\big(E[u(X)|\mathcal{F}_t]\big) = -\log E\big[e^{-X}\big|\mathcal{F}_t\big] \qquad \text{for } X \in L^\infty \tag{4.14}$$

the corresponding $\mathcal{F}_t$-conditional certainty equivalent. Then $\Phi = (\Phi_t)_{0 \le t \le T}$ is a DMCUF, each Φ_t is clearly continuous from below, and the concave conjugate functional of Φ_t is

$$\alpha_t(Q) = -E_Q\left[\log \frac{Z_T^Q}{Z_t^Q}\,\middle|\,\mathcal{F}_t\right] =: -H_t(Q|P), \tag{4.15}$$

i.e., minus the $\mathcal{F}_t$-conditional relative entropy of Q with respect to P. This is shown in Section 4 of DS; see also Example 4.33 in FS. The DMCUF Φ is clearly normalized and time-consistent due to the explicit expression (4.14); hence Φ is strongly time-consistent. Moreover, each Φ_t is well-representable because (2.4) follows easily from (4.14). In fact, Jensen's inequality gives $E[u(X)|\mathcal{F}_t] \le u(E[X|\mathcal{F}_t])$, hence $E[X|\mathcal{F}_t] \ge \Phi_t(X) \ge 0$ for all $X \in \mathcal{A}_t$ and therefore $\inf_{X \in \mathcal{A}_t} E[X] \ge 0 > -\infty$. From Theorem 4 and (4.15), we thus have

$$\Phi_t(X) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}_1^e(P)} \big(E_Q[X|\mathcal{F}_t] + H_t(Q|P)\big).$$

Consider next a financial market as in the present section. Choose $\mathcal{H} = L^a_{\mathrm{loc}}(S)$ so that we have no constraints, define $\mathcal{C}_t$ by (4.8) and $\hat{\Phi} = (\hat{\Phi}_t)_{0 \le t \le T}$ by (4.9) so that $\hat{\Phi}_t$ is by

Theorem 11 the market MCUF induced by $\mathcal{C}_t$. Moreover, $\hat{\Phi}$ is also normalized and strongly time-consistent by Theorem 11. As in Theorem 12, define

$$\Phi_t^{\text{opt}}(X) := \Phi_t \Box \hat{\Phi}_t(X) = \Phi_t \Box (-\mathcal{C}_t)(X) \qquad \text{for } t \in [0,T] \text{ and } X \in L^\infty$$

and assume as in (4.4) that

$$\Phi_t^{\text{opt}}(0) = \operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \Phi_t(g) \in L^\infty. \tag{4.16}$$

We give below a sufficient condition on S to ensure (4.16). By Theorem 7, Φ^{opt} is then again a strongly time-consistent DMCUF, and due to (4.15) and Proposition 13, we have

$$\Phi_t^{\text{opt}}(X) = \operatorname*{ess\,inf}_{Q \in \mathbb{P}^e} \big(E_Q[X|\mathcal{F}_t] + H_t(Q|P) \big). \tag{4.17}$$

Now impose on the financial market the assumptions that $P \notin \mathbb{P}^e$ (so S is not a local P-martingale) and that $\inf_{Q \in \mathbb{P}^e} H(Q|P) = \inf_{Q \in \mathbb{P}^e} H_0(Q|P) < \infty$, so that there exists an equivalent local martingale measure for S with finite relative entropy with respect to P. Then it is well known that the minimal entropy martingale measure $Q^E := \operatorname{argmin}\{H(Q|P) \,|\, Q \in \mathbb{P}^e\}$ exists in $\mathbb{P}^e$ and is unique, and we have

$$H_0\big(Q^E \big| P\big) > 0 \tag{4.18}$$

because P is not in $\mathbb{P}^e$. But (4.18) implies by (4.17) that

$$\Phi_0^{\text{opt}}(0) = \inf_{Q \in \mathbb{P}^e} H_0(Q|P) = H_0\big(Q^E \big| P\big) > 0,$$

and therefore Φ^{opt} is not normalized. Hence this example illustrates that

- a DMCUF may be strongly time-consistent without being normalized.
- the convolution of two normalized DMCUFs may fail to be normalized.

To finish the example, let us briefly discuss how to guarantee the condition (4.16). By the explicit expression (4.14) for Φ_t, (4.16) is equivalent to

$$\operatorname*{ess\,sup}_{g \in \mathcal{C}_t} E[u(g)|\mathcal{F}_t] \in L^\infty, \tag{4.19}$$

and since $g \equiv 0$ is in $\mathcal{C}_t$, it is enough to have an upper bound for $E[u(g)|\mathcal{F}_t]$ uniformly over $g \in \mathcal{C}_t$. Applying Fenchel's inequality

$$u(x) = -e^{-x} \le \sup_{x'>0} \big(u(x') - x'y\big) + xy = y \log y - y + xy$$

with $y = \frac{Z_T^Q}{Z_t^Q}$ for some $Q \in \mathbb{P}^e$ gives

$$E[u(g)|\mathcal{F}_t] \le H_t(Q|P) - 1 + E_Q[g|\mathcal{F}_t] \le H_t(Q|P)$$

because $E_Q[g|\mathcal{F}_t] \le 0$ for any $g \in \mathcal{C}_t$, since $\int_t^\bullet H\,dS$ for $H \in \mathcal{H}_t$ is a Q-supermartingale for any $Q \in I\!P^e$; see Remark 2) after Theorem 12. Hence (4.19) holds as soon as

$$\operatorname*{ess\,inf}_{Q \in I\!P^e} H_t(Q|P) \in L^\infty.$$

One sufficient condition for this is that there exists some $Q \in I\!P^e$ satisfying the reverse Hölder inequality $R_{\mathrm{L\,Log\,L}}(P)$, i.e.,

$$H_t(Q|P) = E\left[\frac{Z_T^Q}{Z_t^Q}\log\frac{Z_T^Q}{Z_t^Q}\,\middle|\,\mathcal{F}_t\right] \le C$$

for all $t \in [0,T]$ with some constant C. This ends the example. □

Let us now examine the relation between arbitrage and our valuation approach. An immediate consequence of Proposition 13 is the following no-arbitrage result for the utility indifference values in the case of unconstrained trading.

Corollary 14. *Under the assumptions of Proposition 13, the valuations p_t and p_t^s are free of arbitrage in the following two senses:*

a) *Both p_t and p_t^s take only values in the interval of arbitrage-free prices, i.e.,*

$$\operatorname*{ess\,inf}_{Q \in I\!P^e} E_Q[X|\mathcal{F}_t] \le p_t(X) \le p_t^s(X) \le \operatorname*{ess\,sup}_{Q \in I\!P^e} E_Q[X|\mathcal{F}_t] \qquad \textit{for all } X \in L^\infty.$$

b) *If $X \in L^\infty$ is attainable from time t, i.e., $X = x_t + \int_t^T H_s\,dS_s$ with $x_t \in L^\infty(\mathcal{F}_t)$ and $H \in L^a_{\mathrm{loc}}(S)$ such that $\int_t^\bullet H_s\,dS_s$ is uniformly bounded, then $p_t(X) = p_t^s(X) = x_t$.*

Proof. a) follows from the remark after Proposition 10 and Proposition 15 below, since $-\mathcal{C}_t$ is a convex cone containing 0. b) follows from (4.5) and (4.13) because $E_Q[X|\mathcal{F}_t] = x_t$ for any $Q \in I\!P^e$ since X is attainable. **q.e.d.**

We now return to the general setup where $-\mathcal{C}_t \subseteq L^\infty$ is an abstract pre-acceptance set at time t. A natural question is then whether the *value* $p_t(X)$ in (4.5) can also be used as a buying *price* for X, and how this is related to issues of arbitrage. First of all, we should like to have the no-arbitrage condition (4.11) that $\Phi_t^{-\mathcal{C}_t}(0) \le 0$. Concavity of $\Phi_t^{-\mathcal{C}_t}$ then yields $\Phi_t^{-\mathcal{C}_t}(X) + \Phi_t^{-\mathcal{C}_t}(-X) \le 2\Phi_t^{-\mathcal{C}_t}(0) \le 0$ so that the interval $\left[\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)\right]$ from the subreplication to the superreplication price for X is non-empty. Because p_t is normalized, we also have from concavity that $0 = 2p_t(0) \ge p_t(X) + p_t(-X) = p_t(X) - p_t^s(X)$ so that the seller value always lies above the buyer value. To exclude arbitrage with "prices" taken from p_t, it is thus enough to have the interlocking inequalities

$$p_t(X) \le -\Phi_t^{-\mathcal{C}_t}(-X) \qquad \text{and} \qquad p_t^s(X) \ge \Phi_t^{-\mathcal{C}_t}(X). \tag{4.20}$$

For equivalent formulations, see Theorem 6.19 of KS. However, it seems more natural to use the stronger condition

$$\big[p_t(X), p_t^s(X)\big] \subseteq \big[\Phi_t^{-\mathcal{C}_t}(X), -\Phi_t^{-\mathcal{C}_t}(-X)\big] \tag{4.21}$$

because even if we had for instance $p_t^s(X) > -\Phi_t^{-\mathcal{C}_t}(-X)$, that seller would be unable to find a buyer at that price $p_t^s(X)$.

Proposition 15. *Let Φ_t be an MCUF at time t and $-\mathcal{C}_t \subseteq L^\infty$ a pre-acceptance set at time t such that (4.4) holds. The corresponding utility indifference values p_t from (4.5) are then free of arbitrage in the sense of (4.21) if one of the following conditions holds:*

a) *$-\mathcal{C}_t$ is a convex cone containing 0.*

b) *0 is in the acceptance set of Φ_t and the MCUF $\Phi_t^{\text{opt}} = \Phi_t \square (-\mathcal{C}_t)$ is normalized, i.e., $\Phi_t(0) \ge 0$ and $\operatorname*{ess\,sup}_{g \in \mathcal{C}_t} \Phi_t(g) = 0$. (Note that this implies (4.11).)*

In particular, if a) or b) holds and if X satisfies $\Phi_t^{-\mathcal{C}_t}(X) = -\Phi_t^{-\mathcal{C}_t}(-X)$, then

$$\Phi_t^{-\mathcal{C}_t}(X) = p_t(X) = p_t^s(X) = -\Phi_t^{-\mathcal{C}_t}(-X).$$

Thus for an asset which can be traded in the market, value and market price must coincide.

Proof. Since $p_t^s(X) = -p_t(-X)$, it suffices to show that $\Phi_t^{-\mathcal{C}_t}(X) \le p_t(X)$. If a) holds, $\Phi_t^{-\mathcal{C}_t}$ is by Lemma 8 positively homogeneous and therefore superadditive, i.e., $\Phi_t^{-\mathcal{C}_t}(X+Y) \ge \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t^{-\mathcal{C}_t}(Y)$. Hence (4.5) and (3.1) imply that

$$\begin{aligned} p_t(X) &= \operatorname*{ess\,sup}_{Y \in L^\infty} \big(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y)\big) - \Phi_t \square \Phi_t^{-\mathcal{C}_t}(0) \\ &\ge \Phi_t^{-\mathcal{C}_t}(X) + \operatorname*{ess\,sup}_{Y \in L^\infty} \big(\Phi_t^{-\mathcal{C}_t}(Y) + \Phi_t(-Y)\big) - \Phi_t \square \Phi_t^{-\mathcal{C}_t}(0) \\ &= \Phi_t^{-\mathcal{C}_t}(X). \end{aligned}$$

If b) holds, using (4.5), (3.1) and $Y := 0$ similarly yields

$$p_t(X) = \operatorname*{ess\,sup}_{Y \in L^\infty} \big(\Phi_t^{-\mathcal{C}_t}(X+Y) + \Phi_t(-Y)\big) \ge \Phi_t^{-\mathcal{C}_t}(X) + \Phi_t(0) \ge \Phi_t^{-\mathcal{C}_t}(X).$$

q.e.d.

When $\mathcal{C}_t$ is only convex but not a convex cone containing 0, even the weaker no-arbitrage condition (4.20) can be violated. This can be explained as follows. Our definition (4.1) of the utility indifference value uses the same set of gains $\mathcal{C}_t$ from strategies irrespective of whether the agent owns X or not, and so we implicitly assume that buying X does not change the set of possible strategies. Note that X is here viewed as a new financial instrument; like in a market with transaction costs, this must be distinguished from a portfolio generating the same payoff as X, but formed from the primary assets in the market. The following example explicitly illustrates how buying or owning such a portfolio can change the set $\mathcal{C}_0$ of allowed gains into a new set $\mathcal{C}_0^X$, and how this makes it reasonable for the agent to pay more for X than the $\mathcal{C}_0$-superreplication price. Indeed, although $p_0(X)$ is bigger than $-\Phi_0^{-\mathcal{C}_0}(-X)$, the agent cannot increase her maximal attainable utility by superreplicating X via the portfolio instead of buying it directly for $p_0(X)$, because she may only work with $\mathcal{C}_0^X$ after the superreplication.

The above discussion shows that one must be very careful when introducing a new instrument X in the market, because (especially with constraints) this may affect the set of allowed trades. However, we do not pursue this delicate issue any further.

Example 2. Consider a one-step discrete-time model with two possible states, a bank account (as numeraire) with zero interest rate and one risky asset S with net payoff $S_1 - S_0 = \big(-1, \frac{1}{4}\big)$. Trading is restricted in that the agent may not go short more than 1 unit of S. The set of payoffs that can be superreplicated by trading from zero capital is thus

$$\mathcal{C}_0 = \big\{\beta\big(-1, \tfrac{1}{4}\big) \,\big|\, \beta \geq -1\big\} - I\!R^2_+.$$

The superreplication price for the payoff $X := \big(\frac{1}{2}, -\frac{1}{4}\big)$ is

$$\begin{aligned} -\Phi_0^{-\mathcal{C}_0}(-X) &= \inf\big\{c \in I\!R \,\big|\, c + \beta(S_1 - S_0) \geq X \text{ for some } \beta \geq -1\big\} \\ &= \inf_{\beta \geq -1} \max\big(\tfrac{1}{2} + \beta, -\tfrac{1}{4} - \tfrac{1}{4}\beta\big) \\ &= -\tfrac{1}{10}; \end{aligned}$$

it is attained for $\beta^* = -\frac{3}{5}$, and the corresponding strategy even replicates X. The preferences of the agent are given by the exponential certainty equivalent with risk aversion $\frac{1}{4}$ so that

$$\Phi_0(X) = -4 \log E\big[e^{-\frac{1}{4}X}\big],$$

and P assigns to both states probability $\frac{1}{2}$. Hence the maximal attainable monetary utility without owning X is

$$\Phi_0^{\mathrm{opt}}(0) = \sup_{g \in \mathcal{C}_0} \Phi_0(g) = \sup_{\beta \geq -1} -4 \log E\big[e^{-\frac{1}{4}\beta(S_1 - S_0)}\big] = -4 \log\Big(\tfrac{1}{2}\big(e^{-\frac{1}{4}} + e^{\frac{1}{16}}\big)\Big) \approx 0.3264,$$

attained for $\beta = -1$. Similarly, the maximal attainable monetary utility when holding X is

$$\sup_{g\in\mathcal{C}_0} \Phi_0(X+g) = \sup_{\beta\geq -1} -4\log E\Big[e^{-\frac{1}{4}\left(X+\beta(S_1-S_0)\right)}\Big] = -4\log\Big(\tfrac{1}{2}\big(e^{-\frac{3}{8}}+e^{\frac{1}{8}}\big)\Big) \approx 0.3763,$$

which is again attained for $\beta = -1$. By (4.5),

$$p_0(X) = \sup_{g\in\mathcal{C}_0} \Phi_0(X+g) - \sup_{g\in\mathcal{C}_0} \Phi_0(g) \approx 0.050 > -\tfrac{1}{10} = -\Phi_0^{-\mathcal{C}_0}(-X)$$

so that even the weak no-arbitrage condition (4.20) is violated.

In the present situation, we can clearly see why this happens. The usual argument why prices should conform with no-arbitrage bounds is that buying X for more than its superreplication price is irrational because it is cheaper to buy the assets required to superreplicate X. However, this does not apply to the values here. For superreplicating X, we need to sell short $|\beta^*| = \frac{3}{5}$ units of S, and then we can go short only $\frac{2}{5}$ further units. Therefore the maximal attainable monetary utility after (super-)replicating $X = -\Phi_0^{-\mathcal{C}_0}(-X) + \beta^*(S_1 - S_0)$ is

$$\sup_{\beta\geq -\frac{2}{5}} \Phi_0\Big(X - \big(-\Phi_0^{-\mathcal{C}_0}(-X)\big) + \beta(S_1-S_0)\Big) = \sup_{\beta\geq -\frac{2}{5}} \Phi_0\big((\beta^*+\beta)(S_1-S_0)\big) = \Phi_0^{\mathrm{opt}}(0).$$

Note how the initial trade to superreplicate X has changed the set of strategies from $\mathcal{C}_0 \mathrel{\widehat{=}} \{\beta \geq -1\}$ to $\mathcal{C}_0^X \mathrel{\widehat{=}} \big\{\beta \geq -\frac{2}{5}\big\}$. On the other hand, if directly buying X for $p_0(X)$ does not change the set of allowed strategies, the maximal monetary utility after that purchase is

$$\sup_{g\in\mathcal{C}_0} \Phi_0\big(X - p_0(X) + g\big) = \Phi_0^{\mathrm{opt}}(X) - p_0(X) = \Phi^{\mathrm{opt}}(0).$$

Hence acting upon the apparent arbitrage opportunity does not yield a higher utility than buying X for $p_0(X)$, since the former trade changes the set of allowed strategies. □

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Appendix

This appendix contains some proofs that have been omitted from the main body of the paper to streamline the presentation.

Some of the results in Section 2 can be obtained as special cases from CDK. This is not entirely obvious for two reasons. Like DS, CDK impose in their axioms for MCUFs normalization and $\mathcal{F}_t$-concavity instead of C); see Remark 2) before Lemma 1. This difference has no effect for those results we want to quote. More importantly, CDK work more generally with DMCUFs defined on processes instead of random variables and therefore use more elaborate notations than we need here. To help readers in making the connection, we very briefly sketch here the main translations between CDK and our setting.

When using CDK results for random variables, replace $\mathcal{R}^\infty$ and $\mathcal{R}^\infty_{\tau,\theta}$ by L^∞; replace $\|\cdot\|_{\tau,\theta}$ by $\|\cdot\|_t$; replace $L^\infty(\mathcal{F}_\tau)$ by $L^\infty(\mathcal{F}_t)$; and omit all $1_{[\tau,\infty)}$. Moreover, replace $\mathcal{D}_{\tau,\theta}$ by $\big\{Z \in L^1_+(P)\,\big|\,E[Z|\mathcal{F}_t]=1\big\}$ which corresponds to the set of densities of the elements in $\mathcal{P}^{=}_t$. Then $\mathcal{D}^{\mathrm{rel}}_{\tau,\theta}$ corresponds to $\big\{Z \in L^1_+(P)\,\big|\,E[Z|\mathcal{F}_t]=1,\ Z>0\ P\text{-a.s.}\big\}$ which can also be written as

$$\left\{\frac{Z^Q_T}{Z^Q_t}\,\middle|\, Z^Q=(Z^Q_t)_{0\le t\le T} \text{ is the density process of some } Q\in\mathcal{M}^e_1(P)\right\}.$$

Finally, $\langle X,a\rangle_{\tau,\theta}$ with $X\in\mathcal{R}^\infty$ and $a\in\mathcal{A}^1$ must be replaced by $E[Xa|\mathcal{F}_t]$ with $X\in L^\infty$ and $a\in\big\{Z\in L^1_+(P)\,\big|\,E[Z|\mathcal{F}_t]=1\big\}$.

Proof of Theorem 4. As discussed after Lemma 5, there are some differences between Theorem 4 and the results in DS and CDK. Nevertheless, the proof mostly parallels the arguments in DS, and this allows us to be brief.

"III) $\Rightarrow$ II)" is clear due to Lemma 3.

"II) $\Rightarrow$ I)" goes as in DS for the continuity. To prove (2.4), choose a sequence (Q^n) in $\mathcal{M}^e_1(P)$ and for $\varepsilon>0$ an $\mathcal{F}_t$-partition (A_n) of Ω such that

$$-\Phi_t(0)=\operatorname*{ess\,sup}_{Q\in\mathcal{M}^e_1(P)}\alpha^0_t(Q)=\sup_{n\in I\!N}\alpha^0_t(Q^n)\le\sum_{n=1}^\infty \mathbf{1}_{A_n}\alpha^0_t(Q^n)+\varepsilon.$$

Define $\tilde Q\in\mathcal{M}^e_1(P)$ by $\frac{d\tilde Q}{dP}:=\sum\limits_{n=1}^\infty \mathbf{1}_{A_n}\frac{Z^n_T}{Z^n_t}$ and note that the σ-pasting property of α^0_t gives $\alpha^0_t(\tilde Q)+\varepsilon\ge-\Phi_t(0)\in L^\infty$. Using (2.3) and (2.5) yields

$$\operatorname*{ess\,inf}_{X\in\mathcal{A}_t}E_{\tilde Q}[X|\mathcal{F}_t]=\operatorname*{ess\,inf}_{X\in L^\infty}\Big\{E_{\tilde Q}[X|\mathcal{F}_t]-\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_1(P)}\big\{E_Q[X|\mathcal{F}_t]-\alpha^0_t(Q)\big\}\Big\}\ge\alpha^0_t(\tilde Q)$$

and therefore $\inf\limits_{X\in\mathcal{A}_t}E_{\tilde Q}[X]\ge E_{\tilde Q}[\alpha^0_t(\tilde Q)]\ge -E_{\tilde Q}[\Phi_t(0)]-\varepsilon>-\infty$, which is (2.4).

"I) ⇒ III)": This is similar to DS, but with some subtle differences. We immediately get

$$\Phi_t(X) \leq \operatorname*{ess\,inf}_{Q\in\mathcal{M}_1^e(P)} \big\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\big\} \tag{A.1}$$

and want to prove equality by showing equality of expectations. To that end, we define $\tilde{\Phi}_0(X) := E_{\tilde{Q}}[\Phi_t(X)]$ and represent this MCUF $\tilde{\Phi}_0$ at time 0 by Theorem 4.31 in FS as

$$\tilde{\Phi}_0(X) = \inf_{Q\in\mathcal{M}_1(P)} \big(E_Q[X] - \tilde{\alpha}_0(Q)\big)$$

with

$$\tilde{\alpha}_0(Q) = \inf_{Y\in L^\infty} \big(E_Q[Y] - \tilde{\Phi}_0(Y)\big). \tag{A.2}$$

We argue below that $\tilde{\alpha}_0(\tilde{Q}) > -\infty$, and because $\tilde{Q} \in \mathcal{M}_1^e(P)$, this implies that we also have

$$\tilde{\Phi}_0(X) = \inf_{Q\in\mathcal{M}_1^e(P)} \big(E_Q[X] - \tilde{\alpha}_0(Q)\big).$$

In analogy to DS, we now define $\tilde{\mathcal{Q}}_t := \{Q \in \mathcal{M}_1^e(P) \,|\, Q = \tilde{Q} \text{ on } \mathcal{F}_t\}$; note the *two* slight differences to $\mathcal{P}_t^=$ from DS. Exactly as in DS, we then can show first

$$\tilde{\Phi}_0(X) = \inf_{Q\in\tilde{\mathcal{Q}}_t} \big(E_Q[X] - \tilde{\alpha}_0(Q)\big)$$

and then

$$E_{\tilde{Q}}\left[\operatorname*{ess\,inf}_{Q\in\mathcal{M}_1^e(P)} \big\{E_Q[X|\mathcal{F}_t] - \alpha_t(Q)\big\}\right] \leq \tilde{\Phi}_0(X) = E_{\tilde{Q}}[\Phi_t(X)].$$

Hence we have (2.6) in view of (A.1).

To see that $\tilde{\alpha}_0(\tilde{Q}) > -\infty$, note that $Y - \Phi_t(Y) \in \mathcal{A}_t$ for any $Y \in L^\infty$. Hence (A.2) gives due to (2.4) that

$$\tilde{\alpha}_0(\tilde{Q}) = \inf_{Y\in L^\infty} E_{\tilde{Q}}[Y - \Phi_t(Y)] \geq \inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] > -\infty.$$

"I) ⇒ IV)" goes as in the static case; see FS, Theorem 4.31, (c) ⇒ (e).

"IV) ⇒ I)": If $X_n \searrow X \in L^\infty$, then $\Phi_t(X_n) \searrow Z$ for some $Z \in L^\infty(\mathcal{F}_t)$ and $Y_n := X_n - \Phi_t(X_n) \longrightarrow X - Z$ P-a.s. Moreover, (Y_n) is like (X_n) uniformly bounded and hence by dominated convergence tends to $X - Z$ in $\sigma(L^\infty, L^1)$. Because Y_n is in $\mathcal{A}_t$ for all n, so is $X - Z$ since $\mathcal{A}_t$ is closed in $\sigma(L^\infty, L^1)$. Thus $\Phi_t(X) \geq Z$ and monotonicity yields

$$\lim_{n\to\infty} \Phi_t(X_n) = Z \leq \Phi_t(X) = \Phi_t\Big(\lim_{n\to\infty} X_n\Big) \leq \lim_{n\to\infty} \Phi_t(X_n).$$

The assertion for MCohUFs is proved as in DS; see their comment after Remark 10. Finally $\mathcal{Q}$ can be chosen convex and closed in L^1 as in Delbaen (2000). **q.e.d.**

Proof of Lemma 5. b) (2.7) gives $\Phi_t(-\mathbf{1}_B) \le -E_Q[\mathbf{1}_B|\mathcal{F}_t]$ for some $Q \in \mathcal{M}_1^e(P)$, and $\Phi_t(0) = 0$. Hence Φ_t is relevant.

a) Almost like in the proof of Theorem 4, "I) $\Longrightarrow$ III)", we define and represent an MCUF $\tilde{\Phi}_0$ at time 0 by

$$\tilde{\Phi}_0(X) := E[\Phi_t(X)] = \inf_{Q\in\mathcal{M}_1(P)} \big(E_Q[X] - \tilde{\alpha}_0(Q)\big) = \inf_{\substack{Q\in\mathcal{M}_1(P),\\ \tilde{\alpha}_0(Q)>-\infty}} \big(E_Q[X] - \tilde{\alpha}_0(Q)\big) \tag{A.3}$$

with

$$\tilde{\alpha}_0(Q) = \inf_{Y\in L^\infty} \big(E_Q[Y] - \tilde{\Phi}_0(Y)\big);$$

the last equality in (A.3) holds since $\tilde{\Phi}_0$ is finite-valued. Because Φ_t is relevant, so is $\tilde{\Phi}_0$. To construct $\tilde{Q} \in \mathcal{M}_1^e(P)$ with

$$\tilde{\alpha}_0(\tilde{Q}) > -\infty, \tag{A.4}$$

we define $B \in \mathcal{F}$ up to nullsets by

$$\mathbf{1}_B := \operatorname{ess\,sup}\Big\{\mathbf{1}_{\{Z_T^Q>0\}}\ \Big|\ Q \in \mathcal{M}_1(P) \text{ and } \tilde{\alpha}_0(Q) > -\infty\Big\}.$$

Then we get for $Q \in \mathcal{M}_1(P)$ with $\tilde{\alpha}_0(Q) > -\infty$ that $E_Q[\mathbf{1}_{B^c}] = 0$. Hence (A.3) yields $\tilde{\Phi}_0(-\mathbf{1}_{B^c}) = \tilde{\Phi}_0(0)$ and so $P[B] = 1$ by relevance of $\tilde{\Phi}_0$. Now choose $Q^n \in \mathcal{M}_1(P)$ with density processes Z^n and $\tilde{\alpha}_0(Q^n) > -\infty$ such that $\sup_{n\in\mathbb{N}} \mathbf{1}_{\{Z_T^n>0\}} = \mathbf{1}_B = 1$ P-a.s., and $\beta_n > 0$ with $\sum_{n=1}^\infty \beta_n = 1$ and $\sum_{n=1}^\infty \beta_n\tilde{\alpha}_0(Q^n) > -\infty$. Then $\frac{d\tilde{Q}}{dP} := \sum_{n=1}^\infty \beta_n Z_T^n$ defines a $\tilde{Q} \in \mathcal{M}_1^e(P)$ which satisfies (A.4). Like for the proof of "I $\Longrightarrow$ III)" for Theorem 4, one now first proves as in DS that $\tilde{\alpha}_0(Q) = -\infty$ for any $Q \in \mathcal{M}_1^e(P) \setminus \mathcal{P}_t^=$, which implies that $\tilde{Q} \in \mathcal{P}_t^=$, and then shows that $\tilde{\alpha}_0(\tilde{Q}) = E_P\big[\alpha_t(\tilde{Q})\big]$. Therefore (2.3) yields

$$\inf_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X] \ge E_{\tilde{Q}}\Big[\operatorname*{ess\,inf}_{X\in\mathcal{A}_t} E_{\tilde{Q}}[X|\mathcal{F}_t]\Big] = E_P\big[\alpha_t(\tilde{Q})\big] = \tilde{\alpha}_0(\tilde{Q}) > -\infty,$$

and so $\tilde{Q}$ does the job. **q.e.d.**

Proof of Lemma 6. a) implies d) and by 3) of Lemma 2 is equivalent to b). If $\Phi_t(0) \equiv 0$, take $X \in L^\infty$ and define $Y := \Phi_t(X)$ to get by translation invariance $\Phi_t(Y) = \Phi_t\big(0 + \Phi_t(X)\big) = \Phi_t(X)$. Time-consistency then yields $\Phi_s(X) = \Phi_s(Y) = \Phi_{sot}(X)$ so that d) implies a).

"b) $\Rightarrow$ c)": To show the inclusion "$\supseteq$", let $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$, $X_2 \in \mathcal{A}_t$ and use translation invariance and $X_2 \in \mathcal{A}_t$ to get $\Phi_t(X) = X_1 + \Phi_t(X_2) \ge X_1$. Monotonicity and $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ thus yield $\Phi_{sot}(X) \ge \Phi_s(X_1) \ge 0$ so that $X \in \mathcal{A}_{sot} = \mathcal{A}_s$ by b). For the converse inclusion, write $X \in \mathcal{A}_s$ as $X = \Phi_t(X) + \big(X - \Phi_t(X)\big)$. The second summand is in $\mathcal{A}_t$, and the first is in $\mathcal{A}_s(\mathcal{F}_t)$ since $\Phi_s\big(\Phi_t(X)\big) = \Phi_{sot}(X) \ge 0$ because $X \in \mathcal{A}_s = \mathcal{A}_{sot}$ by b).

"c) $\Rightarrow$ b)": To show "$\subseteq$", write $X \in \mathcal{A}_s$ by c) as $X = X_1 + X_2$ with $X_1 \in \mathcal{A}_s(\mathcal{F}_t)$ and $X_2 \in \mathcal{A}_t$. As above, this yields $\Phi_t(X) \geq X_1$ and hence by monotonicity of Φ_s that $\Phi_s\big(\Phi_t(X)\big) \geq \Phi_s(X_1) \geq 0$ since $X_1 \in \mathcal{A}_s$. Thus $\Phi_t(X) \in \mathcal{A}_s$ which is equivalent to $X \in \mathcal{A}_{sot}$. To obtain "$\supseteq$", note that $X \in \mathcal{A}_{sot}$ gives $\Phi_t(X) \in \mathcal{A}_s(\mathcal{F}_t)$ so that $X = \big(X - \Phi_t(X)\big) + \Phi_t(X) \in \mathcal{A}_t + \mathcal{A}_s(\mathcal{F}_t) = \mathcal{A}_s$ by c). **q.e.d.**

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