Mathematical Finance, Vol. 2, No. 3 (July 1992), 153–187

OPTION PRICING UNDER INCOMPLETENESS AND STOCHASTIC VOLATILITY

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We consider a very general diffusion model for asset prices which allows the description of stochastic and past-dependent volatilities. Since this model typically yields an incomplete market, we show that for the purpose of pricing options, a small investor should use the minimal equivalent martingale measure associated to the underlying stock price process. Then we present stochastic numerical methods permitting the explicit computation of option prices and hedging strategies, and we illustrate our approach by specific examples.

KEY WORDS: option pricing, stochastic volatility, incomplete markets, equivalent martingale measures, stochastic numerical methods

1. INTRODUCTION

Ever since the path-breaking paper by Black and Scholes (1973) on the pricing of options and corporate liabilities appeared, there has been concern about the assumptions imposed on the behavior of the underlying stock price process. The most criticized of these, besides the absence of transactions costs, is probably the assumption of a constant volatility. Our goal in this paper is to provide an approach to option pricing which allows one to specify very general patterns of volatility behavior and which at the same time still permits a computation of option prices and hedging strategies. This is achieved by combining stochastic numerical methods, on one hand, with a high-dimensional Markovian model, on the other. Since our models will usually yield an incomplete market, we also provide a result on the pricing measure to be used: we prove that a small investor in our model should price options by their expected discounted payoffs, where the expectation is taken with respect to the minimal equivalent martingale measure associated to the underlying stock price process.

We begin in Section 2 with a brief survey of previous results on models with a nonconstant volatility. Basically, there are two directions of generalization in this context; they can be summarized by the key words "stochastic volatility" and "past-dependence."

[1]Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged.

Manuscript received January 1992; final revision received April 1992.

After discussing the merits and disadvantages of some important approaches in the existing literature, we present the general model that we propose to study in this paper. By means of a specific example, we show how one can easily incorporate in this framework an asset price whose volatility is at the same time stochastic and past-dependent. But of course there is a price to pay for the generality of our model. Typically, we are dealing with an incomplete market which implies that there is no unique equivalent martingale measure or risk-neutral probability for the underlying stock price process. We attack this problem in Section 3, where we show that, for the purpose of computing option prices, a small investor in our model should use the minimal equivalent martingale measure. To this end, we first decompose the space of all those assets compatible with the given stock as the direct sum of two subspaces, namely the purely tradable assets and the totally nontradable assets. Using this decomposition allows us to prove a new characterization of the minimal equivalent martingale measure, which in particular yields the prescription given above.

The very generality and flexibility of our model gives rise to a second problem. In addition to incompleteness, we are also faced with the difficulty that our asset prices are given by stochastic differential equations which typically have no explicit solution. To compute option prices and hedging strategies, we therefore have to resort to numerical methods. A survey of the required results and techniques is presented in Section 4, while Section 5 illustrates these methods by means of explicit examples. In particular, our approach permits the simulation of actual trajectories with a high accuracy, and this enables us to study the performance of our hedging strategies under various possible scenarios. The resulting plots of some of our simulations are exhibited and discussed in Section 5.

2. FORMULATION OF THE MODEL

This section introduces our basic model for the asset prices. We begin with a brief survey of some important approaches in the literature before we describe the general class of models studied in this paper. We explain the two basic problems arising in this context, and we conclude the section with a specific example illustrating how one can easily incorporate in our framework an asset price with a stochastic and past-dependent volatility.

2.1. A Model with Stochastic Volatility

Hull and White (1987) consider the following model:

$$
\begin{aligned}
dB_t &= rB_t\,dt, \\
dS_t &= \mu(S_t,\sigma_t,t)S_t\,dt + \sigma_t S_t\,dW_t^1, \\
dv_t &= \gamma(\sigma_t,t)v_t\,dt + \delta(\sigma_t,t)v_t\,dW_t^2,
\end{aligned}
\tag{2.1}
$$

where S_t denotes the stock price at time t, $v_t = \sigma_t^2$ its instantaneous variance, and r the riskless interest rate, which is assumed to be constant. W^1 and W^2 are Brownian motions under P. Similar models have been studied by Hull and White (1988), Johnson and Shanno (1987), Scott (1987), and Wiggins (1987), among others.

The basic idea to price a call option in this model is to form a riskless portfolio containing the option, the stock, and a second call option with the same strike price, but a different expiration date. If we denote by $u(t, S_t)$ the value of the (first) call option at time t and stock price S_t, this approach yields a certain partial differential equation for the option pricing function $u(t, x)$. But the solution of this equation is not unique unless one already knows the price function for the second call option.

To recover uniqueness, Hull and White (1987) make the additional assumptions that W^1 and W^2 are independent, and that the variance v has no systematic risk. This yields a unique option price which can be computed as the (conditional) expectation of the discounted terminal payoff under a risk-neutral probability measure $\tilde{P}$. Put differently, $\tilde{P}$ is obtained from P by means of a Girsanov transformation such that

$$\begin{aligned} dB_t &= rB_t\,dt, \\ dS_t &= rS_t\,dt + \sigma_t S_t\,d\tilde{W}_t^1, \\ dv_t &= \gamma(\sigma_t, t)v_t\,dt + \delta(\sigma_t, t)v_t\,d\tilde{W}_t^2 \end{aligned} \tag{2.2}$$

under $\tilde{P}$, where $\tilde{W}^1$, $\tilde{W}^2$ are independent Brownian motions under $\tilde{P}$. The option price is then given by

$$u(t, S_t) = \tilde{E}\left[\frac{B_t}{B_T}(S_T - K)^+ \;\middle|\; \mathcal{F}_t\right] = e^{-r(T-t)}\tilde{E}[(S_T - K)^+ \mid \mathcal{F}_t].$$

To obtain a more specific form for u, Hull and White (1987) then use the additional assumption contained in (2.1) and the independence of $\tilde{W}^1$, $\tilde{W}^2$ that the instantaneous variance v is not influenced by the stock price S. Setting

$$\bar{v}_{t,T} := \frac{1}{T-t}\int_t^T v_s\,ds,$$

they show that the conditional distribution of S_T/S_t under $\tilde{P}$, given $\bar{v}_{t,T}$, is lognormal with parameters $r(T - t)$ and $\bar{v}_{t,T}(T - t)$. This allows them to reexpress u as

$$u(t, S_t, \sigma_t^2) = \int_0^\infty u_{BS}(t, S_t, \bar{v}_{t,T})\,dF(\bar{v}_{t,T} \mid S_t, \sigma_t^2), \tag{2.3}$$

where u_{BS} denotes the usual Black-Scholes price corresponding to the variance $\bar{v}_{t,T}$ and F is the conditional distribution under $\tilde{P}$ of $\bar{v}_{t,T}$, given S_t and σ_t^2.

While this result is very pleasant from a theoretical viewpoint, it has the practical drawback that F cannot be determined in general. In the very special case of constant coefficients γ and δ, $\bar{v}_{t,T}$ is an integral over lognormal variables which allows one to compute all moments of F. This is used by Hull and White (1988) to give a Taylor expansion of (2.3). In the general case, however, (2.3) can only be computed by a numerical approximation, despite the rather strong assumptions imposed on the model. In particular, (2.1) does not allow a genuine interaction between the stock price S and its volatility σ, since v is not allowed to depend on S.

2.2. A Model with Past-Dependent Volatility

Recently, Kind, Liptser, and Runggaldier (1991) attempted to explicitly account for the dependence of the volatility on past stock prices. They consider a family $(S^\varepsilon, v^\varepsilon)$ of discrete-time processes, where ε denotes the grid size of a partition of the time axis. The discrete-time stock price $(S_t^\varepsilon)_{0 \le t \le T}$ and the corresponding instantaneous variance $(v_t^\varepsilon)_{0 \le t \le T}$ are recursively described in such a way that each v_t^ε depends on the past stock prices S_{t-s}^ε $(0 \le s \le I)$ observed on a time interval of fixed length I. Their main result is then a diffusion approximation: for $\varepsilon \searrow 0$, one has

$$v^\varepsilon \longrightarrow v \quad \text{uniformly on compacts in probability}$$

and

$$S^\varepsilon \longrightarrow S \quad \text{weakly in } D[0, T],$$

where v satisfies a deterministic delay equation and S is given by a stochastic differential equation involving v. Furthermore, v coincides with the quadratic variation process of S. Since v is deterministic and can be obtained from the delay equation and the input values v_s $(-I \le s \le 0)$, one has a Black-Scholes type formula for the limit model as soon as v is known. By the convergence result, this yields an approximation for the option price in the discrete-time model, and this model in turn is a reasonable description of a stock price with past-dependent volatility.

While the convergence result in itself is certainly of interest, its application to option pricing lacks some important features. First of all, the limiting model (which is used to compute prices) does not really exhibit past dependence: once we know the volatility v on any interval of length I, it is completely determined for all future times and does not react to S any more. In particular, v is no longer stochastic. Furthermore, it seems unhandy to approximate a discrete-time model by its continuous-time limit; since one has to fix a particular time partition when setting up the model, it will probably become quite difficult to control the quality of the approximation.

2.3. A General Markovian Model

Let us now present a general model which can include volatilities both stochastic and past dependent. This kind of model goes back to Merton (1969; 1971; 1973) who studied various problems in this framework; see Merton (1990). We consider the following multidimensional diffusion process:

$$dX_t^i = a^i(t, X_t)\, dt + \sum_{j=1}^{n} b^{ij}(t, X_t)\, dW_t^j \tag{2.4}$$

for $i = 0, \dots, m$, where a^i $(i = 0, \dots, m)$ and b^{ij} $(i = 0, \dots, m; j = 1, \dots, n)$ are measurable functions from $[0, T] \times \mathbb{R}^{m+1}$ into $\mathbb{R}$. The process $W = (W^1, \dots, W^n)^*$ (with $*$ denoting transposition) is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}\, Q)$, and $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is the Q-augmentation of the filtration generated by W. We assume that the coefficients a^i and b^{ij} satisfy appropriate growth and Lipschitz conditions so that the solution of (2.4) is a Markov process. We also remark that under suitable continuity and nondegeneracy conditions on the coefficients, $\mathbb{F}$ coincides with the natural filtration $\mathbb{F}^X$ of X; see Harrison and Kreps (1979). This model will be interpreted in the following way. The component X^0 describes the riskless asset; setting $B := X^0$, we shall take $b^{0j} \equiv 0$ for $j = 1, \ldots, n$ and $a^0(t, x) = r(t, x)x^0$, so

$$dB_t = r(t, X_t)B_t\, dt,$$

and we assume that

$$\int_0^T |r(s, X_s)|\, ds \leq L < \infty \quad Q\text{-a.s.} \qquad \text{for some } L > 0.$$

For notational simplicity, we shall work with only one stock. The component X^1 describes its price process and is denoted by S. The other components of X can then be used to model the additional structure of the market in which S is embedded. For instance, they could include a specification of a stochastic volatility and of its dependence on the past. We shall give a concrete example in the next subsection, but at present we refrain quite deliberately from specifying X any further.

In this general framework, an *option* or *contingent claim* will be a random variable of the form $g(X_T)$. The classical example is provided by a European call option with strike price K which corresponds to the claim $(S_T - K)^+$. Since the process X will usually contain more components than just the bond $B = X^0$ and the stock price $S = X^1$, a claim can depend on many things other than just the terminal stock price S_T. In fact, the only serious restriction is that the underlying process X—but not necessarily S—should be Markovian. This implies that (subject to some integrability conditions) we can associate to any contingent claim $g(X_T)$ an *option pricing function* u: $[0, T] \times \mathbb{R}^{m+1} \to \mathbb{R}$ defined by

$$u(t, x) := E_Q\left[\exp\left(-\int_t^T r(s, X_s^{t,x})\, ds\right)g(X_T^{t,x})\right], \tag{2.5}$$

where $(X_s^{t,x})_{t \leq s \leq T}$ denotes the solution of (2.4) starting from x at time t, i.e., with $X_t^{t,x} = x \in \mathbb{R}^{m+1}$.

At this point, we are faced with the first basic problem in our general model: Why should u in (2.5) be called an option pricing function? We shall answer this question in Section 3 by showing that for a suitable choice of the probability measure Q, the process

$$V_t := u(t, X_t) \tag{2.6}$$

can be interpreted as the value or price of the claim $g(X_T)$ at time t. Furthermore, the process

$$\xi_t := \frac{\partial u}{\partial x^1}(t, X_t), \qquad 0 \leq t \leq T, \tag{2.7}$$

defines a hedging strategy in the following way: If we set

$$\eta_t := \frac{V_t - \xi_t S_t}{B_t}, \qquad 0 \le t \le T, \tag{2.8}$$

and interpret ξ_t and η_t as the numbers of shares of stock and unit bonds to be held at time t, respectively, the value of this dynamic portfolio (ξ, η) at any time t is given by

$$\xi_t S_t + \eta_t B_t = V_t = u(t, X_t).$$

In particular, the terminal value of the portfolio will be

$$V_T = u(T, X_T) = g(X_T) \quad Q\text{-a.s.},$$

which means that the hedging strategy duplicates the claim's payoff at the terminal time T.

Except in special cases, however, the strategy (ξ, η) defined by (2.7) and (2.8) will not be riskless. This means that the process

$$C_t := V_t - \int_0^t \xi_u \, dS_u - \int_0^t \eta_u \, dB_u, \qquad 0 \le t \le T,$$

of cumulative costs (current value minus cumulative gains from trade) will not be constant, but fluctuate randomly. Thus, the strategy will in general not be self-financing. But it turns out that for the above-mentioned judicious choice of Q, (ξ, η) will still possess a certain local optimality property; in particular, the cost process C then becomes a martingale; i.e., the strategy is mean-self-financing. This choice of measure will be discussed in more detail in Section 3. For a more precise description of the optimality properties of (ξ, η), we refer to Schweizer (1991a).

The second basic problem is the actual computation of the option pricing function u. It is clear that there can be no hope of obtaining a closed-form expression in general. We shall therefore concentrate on a numerical approach, with special emphasis on precise statements about the accuracy and speed of convergence of this approximation. General results will be presented in Section 4 and illustrated in detail by examples in Section 5.

REMARK. It is worth pointing out that the entire discussion in this subsection is formulated in terms of the probability measure Q. The usual approach is somewhat different. One begins with a probability measure P and a process satisfying some stochastic differential equation under P. Optimization or minimization problems are then also formulated with respect to this basic measure P. The computation of option prices or optimal hedging strategies, however, is often considerably simplified by switching to a suitably chosen equivalent measure Q. We have decided to start here directly with Q for two reasons. First of all, there is no general agreement about the choice of Q except in the special case of so-called complete markets or, more generally, for the pricing of attainable claims. We shall explain the reasons for our choice of Q in Section 3, but we should prefer to avoid this discussion here. However—and this is the second reason—it should also be emphasized that the approximation techniques presented in Section 4 can be applied to any diffusion model of the form (2.4). In particular, they do not depend at all on the economic significance of the measure Q.

2.4. Two Examples

EXAMPLE 2.1. Let us first illustrate by a specific example of the model (2.4) how we can incorporate in our framework a stochastic and past-dependent volatility. We consider the process $X = (X^0, X^1, X^2, X^3)^* = (B, S, \sigma, \zeta)^*$ satisfying the following stochastic differential equation:

$$
\begin{aligned}
dB_t &= r(t, X_t)B_t\,dt, \\
dS_t &= r(t, X_t)S_t\,dt + \sigma_t S_t\,dW_t^1, \\
d\sigma_t &= -q(\sigma_t - \zeta_t)\,dt + p\sigma_t\,dW_t^2, \\
d\zeta_t &= \frac{1}{\alpha}(\sigma_t - \zeta_t)\,dt,
\end{aligned} \tag{2.9}
$$

with $p > 0$, $q > 0$, $\alpha > 0$, where W^1, W^2 are independent Brownian motions under Q. Let us explain and comment on the features of this model.

The bond price B is of the usual structure with a Markovian instantaneous interest rate r. The stock price S follows a generalized geometric Brownian motion since drift and volatility are not constant. Taking the drift to be $r(t, X_t)$ means that the discounted stock price process S/B is a martingale under the measure Q which we use for pricing; this is also quite standard.

The processes σ and ζ should be interpreted as the instantaneous and the weighted average volatility of the stock, respectively. The equation for σ shows that the instantaneous volatility σ_t is disturbed by some external noise (with an intensity parameter p) and at the same time continuously pulled back toward the average volatility ζ_t. The parameter q measures the strength of this restoring force or speed of adjustment.

The equation for the average volatility ζ can be solved explicitly to give

$$
\zeta_t = \zeta_0 \exp\left(-\frac{t}{\alpha}\right) + \frac{1}{\alpha}\int_0^t \exp\left(-\frac{t-s}{\alpha}\right)\sigma_s\,ds, \qquad 0 \le t \le T.
$$

This shows that ζ_t is an average of the values σ_s $(0 \le s \le t)$, weighted with an exponential factor. For very large α, we obtain $\zeta_t \approx \zeta_0$, while a very small value of α yields $\zeta_t \approx \sigma_t$. Thus, the parameter α measures the strength of the past dependence of the average volatility.

EXAMPLE 2.2. As a second example, we briefly indicate how (2.4) can be adapted to give a generalized form of the Hull and White model (2.2). For this purpose, we consider the process $X = (B, S, v)^*$ given by

$$
\begin{aligned}
dB_t &= r(t, X_t)B_t\,dt, \\
dS_t &= r(t, X_t)S_t\,dt + \sqrt{v_t}S_t\,dW_t^1, \\
dv_t &= \gamma(t, X_t)v_t\,dt + \delta(t, X_t)v_t(\rho(t, X_t)\,dW_t^1 + \sqrt{1 - (\rho(t, X_t))^2}\,dW_t^2),
\end{aligned} \tag{2.10}
$$

with independent Brownian motions W^1, W^2 under Q. Here, ρ explicitly accounts for the correlation between the stock price S_t and its instantaneous variance v_t. A particular case of this model was studied by Johnson and Shanno (1987).

3. INCOMPLETE MARKETS AND THE MINIMAL EQUIVALENT MARTINGALE MEASURE

In this section, we present an argument for the choice of a particular probability measure Q to be used for computing option prices in an incomplete market. Our starting point is a very general diffusion model for a multidimensional stock price process S under an initial measure P. Under some integrability conditions, there exists a unique minimal equivalent martingale measure $\hat{P}$ for S. We prove that $\hat{P}$ is characterized among all equivalent martingale measures $\tilde{P}$ for S by the property that it leaves the returns of all nontraded assets unchanged. This result is then used to provide an argument why a small investor should work with $\hat{P}$ instead of P for the purpose of pricing options. The measure $\hat{P}$ will be our choice for Q.

In contrast to the rest of the paper, the results of this section do not require a Markovian structure of our price process. We shall therefore work with a model considerably more general than (2.4). It contains one riskless asset B and m risky assets S^i, $i = 1, \ldots, m$. The bond price B and the stock prices S^i are given by the stochastic differential equation

$$dB_t = r_t B_t\, dt, \tag{3.1}$$

$$dS_t^i = S_t^i \mu_t^i\, dt + S_t^i \sum_{j=1}^{n} \sigma_t^{ij}\, dW_t^j.$$

Here, $W = (W^1, \ldots, W^n)^*$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ denotes the P-augmentation of the filtration generated by W. We take $n \ge m$ so that there are at least as many sources of uncertainty as there are stocks available for trading. All processes will be defined on $[0, T]$, where the constant $T > 0$ denotes the terminal time for our problem. We assume that the *interest rate* $r = (r_t)_{0 \le t \le T}$, the vector $\mu = (\mu_t)_{0 \le t \le T} = (\mu_t^1 \ldots, \mu_t^m)_{0 \le t \le T}$ of *stock appreciation rates*, and the *volatility matrix* $\sigma = (\sigma_t)_{0 \le t \le T} = (\sigma_t^{ij})_{0 \le t \le T; i=1,\ldots,m; j=1,\ldots,n}$ are progressively measurable with respect to $\mathbb{F}$. The interest rate r satisfies

$$\int_0^T |r_u|\, du \le L < \infty \quad P\text{-a.s.} \qquad \text{for some } L > 0,$$

which implies that the bond price process B is bounded above and away from 0, uniformly in t and ω. We also assume that the matrix σ_t has *full rank* m for every t so that the matrix $(\sigma_t \sigma_t^*)^{-1}$ is well defined. Essentially, this means that the basic assets, namely the stock prices, have been chosen in such a way that they are all nonredundant.

This generalization of the Merton model (2.4) used in Section 2.3 was introduced by Bensoussan (1984) and studied in great detail in a series of papers by several authors. Problems solved in this framework include maximization of expected utility—both in the case $m = n$ of *complete markets* (see Cox and Huang 1989; Karatzas et al. 1986; Karatzas, Lehoczky, and Shreve 1987) and in the case $m < n$ of *incomplete markets* (see He and Pearson 1990; Karatzas et al. 1991; Pagès 1989)—existence and uniqueness of equilibrium (see Karatzas, Lehoczky, and Shreve, 1990) and for the case of complete markets also pricing and hedging of options (see Bensoussan 1984; Karatzas and Shreve 1988). An excellent survey of the results for complete markets is given by Karatzas (1989). Here, we want to add some new results about the valuation of options in the incomplete case.

Consider a "small investor," i.e., an economic agent whose actions do not influence prices, who trades in the stocks and the bond. His trading strategy can be described at any time t by his total wealth V_t and by the amounts π_t^i invested in the ith stock for $i = 1, \ldots, m$. The amount invested in the bond is then given by $V_t - \Sigma_{i=1}^m \pi_t^i$. We shall call $\pi = (\pi_t)_{0 \le t \le T} = (\pi_t^1, \ldots, \pi_t^m)^*_{0 \le t \le T}$ a *portfolio process* if π is progressively measurable with respect to $\mathbb{F}$ and satisfies

$$\int_0^T \|\sigma_u^* \pi_u\|^2 \, du < \infty \quad P\text{-a.s.}$$

and

$$\int_0^T |\pi_u^*(\mu_u - r_u \mathbf{1})| \, du < \infty \quad P\text{-a.s.},$$

where $\mathbf{1} = (1, \ldots, 1)^* \in \mathbb{R}^m$. A trading strategy is called *self-financing* if all changes in the wealth process are entirely due to gains or losses from trading in stocks and bond. For such a strategy, the wealth process V must satisfy the equation

$$\begin{aligned} (3.2) \qquad dVt &= \sum_{i=1}^m \pi_t^i \left(\mu_t^i \, dt + \sum_{j=1}^n \sigma_t^{ij} \, dW_t^j \right) + \left(V_t - \sum_{i=1}^m \pi_t^i \right) r_t \, dt \\ &= (\pi_t^*(\mu_t - r_t \mathbf{1}) + r_t V_t) \, dt + \pi_t^* \sigma_t \, dW_t, \qquad 0 \le t \le T. \end{aligned}$$

The discounted wealth process $V' = V/B$ is then given by

$$dV_t' = \pi_t'^* (\mu_t - r_t \mathbf{1}) \, dt + \pi_t'^* \, \sigma_t \, dW_t$$

with $\pi_t' = \pi_t / B_t$. Thus, any portfolio process π uniquely determines a wealth process V such that π and V together constitute a self-financing strategy. If we now interpret such a wealth process V as the price process of some financial asset, this motivates the following

DEFINITION. A *purely tradable asset* is any asset whose value process V is given by (3.2) for some portfolio process π.

Note that this definition involves the underlying stocks $S^1, \ldots, S^m$ in a crucial way; hence it would be more accurate to say "purely tradable with respect to S." Clearly, a purely tradable asset is a particular case of a general asset in the sense of the following

DEFINITION. A *general asset* is any asset whose value process A is a semimartingale with respect to P and $\mathbb{F}$.

REMARK. Any general asset in our model has the form

$$dA_t = \gamma_t^* \, dW_t + dF_t, \qquad 0 \leq t \leq T, \tag{3.3}$$

where F is an $\mathbb{F}$-adapted process with paths of finite variation and the process $\gamma = (\gamma^1, \ldots, \gamma^n)^*$ is progressively measurable with respect to $\mathbb{F}$ and satisfies

$$\int_0^T \|\gamma_u\|^2 \, du < \infty \qquad P\text{-a.s.}$$

This follows immediately from the fact that every local martingale over a Brownian filtration is a stochastic integral of the underlying Brownian motion. Note that F and γ are not necessarily unique unless A is a special semimartingale.

For our purposes, general assets in this wide sense are too general. In order to explain why, we first recall the concept of an equivalent martingale measure for S.

DEFINITION. A probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ is called an *equivalent martingale measure* for S if

1. $\tilde{P}$ and P have the same null sets; i.e., $\tilde{P} \approx P$. In particular, this implies $\tilde{P} = P$ on $\mathcal{F}_0$.
2. The discounted price process $S' = S/B$ is a (vector) martingale under $\tilde{P}$.

Any probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ induces a certain price system on random variables in the following way: if U_{t+h} is a payoff to be made or received at time $t + h$, then the price of U_{t+h} at time t under $\tilde{P}$ is computed as

$$\tilde{E}\left[\frac{B_t}{B_{t+h}} U_{t+h} \,\middle|\, \mathcal{F}_t\right],$$

provided that $U_{t+h}/B_{t+h} \in \mathcal{L}^1(\tilde{P})$. If we assume that the observed stock price process S is in equilibrium under the price system $\tilde{P}$, then the price of S_{t+h} at time t must be S_t. This implies

$$\frac{S_t}{B_t} = \tilde{E}\left[\frac{S_{t+h}}{B_{t+h}} \,\middle|\, \mathcal{F}_t\right];$$

i.e., S' is a martingale under $\tilde{P}$. Thus, an equivalent martingale measure can be interpreted as a price system which is consistent with having S as an equilibrium stock price.

DEFINITION. An equivalent martingale measure $\hat{P}$ for S is called *minimal* if any local P-martingale L which is orthogonal to S^i for $i = 1, \ldots, m$ (in the sense that $\langle L, S^i \rangle = 0$) remains a local martingale under $\hat{P}$.

REMARKS. (1) The concept of a minimal equivalent martingale measure in the above sense was introduced by Föllmer and Schweizer (1991) and used there and in Schweizer (1990) and Schweizer (1991a) to compute certain optimal hedging strategies; see also He and Pearson (1990) and Pagès (1989) for related work. As a matter of fact, the preceding definition is slightly different from the one given in Föllmer and Schweizer (1991), where it was required that all square-integrable P-martingales L orthogonal to S should also be $\hat{P}$-martingales. But since we are working here with a Brownian filtration, we know that every local martingale must be continuous, and this will enable us to use localized versions of the results in Föllmer and Schweizer (1991).

(2) Intuitively, $\hat{P}$ is that equivalent martingale measure which is closest to P in a certain sense. This is made more precise in Föllmer and Schweizer (1991), where the distance between P and $\hat{P}$ is measured in terms of a relative entropy. Nevertheless, the definition of $\hat{P}$ is rather technical and lacks a clear economic interpretation. The main purpose of this section is therefore to provide a more intuitive characterization of $\hat{P}$.

We begin by describing more precisely the equivalent martingale measures for S. If $\tilde{P}$ is any equivalent martingale measure for S and

$$\tilde{Z}_t := E_P\left[\frac{d\tilde{P}}{dP}\,\middle|\,\mathcal{F}_t\right] = \left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t}, \qquad 0 \le t \le T,$$

denotes a continuous version of the density process of $\tilde{P}$ with respect to P, then $\tilde{Z}$ can be written as

$$\tilde{Z}_t = \exp\left(-\int_0^t \tilde{\lambda}_u^* \, dW_u - \frac{1}{2}\int_0^t \|\tilde{\lambda}_u\|^2 \, du\right), \qquad 0 \le t \le T, \tag{3.4}$$

where $\tilde{\lambda} = (\tilde{\lambda}^1, \ldots, \tilde{\lambda}^n)^*$ is adapted to $\mathbb{F}$ and satisfies

$$\int_0^T \|\tilde{\lambda}_u\|^2 \, du < \infty \quad P\text{-a.s.} \tag{3.5}$$

and

$$\sigma_t \tilde{\lambda}_t = \mu_t - r_t \mathbf{1}, \qquad 0 \le t \le T. \tag{3.6}$$

In fact, (3.4) and (3.5) follow from Itô's representation theorem and the fact that $\tilde{Z}$ is strictly positive, and the martingale property under P of $\tilde{Z}\,S'$ implies (3.6). Note that (3.6) does in general not determine $\tilde{\lambda}$ uniquely unless each σ_t is invertible; i.e., $m = n$.

Using the results of Föllmer and Schweizer (1991), we next proceed to describe the minimal equivalent martingale measure $\hat{P}$ in more detail. First of all, $\hat{P}$ is unique if it exists. Furthermore, there is only one natural candidate: if we set

$$\hat{\lambda}_t := \sigma_t^*(\sigma_t \sigma_t^*)^{-1}(\mu_t - r_t \mathbf{1}), \qquad 0 \le t \le T, \tag{3.7}$$

and

$$\hat{Z}_t := \exp\left(-\int_0^t \hat{\lambda}_u^* \, dW_u - \frac{1}{2}\int_0^t \|\hat{\lambda}_u\|^2 \, du\right), \qquad 0 \le t \le T, \tag{3.8}$$

where we assume that

$$\int_0^T \|\hat{\lambda}_u\|^2 \, du < \infty \quad P\text{-a.s.}, \tag{3.9}$$

then the minimal equivalent martingale measure exists if and only if

$$d\hat{P} = \hat{Z}_T \, dP$$

defines an equivalent martingale measure $\hat{P}$ for S. This is the case if and only if $\hat{Z}$ is a martingale, which is equivalent to the condition $E[\hat{Z}_T] = 1$. For instance, it would be sufficient to impose a Novikov condition on $\hat{\lambda}$.

ASSUMPTION. *For the rest of this section, we shall suppose that (3.9) and the condition $E[\hat{Z}_T] = 1$ are satisfied, so that the minimal equivalent martingale measure $\hat{P}$ exists.*

REMARKS. (1) The structure of equivalent martingale measures has been studied by several authors in frameworks of varying generality. We refer to Ansel and Stricker (1991), El Karoui and Quenez (1991), He and Pearson (1990), Karatzas et al. (1991), and Schweizer (1991b) for results similar to (3.4)–(3.6). A construction of the minimal equivalent martingale measure $\hat{P}$ in a general semimartingale model can be found in Schweizer (1991b).

(2) Assuming the existence of $\hat{P}$ (which in particular implies the standard assumption that the set of equivalent martingale measures for S is nonempty) is not really crucial for the subsequent arguments. From a technical point of view, it is an integrability condition which could be considerably weakened by recasting the discussion below in terms of martingale densities instead of equivalent martingale measures; see Schweizer (1991b). On the other hand, the slight loss of generality incurred by this assumption is amply compensated by the fact that the interpretation of our results is much easier to express in terms of martingale measures.

Now consider a general asset with value process A. If A is an equilibrium price under some pricing measure $\tilde{P}$, then $A' = A/B$ should be a martingale under this measure; i.e., $\tilde{P}$ should be an equivalent martingale measure for A. But of course we only want to consider those pricing measures under which the given stock price S can occur as an equilibrium price. Hence, $\tilde{P}$ should also be an equivalent martingale measure for S, and this motivates the following

DEFINITION. A *compatible asset* is any general asset whose discounted value process $A' = A/B$ is a local martingale under some equivalent martingale measure for S.

Note that this definition also involves S; again it would be more precise to say "compatible with S."

REMARKS. (1) One could also define a compatible asset by the stipulation that A' should be a true martingale under some equivalent martingale measure for S. To avoid integrability conditions, however, it is more convenient to work with local martingales.

(2) The idea behind the definition of a compatible asset is that an incompatible asset should not exist in an equilibrium situation with stock prices given by S. As pointed out by D. Sondermann, one can therefore ask if an incompatible asset gives rise to an arbitrage opportunity in some sense. We have been unable to answer this question so far.

Let us now examine the structure of a compatible asset more closely. To this end, we consider the space $L_a^2[0, T]$ of all $\mathbb{F}$-adapted $\mathbb{R}^n$-valued processes ν satisfying

$$\int_0^T \|\nu_u\|^2 \, du < \infty \quad P\text{-a.s.}$$

Following Karatzas et al. (1991), we decompose $L_a^2[0, T]$ into the orthogonal subspaces

$$K(\sigma) := \{\nu \in L_a^2[0, T] \mid \sigma_t \nu_t = 0 \quad \text{for } 0 \le t \le T, \ P\text{-a.s.}\}$$

and

$$K^{\perp}(\sigma) := \{\nu \in L_a^2[0, T] \mid \nu_t \in \text{Range}(\sigma_t^*) \quad \text{for } 0 \le t \le T, \ P\text{-a.s.}\}.$$

If $\tilde{\lambda} \in L_a^2[0, T]$ satisfies (3.6), then $\tilde{\lambda}$ can be written as

$$\tilde{\lambda} = \hat{\lambda} + \vartheta \qquad \text{for some } \vartheta \in K(\sigma). \tag{3.10}$$

Indeed, decomposing $\tilde{\lambda}$ as $\tilde{\lambda} = \vartheta + \sigma^* \pi$ with $\vartheta \in K(\sigma)$ yields by (3.6)

$$\mu - r\mathbf{1} = \sigma \tilde{\lambda} = \sigma \sigma^* \pi,$$

and therefore by (3.7)

$$\tilde{\lambda} = \vartheta + \sigma^* (\sigma \sigma^*)^{-1} (\mu - r\mathbf{1}) = \vartheta + \hat{\lambda}.$$

This allows us to prove the following result.

LEMMA 3.1. *Every compatible asset has a value process A of the form*

$$dA_t = (\pi_t^*(\mu_t - r_t\mathbf{1}) + A_t r_t) \, dt + \pi_t^* \sigma_t \, dW_t + \nu_t^* \, dW_t + \nu_t^* \vartheta_t \, dt \tag{3.11}$$

for some portfolio process π and some processes ν, $\vartheta \in K(\sigma)$.

Proof. First we observe that A or, equivalently, A' must be continuous. In fact, let $\tilde{P}$ be an equivalent martingale measure for S such that A' is a local $\tilde{P}$-martingale. Then $A'\tilde{Z}$

is a local P-martingale and therefore continuous, since $\mathbb{F}$ is a Brownian filtration. Strict positivity and continuity of $\tilde{Z}$ then imply the continuity of A'.

Now denote by $\tilde{\lambda}$ the process corresponding to $\tilde{P}$ by (3.4). Under P, $A' = A/B$ has the form

$$dA'_t = \frac{1}{B_t}\,dF_t - A'_t r_t\,dt + \frac{\gamma_t^*}{B_t}\,dW_t,$$

where we have used (3.3). If we decompose $\gamma \in L^2_d[0, T]$ as

$$\gamma = \nu + \sigma^* \pi \qquad \text{with } \nu \in K(\sigma),$$

then applying Girsanov's theorem to W shows that A' can be written under $\tilde{P}$ as

$$dA'_t = \frac{1}{B_t}\,dF_t - \left(A'_t r_t + \frac{\gamma_t^*}{B_t}\,\tilde{\lambda}_t\right) dt + \frac{\gamma_t^*}{B_t}\,d\tilde{W}_t$$

for some $\tilde{P}$-Brownian motion $\tilde{W}$. But since A' is a continuous local $\tilde{P}$-martingale, we conclude that

$$\begin{aligned} dF_t &= (A_t r_t + \gamma_t^* \tilde{\lambda}_t)\,dt \\ &= (A_t r_t + (\nu_t^* + \pi_t^* \sigma_t)(\hat{\lambda}_t + \vartheta_t))\,dt \\ &= (A_t r_t + \nu_t^* \hat{\lambda}_t + \nu_t^* \vartheta_t + \pi_t^*(\mu_t - r_t \mathbf{1}))\,dt \end{aligned}$$

by (3.10) and (3.7). Since $\nu^* \hat{\lambda} = 0$ by (3.7), this yields the assertion by (3.3). □

REMARK. Lemma 3.1 is closely related to a result of Ansel and Stricker (1991) on the structure of continuous asset prices admitting an equivalent martingale measure; see also Schweizer (1991b). However, the representation (3.11) gives us more information about the drift term of A than in Ansel and Stricker (1991) since we only consider those measures $\tilde{P}$ which are at the same time equivalent martingale measures for S.

DEFINITION. A *totally nontradable asset* is any general asset with a value process A of the form

$$dA_t = \nu_t^*\,dW_t + \nu_t^* \vartheta_t\,dt, \qquad 0 \le t \le T, \tag{3.12}$$

for some processes $\nu, \vartheta \in K(\sigma)$.

Note that this definition involves σ (but not the drift μ) and therefore again the underlying stock S; "totally nontradable with respect to S" would thus be more accurate. Intuitively, a totally nontradable asset should be an asset which cannot be generated in a self-financing way by trading only in the available stocks and the bond. The next result shows that, up to constants, the space of all compatible assets is the direct sum of the space of purely tradable assets and the space of totally nontradable assets. In particular, this justifies the terminology of the preceding definition.

PROPOSITION 3.1. (1) *Any compatible asset admits a decomposition as the sum of a purely tradable asset and a totally nontradable asset. This decomposition is unique up to a constant.*

(2) *A compatible asset with nonconstant value process A is totally nontradable if and only if there exists no portfolio process π whose corresponding wealth process V coincides with A.*

Proof. (1) The existence of the decomposition is obvious from Lemma 3.1 and the definitions. To prove uniqueness, suppose we have $A = V$ with

$$dA_t = \nu_t^* \, dW_t + \nu_t^* \vartheta_t \, dt$$

and

$$dV_t = (\pi_t^*(\mu_t - r_t\mathbf{1}) + V_t r_t) \, dt + \pi_t^* \sigma_t \, dW_t.$$

This implies $\nu = \sigma^* \pi$ and therefore $\pi = 0$, since $\nu \in K(\sigma)$ and $\sigma\sigma^*$ is invertible. Thus we obtain $\nu = 0$, so A must be constant.

Statement (2) follows immediately from (1). □

The central result of this section now provides a new characterization of the minimal equivalent martingale measure $\hat{P}$.

THEOREM 3.1. *Among all equivalent martingale measures $\tilde{P}$ for S, $\hat{P}$ is characterized by the property that every totally nontradable asset satisfies the same stochastic differential equation under P as under $\hat{P}$. In other words, $\hat{P}$ is the only equivalent martingale measure $\tilde{P}$ for S with the property that for every totally nontradable asset with value process A we have*

$$dA_t = \nu_t^* \, dW_t + \nu_t^* \vartheta_t \, dt \qquad \textit{under } P$$

for some P-Brownian motion W if and only if

$$dA_t = \nu_t^* \, d\tilde{W}_t + \nu_t^* \vartheta_t \, dt \qquad \textit{under } \tilde{P}$$

for some $\tilde{P}$-Brownian motion $\tilde{W}$, with the same coefficients ν, ϑ.

Proof. (1) Take any totally nontradable asset with value process A. Then by definition

$$dA_t = \nu_t^* \, dW_t + \nu_t^* \vartheta_t \, dt \qquad \text{under } P.$$

Applying Girsanov's theorem to W and $\hat{P}$ yields

$$\begin{aligned} dA_t &= \nu_t^* \, d\hat{W}_t - \nu_t^* \hat{\lambda}_t \, dt + \nu_t^* \vartheta_t \, dt \\ &= \nu_t^* \, d\hat{W}_t + \nu_t^* \vartheta_t \, dt \qquad \text{under } \hat{P}, \end{aligned}$$

since $\nu \in K(\sigma)$ implies $\nu^* \hat{\lambda} = 0$ by (3.7).

(2) Conversely, let $\tilde{P}$ be any equivalent martingale measure associated with $\tilde{\lambda}$ by (3.4). Due to Girsanov's theorem and (3.10), a totally nontradable asset with value process

$$dA_t = \nu_t^* \, dW_t + \nu_t^* \vartheta_t \, dt \qquad \text{under } P$$

is then given by

$$\begin{aligned} dA_t &= \nu_t^* \, d\tilde{W}_t - \nu_t^* \tilde{\lambda}_t \, dt + \nu_t^* \, \vartheta_t \, dt \\ &= \nu_t^* \, d\tilde{W}_t - \nu_t^* \tilde{\vartheta}_t \, dt + \nu_t^* \vartheta_t \, dt \qquad \text{under } \tilde{P}, \end{aligned}$$

since $\nu^* \hat{\lambda} = 0$ by (3.7). But if $\tilde{P}$ has the property that every totally nontradable asset satisfies the same stochastic differential equation under $\tilde{P}$ as under P, then we must have

$$\nu^* \tilde{\vartheta} = 0 \qquad \text{for all } \nu \in K(\sigma),$$

and this implies $\tilde{\vartheta} = 0$, so $\tilde{\lambda} = \hat{\lambda}$ by (3.10) and therefore $\tilde{P} = \hat{P}$. □

Interpretation. Let us explain how Theorem 3.1 may be used to justify in a certain sense the choice of $\hat{P}$ for the purpose of pricing options. First we note that as in Karatzas et al. (1991), our incomplete model for the stock prices $S^1, \ldots, S^m$ can be embedded in a larger complete model by adding $n - m$ suitably chosen assets. In fact, it is sufficient to take any $n - m$ totally nontradable assets whose coefficients $\nu(1), \ldots, \nu(n - m)$ in (3.12) form a basis of $K(\sigma)$; if one wants to avoid redundancies in the embedding, this choice is essentially also necessary. In particular, we lose no generality in assuming these completing assets to be totally nontradable with respect to $S^1, \ldots, S^m$.

Now consider again a small investor who wants to price options, but who only has the possibility to trade in S and B. From his subjective point of view, he is therefore dealing with an incomplete market. If he assumes $S^1, \ldots, S^m$ to be equilibrium prices, the pricing measure he is eventually going to use must certainly be an equivalent martingale measure for S. In addition to S and B, there are also assets in the (large) market which are totally nontradable from his own perspective. He may have formed an opinion about these assets, and this opinion is modeled by the probability measure P. But if he believes that the market as a whole is in equilibrium, then he must in particular accept the prices of the nontradable assets such as they are. By Theorem 3.1, this implies that he should use the minimal equivalent martingale measure $\hat{P}$ for pricing.

We emphasize that this argument hinges on two points. The first is the assumption of equilibrium for the large complete market, and the second is the interpretation of P as our small agent's subjective assessment of the various assets in the market. Thus, one cannot say that "the minimal equivalent martingale measure is the correct pricing measure" in any absolute sense; this is only true from the perspective of an agent with beliefs given by P. Furthermore, $\hat{P}$ will in general not be a martingale measure for the nontradable assets; this reflects the fact that the subjective assessment P of these assets may well differ from their actual equilibrium prices. Note, however, that this discrepancy will not give rise to arbitrage opportunities since our agent cannot trade in the seemingly mispriced assets.

What is now the actual effect of using $Q = \hat{P}$ instead of P? If we start from any model of the type (3.1) under P, then switching to $\hat{P}$ changes the appreciation rates of all traded stocks S^i from μ^i to the riskless interest rate r. All other (nontraded) assets are left completely unchanged. Examples are given by (2.9), where we start directly with Q, and more explicitly by the change from (2.1) to (2.2).

REMARK. There is an alternative interpretation of the minimal equivalent martingale measure: using $\hat{P}$ for the purpose of pricing options is equivalent to assuming that all nontraded risks can be diversified away and thus are unpriced. This approach is taken in Hull and White (1987) and Scott (1987) for a model of the type (2.1). As Wiggins (1987) has pointed out, however, such an assumption may be rather questionable from an empirical point of view. The results of this section grew out of an attempt to provide a more systematic method for choosing a pricing measure Q, given an a priori measure P.

4. NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS

This section gives a short review of numerical methods for solving stochastic differential equations. They will be applied in Section 5 to compute the option pricing function u and the hedging strategy ξ in two specific examples.

4.1. General Remarks

The classical Black-Scholes model

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

is one of the rare cases where one knows the explicit solution for a stochastic differential equation. In more realistic situations, one is usually less fortunate and therefore has to use approximations. The following numerical methods are also applicable for other problems than option pricing, and they work equally well for different choices of the underlying probability measure Q. To explain the basic approach, we consider the $\mathbb{R}^{m+1}$-valued diffusion process

$$X_t = X_0 + \int_0^t a(s, X_s)\, ds + \sum_{j=1}^{n} \int_0^t b^j(s, X_s)\, dW_s^j \tag{4.1}$$

as defined in (2.4). For a basic monograph on numerical methods for stochastic differential equations, we refer to Kloeden and Platen (1991).

Before one starts to solve (4.1) numerically, one has to answer an important question concerning the purpose of this approximation: Does one need strong or weak approximations? If one wants to simulate approximate sample paths Y^Δ of the solution X of (4.1), then one requires *strong approximations* which converge at every given time instant T with a given strong order $\gamma > 0$ as the step size Δ of the time discretization tends to 0. That is, there exist constants K and $\delta_0 < T$ not depending on Δ such that for all $\Delta \in (0, \delta_0)$, we have the estimate

$$E[|X_T - Y^\Delta(T)|] \le K\Delta^\gamma.$$

Milstein (1974) was one of the first who developed strong approximation schemes. Platen (1981) and Wagner and Platen (1978) proved general results which allow to construct approximations of any desired strong order $\gamma > 0$. Clark (1982), Newton (1986), Pardoux and Talay (1985), Rümelin (1982), and Talay (1983), among others, also studied strong approximations. In our context, we shall use strong approximations to simulate an approximate trajectory of an asset price. This allows us to test the theoretically obtained hedging strategies and option prices.

A completely different situation occurs if one is only interested in computing the value of a functional of a diffusion process, for example an option price. Then it is no longer necessary to have pathwise (i.e., strong) approximations. One only needs a stochastic model which yields a functional with the same expected value as the desired functional. In other words, one only has to approximate the underlying probability law of the diffusion, and this *weak approximation* is a much easier task than the approximation of trajectories. We say that an approximation Y^{Δ} converges with weak order $\beta > 0$ as the grid size Δ tends to 0 if there exist constants $K > 0$ and $\delta_0 < T$ for every function $g\colon \mathbb{R}^{m+1} \to \mathbb{R}$ from a given class C_p of test functions such that for all $\Delta \in (0, \delta_0)$ we have

$$|E[g(X_T)] - E[g(Y^{\Delta}(T))]| \leq K\Delta^{\beta}.$$

As class C_p of test functions one can use for instance the class of smooth functions which together with their derivatives have at most polynomial growth. This choice allows a clear classification of a wide range of numerical schemes and also includes the convergence of all the moments of Y^{Δ}. Weak approximations were proposed and studied in Milstein (1978, 1988), Pardoux and Talay (1985), Platen (1984), and Talay (1984), among others. Mikulevicius and Platen (1988) show how to construct convergence schemes of any desired weak order $\beta > 0$.

4.2. Strong Approximations

For simplicity, we shall consider throughout the paper an equidistant time discretization

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$$

of the interval $[0, T]$ with step size

$$\Delta = T/N, \qquad N = 1, 2, \ldots .$$

The simplest heuristic time-discrete approximation is the stochastic generalization of the *Euler approximation*. It has the form

$$Y_{k+1}^{\Delta} = Y_k^{\Delta} + a(\tau_k, Y_k^{\Delta})\Delta + \sum_{j=1}^{n} b^j(\tau_k, Y_k^{\Delta})\Delta W_k^j \tag{4.2}$$

for $k = 0, 1, \ldots, N - 1$ with initial value $Y_0^{\Delta} = X_0$ and increments

$$\Delta W_k^j = W_{\tau_{k+1}}^j - W_{\tau_k}^j$$

of the jth component of the driving Wiener process. Thus, the ΔW_k^j are independent Gaussian random variables with expectation 0 and variance Δ. Now it is easy to generate by standard methods such normally distributed random numbers and thus to simulate the values of the Euler approximation. It turns out that the Euler approximation converges with strong order $\gamma = 0.5$, which is often too slow for practical investigations. At the cost of assuming more smoothness of the coefficient functions a and b, the stochastic Taylor formula (see Kloeden and Platen 1991; Platen 1982; and Wagner and Platen 1978) provides a general systematic means of deriving higher-order numerical schemes for stochastic differential equations. The Euler scheme (4.2) results from the simplest useful truncation of a stochastic Taylor expansion. Taking more terms from this expansion, one obtains the *Milstein scheme* in the form

$$Y_{k+1}^{\Delta} = Y_k^{\Delta} + a(\tau_k, Y_k^{\Delta})\Delta + \sum_{j=1}^{n} b^j(\tau_k, Y_k^{\Delta})\Delta W_k^j + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{i=0}^{m} b^{i,j_1}(\tau_k, Y_k^{\Delta}) \frac{\partial b^{j_2}}{\partial x^i}(\tau_k, Y_k^{\Delta}) I_{(j_1,j_2)} \tag{4.3}$$

with

$$I_{(j_1,j_2)} := \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_2} dW_{s_1}^{j_1}\, dW_{s_2}^{j_2}. \tag{4.4}$$

Under sufficient regularity conditions on a and b, this scheme converges with strong order $\gamma = 1.0$. One problem to be overcome here is the simulation of the double Wiener integrals in (4.4) arising for $j_1 \neq j_2$. The numerical approximation of these and several further multiple stochastic integrals is considered in Kloeden, Platen, and Wright (1991). For $j_1 = j_2$ one has

$$I_{(j_1,j_1)} = \tfrac{1}{2}\left((\Delta W_k^{j_1})^2 - \Delta\right).$$

In the case $j_1 \neq j_2$, one can use instead of $I_{(j_1,j_2)}$ the approximation

$$I_{(j_1,j_2)}^p := \left(\tfrac{1}{2}\,\xi_{j_1}\xi_{j_2} + \sqrt{\rho_p}(\mu_{j_1,p}\xi_{j_2} - \mu_{j_2,p}\xi_{j_1})\right)\Delta + \frac{\Delta}{2\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\zeta_{j_1,r}(\sqrt{2}\xi_{j_2} + \eta_{j_2,r}) - \zeta_{j_2,r}(\sqrt{2}\xi_{j_1} + \eta_{j_1,r})\right). \tag{4.5}$$

Here,

$$\rho_p = \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2},$$

and $\mu_{j,p}$, $\eta_{j,r}$, and $\zeta_{j,r}$ have to be chosen such that the random variables

$$\xi_j := \frac{1}{\sqrt{\Delta}}\,\Delta W_k^j$$

and $\mu_{j,p}$, $\eta_{j,r}$, and $\zeta_{j,r}$ are independent and $\mathcal{N}(0, 1)$-distributed for $j = 1, \ldots, n$. To ensure the strong order $\gamma = 1.0$, one has to choose

$$p = p(\Delta) \geq \frac{K'}{\Delta}$$

for some constant K'.

We mention another explicit scheme of strong order $\gamma = 1.0$ which has the form

$$(4.6) \qquad Y_{k+1}^{\Delta} = Y_k^{\Delta} + a(\tau_k, Y_k^{\Delta})\Delta + \sum_{j=1}^{n} b^j(\tau_k, Y_k^{\Delta})\Delta W_k^j + \frac{1}{\sqrt{\Delta}} \sum_{j_1, j_2 = 1}^{n} (b^{j_2}(\tau_k, \overline{Y}_k^{j_1}) - b^{j_2}(\tau_k, Y_k^{\Delta}))I_{(j_1, j_2)}$$

with supporting points

$$\overline{Y}_k^j = Y_k^{\Delta} + a(\tau_k, Y_k^{\Delta})\Delta + b^j(\tau_k, Y_k^{\Delta})\sqrt{\Delta}.$$

This scheme converges with strong order $\gamma = 1.0$ under analogous conditions as the Milstein scheme, but its implementation is more convenient since it avoids the computation of the derivatives of b. The multiple stochastic integrals can be approximated as above. Further details and strong schemes of higher order can be found in Kloeden and Platen (1991).

4.3. Weak Approximations

It turns out that the Euler scheme (4.2) converges under sufficient regularity of a and b with weak order $\beta = 1.0$. This also holds if we replace ΔW_k^j by another, much simpler random variable, for example the two-point random variable $\Delta \hat{W}_k^j$ with

$$P[\Delta \hat{W}_k^j = \pm\sqrt{\Delta}] = 1/2.$$

To construct higher-order weak approximations, one can include multiple stochastic integrals from the stochastic Taylor expansion. Using all double multiple stochastic integrals leads for example to the Taylor scheme

$$(4.7) \qquad Y_{k+1}^{\Delta} = Y_k^{\Delta} + a(\tau_k, Y_k^{\Delta})\Delta + \tfrac{1}{2}L^0 a(\tau_k, Y_k^{\Delta})\Delta^2 + \sum_{j=1}^{n} (b^j(\tau_k, Y_k^{\Delta})\Delta W_k^j + L^0 b^j(\tau_k, Y_k^{\Delta})I_{(0,j)} + L^j a(\tau_k, Y_k^{\Delta})I_{(j,0)}) + \sum_{j_1, j_2 = 1}^{n} L^{j_1} b^{j_2}(\tau_k, Y_k^{\Delta})I_{(j_1, j_2)}$$

of weak order $\beta = 2.0$, with the operators

(4.8)
$$L^0 = \frac{\partial}{\partial t} + \sum_{i=0}^{m} a^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,\ell=0}^{m} \sum_{j=1}^{n} b^{i,j} b^{\ell,j} \frac{\partial^2}{\partial x^i \partial x^\ell},$$

and

$$L^j = \sum_{\ell=0}^{m} b^{\ell,j} \frac{\partial}{\partial x^\ell}$$

for $j = 1, \ldots, n$, and the multiple stochastic integrals

$$I_{(j_1,j_2)} = \int_{\tau_k}^{\tau_{k+1}} \int_{\tau_k}^{s_2} dW_{s_1}^{j_1}\, dW_{s_2}^{j_2} \qquad (j_1, j_2 = 0, 1, \ldots, n),$$

where we write dW_s^0 for ds.

This scheme is due to Milstein (1978) and Talay (1984). Its convergence of weak order $\beta = 2.0$ still holds if one replaces the involved random variables by simpler ones. For instance one can choose $\Delta\hat{W}_k^j$ (instead of ΔW_k^j) as three-point random variables with

(4.9)
$$P\big[\Delta\hat{W}_k^j = \pm\sqrt{3\Delta}\big] = 1/6 \qquad \text{and} \qquad P\big[\Delta\hat{W}_k^j = 0\big] = 2/3$$

and replace the multiple stochastic integrals by

(4.10)
$$\hat{I}_{(j,0)} = \hat{I}_{(0,j)} = \tfrac{1}{2}\Delta \cdot \Delta\hat{W}_k^j$$

and

(4.11)
$$\hat{I}_{(j_1,j_2)} = \tfrac{1}{2}(\Delta\hat{W}_k^{j_1}\, \Delta\hat{W}_k^{j_2} + V_{j_1,j_2}) \qquad \text{for } j_1, j_2 = 1, \ldots, n,$$

where V_{j_1,j_2} are two-point random variables with

(4.12)
$$\begin{aligned} P[V_{j_1,j_2} &= \pm\Delta] = \tfrac{1}{2} && \text{for } j_2 = 1, \ldots, j_1 - 1, \\ V_{j_1,j_1} &= -\Delta, \\ V_{j_1,j_2} &= -V_{j_2,j_1} && \text{for } j_2 = j_1 + 1, \ldots, n, \end{aligned}$$

and where all random variables are independent. Actually, these simpler random variables only need to have the same moments as ΔW_k^j and $I_{(j_1,j_2)}$, respectively, up to some order depending on $\beta = 2.0$. Apart from that requirement, they can be chosen quite arbitrarily.

Unfortunately the implementation of a scheme like (4.7) is rather cumbersome because of the derivatives involved. Platen (1984) therefore proposed a derivative free method of weak order $\beta = 2.0$ which has the form

$$\begin{aligned} Y_{k+1}^\Delta &= Y_k^\Delta + \tfrac{1}{2}(a(\bar{Y}) + a(Y_k^\Delta))\,\Delta \\ &\quad + \frac{1}{4} \sum_{j=1}^{n} \Big[(b^j(\bar{R}_+^j) + b^j(\bar{R}_-^j) + 2b^j(Y_k^\Delta))\,\Delta\hat{W}_k^j \end{aligned}$$

$$(4.13)\qquad + \sum_{\substack{r=1\\ r\neq j}}^{n} \left(b^j(\overline{U}^r_+) + b^j(\overline{U}^r_-) - 2b^j(Y^\Delta_k)\right) \Delta\hat{W}^j_k \Delta^{-1/2}\Bigg]$$

$$+ \frac{1}{4}\sum_{j=1}^{n}\Bigg[\left(b^j(\overline{R}^j_+) - b^j(\overline{R}^j_-)\right)\left((\Delta\hat{W}^j_k)^2 - \Delta\right)$$

$$+ \sum_{\substack{r=1\\ r\neq j}}^{n} \left(b^j(\overline{U}^r_+) - b^j(\overline{U}^r_-)\right)\left(\Delta\hat{W}^j_k\Delta\hat{W}^r_k + V_{r,j}\right)\Bigg]\Delta^{-1/2}$$

with supporting points

$$\begin{aligned}\overline{Y} &= Y^\Delta_k + a(Y^\Delta_k)\Delta + \sum_{j=1}^{n} b^j(Y^\Delta_k)\Delta\hat{W}^j_k,\\ \overline{R}^j_\pm &= Y^\Delta_k + a(Y^\Delta_k)\Delta \pm b^j(Y^\Delta_k)\sqrt{\Delta},\\ \overline{U}^j_\pm &= Y^\Delta_k \pm b^j(Y^\Delta_k)\sqrt{\Delta},\end{aligned}$$

where the random variables $\Delta\hat{W}^j_k$ and $V_{r,j}$ are as in (4.9)–(4.12). As a matter of fact, (4.13) gives the scheme for the autonomous version of the diffusion (4.1), but the non-autonomous case is easily covered if one interprets one component of the diffusion as the time t. In Kloeden and Platen (1991), one can find further weak Taylor schemes of higher order and other derivative-free Runge-Kutta type schemes.

4.4. A Variance Reduction Technique

One can use the schemes (4.7) or (4.13) to simulate directly functionals of the type $E[g(X_T)]$ as needed for the computation of the option pricing function (2.5). But in many cases, it turns out that the variances of the random variable $g(X_T)$ and also of the approximating random variable $g(Y^\Delta_N)$ can be rather large. Thus we may need an enormous amount of computer time to obtain reliable results. However, a measure transformation method proposed by Milstein (1988) allows us to consider another diffusion process $\tilde{X}$ which provides us with a functional with the same expectation as $g(X_T)$, but with a much smaller variance. Thus it will be more efficient to simulate the process $\tilde{X}$ and to estimate from its outcomes the corresponding expectation.

To be more precise, let us denote by $(X^{t,x}_s)_{t\leq s\leq T}$ the diffusion process fulfilling the stochastic differential equation (4.1) starting from $x \in \mathbb{R}^{m+1}$ at time $t \in [0, T]$. Our aim is to approximate the functional

$$u(t, x) = E[g(X^{t,x}_T)]$$

for a given function g: $\mathbb{R}^{m+1} \to \mathbb{R}$ and time $t = 0$. If u is sufficiently smooth, then it solves the Kolmogorov backward equation

$$L^0 u(t,x) = 0$$

for $(t, x) \in (0, T) \times \mathbb{R}^{m+1}$ with

$$u(T, y) = g(y)$$

for all $y \in \mathbb{R}^{m+1}$, where L_0 is defined in (4.8). Now consider a diffusion process $\tilde{X}$ satisfying

$$\tilde{X}_t = X_0 + \int_0^t \left(a(s, \tilde{X}_s) - \sum_{j=1}^n b^j(s, \tilde{X}_s) h^j(s, \tilde{X}_s) \right) ds$$
$$+ \sum_{j=1}^n \int_0^t b^j(s, \tilde{X}_s)\, dW_s^j$$

and a correction process

$$\Theta_t = 1 + \sum_{j=1}^n \int_0^t \Theta_s h^j(s, \tilde{X}_s)\, dW_s^j,$$

where h^j: $[0, T] \times \mathbb{R}^{m+1} \to \mathbb{R}$ ($j = 1, \ldots, n$) are sufficiently regular functions. It turns out by the Girsanov transformation that independently of the choice of the functions h^j, one obtains

$$E[g(X_T)] = E[g(\tilde{X}_T)\Theta_T]. \tag{4.14}$$

Now it is a question of experience to find functions h^j such that the variance of $g(\tilde{X}_T)\Theta_T$ is considerably smaller than that of $g(X_T)$. If we knew u already, then the optimal choice for h^j would be

$$h^j(t, x) = -\frac{1}{u(t, x)} \sum_{i=0}^m b^{i,j}(t, x) \frac{\partial u}{\partial x^i}(t, x), \tag{4.15}$$

leading to $g(\tilde{X}_T)\Theta_T = u(0, x)$ whose variance is 0. In general, one can take some approximation $\tilde{u}$ for u and then use (4.15) with u replaced by $\tilde{u}$. Finally one has to simulate a weak approximation of the process $(\tilde{X}, \Theta)$ to estimate the functional (4.14).

5. NUMERICAL EXAMPLES

In this section, we illustrate by two examples how the methods of Section 4 can be applied to compute numerical approximations for the option pricing function u and the hedging strategy ξ.

EXAMPLE 5.1. As our first example, we consider the model (2.9) with its stochastic and past-dependent volatility. For simplicity, we take the interest rate $r(t, x)$ to be identically 0. The other parameters are assigned the following values: $p = 0.3$, $q = 1.0$, expiration time $T = 1$, and past-dependence parameter $\alpha = 0.1$ in the first run and $\alpha =$

1.0 in the second run. Thus we have to consider (for the first run) the three-dimensional stochastic differential equation

$$
\begin{aligned}
(5.1) \qquad dS_t &= \sigma_t S_t \, dW_t^1, \\
d\sigma_t &= -(\sigma_t - \zeta_t) \, dt + 0.3\sigma_t \, dW_t^2, \\
d\zeta_t &= 10(\sigma_t - \zeta_t) \, dt.
\end{aligned}
$$

As starting values, we take $S_0 = 1.0$, $\sigma_0 = 0.1$, and $\zeta_0 = 0.1$. The claim to be considered will be a European call option; i.e.,

$$
g(X_T) = (S_T - K)^+,
$$

with strike price $K = 1.0$.

First of all, we simulate a trajectory of the process $X = (X_t)_{0 \le t \le 1}$. The corresponding trajectories of the instantaneous volatility (σ_t) and the average volatility (ζ_t) are plotted in Figure 5.1, while Figure 5.2 shows the corresponding trajectory (labelled with "price") of the asset price (S_t). We observe that the instantaneous volatility σ fluctuates quite considerably around the average volatility ζ, where both σ and ζ start at 0.1. We also note in Figure 5.2 that in this realization, the simulated trajectory of the asset price S yields a terminal stock price S_T higher than the strike price $K = 1.0$, so that the option ends up in the money. All these trajectories are obtained by an application of the scheme (4.6) (with strong order $\gamma = 1.0$) to the nonlinear stochastic differential equation (5.1).

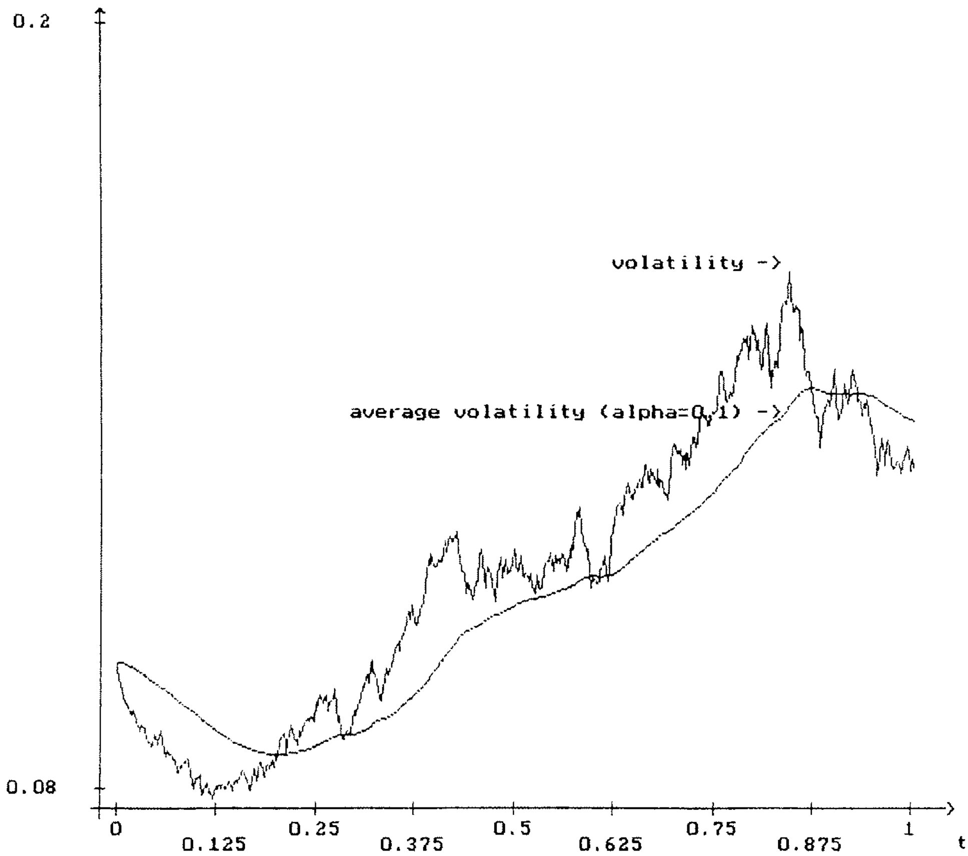

FIGURE 5.1. Volatility σ_t and average volatility $\zeta_t (\alpha = 0.1)$.

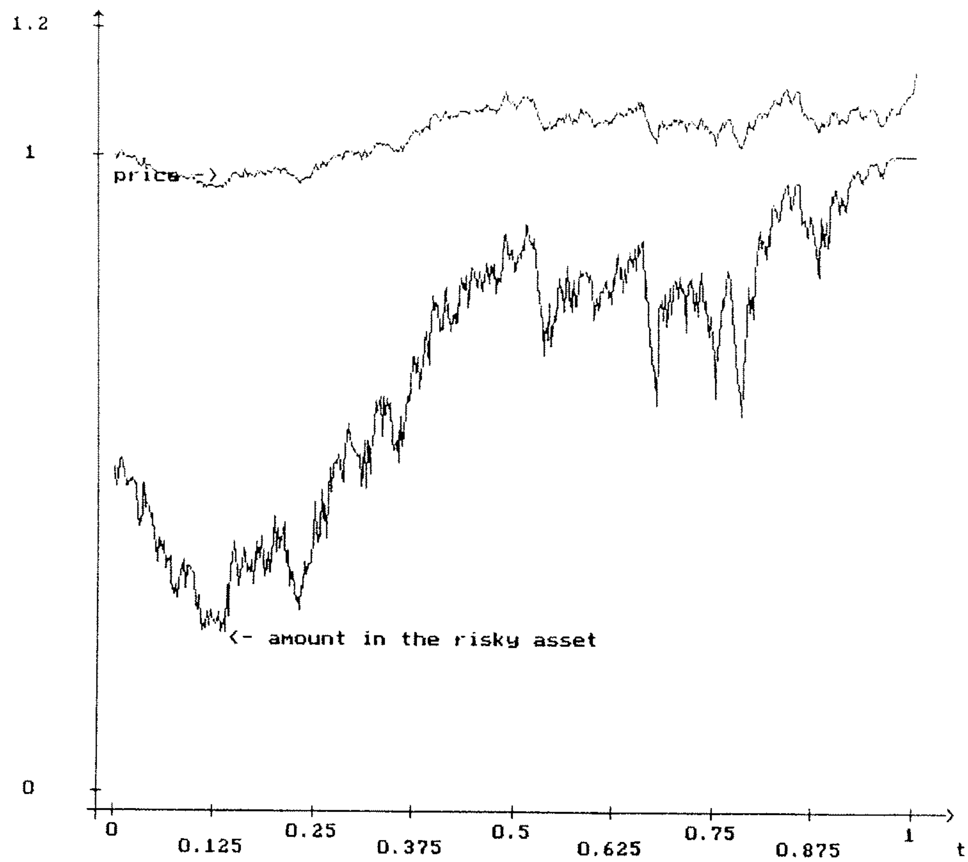

FIGURE 5.2. Stock price S_t and hedging strategy $\xi_t(\alpha = 0.1, S_T > K)$.

We have used a step size $\Delta = 2^{-9}$ and a number $p = 10$ of terms in the approximation (4.5) of the double Wiener integrals. We emphasize that this simulation requires a strong approximation scheme because we are interested in the simulated paths themselves.

In contrast, only a weak approximation is needed for the computation of the option price V_0, since this is given by the expectation

$$V_0 = E[(S_T - K)^+].$$

This approximation is achieved by the simulation of 20 batches containing 50 approximate trajectories each. The trajectories are generated by the derivative-free method (4.13) of weak order $\beta = 2.0$, with step size $\Delta = 2^{-2}$. Within each batch, we first form an average from the 50 simulated realizations of the contingent claim $(S_T - K)^+$. By the central limit theorem, these 20 batch averages can be treated as approximately Gaussian distributed random variables. This allows us to form approximate confidence intervals for the option price on the basis of the Student t-distribution. Figure 5.3 shows the 90% confidence intervals formed using 5, 10, 15, and 20 batches, respectively. The estimate for the option price V_0 is given by 0.041414. Observe in Figure 5.3 that increasing the computational effort by a factor of four (by generating 20 instead of 5 batches) halves the length of the corresponding confidence interval. This illustrates a well-known effect in stochastic numerical methods. As an aside, we remark that forming batches is of course not necessary if we only want to obtain one confidence interval; we should then treat all 1000 trajectories as one single batch.

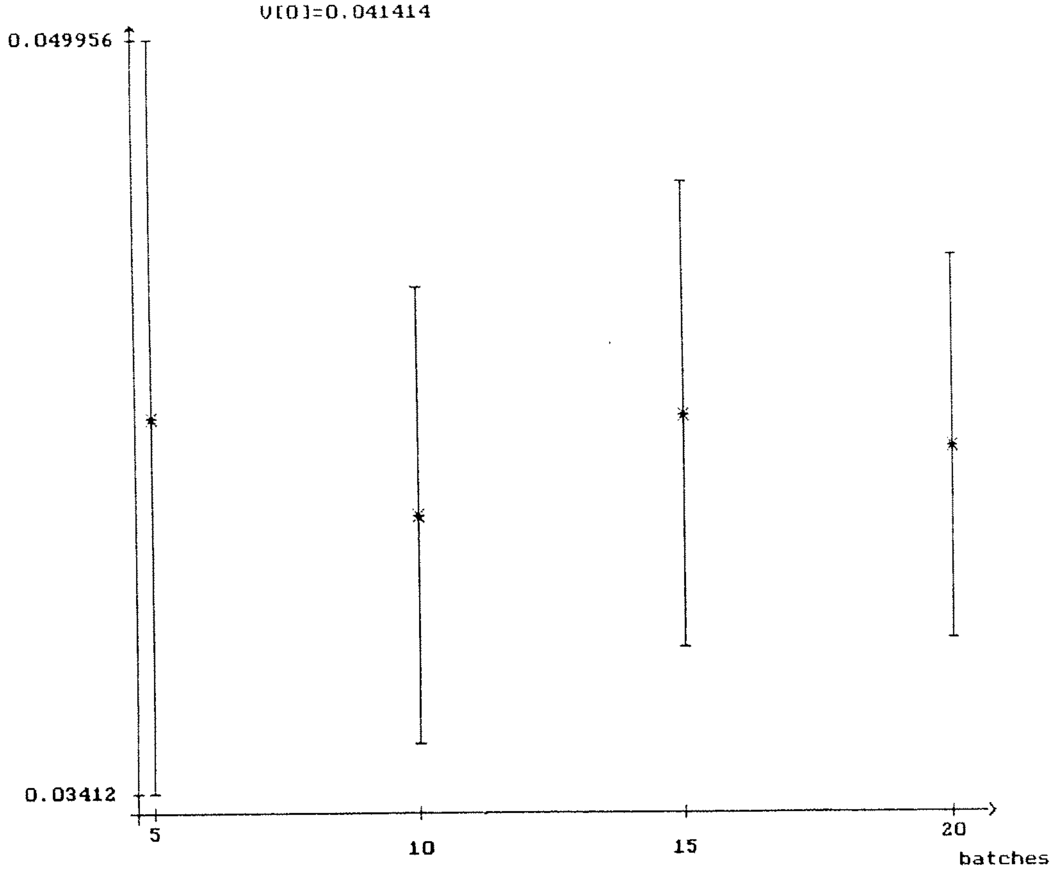

FIGURE 5.3. Confidence intervals for option price V_0 ($\alpha = 0.1$).

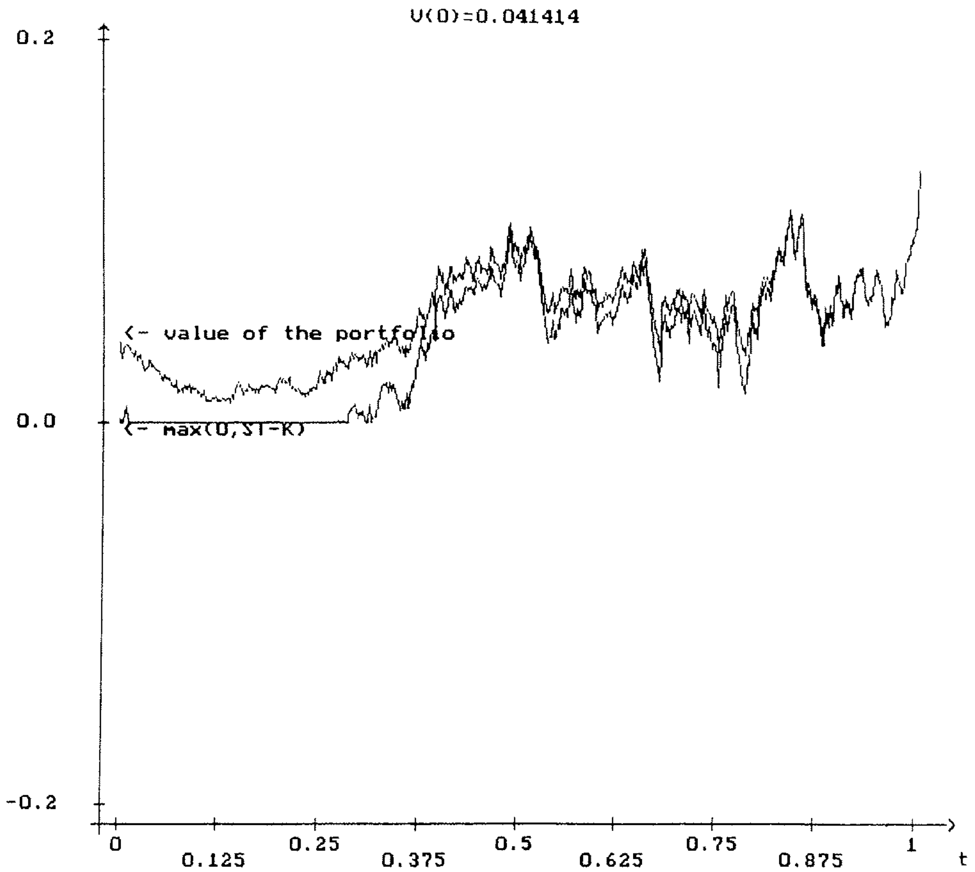

FIGURE 5.4. Value V_t of portfolio and inner value $(S_t - K)^+$ of option ($\alpha = 0.1$, $S_T > K$).

In Figure 5.4, we have plotted the values (V_t) of the hedging portfolio held along the trajectory of S shown in Figure 5.2. Figure 5.4 is obtained by simulating the values $V_{\tau k}$ at the discretization points τ_k (situated on a grid with step size 2^{-9}) in the same way as V_0 and then interpolating linearly. For comparison purposes, we also show in Figure 5.4 the inner value $(S_t - K)^+$ of the option along the trajectory of S. As expected, the value of the hedging portfolio approaches the inner value of the option from above as the time to expiration goes to 0.

Finally, the stock component (ξ_t) (labeled as "amount in the risky asset") of our hedging strategy is shown in Figure 5.2, again along the same trajectory of S (labeled as "price") as before. The value of ξ_t is given by (2.7), and we approximate the derivative of u in (2.7) by the spatial difference quotient of u. As the time to expiration grows shorter, we observe that ξ_t approaches 1; this is just what we expect to happen in this case, since the option ends up in the money. Note also that at the beginning of the time interval, ξ_t fluctuates around the value 0.5 whenever S_t is close to the strike price $K = 1.0$, just like in the case of a constant volatility.

Figures 5.5–5.8 show the corresponding results for the same example with a larger past-dependence parameter of $\alpha = 1.0$. The increase of α is reflected by the fact that the average volatility ζ_t in Figure 5.5 is much smoother. This time, we have chosen a trajectory of S ending below K (see Figure 5.6), so that the option terminates out of the money. Again, our hedging strategy performs very well: Figures 5.6 and 5.8 show that both ξ_t and the value V_t of the hedging portfolio tend to 0 as the expiration date approaches.

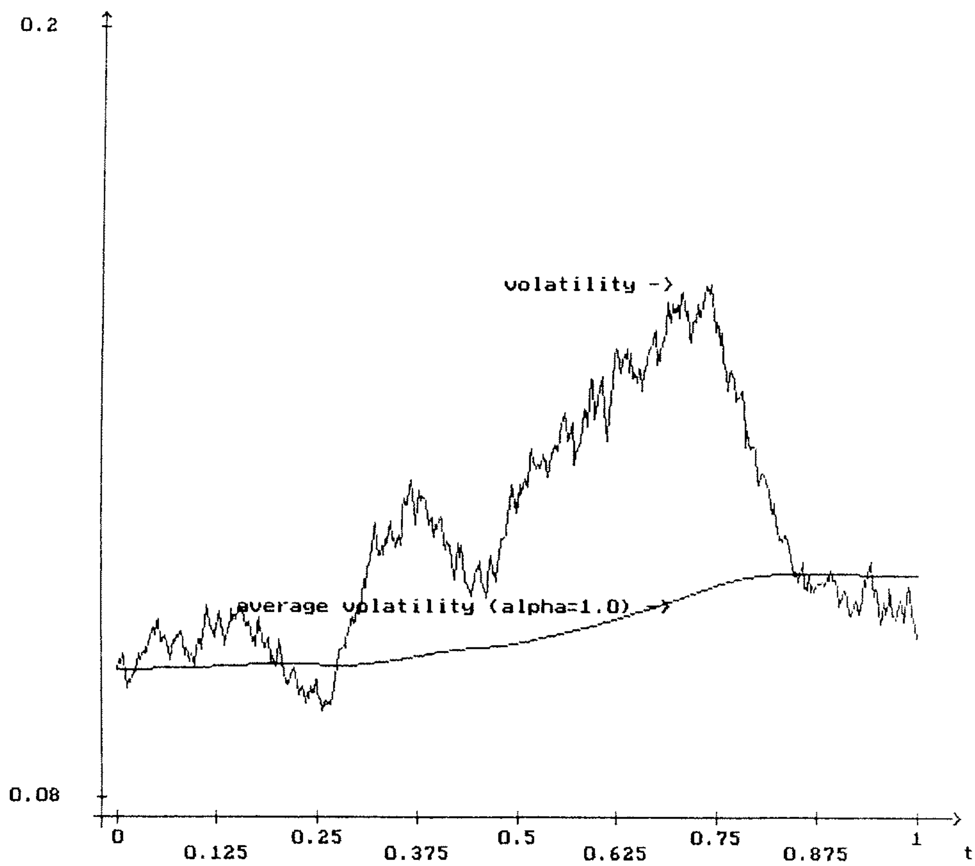

FIGURE 5.5. Volatility σ_t and average volatility $\zeta_t (\alpha = 1.0)$.

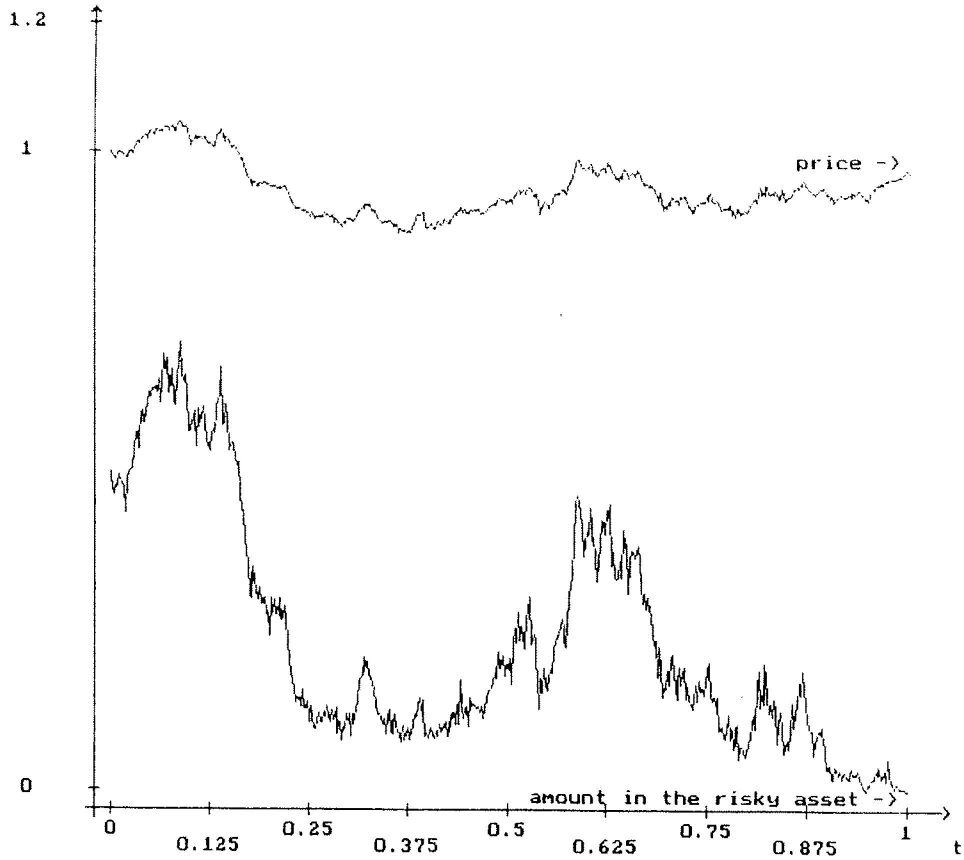

FIGURE 5.6. Stock price S_t and hedging strategy $\xi_t(\alpha = 1.0, S_T < K)$.

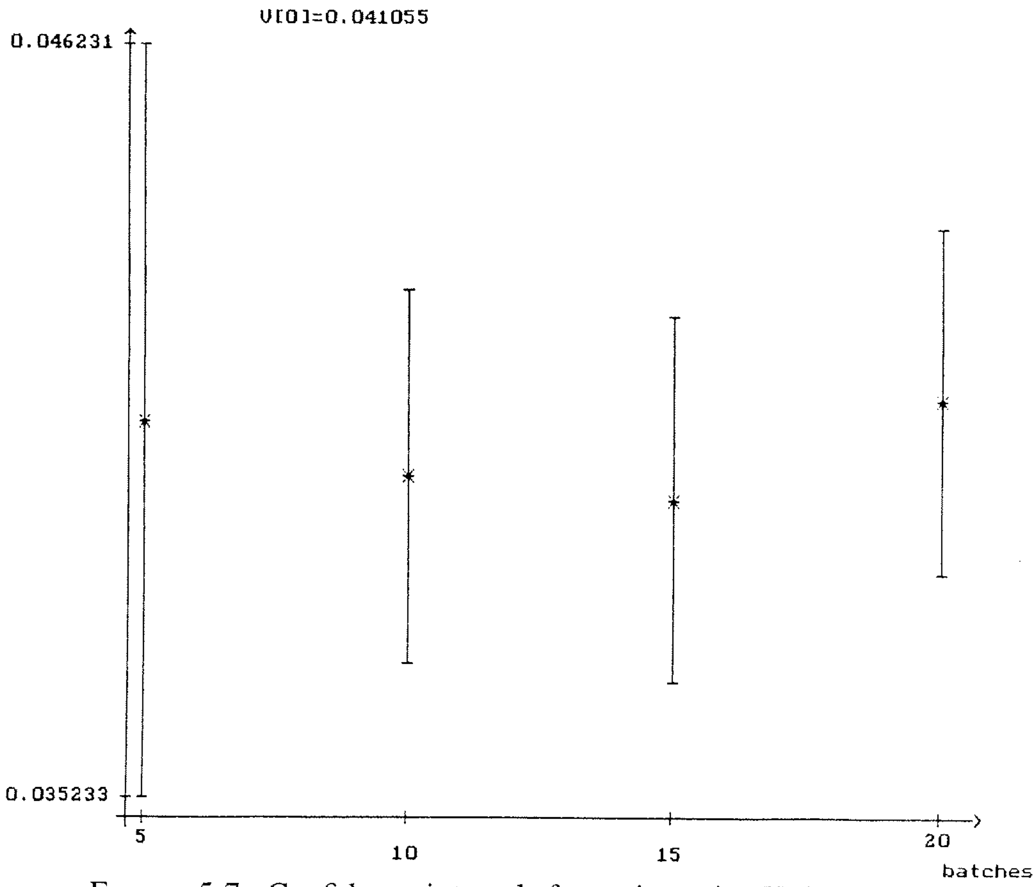

FIGURE 5.7. Confidence intervals for option price $V_0(\alpha = 1.0)$.

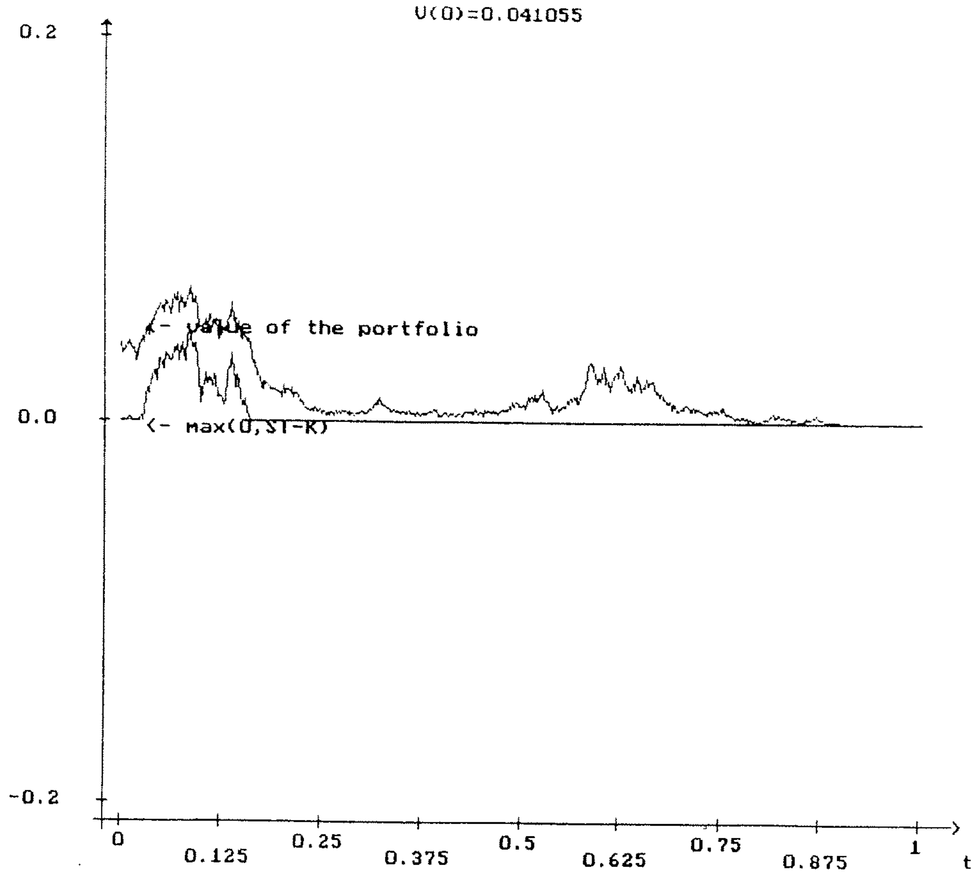

FIGURE 5.8. Value V_t of portfolio and inner value $(S_t - K)^+$ of option ($\alpha = 1.0$, $S_T < K$).

EXAMPLE 5.2. As a second example, we exhibit the results of some simulations for the generalization (2.10) of the Hull-White model (2.2). For simplicity, we again take the interest rate $r(t, x)$ to be identically 0. The other parameters are given by $\gamma(t, x) \equiv 1$, $\delta(t, x) \equiv 1$, correlation $\rho(t, x) \equiv -0.5$, time to expiration $T = 1$, and strike price $K = 1.0$. Thus we consider the two-dimensional stochastic differential equation

$$\text{(5.2)} \qquad dS_t = \sqrt{v_t} S_t \, dW_t^1, \qquad dv_t = v_t \, dt + v_t \left(-\frac{1}{2} \, dW_t^1 + \frac{\sqrt{3}}{2} \, dW_t^2 \right),$$

and we start again with $S_0 = 1.0$ and $v_0 = 0.01$ (note that $v_0 = \sigma_0^2$). For the simulation of the trajectories and the computation of the portfolio values, we use the same schemes and step sizes as in the first example. Figures 5.9–5.12 show the results for a trajectory of the asset price S which ends up above the strike price at the expiration date, while Figures 5.13–5.16 illustrate the case of landing below K. As in the first example, we observe a very good performance of our hedging strategy.

In this example, we also tried to apply the measure transformation technique discussed in Section 4 in an attempt to reduce the variance of the simulated random variable, despite the fact that the required smoothness assumptions are not fulfilled by the payoff function $g(x) = (x - K)^+$. Unfortunately, it turned out that this smoothness is not only a technical condition; the measure transformation method is therefore not applicable in the case of a call option.

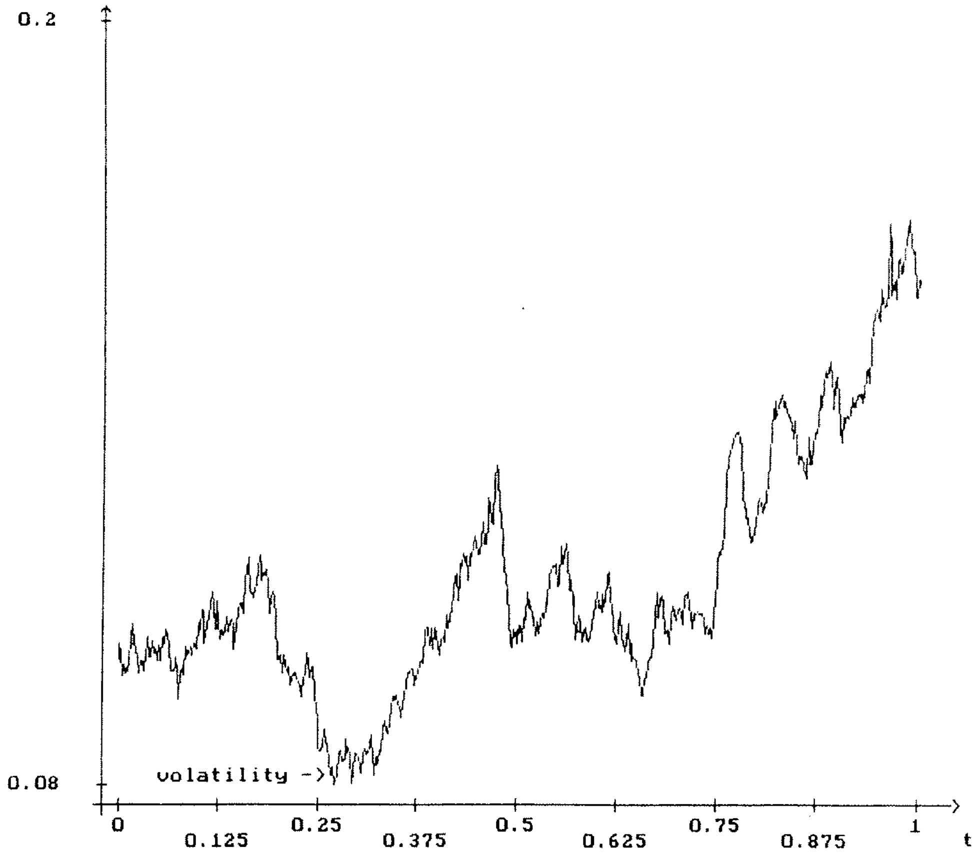

FIGURE 5.9. Volatility $\sqrt{v_t}$.

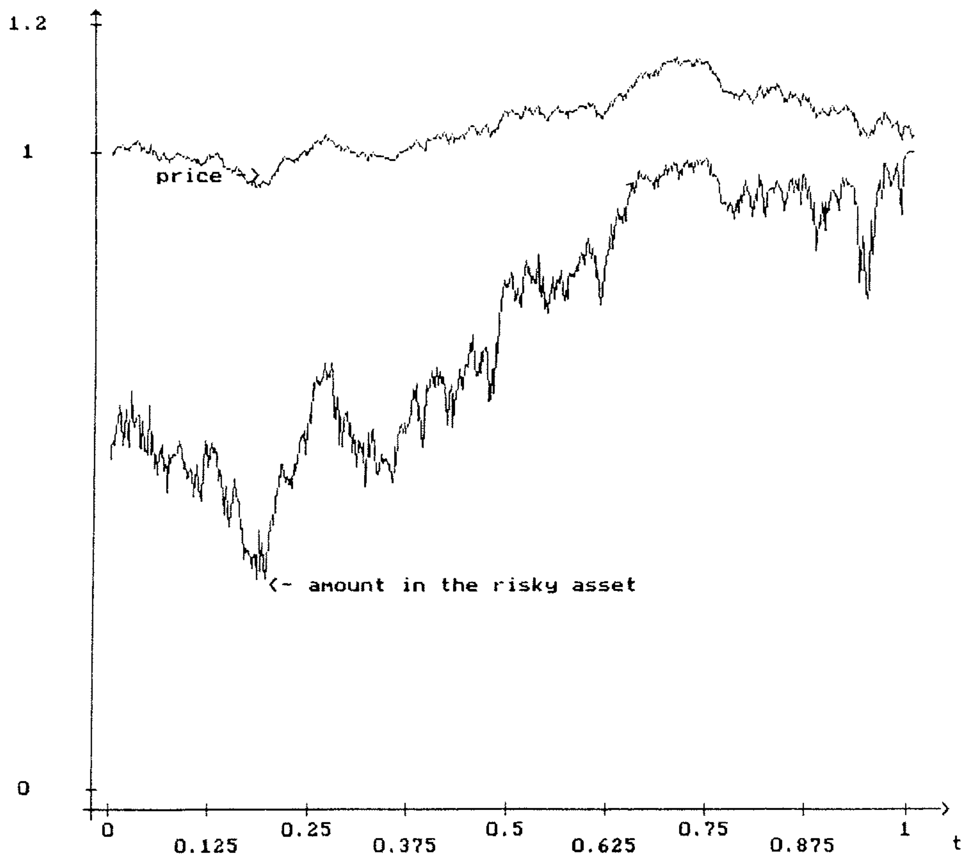

FIGURE 5.10. Stock price S_t and hedging strategy $\xi_t(S_T > K)$.

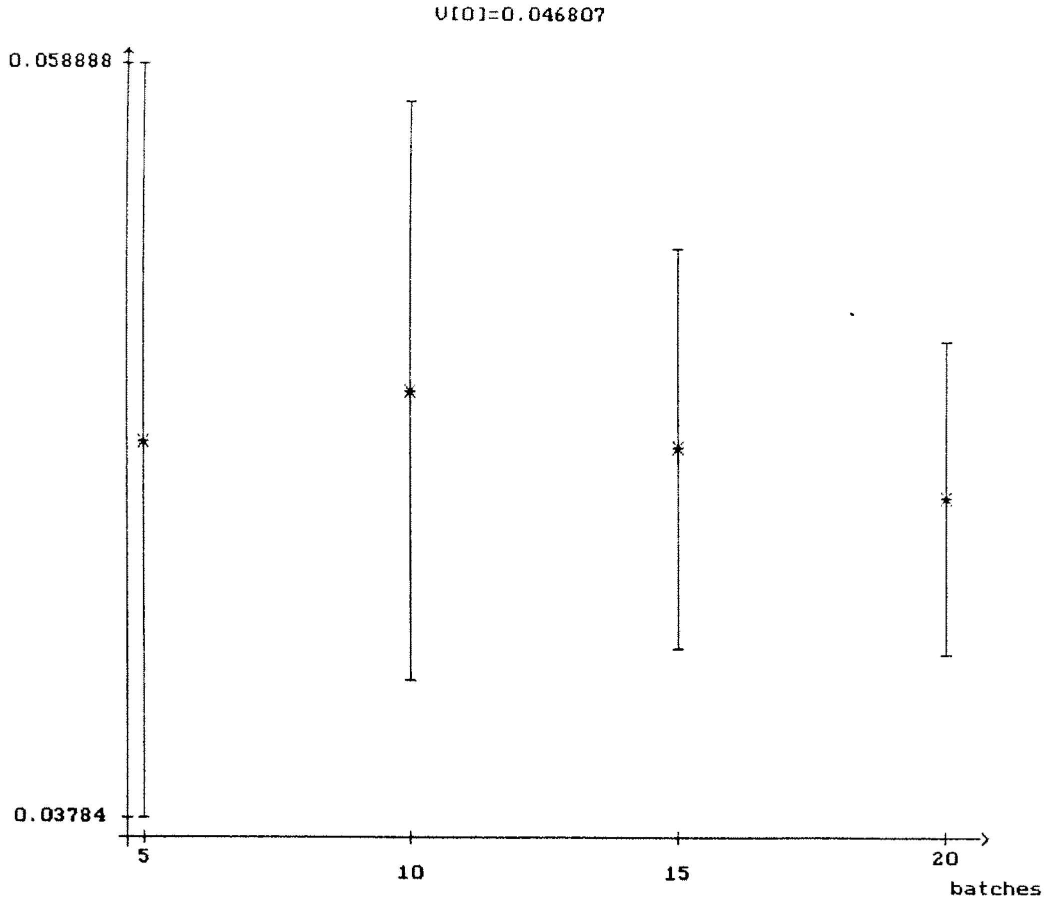

FIGURE 5.11. Confidence intervals for option price V_0.

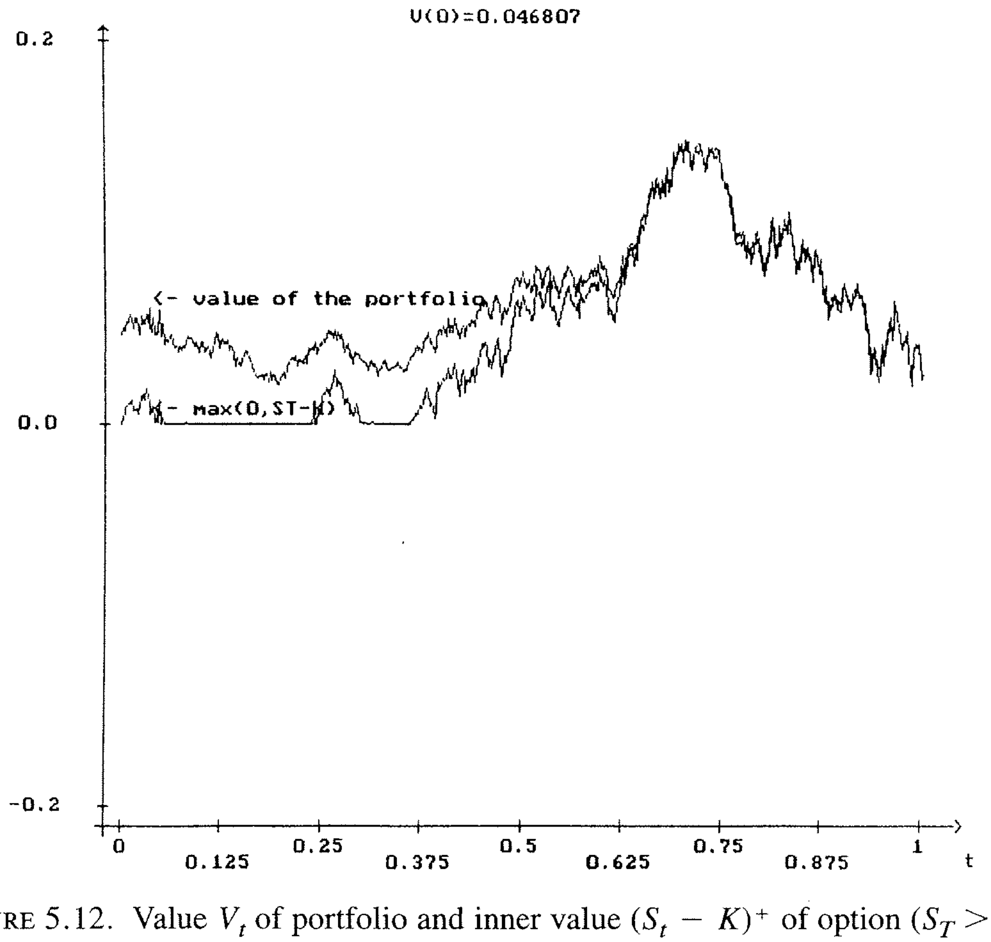

FIGURE 5.12. Value V_t of portfolio and inner value $(S_t - K)^+$ of option $(S_T > K)$.

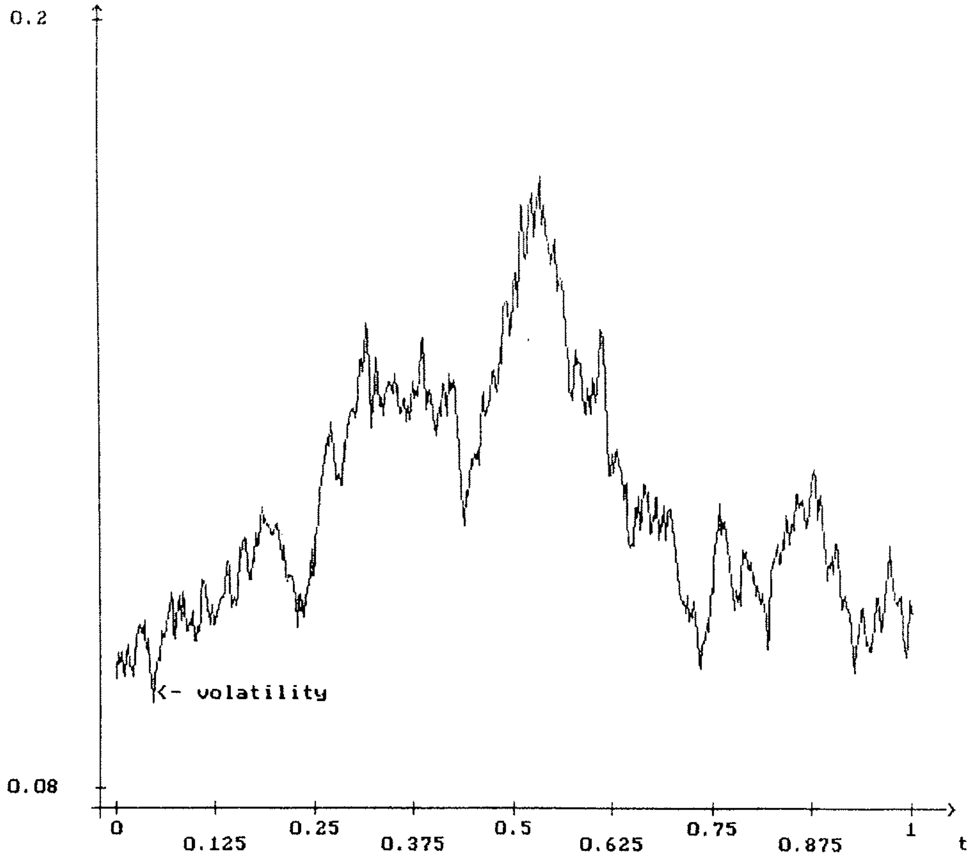

FIGURE 5.13. Volatility $\sqrt{v_t}$.

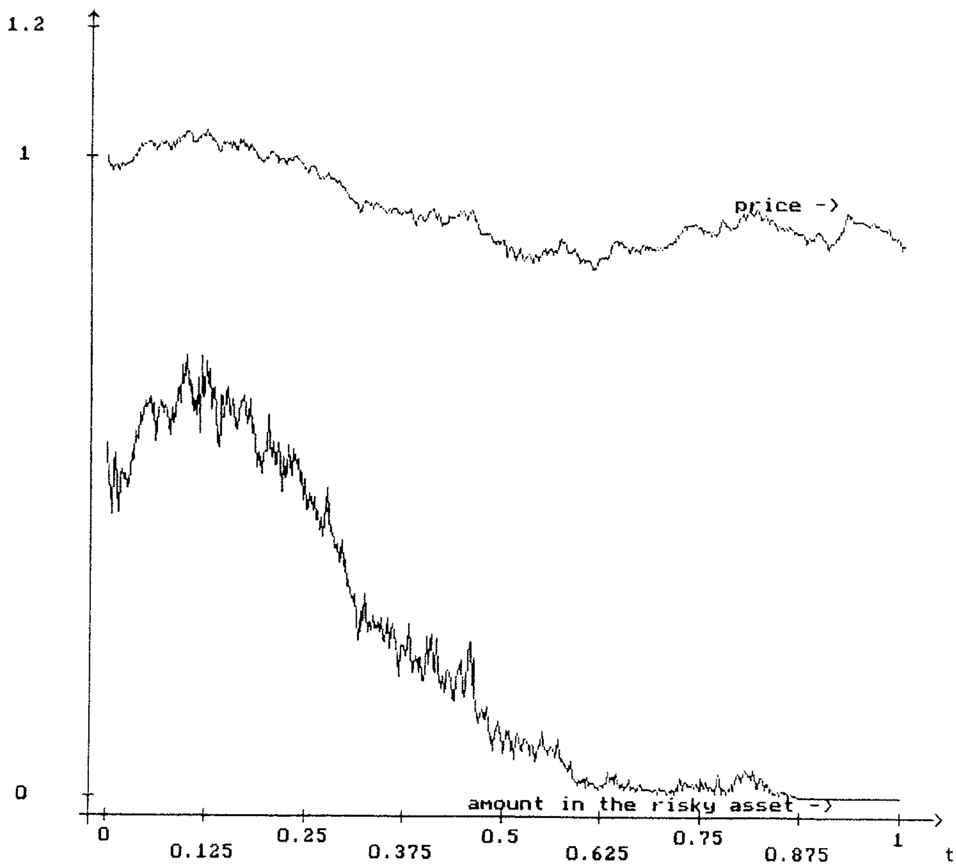

FIGURE 5.14. Stock price S_t and hedging strategy $\xi_t(S_T < K)$.

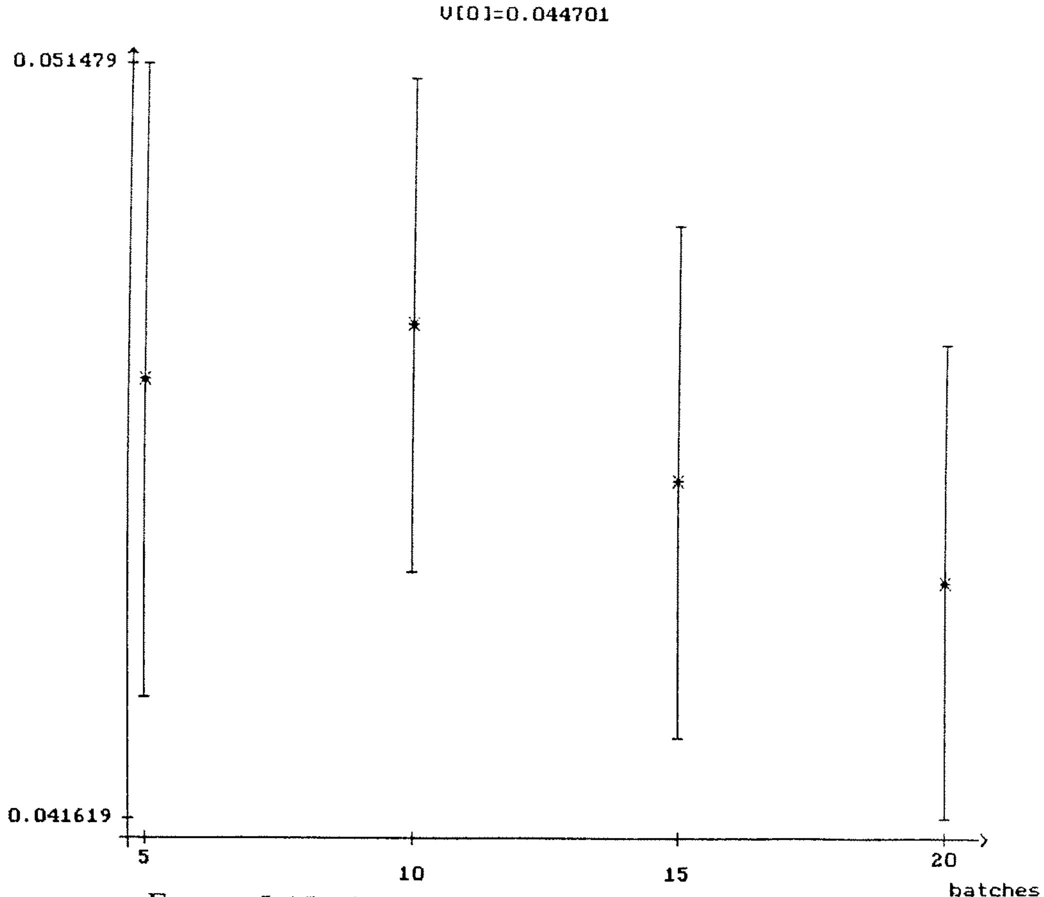

FIGURE 5.15. Confidence intervals for option price V_0.

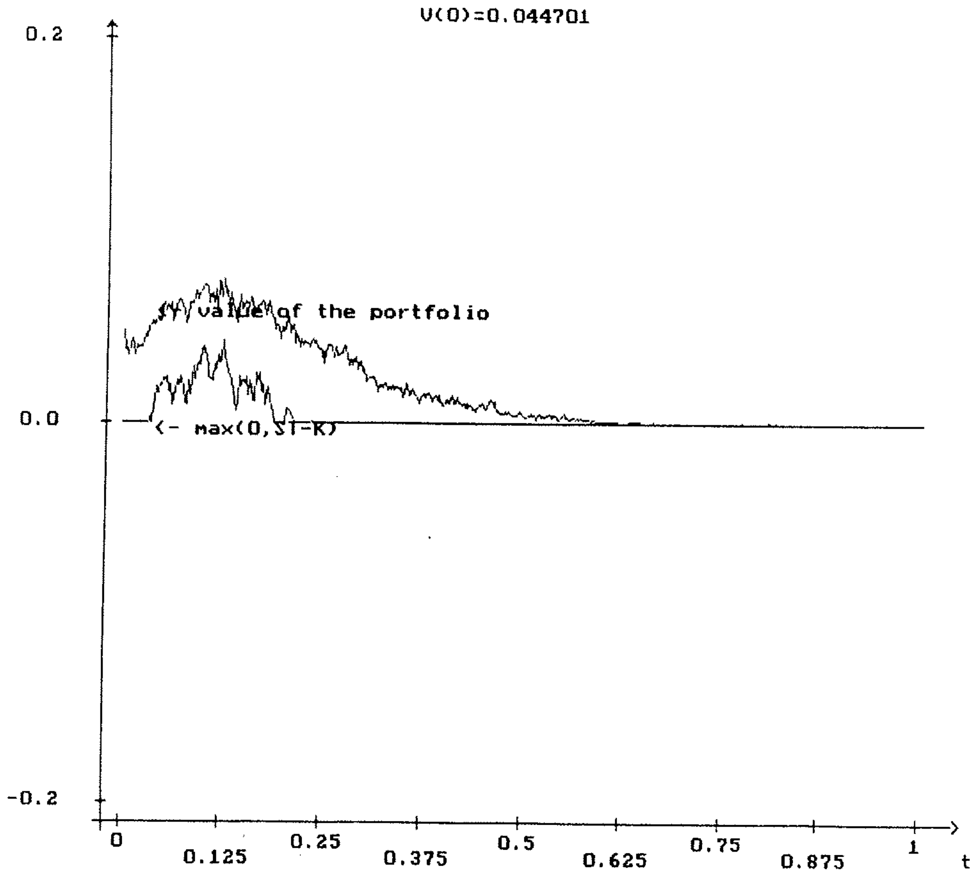

FIGURE 5.16. Value V_t of portfolio and inner value $(S_t - K)^+$ of option $(S_T < K)$.

Summarizing the preceding numerical results obtained on a 386 PC, we conclude that the proposed strong and weak approximation schemes are very well suited to the problem of numerically handling option pricing problems of the given form, even in cases of stochastic and past-dependent volatilities.

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