The minimal martingale measure

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Abstract: Suppose discounted asset prices in a financial market are given by a P-semimartingale of the form $S = S_0 + M + A$. The minimal martingale measure for S is characterised by the properties that it turns S into a local martingale and preserves the martingale property for any local P-martingale strongly P-orthogonal to M. It plays a key role in finding locally risk-minimising strategies, and it comes up in various other contexts as well. Importantly, its density process can be written explicitly in terms of M and A, so that one can use it very generally and broadly. In some specific settings, it also has other optimality properties.

Key words: martingale measure, local risk-minimisation, structure condition, Föllmer–Schweizer decomposition, hedging, option pricing, quadratic hedging criteria

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Let $S = (S_t)$ be a stochastic process on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), P\big)$ that models the discounted prices of primary traded assets in a financial market. An *equivalent local martingale measure (ELMM)* for S is a probability measure Q equivalent to the original (historical) measure P such that S is a local Q-martingale; see [equivalent martingale measure and ramifications]. If S is a nonnegative P-semimartingale, the fundamental theorem of asset pricing says that an ELMM Q for S exists if and only if S satisfies the no-arbitrage condition (NFLVR), i.e. admits no free lunch with vanishing risk; see [fundamental theorem of asset pricing]. By Girsanov's theorem, S is then under P a semimartingale with a decomposition $S = S_0 + M + A$ into a local P-martingale M and an adapted process A of finite variation. If S is special under P, then A can be chosen predictable and the resulting canonical decomposition of S is unique. We say that S satisfies the *structure condition (SC)* if M is locally P-square-integrable and A has the form $A = \int d\langle M\rangle\lambda$ for a predictable process λ such that the increasing process $\int \lambda'\, d\langle M\rangle\lambda$ is finite-valued. In an Itô process model where S is given by a stochastic differential equation $dS_t = S_t\big((\mu_t - r_t)\,dt + \sigma_t\,dW_t\big)$, the latter process is given by $\int \big((\mu_t - r_t)/\sigma_t\big)^2\,dt$, the integrated squared instantaneous Sharpe ratio of S; see [Sharpe ratio].

Definition. Suppose S satisfies (SC). An ELMM $\widehat{P}$ for S with P-square-integrable density $d\widehat{P}/dP$ is called *minimal martingale measure (for S)* if $\widehat{P} = P$ on $\mathcal{F}_0$ and if every local P-martingale L which is locally P-square-integrable and strongly P-orthogonal to M is also a local $\widehat{P}$-martingale. We call $\widehat{P}$ *orthogonality-preserving* if L is also strongly $\widehat{P}$-orthogonal to S.

The basic idea for the minimal martingale measure (MMM) first appeared in [46] in a more specific model, where it was used as an auxiliary technical tool in the context of local risk-minimisation. (See also [hedging, general concepts] for an overview of key ideas on hedging and [mean-variance hedging and portfolio selection] for an alternative quadratic approach.) More precisely, the so-called locally risk-minimising strategy for a given contingent claim H was obtained there (under some specific assumptions) as the integrand from the classical Galtchouk–Kunita–Watanabe decomposition of H under $\widehat{P}$. However, the introduction of $\widehat{P}$ in [46] and also in [47] was still somewhat ad hoc. The above definition was given in [18] where also the main results presented here can be found. In particular, [18] showed that for continuous S, the Galtchouk–Kunita–Watanabe decomposition of H under the minimal martingale measure $\widehat{P}$ provides (under very mild integrability conditions) the so-called Föllmer–Schweizer decomposition of H under the original measure P, and this in turn immediately gives the locally risk-minimising strategy for H. We emphasise that this is no longer true in general if S has jumps. The MMM subsequently found various other applications and uses and has become fairly popular, especially in models with continuous price processes.

Suppose now that S satisfies the structure condition (SC). For every ELMM Q for S with $dQ/dP \in L^2(P)$, the density process then takes the form

$$Z^Q := \frac{dQ}{dP}\bigg|_{\mathbb{F}} = Z_0^Q \mathcal{E}\left(-\int \lambda\, dM + L^Q\right)$$

with some locally P-square-integrable local P-martingale L^Q. If the MMM $\widehat{P}$ exists, then it has $\widehat{Z}_0 = 1$ and $L^{\widehat{P}} \equiv 0$, and its density process is thus given by the stochastic exponential (see [stochastic exponentials])

$$\begin{aligned}\widehat{Z} &= \mathcal{E}\left(-\int \lambda\, dM\right)\\ &= \exp\left(-\int \lambda\, dM - \frac{1}{2}\int \lambda'\, d[M]\,\lambda\right)\prod(1-\lambda'\Delta M)\exp\left(\lambda'\Delta M + \frac{1}{2}(\lambda'\Delta M)^2\right).\end{aligned}$$

The advantage of this explicit representation is that it allows to determine the minimal martingale measure $\widehat{P}$ and its density process $\widehat{Z}$ directly from the ingredients M and λ of the canonical decomposition of S. Conversely, one can start with the above expression for $\widehat{Z}$ to define a candidate for the density process of the MMM. This gives existence of the MMM under the following conditions:

(i) $\widehat{Z}$ is strictly positive; this happens if and only if $\lambda'\Delta M < 1$, i.e. all the jumps of $\int \lambda\, dM$ are strictly below 1.

(ii) The local P-martingale $\widehat{Z}$ is a true P-martingale.

(iii) $\widehat{Z}$ is P-square-integrable.

Condition (i) automatically holds (on any finite time interval) if S, hence also M, is continuous; it typically fails in models where S has jumps. Conditions (ii) and (iii) can fail even if (i) holds and even if there exists some ELMM for S with P-square-integrable density; see [45] or [15] for a counterexample.

The above explicit formula for $\widehat{Z}$ shows that $\widehat{P}$ is minimal in the sense that its density process contains the smallest number of symbols among all ELMMs Q. More seriously, the original idea was that $\widehat{P}$ should turn S into a (local) martingale while having a minimal impact on the overall martingale structure of our setting. This is captured and made precise by the definition. If S is continuous, one can show that $\widehat{P}$ is even orthogonality-preserving; see [18] for this, and note that this usually fails if S has jumps.

To some extent, the naming of the "minimal" martingale measure is misleading since $\widehat{P}$ was not originally defined as the minimiser of a particular functional on ELMMs. However, if S is continuous, Föllmer and Schweizer [18] have proved that $\widehat{P}$ minimises

$$Q \mapsto H(Q|P) - E_Q\left[\int_0^\infty \lambda'_u\, d\langle M\rangle_u \lambda u\right]$$

over all ELMMs Q for S; see also [49]. Moreover, Schweizer [50] has shown that if S is continuous, then $\widehat{P}$ minimises the reverse relative entropy $H(P|Q)$ over all ELMMs Q for S; this no longer holds if S has jumps. Under more restrictive assumptions, other minimality properties for $\widehat{P}$ have been obtained by several authors. But a general result under the sole assumption (SC) is not available so far.

There is a large amount of literature related to the MMM. In fact, a Google Scholar search for "minimal martingale measure" (enclosed in quotation marks) produced in April 2008 a list of well over 400 hits. As a first category, this contains papers where the MMM is studied per se, or used as in the original approach of local risk-minimisation. In terms of topics, the following areas of related work can be found in that category:

- properties, characterisation results and generalisations for the MMM: [1], [4], [9], [11], [14], [19], [33], [36], [37], [49], [51].
- convergence results for option prices (computed under the MMM): [25], [32], [42], [44].
- applications to hedging: [7], [39], [47], [48]. See also [hedging, general concepts].
- uses for option pricing: [8], [13], [55], to name only a very a few; comparison results for option prices are given in [22], [24], [34]. See also [risk neutral pricing].
- problems and counterexamples: [15], [16], [43], [45], [52].
- equilibrium justifications for using the MMM: [26], [40].

A second category of papers contains those where the MMM has (sometimes unexpectedly) come up in connection with various other problems and topics in mathematical finance. Examples include

- classical utility maximisation and utility indifference valuation ([3], [20], [21], [23], [35], [41], [53], [54]); the MMM here often appears because the special structure of a given model implies that it has a particular optimality property. See also [expected utility maximization], [expected utility maximization], [utility indifference valuation] and [minimal entropy martingale measure].
- the numeraire portfolio and growth-optimal investment ([2], [12]); this is related to the minimisation of the reverse relative entropy $H(P|\cdot)$ over ELMMs. See also [Kelly problem].
- the concept of value preservation ([28], [29], [30]); here the link seems to come up because value preservation is like local risk-minimisation a local optimality criterion.
- good deal bounds in incomplete markets ([5], [6]); the MMM naturally shows up here because good deal bounds are formulated via instantaneous quadratic restrictions on the pricing kernel (ELMM) to be chosen. See also [good-deal bounds], [Sharpe ratio] and [pricing kernels].
- local utility maximisation ([27]); again, the link here is due to the local nature of the criterion that is used.
- risk-sensitive control ([17], [31], [38]); this is an area where the connection to the MMM

seems not yet well understood. See also [risk-sensitive asset management].

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