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Arbitrage-free market models for option prices: the multi-strike case

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Abstract This paper studies modeling and existence issues for market models of option prices in a continuous-time framework with one stock, one bond and a family of European call options for one fixed maturity and all strikes. After arguing that (classical) implied volatilities are ill-suited for constructing such models, we introduce the new concepts of *local implied volatilities* and *price level*. We show that these new quantities provide a natural and simple parametrization of all option price models satisfying the natural *static* arbitrage bounds across strikes. We next characterize absence of *dynamic* arbitrage for such models in terms of drift restrictions on the model coefficients. For the resulting infinite system of SDEs for the price level and all local implied volatilities, we then study the question of solvability and provide sufficient conditions for existence and uniqueness of a solution. We give explicit examples of volatility coefficients satisfying the required assumptions, and hence of arbitrage-free multi-strike market models of option prices.

Keywords Option prices · Market model · Implied volatility · Static arbitrage · Dynamic arbitrage · Drift restrictions · Existence result

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1 Introduction

Consider a financial market where the following assets are all traded liquidly: a bank account (bond) paying no interest, a stock S, and a collection of European call options

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$C(K,T)$ on S with various strikes $K \in \mathcal{K}$ and maturities $T \in \mathcal{T}$. Our ultimate goal is to establish a framework for pricing and hedging (possibly exotic) derivatives in an arbitrage-free way, using all the liquid tradables as potential hedging instruments. The present paper takes a step in that direction.

In order to achieve our goal, we want to construct a class of models for bond, stock and options having at least the following features:

(0) Of course, the model should be *arbitrage-free*.
(1) Any initial option price data from the market can be reproduced by the model; this could be called *perfect calibration* or *smile-consistency*.
(2) Empirically observed stylized facts from market time series, i.e., characteristic features of the *joint dynamics* of stock and options, can be incorporated in the model. This requires that explicit expressions for option price processes and their dynamics should be available.

The overwhelming majority of the literature uses the *martingale approach*, where one specifies the dynamics of the underlying S under some pricing (i.e., martingale) measure Q and defines option prices by $C_t(K,T) := \mathbb{E}_Q[(S_T - K)^+ | \mathcal{F}_t]$. This obviously satisfies (0), and a perfect fit of the entire initial option surface as in (1) is for instance possible with the so-called smile-consistent models. However, (2) is usually not feasible, or if it is to some extent, this often comes at the cost that it entails a loss in (1). We discuss this in more detail in the next section.

An alternative approach is the use of *market models* where one specifies the joint dynamics of all tradable assets—here, stock and options. This gives (1) and (2) by construction, and the remaining issue is to ensure the absence of arbitrage to have (0) as well. In interest rate modeling, this leads to the well-known HJM drift conditions; but the case of options is more complicated. In fact, the absence of *dynamic* arbitrage again corresponds to *drift conditions* for the joint dynamics of S and the $C(K,T)$. But in addition, the absence of *static* arbitrage enforces a number of relations between the various $C(K,T)$ and S, and this means that the *state space* of these processes is *constrained* as well. To obtain a tractable model, one must therefore reparametrize the tradables in such a way that the parametrizing processes have a simple state space and yet capture all the static arbitrage constraints. We explain this in more detail in the next section, but the point here is that this (modeling) task is quite difficult.

The literature with actual results on arbitrage-free market models for option prices is quite small and most compactly summarized in terms of the families $\mathcal{K}$ and $\mathcal{T}$. Again, a more thorough discussion is postponed to the next section. For the case $\mathcal{K} = \{K\}$, $\mathcal{T} = \{T\}$ of one single call option available for trade, there are both an existence result and some explicit examples for models. For models with $\mathcal{K} = \{K\}$, $\mathcal{T} = (0,\infty)$ (one fixed strike, all maturities), the drift restrictions are well known, but the existence of models has been proved only very recently. The other extreme $\mathcal{K} = (0,\infty)$, $\mathcal{T} = \{T\}$ (all strikes, one fixed maturity) is the focus of this paper; it is more difficult and has (to the best of our knowledge) no precursors in terms of parametrization or results. Finally, the case $\mathcal{K} = (0,\infty)$, $\mathcal{T} = (0,\infty)$ of the full surface of strikes and maturities is still open despite some recent work by Carmona and Nadtochiy [13]; see Sect. 6 for more details.

The paper is structured as follows. In Sect. 2, we give an overview of the literature that is most closely related to the problem studied here and we explain in more detail the nature of our contribution. Section 3 reviews market models for stochastic implied volatilities. We characterize the absence of arbitrage in terms of drift restrictions and provide a general existence result for the case of a single option $C(K,T)$. This is done in order to illustrate where one meets difficulties with classical implied volatilities when passing to models with multiple strikes. Our main contribution is contained in Sect. 4. Instead of modeling stock price and (classical) implied volatilities, we introduce for a set of maturity-T call prices with strikes $K \geq 0$, *price level* and *local implied volatilities* which parametrize in a natural and simple way all possible arbitrage-free option prices. We provide explicit formulas for these new quantities as functions of stock price and (classical) implied volatilities, and vice versa. In analogy to (classical) implied volatilities, we then characterize their arbitrage-free dynamics in terms of drift restrictions. In Sect. 5, we provide explicit and fairly general examples of arbitrage-free dynamic models for the price level and the local implied volatilities. To prove the existence and uniqueness of a solution to the corresponding infinite system of SDEs, we adapt results from [44] to our setting. Section 6 concludes and points out a number of open questions.

2 Background, motivation, and literature

This section discusses in more detail what we want to and what we can achieve with our approach. Moreover, it also gives an overview of related literature, and for this, a slightly broader perspective is useful. So let us look at models that exploit or produce information about an underlying stock as well as options written on S.

2.1 Martingale models

In the martingale approach, one writes down a dynamic model (usually an SDE) for a stock price martingale S under a probability measure Q and defines

$$C_t(K,T) := \mathbb{E}_Q\big[(S_T - K)^+ \big| \mathcal{F}_t\big], \quad 0 \leq t \leq T,$$

for $K \in \mathcal{K} \subseteq (0,\infty)$ and $T \in \mathcal{T} \subseteq (0,\infty)$. These models by construction satisfy the requirement (0) of being arbitrage-free. Calibration as in (1) to given market option prices is more or less feasible for instance in *stochastic volatility* models (e.g., Hull and White [31], Heston [30], Davis [20]) or in models with *jumps* (e.g., Merton [37], Barndorff-Nielsen and Shephard [3], Carr et al. [15]), and several of the models also match some of the stylized features for S alone. But of course, calibration is limited by the fact that one has only a finite number of parameters to be fitted. A perfect fit of the entire option surface $C_0(K,T)$ for $\mathcal{K} = (0,\infty)$, $\mathcal{T} = (0,\infty)$ is achieved by the so-called smile-consistent models, most prominent among which are the *local volatility* model of Dupire [24] and discrete-time implied tree models like Derman and Kani [22]. A good overview on smile-consistent pricing is given by Skiadopoulos [43] and some recent papers like Carr et al. [17] or Rousseau [39] also produce in addition fairly realistic dynamic behavior for S alone.

Knowing at time 0 all call option prices $C_0(K, T)$ for $K \in (0, \infty)$ is equivalent to knowing the marginal distribution of S_T under Q; this observation goes back to Breeden and Litzenberger [11]. Hence perfect fitting of all $C_0(K, T)$ with $K \in (0, \infty)$, $T \in \mathcal{T}$ can be achieved by constructing a martingale with the corresponding marginals for S_T, $T \in \mathcal{T}$, under Q, and this can be done in many ways and situations; see for instance Madan and Yor [36], Carr et al. [16], Bibby et al. [6], Atlan [1], Hamza and Klebaner [28]. There are also many papers on calibration or empirical analyses of various models, even if we do not quote any of this work here. But from our perspective, all these models suffer from the same fundamental drawback: In general, there are no explicit expressions for the processes $C(K, T)$, and so their joint dynamics with S are not really available.

Another question of interest in the context of models for stocks and options is the link between the implied and the instantaneous volatility. This has been studied both for local and for stochastic volatility models, and the typical results are asymptotic relationships close to maturity and for at-the-money options; see for instance Berestycki et al. [4, 5] or Durrleman [26]. But again, these papers neither provide nor study the joint dynamics of S and $C(K, T)$.

2.2 Market models

As already explained in the introduction, a natural way to construct a model satisfying the requirements (1) of perfect calibration and (2) of joint dynamics is to use a market model, where one specifies the dynamics of all liquid tradables simultaneously. This goes back to ideas from interest rate modeling, and the absence of *dynamic* arbitrage there leads to the well-known *drift conditions* of Heath et al. [29]. The same type of conditions also appears in option price models. But in addition, *static* arbitrage bounds lead to restrictions on the *state space* of the quantities used to describe the model, and so the choice of a suitable *parametrization* becomes a crucial issue. (As a matter of fact, the same problem arises in the interest rate context if one insists on modeling zero-coupon bond prices; but it is easily resolved there by passing to forward rates instead.)

In the literature, some work has been done in special cases. If the option collection consists of a single call $C = C(K, T)$, one has the static arbitrage bounds $(S_t - K)^+ \le C_t \le S_t$ as well as the terminal condition $C_T = (S_T - K)^+$. Specifying directly for the pair (S, C) dynamics which obey these state space constraints is quite delicate. It is much easier to reparametrize the option price C by its implied volatility $\hat{\sigma}$ via $C_t = c(S_t, K, (T - t)\hat{\sigma}_t^2)$, where c is the well-known Black–Scholes [7] function given in (3.1) below. Then the pair $(S, \hat{\sigma})$ may take any value in $(0, \infty)^2$, the static arbitrage bounds and terminal condition are satisfied, and one can proceed to specify and study models for the joint dynamics of $(S, \hat{\sigma})$. Such *market models of implied volatilities* for a single option have first been proposed in Lyons [35] and Schönbucher [40], and arbitrage-free examples have been constructed in Babbar [2]. Even in this apparently simple situation, the construction is not entirely straightforward: in an Itô process framework over a Brownian filtration, the drifts are essentially determined by the volatilities of S and $\hat{\sigma}$, and if one takes these nonlinear drift restrictions into account, the question whether the resulting two-dimensional SDE system for S and $\hat{\sigma}$ admits a solution becomes nontrivial.

The situation becomes much more complicated if our option collection contains more than one single call. In the literature, one can find several variants of *necessary* conditions on the implied volatility dynamics for the resulting model to be arbitrage-free; see, for instance, Schönbucher [40], Brace et al. [8, 9], and Ledoit et al. [33]. However, none of these works provide any explicit example of a multi-option market model; in other words, no *sufficient* conditions are given, and the existence of such models with specified dynamics remains an open issue. The key difficulty is that the well-known static no-arbitrage conditions for calls with different strikes and maturities (see, e.g., Carr and Madan [14] or Davis and Hobson [21]) entail rather complicated relations between the implied volatilities of these options; this is illustrated in some more detail in Sect. 3. We believe that there is a fundamental reason for this problem: despite their importance as a market standard to quote option prices, (classical) implied volatilities are unsuited for modeling call prices in a multi-option model. Put bluntly, they give the wrong parametrization.

Of course, the idea of replacing implied volatilities by another parametrization of call prices in option market models is not entirely new. For the case $\mathcal{K} = \{K\}$, $\mathcal{T} = (0, \infty)$ of a family with one fixed strike K and all maturities $T > 0$, Schönbucher [40] has introduced the *forward implied volatilities*

$$\hat{\sigma}^2_{\mathrm{fw}}(T) := \frac{\partial}{\partial T}\big((T-t)\hat{\sigma}^2(T)\big), \tag{2.1}$$

and we have recently used in [41] new techniques from [44] for infinite-dimensional SDE systems to prove existence results for this class of models. The main contribution in [41] is to show how one can handle the complicated SDE systems that arise via the drift restrictions coming from absence of dynamic arbitrage. The choice of the parametrization (2.1) is taken from Schönbucher [40] and has its roots in the obvious analogy to the well-known forward rates for interest rate modeling. As a matter of fact, the results in [41] are more generally given for a maturity term structure of options with one fixed (convex or concave) payoff function h and all maturities $T > 0$. The special case $h = \log$ corresponds to a market model for variance swaps, where the drift conditions take a particularly simple form; the resulting model has been explicitly analyzed in Bühler [12]. Jacod and Protter [32] also study models for options with one fixed payoff function and all maturities and parametrize via the maturity derivatives $\frac{\partial}{\partial T} C_t(T)$. However, they do not specify $C(T)$ by joint dynamics with S, and so their work falls into the realm of the martingale approach discussed in Sect. 2.1.

In this paper, we consider the other extreme of the spectrum. We want to construct arbitrage-free market models for call option prices in the case $\mathcal{K} = (0, \infty)$, $\mathcal{T} = \{T\}$ of a family with one fixed maturity T and all strikes $K > 0$. This is substantially more difficult than the case of all maturities with one fixed strike because it requires new ideas already at the modeling level. Our main achievement is to introduce a new parametrization of option prices for the multi-strike case in such a way that arbitrage-free dynamic modeling becomes tractable. We define these new quantities, called “local implied volatilities,” in Sect. 4. They have no comparable precursor or analogue in interest rate theory because the traded assets in interest rate market models, the zero-coupon bonds, simply do not have any “strike structure.” The key feature of these new parameters is that they have a simple state space and yet capture precisely all the static arbitrage restrictions. Once they have been constructed, dynamic arbitrage conditions and existence results for the corresponding dynamic option models still need to be dealt with, but this can be achieved by using our techniques developed in [44] and [41].

3 Market models for implied volatilities

In this section, we review market models for implied volatilities and explain why their usefulness in arbitrage-free modeling is mostly limited to the case of a single traded option $C(K,T)$. The general setup along with the concept of implied volatilities is introduced in Sect. 3.1. In Sect. 3.2, we characterize the absence of dynamic arbitrage in such models in terms of drift restrictions. This recovers the results in Sect. 3.3 of Schönbucher [40]. Our Sect. 3.3 provides sufficient conditions for the existence and uniqueness of the corresponding dynamics of S and $\hat{\sigma}(K,T)$ in the case of only one pair (K,T) and discusses the difficulties that arise if one tries to generalize this approach to a model of implied volatilities for more than one strike.

3.1 Implied volatilities of call options

Throughout this paper, we work with the following setup. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T>0$ a fixed maturity. Let $(S_t)_{0\le t\le T}$ be a positive process modeling a stock price, and $(B_t)_{0\le t\le T}$ a positive process with $B_T=1$ P-a.s., modeling the price of a (nondefaultable) zero-coupon bond with maturity T. Moreover, for $K>0$, let $(C_t(K))_{0\le t\le T}$ be a nonnegative process modeling the price of a European call option on S paying $(S_T-KB_T)^+=(S_T-K)^+$ at time T. Finally, let $c(S,K,\Upsilon)$ be the Black–Scholes function

$$\left.\begin{aligned} c(S,K,\Upsilon) &= SN\left(\frac{\log(S/K)+\frac{1}{2}\Upsilon}{\Upsilon^{\frac{1}{2}}}\right) - KN\left(\frac{\log(S/K)-\frac{1}{2}\Upsilon}{\Upsilon^{\frac{1}{2}}}\right) \quad (\Upsilon>0),\\ c(S,K,0) &= (S-K)^+, \end{aligned}\right\} \tag{3.1}$$

where $N(\cdot)$ denotes the standard normal distribution function. Clearly, c is strictly increasing in Υ with $\lim_{\Upsilon\to\infty} c(S,K,\Upsilon)=S$ for all $S,K\ge 0$. If the model consisting of B, S, and $C(K)$ for a fixed $K>0$ does not admit an elementary arbitrage opportunity, then it is well known that we have, for all $0\le t\le T$,

$$(S_t-KB_t)^+ \le C_t(K) \le S_t$$

(see, e.g., [41, Proposition 2.1]). This allows us to give the following:

Definition 3.1 The *implied volatility* of the price $C_t(K)$ is the unique parameter $\hat{\sigma}_t(K)\ge 0$ satisfying

$$c\big(S_t, KB_t, (T-t)\hat{\sigma}_t^2(K)\big) = C_t(K). \tag{3.2}$$

Since the Black–Scholes function c has the homogeneity property $c(S,K,\Upsilon)=S\tilde{c}(\frac{K}{S},\Upsilon)$ for a suitable function $\tilde{c}$, the implied volatility is invariant under a change of numeraire; in other words, the defining condition (3.2) can be rewritten, for every positive numeraire process M, as

$$c\big(S_t/M_t, KB_t/M_t, (T-t)\hat{\sigma}_t^2(K)\big)=C_t(K)/M_t.$$

Therefore, we always use from now on the bond B as numeraire, and in the sequel, all price processes B, S, $C(K)$ denote B-discounted price processes, so that $B\equiv 1$.

3.2 Drift restrictions for implied volatilities

Let W be an m-dimensional Brownian motion on $(\Omega,\mathcal{F},P)$, $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ the P-augmented filtration generated by W, and $\mathcal{F}=\mathcal{F}_T$. We denote, for $d\in\mathbb{N}$ and $p\ge 1$, by $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ the space of all $\mathbb{R}^d$-valued, progressively measurable, and locally p-integrable (in t, P-a.s.) processes on $[0,T]$. We model a stock price process $(S_t)_{0\le t\le T}$ and, for some set of strikes $\mathcal{K}\subseteq(0,\infty)$, a family of price processes $(C_t(K))_{0\le t\le T}$ $(K\in\mathcal{K})$ of call options paying $(S_T-K)^+$ at time T by

$$C_t(K)=c\big(S_t,K,(T-t)\mathcal{X}_t(K)\big) \tag{3.3}$$

with dynamics

$$dS_t=\mu_t S_t\,dt+\sigma_t S_t\,dW_t, \qquad S_0=s_0, \tag{3.4}$$

$$d\mathcal{X}_t(K)=u_t(K)\mathcal{X}_t(K)\,dt+v_t(K)\mathcal{X}_t(K)\,dW_t, \qquad \mathcal{X}_0(K)=x_0(K) \tag{3.5}$$

for $0\le t\le T$. Here c is the Black–Scholes function from (3.1), $\mu,u(K)$ are in $L^1_{\mathrm{loc}}(\mathbb{R})$, and $\sigma,v(K)$ are in $L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Each $\mathcal{X}(K)=\hat{\sigma}^2(K)$ is a positive process modeling the square of the implied volatility of $C(K)$.

It is now natural to ask under which conditions there exists a common equivalent local martingale measure for the (discounted) price processes S, $C(K)$ for all $K\in\mathcal{K}$. The existence of such a measure is essentially equivalent to the *drift restrictions*

$$\mu_t=-\sigma_t b_t, \tag{3.6}$$

$$\begin{aligned} u_t(K)={}&\frac{1}{T-t}\left(1-\frac{1}{\mathcal{X}_t(K)}\left|\sigma_t+\frac{1}{2}\log\left(\frac{K}{S_t}\right)v_t(K)\right|^2\right)\\ &+\left(\frac{1}{16}(T-t)\mathcal{X}_t(K)+\frac{1}{4}\right)\left|v_t(K)\right|^2-\left(\frac{\sigma_t}{2}+b_t\right)\cdot v_t(K) \end{aligned} \tag{3.7}$$

for all $K\in\mathcal{K}$ and a *market price of risk* process $b\in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. More precisely, we have the following result.

Theorem 3.2 (a) *If there exists a common equivalent local martingale measure Q for S and $C(K)$ for all $K\in\mathcal{K}$, then there exists a market price of risk process $b\in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ such that* (3.6) *and* (3.7) $(K\in\mathcal{K})$ *hold for a.e. $t\in[0,T]$, P-a.s.*

(b) *Conversely, suppose that the coefficients* μ, σ, $u(K)$, *and* $v(K)$ *satisfy, as functions of* S *and* $\mathcal{X}(K)$, *relations* (3.6) *and* (3.7) ($K \in \mathcal{K}$) *for a.e.* $t \in [0,T]$, P*-a.s., for some bounded* (*uniformly in* t, ω) *process* $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. *Also suppose that there exists a family of positive continuous adapted processes* S, $\mathcal{X}(K)$ *on* $[0,T]$ *satisfying the system* (3.4), (3.5). *Then there exists a common equivalent local martingale measure* Q *on* $\mathcal{F}_T$ *for* $(S_t)_{0\le t\le T}$, $(C_t(K))_{0\le t\le T}$ *for all* $K \in \mathcal{K}$. *One such measure is given by*

$$\frac{dQ}{dP} := \mathcal{E}\Big(\int b\,dW\Big)_T,$$

where $\mathcal{E}$ *denotes the stochastic exponential. Moreover, if* σ *is bounded, then* S *and* $C(K)$ *for all* $K \in \mathcal{K}$ *are martingales under* Q.

This result is essentially proved in Sect. 3.3 of Schönbucher [40]. Note that the free input parameters in the model are the stock volatility σ and the family of processes $v(K)$ for all $K \in \mathcal{K}$, i.e., the *volatilities of the implied volatilities* $\mathcal{X}(K)$; they determine (together with the market price of risk b) the drifts μ and $u(K)$ via (3.6) and (3.7). The $v(K)$ are often called volvols.

3.3 Existence problems in arbitrage-free implied volatility models

We now turn to the question of existence and uniqueness of solutions for arbitrage-free implied volatility models. This is an important issue; without an existence result, it is not possible to specify a concrete model, and the uniqueness is the basis for any convergence result of an eventual numerical implementation. For the case of one single call option, a positive result can be found in Babbar [2]. In order to point out the major difficulties arising in the general case of several calls, we review here the case of a single option by discussing a slightly more general version of the basic result of Babbar [2].

Consider the model (3.4), (3.5), where μ and $u(K)$ are given by (3.6), (3.7), and take the case where the coefficients $v(K)$ are nonzero constants and σ is in $L^2_{\mathrm{loc}}(\mathbb{R}^m)$. In general, two problems will arise:

1. Because of the nonlinear dependence on $\mathcal{X}(K)$ of the drifts $u(K)$ in (3.7), a solution of (3.5) will in general only exist up to an explosion time which may be strictly less than T with positive probability.
2. Due to the factor $\frac{1}{T-t}$, the drifts $u(K)$ will typically not be in $L^1_{\mathrm{loc}}(\mathbb{R})$. The solution of (3.5) will explode at maturity, i.e., for $t \nearrow T$, and $C(K)$ will no longer be a local martingale on $[0,T]$.

So for a general specification of the coefficients σ, $v(K)$, we must expect that the system (3.4), (3.5) does not have a (nonexploding) solution on $[0,T]$ and that there does not exist an arbitrage-free model with these coefficients. See also Schönbucher [40], Sect. 3.5, for a discussion of the second problem.

For a positive result, we need to make a choice for σ, $v(K)$ which avoids the above difficulties. As in Babbar [2], we restrict to the case $\mathcal{K} = \{K\}$ of a single call option and, to simplify the notation, we drop the dependence on K in the quantities $\mathcal{X}$, u, v.

We choose the processes v and σ as functions of the state variables $t, S, \mathcal{X}$, writing (with a slight abuse of notation)

$$v_t = v(t, S, \mathcal{X}),$$
$$\sigma_t = \sigma(t, S, \mathcal{X}),$$

and define for a fixed market price of risk process b the processes

$$\mu_t = \mu(t, S, \mathcal{X}),$$
$$u_t = u(t, S, \mathcal{X})$$

by the drift restrictions (3.6), (3.7). In order to obtain a unique strong solution to our SDEs, we have to impose some sort of Lipschitz condition on the coefficients. To that end, let $U \subseteq \mathbb{R}^d$ and Θ be a (possibly empty) set. We say that a function $f : \Theta \times U \to \mathbb{R}^k$ is *locally Lipschitz on* U if $f(\cdot, x)$ is bounded for fixed $x \in U$ and if there exists a continuous function $C(\cdot, \cdot)$ on U^2 such that

$$\left| f(\theta, x) - f(\theta, x') \right| \le C(x, x')|x - x'| \quad \forall x, x' \in U,\ \theta \in \Theta.$$

One easily checks that if f, g are locally Lipschitz on U and $h : f(U) \to \mathbb{R}^k$ is locally Lipschitz on $f(U)$, then $f + g$, fg, and $h \circ f$ are locally Lipschitz on U. In the following, *const* denotes a generic positive constant whose value can change from one line to the next. We now have the following result.

Proposition 3.3 *Let b be a progressively measurable process which is uniformly bounded and*

$$\sigma(t, s, x) := -\frac{1}{2} \log(K/s) v(t, s, x) + \sqrt{x}\big(f(t, s, x) + (T - t) g(t, s, x)\big), \tag{3.8}$$

where $f, g, v : [0, T] \times (0, \infty)^2 \to \mathbb{R}^m$ are locally Lipschitz on $(0, \infty)^2$ and satisfy

$$\left.\begin{aligned} \left| f(t, s, x) \right| &= 1, \\ \left| g(t, s, x) \right| &\le const, \\ \left| v(t, s, x) \right| &\le const \frac{1}{1 + \sqrt{x} + |\log(K/s)|}. \end{aligned}\right\} \tag{3.9}$$

Then the system (3.4), (3.5) *with* (3.6), (3.7) *has a unique positive* (*nonexploding*) *solution* $(S, \mathcal{X})$ *on* $[0, T]$.

For the proof, we need an existence and uniqueness result for strong solutions of SDEs with locally Lipschitz coefficients.

Proposition 3.4 *Let $x_0 \in \mathbb{R}^d$ and $\beta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be functions which are locally Lipschitz on $\mathbb{R}^d$. Suppose that, for $f \in \{\beta, \gamma\}$,*

$$\left| f(t, x) \right| \le const\big(1 + |x|\big) \quad \forall t \ge 0, x \in \mathbb{R}^d. \tag{3.10}$$

Then there exists a unique strong solution on $[0, \infty)$ *to the SDE*

$$dX_t = \beta(t, X_t)\, dt + \gamma(t, X_t)\, dW_t, \qquad X_0 = x_0.$$

Proof This follows from Chap. 5 and Theorems 3.1 and 3.2 in Durrett [25] for the time-homogeneous case. The time-inhomogeneous case is proved completely analogously. □

We now come to the

Proof of Proposition 3.3 In order to deal with the positivity requirement for S and $\mathcal{X}$, we work with a suitable transformation of the state variables. Let $a \geq 1$ be sufficiently large. Then there exists a convex smooth strictly increasing function $\psi : \mathbb{R} \to (0, \infty)$ such that

$$\psi(z) = \begin{cases} -\frac{1}{z} & \text{for } z \leq -a, \\ z & \text{for } z \geq a. \end{cases} \tag{3.11}$$

Let $\varphi : (0, \infty) \to \mathbb{R}$ be the inverse of ψ. We apply Proposition 3.4 with $d = 2$ to the SDE system for the processes $Y = \log S$, $Z = \varphi(\mathcal{X})$, that is, to the system

$$\left.\begin{aligned} dY &= \Big(\mu\big(t, e^Y, \psi(Z)\big) - \frac{1}{2}\big|\sigma\big(t, e^Y, \psi(Z)\big)\big|^2\Big)\, dt + \sigma\big(t, e^Y, \psi(Z)\big)\, dW, \\ Y(0) &= \log s_0, \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} dZ &= \bar{u}(t, Y, Z)\, dt + \bar{v}(t, Y, Z)\, dW, \\ Z(0) &= \varphi(x_0), \end{aligned}\right\} \tag{3.13}$$

where

$$\begin{aligned} \bar{u}(t, y, z) &= u\big(t, e^y, \psi(z)\big)\psi(z)\varphi'\big(\psi(z)\big) + \frac{1}{2}\big|v\big(t, e^y, \psi(z)\big)\big|^2 \psi(z)^2 \varphi''\big(\psi(z)\big), \\ \bar{v}(t, y, z) &= v\big(t, e^y, \psi(z)\big)\psi(z)\varphi'\big(\psi(z)\big). \end{aligned}$$

If we have a solution (Y, Z) to (3.12), (3.13), then by Itô's lemma $(S, \mathcal{X}) := (e^Y, \psi(Z))$ is a solution to (3.4), (3.5), and vice versa.

It now only remains to check the conditions of Proposition 3.4 for the coefficients in the system (3.12), (3.13). The local Lipschitz condition is clearly satisfied, since

$f, g, v, \psi, \varphi', \varphi''$ are locally Lipschitz. To check (3.10), first note that we have

$$\left.\begin{aligned} \big|\psi(z)\varphi'\big(\psi(z)\big)\big| &\le const\big(1+|z|\big) \quad \forall z\in\mathbb{R},\\ \big|\psi(z)^2\varphi''\big(\psi(z)\big)\big| &\le const\big(1+|z|\big) \quad \forall z\in\mathbb{R}. \end{aligned}\right\} \tag{3.14}$$

This follows for $z \notin [-a, a]$ by direct computation from (3.11) and for $z \in [-a, a]$ by the continuity of the functions on the left-hand sides of (3.14). Next, note that from (3.7) and (3.8) we obtain

$$\begin{aligned} u(t,s,x) &= -2f(t,s,x)\cdot g(t,s,x) - (T-t)\big|g(t,s,x)\big|^2 \\ &\quad + \left(\frac{1}{16}(T-t)x + \frac{1}{4}\right)\big|v(t,s,x)\big|^2 - \left(\frac{1}{2}\sigma(t,s,x) + b\right)\cdot v(t,s,x), \end{aligned}$$

and now (3.8) and (3.9) imply that $u(t,s,x)$ and $v(t,s,x)$ are bounded. Together with (3.14), this yields (3.10) for the coefficients of (3.13). Finally, (3.8) and (3.9) imply

$$\big|\sigma\big(t, e^y, \psi(z)\big)\big| \le const\big(1+\sqrt{\psi(z)}\,\big) \le const\big(1+\sqrt{a+|z|}\,\big).$$

Together with (3.6), this yields (3.10) for the coefficients of (3.12), and the proof is complete. □

The main ideas for the volatility specifications in Proposition 3.3 are the following. First, we choose v in a form which ensures that u from (3.7) is bounded in $\mathcal{X}$; this is why we need the asymptotic behavior $v(t,s,x) \sim \frac{1}{\sqrt{x}}$ for $x \to \infty$ in (3.9). Once v is given, the choice of σ in (3.8) is then necessary to remove the singularity and ensure the boundedness of u from (3.7) near maturity, i.e., for $T - t \searrow 0$.

We can now also illustrate one of the main difficulties in extending this result to $|\mathcal{K}| > 1$. In the market model (3.4), (3.5) for a stock S and several call options and squared implied volatilities $\mathcal{X}(K)$ satisfying the no-arbitrage conditions (3.6), (3.7), it is not clear how to choose the coefficients σ, $v(K)$ to ensure nonexplosion of the drifts $u(K)$ (and thus the absence of arbitrage) for $T - t \searrow 0$. Already for $|\mathcal{K}| = 2$, the above method would force us to choose σ for given $v(K_1)$ and $v(K_2)$ in such a way that we keep both $u(K_1)$ and $u(K_2)$ under control, and it is not clear if or how this could be achieved. It does not help either if one tries to first specify σ and then find suitable $v(K_1)$ and $v(K_2)$; getting simultaneous control over both $u(K_1)$ and $u(K_2)$ looks equally hard.

The true reason behind these difficulties is the fact that implied volatilities cannot take arbitrary values across a spectrum of strikes: the absence of arbitrage between different options enforces awkward constraints and relations between the corresponding implied volatilities. For an easy example, take strikes $0 < K_1 < K_2$ and suppose that the call price $C(K_1)$ is given by some implied volatility $\hat{\sigma}(K_1) > 0$. If the implied volatility $\hat{\sigma}(K_2)$ for the strike K_2 now exceeds a certain finite bound (depending on S, $\hat{\sigma}(K_1)$, K_1, K_2), the price curve $K \mapsto c(S, K, (T-t)\hat{\sigma}^2(K))$ is no longer decreasing on $[K_1, K_2]$, which leads to an immediate arbitrage opportunity. This is the same effect we have already seen in Sect. 2, where modeling (S, C) was very delicate, whereas modeling $(S, \hat{\sigma})$ was straightforward. In the present two-strike situation, we now find that modeling $(S, \hat{\sigma}(K_1), \hat{\sigma}(K_2))$ is unpleasant, and the reason is again that the state space of this process is very complicated due to static arbitrage constraints. We conclude again that we need another, better suited parametrization—and this is why we think that (classical) implied volatilities are not the right choice for constructing market models of option prices.

To overcome these problems, we introduce in the next section a transformation of the variables S, $\mathcal{X}(K)$, $K \in \mathcal{K}$, to a new parametrization which has a very simple state space, and for which a nonsingular specification of the arbitrage-free dynamics consequently becomes straightforward.

4 Local implied volatilities and price level of option price curves

It has been a market standard for a long time to use implied volatilities for quoting option prices, and they have been extensively analyzed in many respects; a good overview can be found in Lee [34]. In particular, the statistical behavior of $\hat{\sigma}$ is well known from many empirical studies on the dynamics of the surface $\hat{\sigma}(K, T)$ (see, for example, Cont and da Fonseca [18] for a list of references in this area). But for the purpose of a theoretical analysis, implied volatilities in an arbitrage-free setting suffer from the serious drawback that they cannot take arbitrary positive values across different strikes, and we have just seen in Sect. 3.3 that this makes the construction of option market models via implied volatilities a tremendous (if not impossible) challenge.

In this section, we therefore propose to parametrize call prices by a new set of quantities which do not suffer from the above problems. In Sect. 4.1, we introduce *local implied volatilities* and the *price level* which model an arbitrage-free set of call prices in a natural way. We provide an interpretation for these quantities and show how they are related to classical implied volatilities and the stock price. Examples are given in Sect. 4.2. In Sect. 4.3, we then derive the arbitrage-free dynamics of local implied volatilities and the price level. The resulting infinite system of SDEs is studied in more detail in Sect. 5; it is still complicated but tractable.

4.1 The new parametrization: definitions and basic properties

We resume the setup of Sect. 3.1. On some probability space $(\Omega, \mathcal{F}, P)$, we have a (discounted) bond price process $B \equiv 1$ and positive processes $(S_t)_{0 \le t \le T}$ and $(C_t(K))_{0 \le t \le T}$, $K > 0$, modeling the (discounted) prices of a stock S and European call options on S with one fixed maturity $T > 0$ and all strikes $K > 0$. In the notation of Sect. 3, we have here $\mathcal{K} = (0, \infty)$. By setting $C_t(0) := S_t$, the model is specified through the processes $C(K)$, $K \ge 0$, on the interval $[0, T]$.

Definition 4.1 A function $\Gamma : [0, \infty) \to [0, \infty)$ is called a *price curve*. A price curve is called *statically arbitrage-free* if it is convex and satisfies $-1 \le \Gamma'_+(K) \le 0$ for all $K \ge 0$.

This definition is motivated by the following:

Proposition 4.2 *If the above market $C(K)$, $K \geq 0$ does not admit an elementary arbitrage opportunity, then, for each $t \in [0, T)$, the price curve $K \mapsto \Gamma(K) := C_t(K)$ is statically arbitrage-free.*

Proof See Davis and Hobson [21], Theorem 3.1. □

Since $C_t(0) = S_t$, one easily checks that any statically arbitrage-free price curve satisfies the elementary static arbitrage bounds $(S_t - K)^+ \leq C_t(K) \leq S_t$. Besides absence of arbitrage, a further economically reasonable requirement for a European call option model is that $\lim_{K\to\infty} C_t(K) = 0$ for all $t \in [0, T)$, and, under some nondegeneracy conditions on S, one can also show that the inequalities in Definition 4.1 must be strict. This motivates that we restrict ourselves to the following class of models.

Definition 4.3 An option model $C(K)$, $K \geq 0$ is called *admissible* if the price curve $\Gamma(K) := C_t(K)$ has an absolutely continuous derivative with $\Gamma''(K) > 0$ for a.e. $K > 0$, $-1 < \Gamma'(K) < 0$ for all $K > 0$, and $\lim_{K\to\infty} \Gamma(K) = 0$ for each $t \in [0, T)$, and if we have $C_T(K) = (S_T - K)^+$ for all $K \geq 0$, P-a.s.

We now introduce a new set of fundamental quantities which allow a straightforward parametrization of admissible option models. Let $N^{-1}(\cdot)$ denote the quantile function and $n(\cdot) = N'(\cdot)$ the density function of the standard normal distribution. To motivate our subsequent definition, recall the Black–Scholes function $c(S, K, \Upsilon)$ in (3.1) and note that its first and second partial derivatives with respect to the strike are given by $c_K(S, K, (T-t)\sigma^2) = -N(d_2)$ and $c_{KK}(S, K, (T-t)\sigma^2) = n(d_2)\frac{1}{K}\frac{1}{\sqrt{T-t}\,\sigma}$ with $d_2 = \frac{\log(S/K)-(T-t)\sigma^2/2}{\sqrt{T-t}\,\sigma}$. Hence we have the identity

$$\sigma = \frac{n(N^{-1}(-c_K(S, K, (T-t)\sigma^2)))}{\sqrt{T-t}\,K\, c_{KK}(S, K, (T-t)\sigma^2)}. \tag{4.1}$$

This motivates the following:

Definition 4.4 Let $C_t(K)_{0\leq t\leq T}$ be admissible. The *local implied volatility* of the price curve at time $t \in [0, T)$ is the measurable function $K \mapsto X_t(K)$ given by

$$X_t(K) := \frac{1}{\sqrt{T-t}\,K C_t''(K)} n\big(N^{-1}\big(-C_t'(K)\big)\big) \quad \text{for a.e. } K > 0. \tag{4.2}$$

Next, for a fixed constant $K_0 \in (0, \infty)$, we define the *price level* of the price curve at time $t \in [0, T)$ as

$$Y_t := \sqrt{T-t}\,N^{-1}\big(-C_t'(K_0)\big). \tag{4.3}$$

By (4.1), the quantity $X_t(K)$ can be interpreted as an "implied volatility" in the sense of a functional of the call price curve that yields back the volatility parameter for option prices given by Black–Scholes prices. The terminology *local* implied volatility is justified by the following result.

Proposition 4.5 *Let $X(K)$ and Y be the local implied volatilities and price level of an admissible model $C(K)$, $K \geq 0$. Suppose that, for a small interval $I = [a,b] \subseteq (0,\infty)$ and fixed $t < T$, we have $X_t(K) = X_t(a)$ for all $K \in I$. Then there exists a unique pair $(x_t, z_t) \in (0,\infty)^2$ such that*

$$c\big(z_t, K_1, (T-t)x_t^2\big) - c\big(z_t, K_2, (T-t)x_t^2\big) = C_t(K_1) - C_t(K_2) \tag{4.4}$$

holds for all $K_1, K_2 \in I$. It is given by

$$\begin{aligned} x_t &= X_t(a), \\ z_t &= \exp\left(X_t(a)Y_t - X_t(a) \int_{K_0}^{a} \frac{dh}{X_t(h)h} + \log a + \frac{1}{2}(T-t)X_t(a)^2 \right). \end{aligned}$$

The proof is given at the end of this section. By Proposition 4.5, the local implied volatility $X_t(a)$ at the strike a is that unique implied volatility parameter x_t for which the Black–Scholes formula prices all call option differences $C_t(K_1) - C_t(K_2)$ with strikes in the small interval $I = [a,b]$ consistently, i.e., with the same implied volatility parameter x_t and the same "implied stock price" z_t. Note that this interpretation only holds locally, in the sense that the assumption $X_t(K) = X_t(a)$ for $K \in [a,b]$ can only hold (approximately) if the interval $[a,b]$ is very small. The "implied stock price" z_t in general depends on I and differs from the underlying stock price $C_t(0) = S_t$, unless the price curve $C_t(K)$ is generated by a Black–Scholes model; see Example 4.10 below. As we shall see later, the price level Y_t (which does not depend on K or I by construction) serves as a convenient substitute for the implied stock price z_t in our option model framework.

For an admissible option model, we clearly have by definition $X_t(K) > 0$, $Y_t \in \mathbb{R}$. The main motivation for Definition 4.4 is that the set of positive local implied volatility curves and real-valued price levels is (up to some integrability conditions) in a *one-to-one relation* to admissible option price models, as is shown in the following:

Theorem 4.6 *Let $X(K), Y$ be the local implied volatilities and price level of an admissible model $C(K)$. Then*

$$C_t(K) = \int_K^{\infty} N\left(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}} \right) dk, \qquad K \in [0,\infty), \tag{4.5}$$

$$C_t'(K) = -N\left(\frac{Y_t - \int_{K_0}^{K} \frac{dh}{X_t(h)h}}{\sqrt{T-t}} \right), \qquad K \in (0,\infty), \tag{4.6}$$

$$C_t''(K) = n\left(\frac{Y_t - \int_{K_0}^{K} \frac{dh}{X_t(h)h}}{\sqrt{T-t}} \right) \frac{1}{X_t(K)K\sqrt{T-t}}, \qquad \text{a.e. } K \in (0,\infty). \tag{4.7}$$

Conversely, for continuous adapted processes $X(K) > 0$, Y on $[0,T]$ for which the right-hand side of (4.5) is finite P-a.s., define $C_t(K)$ via (4.5). Then $C(K)$, $K \geq 0$ is an admissible model having local implied volatilities $X(K)$ and price level Y.

Proof We start with the second assertion. Equation (4.5) implies (4.6) and (4.7), so the price curves $\Gamma(K) := C_t(K)$ are strongly arbitrage-free. The finiteness of the integral in (4.5) ensures $\lim_{K\to\infty} \Gamma(K) = 0$, and using the behavior of the integrand $N\left(\frac{Y_t - \int_{K_0}^k \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\right)$ in (4.5) for $t \nearrow T$ gives

$$C_T(K) = \int_0^\infty I_{\left\{Y_T > \int_{K_0}^k \frac{dh}{X_T(h)h}\right\}}(k)\, dk - \int_0^K I_{\left\{Y_T > \int_{K_0}^k \frac{dh}{X_T(h)h}\right\}}(k)\, dk.$$

Because $X(h) > 0$, the integrand $k \mapsto g(k) := I_{\left\{Y_T > \int_{K_0}^k \frac{dh}{X_T(h)h}\right\}}$ is decreasing, has values 0 and 1, and its integral from 0 to ∞ is $C_T(0) = S_T$. So either $g \equiv 1$ on $[0, K]$, in which case $C_T(K) = S_T - K$, or g drops to 0 before K, in which case $S_T = \int_0^K g(k)dk < K$. This shows that $C_T(K) = (S_T - K)^+$. Finally, solving (4.6) at $K = K_0$ for Y_t gives (4.3), and solving (4.7) for $X_t(K)$ via plugging in (4.6) yields (4.2).

For the first assertion, note that $C_t'(K)$ defined by (4.6) solves the first-order ODE (4.2) for $C_t'(K)$ with initial condition (4.3). Since $x \mapsto n(N^{-1}(-x))$ is locally Lipschitz on $(0, 1)$, the solution is unique and thus must be given by (4.6). Finally, (4.5) follows by integrating (4.6) and using $\lim_{K\to\infty} C_t(K) = 0$. □

Remark Definition 4.4 and Theorem 4.6 can be extended to the following setting. Let $C(K)$, $K \geq 0$ be an option model such that the price curve $\Gamma(K) := C_t(K)$ is statically arbitrage-free, has absolutely continuous derivative, and satisfies $\lim_{K\to\infty} \Gamma(K) = 0$ for each $t \in [0, T)$. We also suppose that there exists a constant $K_0 \in (0, \infty)$ with $K_0 \in I_t := \{K > 0 \mid C_t'(K) \in (-1, 0)\}$ for all $t \in [0, T)$. Note that I_t is an interval since $C_t'(\cdot)$ is an increasing function. Then define, for $t \in [0, T)$,

$$X_t(K) := \begin{cases} \frac{1}{\sqrt{T-t}\,K C_t''(K)} n(N^{-1}(-C_t'(K))) & \text{if } C_t''(K) > 0, \\ 0 & \text{if } C_t''(K) = 0,\ K \notin I_t, \\ \infty & \text{if } C_t''(K) = 0,\ K \in I_t, \end{cases}$$

and Y_t by (4.3). Then in analogy to Theorem 4.6, the option prices can be written as in (4.5)–(4.7) with the obvious interpretation of the integrals $\int_{K_0}^k \frac{dh}{X_t(h)h}$ for $[0, \infty]$-valued functions $X_t(h)$. Conversely, for a process $X(K)$ satisfying $X_t(K) \in (0, \infty]$ for $K \in I_t$ and $X_t(K) = 0$ for $K \notin I_t$ with intervals I_t containing K_0, and a real-valued process Y, define $C_t(K)$ again via (4.5). Then we obtain a model with statically arbitrage-free price curves having local implied volatilities $X(K)$ and price level Y. This extended definition of the local implied volatilities can be used, for example, to construct models in which the stock price S only takes values in some interval $I \subseteq [0, \infty)$; in this case, the absence of arbitrage requires that $C_t''(K) = 0$ for $K \notin I$.

Here is a sufficient criterion for the finiteness of the right-hand side of (4.5). Note that this is a condition on $X(K)$.

Proposition 4.7 *If there exists $K_1 > 0$ such that $X_t(K) \leq \sqrt{\frac{1}{2}\frac{\log K}{T-t}}$ for a.e. $K \geq K_1$, then the outer integral in (4.5) is finite.*

Proof We may assume that $K_1 > K_0$. Then, for sufficiently large k (depending on K_1 and Y_t), we have

$$\int_{K_0}^{k} \frac{dh}{X_t(h)h} \geq \int_{K_1}^{k} \frac{dh}{2\sqrt{\log h}\, h}\sqrt{8(T-t)} = \left(\sqrt{\log k} - \sqrt{\log K_1}\right)\sqrt{8(T-t)},$$

$$\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}} \leq -\sqrt{\log k}\sqrt{8} + \sqrt{\log K_1}\sqrt{8} + \frac{Y_t}{\sqrt{T-t}} \leq -\sqrt{4\log k},$$

$$N\left(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\right) \leq N\left(-\sqrt{4\log k}\right) \leq n\left(-\sqrt{4\log k}\right) = \frac{1}{\sqrt{2\pi}}e^{-2\log k} = \frac{1}{\sqrt{2\pi}}\frac{1}{k^2}.$$

So the integral is finite (P-a.s., for each t). □

Remarks (1) If there exists an equivalent martingale measure $Q \approx P$ for the model $C(K)$, $K \geq 0$, then we have $C_t'(K) = -Q[S_T > K \,|\, \mathcal{F}_t]$, and hence (4.6) implies the relation

$$Q[S_T > K_0 \,|\, \mathcal{F}_t] = N\left(\frac{Y_t}{\sqrt{T-t}}\right)$$

between the price level and the risk-neutral probability of the call $C(K_0)$ being in the money at maturity. In particular, this relation shows that we have to exclude the choice $K_0 = 0$ in the definition (4.3) of the price level unless we are dealing with a defaultable stock model.

(2) Under the assumption in (1), the default probability $P[S_T = 0 \,|\, \mathcal{F}_t]$ is zero if and only if we have $C_t'(0) = -1$. This condition can be ensured in a local implied volatility model, for instance, by demanding that, for some $\epsilon > 0$ and $x > 0$, we have

$$X_t(K) \leq x|\log K| \quad \text{for a.e. } K \leq \epsilon;$$

this implies that $\int_{K_0}^{K} \frac{dh}{X_t(h)h} \to -\infty$ for $K \to 0$ and hence $C_t'(0) = -1$ by (4.6).

In addition to providing an interpretation for the local implied volatilities and price level, we can also express these new quantities in terms of the classical implied volatilities $\hat{\sigma}_t(K)$ of the admissible option prices $C_t(K)$ and the stock price S_t. These formulas are a bit lengthy but explicit.

Proposition 4.8 *Define* $\Upsilon_t(K) := (T-t)\hat{\sigma}_t^2(K)$ *and* $d_2(t,K) := \frac{\log(S_t/K) - \frac{1}{2}\Upsilon_t(K)}{\sqrt{\Upsilon_t(K)}}$. *Then the local implied volatilities and price level are given by*

$$Y_t = \sqrt{T-t}\; N^{-1}\left(N\left(d_2(t,K_0)\right) - \frac{1}{2}n\left(d_2(t,K_0)\right)\sqrt{\Upsilon_t(K_0)}\frac{K_0}{\hat{\sigma}_t^2(K_0)}\frac{d}{dK}\hat{\sigma}_t^2(K_0)\right). \tag{4.8}$$

and

$$
\begin{aligned}
X_t(K) = \hat{\sigma}_t(K)\, n\Bigg(N^{-1}\Bigg(& N\big(d_2(t,K)\big) \\
& - \frac{1}{2} n\big(d_2(t,K)\big)\sqrt{\Upsilon_t(K)}\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K)\Bigg)\Bigg) \\
& \times \frac{1}{n(d_2(t,K))}\Bigg[1 + \bigg(\log\bigg(\frac{S_t}{K}\bigg) + \frac{1}{2}\Upsilon_t(K)\bigg)\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K) \\
& + \frac{1}{4}\bigg(\log^2\bigg(\frac{S_t}{K}\bigg) - \Upsilon_t(K) - \frac{1}{4}\Upsilon_t^2(K)\bigg)\bigg(\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K)\bigg)^2 \\
& + \frac{1}{2}\Upsilon_t(K)\frac{K^2}{\hat{\sigma}_t^2(K)}\frac{d^2}{dK^2}\hat{\sigma}_t^2(K)\Bigg]^{-1}
\end{aligned} \tag{4.9}
$$

Proof From (3.2) we obtain

$$
\begin{aligned}
C_t'(K) &= c_K\big(S_t, K, \Upsilon_t(K)\big) + c_\Upsilon\big(S_t, K, \Upsilon_t(K)\big)(T-t)\frac{d}{dK}\hat{\sigma}_t^2(K), \\
C_t''(K) &= c_{KK}\big(S_t, K, \Upsilon_t(K)\big) + 2c_{K\Upsilon}\big(S_t, K, \Upsilon_t(K)\big)(T-t)\frac{d}{dK}\hat{\sigma}_t^2(K) \\
&\quad + c_{\Upsilon\Upsilon}\big(S_t, K, \Upsilon_t(K)\big)\bigg((T-t)\frac{d}{dK}\hat{\sigma}_t^2(K)\bigg)^2 \\
&\quad + c_\Upsilon\big(S_t, K, \Upsilon_t(K)\big)(T-t)\frac{d^2}{dK^2}\hat{\sigma}_t^2(K).
\end{aligned}
$$

Computing the partial derivatives of the Black–Scholes function $c(S, K, \Upsilon)$ yields

$$
C_t'(K) = -N\big(d_2(t,K)\big) + \frac{1}{2} n\big(d_2(t,K)\big)\sqrt{\Upsilon_t(K)}\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K), \tag{4.10}
$$

$$
\begin{aligned}
C_t''(K) = \frac{1}{K} n\big(d_2(t,K)\big)\frac{1}{\sqrt{\Upsilon_t(K)}}\Bigg[& 1 + \bigg(\log\bigg(\frac{S_t}{K}\bigg) + \frac{1}{2}\Upsilon_t(K)\bigg)\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K) \\
& + \frac{1}{4}\bigg(\log^2\bigg(\frac{S_t}{K}\bigg) - \Upsilon_t(K) - \frac{1}{4}\Upsilon_t^2(K)\bigg)\bigg(\frac{K}{\hat{\sigma}_t^2(K)}\frac{d}{dK}\hat{\sigma}_t^2(K)\bigg)^2 \\
& + \frac{1}{2}\Upsilon_t(K)\frac{K^2}{\hat{\sigma}_t^2(K)}\frac{d^2}{dK^2}\hat{\sigma}_t^2(K)\Bigg].
\end{aligned} \tag{4.11}
$$

Now the result follows by inserting (4.10), (4.11) into (4.2), (4.3). □

Note that by using (4.10), (4.11), we could express the (strong) static no-arbitrage restrictions $C_t'(K) \in (-1, 0)$ and $C_t''(K) > 0$ of Proposition 4.2 in terms of the implied volatilities $\hat{\sigma}(K)$. However, it is not at all obvious how one could parametrize those implied volatility curves $K \mapsto \hat{\sigma}_t(K)$ which satisfy the resulting conditions. In contrast, Theorem 4.6 says that any positive-valued local implied volatility curve $X(K)$ satisfying the simple bound in Proposition 4.7 is compatible with the static no-arbitrage restrictions in an admissible model. In other words, the absence of dynamic arbitrage can be characterized and ensured by imposing only conditions on the coefficients in the dynamics; there are no restrictions on the state space of the $X(K)$ and Y, which can take arbitrary values in $(0,\infty)$ and $\mathbb{R}$, respectively. We therefore argue that this new parametrization is very natural and convenient for constructing models.

Remark In [41], we have shown how one can parametrize an infinite family of option price processes for one fixed payoff function (e.g., calls with one fixed strike) and all maturities $T>0$. The parametrization chosen in [41] is via the *forward implied volatilities* over the remaining time to maturity, and we have shown that this is very convenient to analyze in terms of absence of dynamic arbitrage. The local implied volatilities and price level introduced here play the analogous role for models of calls with one fixed maturity and all strikes $K>0$.

It remains to give the

Proof of Proposition 4.5 By (4.5) and the assumption in Proposition 4.5, we have, for any $K_1, K_2 \in I$,

$$\begin{aligned} C_t(K_1) - C_t(K_2) &= \int_{K_1}^{K_2} N\bigg(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\bigg) dk \\ &= \int_{K_1}^{K_2} N\bigg(\frac{Y_t - \int_{K_0}^{a} \frac{dh}{X_t(h)h} + \frac{\log a}{X_t(a)} - \frac{\log k}{X_t(a)}}{\sqrt{T-t}}\bigg) dk. \end{aligned}$$

For any $x_t>0$ and $z_t>0$, we have $\frac{d}{dK}c(z_t,K,(T-t)x_t^2) = -N\big(\frac{\log(z_t/K)-\frac{1}{2}(T-t)x_t^2}{\sqrt{T-t}\,x_t}\big)$ for all K, and therefore

$$\begin{aligned} & c\big(z_t, K_1, (T-t)x_t^2\big) - c\big(z_t, K_2, (T-t)x_t^2\big) \\ &\quad = \int_{K_1}^{K_2} N\bigg(\frac{\frac{\log z_t}{x_t} - \frac{1}{2}(T-t)x_t - \frac{\log k}{x_t}}{\sqrt{T-t}}\bigg) dk. \end{aligned} \tag{4.12}$$

So clearly $z_t = \exp(X_t(a)(Y_t - \int_{K_0}^{a} \frac{dh}{X_t(h)h} + \frac{\log a}{X_t(a)} + \frac{1}{2}(T-t)X_t(a)))$ and $x_t = X_t(a)$ satisfy (4.4). To see the uniqueness of (x_t, z_t), note that by using (4.2), then (4.4), and finally (4.1), we obtain, for $K \in I$,

$$\begin{aligned} X_t(a) = X_t(K) &= \frac{1}{\sqrt{T-t}\,K C_t''(K)} n\big(N^{-1}\big(-C_t'(K)\big)\big) \\ &= \frac{n(N^{-1}(-c_K(z_t,K,(T-t)x_t^2)))}{\sqrt{T-t}\,K\, c_{KK}(z_t,K,(T-t)x_t^2)} = x_t. \end{aligned}$$

Moreover, (4.12) shows that $c(z_t, K_1, (T-t)x_t^2) - c(z_t, K_2, (T-t)x_t^2)$ is strictly increasing in z_t, so the uniqueness of z_t follows. □

4.2 Examples

We now illustrate the concept of local implied volatilities in several stock models. By Theorem 4.6, the local implied volatilities and price level could have been equivalently defined via (4.5). This equation (and along with it also the new parametrization) can be motivated quite naturally as an option pricing formula in a certain stock martingale model driven by a one-dimensional Brownian motion, as explained in our first example.

Example 4.9 Suppose that the stock price process $S = (S_t)_{0\le t\le T}$ satisfies

$$S_T = F\big(\widetilde{W}_T\big) \tag{4.13}$$

for a Brownian motion $\widetilde{W}$ under a pricing measure Q and a strictly increasing bijective differentiable function $F : \mathbb{R} \to I \subseteq (0,\infty)$. Suppose further that both S and $\widetilde{W}$ generate the same filtration $(\mathcal{F}_t)_{0\le t\le T}$. Our aim is now to fit the function F to a given admissible price curve $C_0(K)$ of call option prices at time 0. Under the absence of arbitrage, we obtain, for the call option prices $C_t(K)$ with $K \ge 0$,

$$\begin{aligned}
C_t(K) = \mathbb{E}_Q\big[(S_T-K)^+\big|\mathcal{F}_t\big] &= \mathbb{E}_Q\bigg[\bigg(F\bigg(\frac{\widetilde{W}_T-\widetilde{W}_t}{\sqrt{T-t}}\sqrt{T-t}+\widetilde{W}_t\bigg)-K\bigg)^+\bigg|\mathcal{F}_t\bigg]\\
&= \mathbb{E}_Q\big[\big(F\big(\widetilde{W}_1\sqrt{T-t}+y\big)-K\big)^+\big]\big|_{y=\widetilde{W}_t}\\
&= \mathbb{E}_Q\bigg[\int_K^\infty I_{\{F(\widetilde{W}_1\sqrt{T-t}+y)\ge k\}}dk\bigg]\bigg|_{y=\widetilde{W}_t}\\
&= \int_K^\infty Q\big[F\big(\widetilde{W}_1\sqrt{T-t}+y\big)\ge k\big]dk\,\bigg|_{y=\widetilde{W}_t}\\
&= \int_K^\infty Q\bigg[\widetilde{W}_1 \ge \frac{F^{-1}(k)-y}{\sqrt{T-t}}\bigg]dk\,\bigg|_{y=\widetilde{W}_t}\\
&= \int_K^\infty N\bigg(\frac{\widetilde{W}_t-F^{-1}(k)}{\sqrt{T-t}}\bigg)dk
\end{aligned} \tag{4.14}$$

for $t\in[0,T]$. In particular, for $K=0$, we obtain the underlying stock price model $S_t = C_t(0)$. The local implied volatilities are now introduced as a straightforward parametrization of the function F^{-1}. Since $F : \mathbb{R}\to I$ is strictly increasing, bijective, and differentiable, for a fixed $K_0 > 0$, there exists a unique integrable function $f : I \to [0,\infty]$ with $F^{-1}(k) = \int_{K_0}^k f(h)\,dh$. If we define

$$X(k) := \frac{1}{f(k)k} \quad (k\ge 0) \tag{4.15}$$

and $Y_t := \widetilde{W}_t$, then $F^{-1}(k) = \int_{K_0}^{k} \frac{dh}{X(h)h}$, and (4.14) becomes (4.5). Solving this for $X(k)$ and Y then leads to Theorem 4.6 and shows how one can fit local implied volatilities, and hence the function F, to the initial price curve $C_0(K)$.

Theorem 4.6 says that the admissible price curves $C_0(K)$ at time 0 are in a one-to-one relation to positive local implied volatilities $X(k)$ and real-valued price levels Y_0. The above calculations show that the admissible price curves $C_0(K)$ are also in a one-to-one relation to strictly increasing bijective differentiable functions $F : \mathbb{R} \to (0, \infty)$ as in (4.13). This reflects the well-known result that modeling statically arbitrage-free price curves of T-maturity calls is equivalent to specifying the distribution of the stock price S_T under the pricing measure. Our parametrization of the arbitrage-free option prices then arises quite naturally via the nonnegative quantities $f(k)$ parametrizing the set of functions $F^{-1}(k)$ as above. Once the representation (4.14) is found, the scaling in the definition (4.15) of the local implied volatilities $X(k)$ is chosen to establish compatibility with the Black–Scholes model, in the sense that $X(k)$ is constant and yields the volatility parameter if S follows a geometric Brownian motion.

Example 4.10 For the Black–Scholes model with volatility $\sigma > 0$, we have $S_t = S_0 \exp(\sigma \widetilde{W}_t - \frac{1}{2}\sigma^2 t)$, where $\widetilde{W}$ is a Brownian motion under the risk-neutral measure, and $\hat{\sigma}_t(K) = \sigma$ for all $K > 0$. So (4.9) simplifies to $X_t(K) = \sigma$, recovering identity (4.1) again, and (4.8) yields

$$\begin{aligned} Y_t = \sqrt{T-t}\, d_2(t, K_0) &= \frac{1}{\sigma}\log S_t - \frac{1}{\sigma}\log K_0 - \frac{1}{2}(T-t)\sigma \\ &= \widetilde{W}_t + \frac{1}{\sigma}\left(\log(S_0/K_0) - \frac{1}{2}T\sigma^2\right). \end{aligned}$$

Plugging this into the expression for z_t in Proposition 4.5 readily shows that $z_t = S_t$ in the Black–Scholes model.

Example 4.11 In Heston's [30] stochastic volatility model, the stock and instantaneous variance are modeled by the 2-dimensional diffusion (S, ζ) given by

$$\begin{aligned} dS_t &= S_t\sqrt{\zeta_t}\, d\widetilde{W}_t^1, \\ d\zeta_t &= \kappa(\theta - \zeta_t)dt + \nu\sqrt{\zeta_t}\big(\rho\, d\widetilde{W}_t^1 + \sqrt{1-\rho^2}\, d\widetilde{W}_t^2\big) \end{aligned}$$

for a 2-dimensional Brownian motion $(\widetilde{W}^1, \widetilde{W}^2)$ under the risk-neutral measure, with constants $\kappa, \theta, \nu > 0$, and $\rho \in (-1, 1)$. Stochastic volatility models can reproduce implied volatility smiles and skews, and Heston's model is a popular choice in practice, since there exists a semi-closed formula for call prices (see [30]) which allows fast calibration of the model parameters to market prices. Because of the complicated structure of the price formula, however, no simple expression for local implied volatilities in Heston's model seems to be available. Numerical calculations show that, for typical parameter values, local implied volatilities exhibit a similar but more pronounced smile and skew structure than classical Black–Scholes implied volatilities; see Fig. 1.

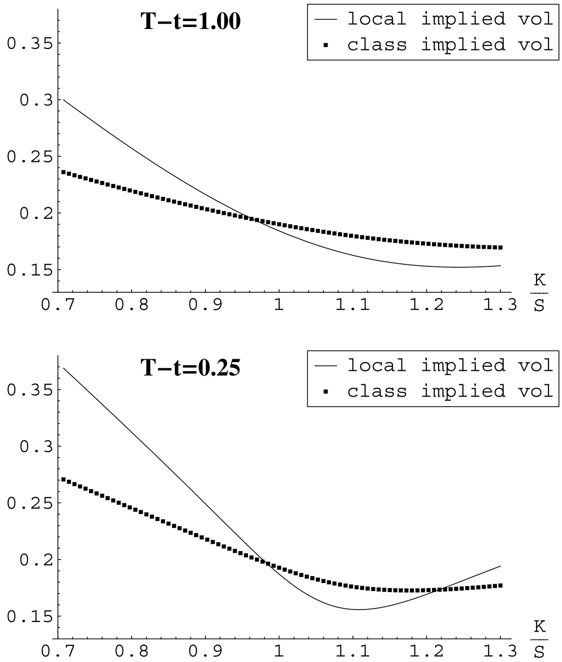

Fig. 1 Classical versus local implied volatility as function of moneyness $\frac{K}{S_t}$ in Heston's model for different times to maturity. Parameter values are $\kappa = 3$, $\theta = 0.2^2$, $\nu = 0.45$, $\rho = -0.5$ and $\zeta_t = 0.2^2$

4.3 Arbitrage-free dynamics of the local implied volatilities

In this section, we derive the dynamics of the local implied volatilities under the absence of arbitrage. Let W be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the P-augmented filtration generated by W, and $\mathcal{F} = \mathcal{F}_T$. We suppose that we have positive processes $X_t(K)$ for a.e. $K > 0$, satisfying the condition in Proposition 4.7, and a real valued process Y_t with P-dynamics

$$dX_t(K) = u_t(K)X_t(K)\,dt + v_t(K)X_t(K)\,dW_t \quad (0 \le t \le T), \tag{4.16}$$

$$dY_t = \beta_t\,dt + \gamma_t\,dW_t \quad (0 \le t \le T), \tag{4.17}$$

where $\beta, u(K) \in L^1_{\mathrm{loc}}(\mathbb{R})$, and $\gamma, v(K) \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ for a.e. K. We also suppose that u, v are uniformly bounded in ω, t, K and that the initial local implied volatility curve satisfies $\int_{K_0}^{K} \frac{dh}{X_0(h)^2} < \infty$ for all $K > 0$. Now define the processes $C_t(K)$, $K \ge 0$ by (4.5), so that $X_t(K)$, Y_t are by construction and Theorem 4.6 the local implied volatilities and price level of the option prices $C_t(K)$, $K \ge 0$. Remember that $S_t = C_t(0)$ and note that, for defining $C_t(0)$ via (4.5), the values $X_t(0)$ are not needed.

Our aim is now to show that the existence of a common equivalent local martingale measure for $C(K)$ for all $K \ge 0$ is essentially equivalent to the *drift restrictions*

$$\beta_t = \frac{1}{2}\frac{Y_t}{T-t}\big(|\gamma_t|^2 - 1\big) - \gamma_t \cdot b_t, \tag{4.18}$$

$$
\begin{aligned}
u_t(K) = \frac{1}{T-t}\Bigg[\frac{1}{2} - \frac{1}{2}\left|\gamma_t + \int_{K_0}^{K} \frac{v_t(h)}{X_t(h)h}\,dh\right|^2 \\
+ \left(Y_t - \int_{K_0}^{K} \frac{dh}{X_t(h)h}\right)\left(\gamma_t + \int_{K_0}^{K} \frac{v_t(h)}{X_t(h)h}dh\right) \cdot v_t(K)\Bigg] \\
+ \left|v_t(K)\right|^2 - v_t(K) \cdot b_t
\end{aligned}
\tag{4.19}
$$

for a *market price of risk* process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. More precisely, we have the following result.

Theorem 4.12 (a) *If there exists a common equivalent local martingale measure Q for all $C(K)$ $(K \geq 0)$, then there exists a market price of risk process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$ such that* (4.18), (4.19) (*for a.e.* $K > 0$) *hold for a.e.* $t \in [0, T]$, *P-a.s.*

(b) *Conversely, suppose that the coefficients $\beta, \gamma, u(K)$, and $v(K)$ satisfy, as functions of Y_t and $X_t(K)$, relations* (4.18), (4.19) (*for a.e.* $K > 0$) *for a.e.* $t \in [0, T]$, *P-a.s. for some bounded* (*uniformly in* t, ω) *process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Also suppose that there exists a family of continuous adapted processes $X(K) > 0$, Y satisfying the system* (4.16) (*for a.e.* $K > 0$) *and* (4.17). *Then there exists a common equivalent local martingale measure Q on $\mathcal{F}_T$ for $C(K)$ $(K \geq 0)$. One such measure is given by*

$$
\frac{dQ}{dP} := \mathcal{E}\left(\int b\,dW\right)_T, \tag{4.20}
$$

where $\mathcal{E}$ is again the stochastic exponential.

(c) *In the situation of* (a) *or* (b), *the dynamics of $C(K)$ under Q are given by*

$$
dC_t(K) = \int_K^{\infty} n\left(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\left(\gamma_t + \int_{K_0}^{k} \frac{v_t(h)}{X_t(h)h}dh\right) dk \cdot d\widetilde{W}_t \tag{4.21}
$$

for $K \geq 0$ and a Q-Brownian motion $\widetilde{W} = W - \int b_s\,ds$.

Equations (4.18), (4.19) for the local implied volatility setting are the analogues to the drift restrictions (2.13), (2.14) in [41] for the forward implied volatility modeling. Note that the *free input parameters* are the market price of risk process b as well as γ and the family of processes $v(K)$ for all K, i.e., the volatilities of the state variables Y and $X(K)$; they determine the drifts β and $u(K)$ via (4.18), (4.19). Note also that since $S_t = C_t(0)$, the volatility σ_t of the stock price process $dS_t = \sigma_t S_t d\widetilde{W}_t$ can easily be derived from (4.21) and (4.5) as

$$
\begin{aligned}
\sigma_t = \int_0^{\infty} n\left(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\left(\gamma_t + \int_{K_0}^{k} \frac{v_t(h)}{X_t(h)h}dh\right) dk \\
\times \left(\int_0^{\infty} N\left(\frac{Y_t - \int_{K_0}^{k} \frac{dh}{X_t(h)h}}{\sqrt{T-t}}\right) dk\right)^{-1}.
\end{aligned}
$$

This implies that if γ or the "volvols" $v(K)$ are random, we obtain for the stock price S a model with a certain (quite specific) stochastic volatility. Whether or not this is Markovian depends on $\gamma, v(K)$.

Example 4.13 Let $X_0(\cdot)$ be a positive measurable function on $(0,\infty)$ satisfying the condition in Proposition 4.7 and $Y_0 \in \mathbb{R}$. Take $m=1$, $\gamma \equiv 1$, and $v(K) \equiv 0$ for all K. Then Theorem 4.12 yields $\beta = -b$, $u(K)=0$ for all K, and thus $Y_t = W_t - \int_0^t b_s\,ds$ and $X_t(K) = X_0(K)$ for all K. Hence we recover the arbitrage-free one-factor model with constant (strike-dependent) local implied volatility of Example 4.9. In Sect. 5, we construct more generally arbitrage-free option price models with stochastic (and thus potentially more realistic) local implied volatility processes.

The remainder of this section is devoted to the proof of Theorem 4.12. We use the following:

Proposition 4.14 *Let $Z_t(k) := Y_t - \int_{K_0}^k \frac{dh}{X_t(h)h}$. Under P, the dynamics of $C_t(K)$ for each fixed $K \geq 0$ are then given by*

$$dC_t(K) = \int_K^\infty n\left(\frac{Z_t(k)}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\left[\frac{1}{2}\frac{Z_t(k)}{T-t}\left(1-\left|\gamma_t + \int_{K_0}^k \frac{v_t(h)}{X_t(h)h}dh\right|^2\right)\right.$$
$$\left. + \beta_t - \int_{K_0}^k \frac{v_t^2(h)-u_t(h)}{X_t(h)h}\,dh\right]dk\,dt$$
$$+ \int_K^\infty n\left(\frac{Z_t(k)}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\left(\gamma_t + \int_{K_0}^k \frac{v_t(h)}{X_t(h)h}\,dh\right)dk\cdot dW_t.$$

Proof Formally this follows from applying Itô's lemma under the integral in (4.5) and then using (4.16), (4.17). Using the condition in Proposition 4.7, one can show that we may apply Fubini's theorem for stochastic integrals (see Protter [38], Chap. IV, Theorem 65) to justify interchanging the dk-integral and the stochastic integral. A detailed proof can be found in [46], Sect. 4.7.3. □

Proof of Theorem 4.12 (a) Since $\mathbb{F}$ is generated by W, Itô's representation theorem implies that we have $\mathbb{E}[\frac{dQ}{dP}|\mathcal{F}_t] = \mathcal{E}(\int b\,dW)_t$ for some process $b \in L^2_{\mathrm{loc}}(\mathbb{R}^m)$, and

$$\widetilde{W} := W - \int b_t\,dt$$

is a Q-Brownian motion by Girsanov's theorem. Now Proposition 4.14 yields

$$dC_t(K) = \int_K^\infty n\left(\frac{Z_t(k)}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\mu_t(k)\,dk\,dt$$
$$+ \int_K^\infty n\left(\frac{Z_t(k)}{\sqrt{T-t}}\right)\frac{1}{\sqrt{T-t}}\left(\gamma_t + \int_{K_0}^k \frac{v_t(h)}{X_t(h)h}\,dh\right)dk\cdot d\widetilde{W}_t, \quad (4.22)$$

where

$$\mu_t(k) := \frac{1}{2}\frac{Z_t(k)}{T-t}\left(1 - \left|\gamma_t + \int_{K_0}^k \frac{v_t(h)}{X_t(h)h}\,dh\right|^2\right) + \beta_t - \int_{K_0}^k \frac{v_t^2(h) - u_t(h)}{X_t(h)h}\,dh$$
$$+ \left(\gamma_t + \int_{K_0}^k \frac{v_t(h)}{X_t(h)h}\,dh\right) \cdot b_t$$

for $k > 0$. Since $C(K)$ are local Q-martingales for all K, by Fubini's theorem we have P-a.s., for a.e. t,

$$\mu_t(k) = 0 \quad \text{for a.e. } k \tag{4.23}$$

and then for all k by the continuity of μ_t in k. Letting $k \to K_0$ in (4.23), we obtain (4.18). Finally, (4.19) follows after a straightforward calculation if we differentiate (4.23) with respect to k.

(b) Define $\frac{dQ}{dP} := \mathcal{E}(\int b\,dW)_T$ on $\mathcal{F}_T$; then $\widetilde{W} := W - \int b_t\,dt$ is a Q-Brownian motion on $[0, T]$ by Girsanov's theorem. Another lengthy but straightforward calculation shows that (4.18) and (4.19) imply $\mu_t(k) = 0$ for all k. Plugging this and $dW_t = d\widetilde{W}_t + b_t\,dt$ into Proposition 4.14, we obtain (c) under (b). It now easily follows from (c) that $C(K)$ for all $K \geq 0$ are Q-local martingales on $[0, T]$.

(c) The assertion under (b) has been proved together with (b) above. Under (a), the assertion follows from (4.22) and (4.23). □

5 A class of arbitrage-free local implied volatility models

In this section, we apply the existence and uniqueness results of [44] to the infinite system (4.16), (4.17) of SDEs arising in Sect. 4, providing explicit examples of arbitrage-free local implied volatility models. This requires some additional work: An existence result for general SDEs like in [44] uses assumptions on both the drift and the volatility coefficients, but in the case of our system (4.16), (4.17), we may only choose the volatility coefficients γ, v. Our aim is therefore to find conditions on the coefficients γ, v such that the drift coefficients β, u given by (4.18), (4.19) behave nicely and the results of [44] can be applied.

We first adapt the framework for infinite systems of SDEs developed in [44] to the present setup in Sect. 5.1. Then we provide an existence result in Sect. 5.2. We generalize Example 4.13 to nonzero v and hence to stochastic local implied volatilities. Our approach is similar in spirit to the existence results in [44, Sect. 5] for interest rate term structure models or in [41, Sect. 3] for forward implied volatility term structure models.

5.1 Construction of the solution space

In this section, we define the spaces in which we construct the SDE solutions in Sect. 5.2 below. This is done broadly in parallel to Sect. 3.1 in [41]. Some rather technical concepts from [44] (including an existence result for infinite-dimensional SDEs) which are only used in proofs have been shifted into the Appendix; it can be skipped by those readers who are mainly interested in the results and less in the details of the proofs. An alternative approach to our existence problem could be based on the theory of Hilbert space-valued SDEs by Da Prato and Zabczyk [19]; this is sketched at the end of this section.

Our key trick to obtaining results for the infinite family of real-valued processes $X(K)$, $K>0$, on Ω is to view this as one real-valued process on an extended space $\widetilde{\Omega}$ whose elements are pairs (K,ω). Since we only need our processes up to the fixed maturity T, we work throughout this section on $[0,T]$. So let $(\Omega,\mathcal{F},P)$ be a probability space, $T>0$, $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ a filtration on this space satisfying the usual conditions, W an m-dimensional Brownian motion with respect to P and $\mathbb{F}$, and $K_0>0$ the constant in Definition 4.4. Let λ be a strictly positive probability density on $(0,\infty)$ with $\lambda(K_0)<\infty$, and define

$$\zeta(K):=\inf_{h\in[K_0\wedge K,K_0\vee K]}\lambda(h)h,\qquad K\in(0,\infty).$$

Then $\zeta^*:=\sup_{K\in(0,\infty)}\zeta(K)=\lambda(K_0)K_0<\infty$. Let ν be the probability on $(0,\infty)$ corresponding to λ, and set

$$(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathcal{G}},\tilde{P}):=\Big((0,\infty)\times\Omega,\ \big(\big\{\emptyset,(0,\infty)\big\}\otimes\mathcal{F}\big)\vee\widetilde{\mathcal{N}},\ \mathcal{B}(0,\infty)\otimes\mathcal{F},\ \nu\otimes P\Big),$$

where $\widetilde{\mathcal{N}}$ is the family of $(\nu\otimes P)$-zero sets in $\mathcal{B}(0,\infty)\otimes\mathcal{F}$. Also define

$$\widetilde{\mathbb{G}}=(\widetilde{\mathcal{G}}_t)_{t\in[0,T]}\quad\text{with }\ \widetilde{\mathcal{G}}_t:=\big(\mathcal{B}(0,\infty)\otimes\mathcal{F}_t\big)\vee\widetilde{\mathcal{N}},\ t\in[0,T],$$

$$\widetilde{W}=(\widetilde{W}_t)_{t\in[0,T]}\quad\text{with }\ \widetilde{W}_t(k,\omega):=W_t(\omega)\ \forall t\in[0,T],\ (k,\omega)\in\widetilde{\Omega}.$$

It is straightforward to check that $\widetilde{W}$ is a $(\widetilde{\mathbb{G}},\tilde{P})$-Brownian motion on $\widetilde{\Omega}$.

We can now introduce the spaces in which we construct our SDE solutions. The following definition coincides with the corresponding one in [44].

Definition 5.1 For $p\ge 1$ and $d\in\mathbb{N}$, $\mathcal{S}_c^{p,d}$ or shortly $\mathcal{S}_c^p$ is the space of all (equivalence classes of) $\mathbb{R}^d$-valued, $\widetilde{\mathbb{G}}$-adapted, $\tilde{P}$-a.s. continuous processes $X=((X(t))_{0\le t\le T}$ on $\widetilde{\Omega}$ which satisfy

$$\|X\|^p:=\mathbb{E}^{\tilde{P}}\Big[\sup_{0\le t\le T}\big|X(t)\big|^p\Big]=\int_0^\infty\mathbb{E}\Big[\sup_{0\le t\le T}\big|X(t,k)\big|^p\Big]d\nu(k)<\infty;$$

we identify X and X' in $\mathcal{S}_c^p$ if $\|X-X'\|=0$.

The following simple result says that stochastic integrals with respect to $\widetilde{W}$ can be interpreted as stochastic integrals with respect to W in the natural way; it is proved exactly like Proposition 5.1 in [44].

Proposition 5.2 *Let h be a $\widetilde{\mathbb{G}}$-progressively measurable process on $\widetilde{\Omega}$ such that $\int_0^T h_u^2\,du<\infty$ $\tilde{P}$-a.s. Then we have $\int_0^T h_u(k)^2\,du<\infty$ P-a.s. for a.e. $k\in(0,\infty)$, and the stochastic integral $\int h\,d\widetilde{W}$ satisfies*

$$\Big(\int_0^t h_u\,d\widetilde{W}_u\Big)(k)=\Big(\int_0^t h_u(k)\,dW_u\Big)\quad\forall t\ P\text{-a.s. for a.e. }k\in(0,\infty).$$

From now on, we identify $\mathbb{F}$-progressively measurable (or $\mathbb{F}$-adapted) processes h on Ω with $\widetilde{\mathbb{G}}$-progressively measurable (or $\widetilde{\mathbb{G}}$-adapted) processes $\widetilde{h}$ on $\widetilde{\Omega}$ by setting $\widetilde{h}(t,k,\omega) := h(t,\omega)$, and similarly $\mathbb{F}$-stopping times τ on Ω with $\widetilde{\mathbb{G}}$-stopping times $\widetilde{\tau}$ on $\widetilde{\Omega}$ by setting $\widetilde{\tau}(k,\omega) := \tau(\omega)$. In other words, we extend quantities from Ω to $\widetilde{\Omega} = (0,\infty) \times \Omega$ by letting them be constant in the k-argument, $k \in (0,\infty)$. With a slight abuse of notation, we write τ for $\widetilde{\tau}$ and h for $\widetilde{h}$, in particular W for $\widetilde{W}$.

In Sect. 5.2 below, we consider 2-dimensional processes (X,Y) on the space $\widetilde{\Omega}$ such that $X(t,k,\omega)$ represents the local implied volatility at strike k and $Y(t,k,\omega)$ does not depend on k and represents the price level of the underlying option curve at time t when the market is in state $\omega \in \Omega$. Proposition 5.2 then implies that, for a.e. k, $X(\cdot,k)$ can be interpreted as an Itô process on Ω.

Let us conclude this section with some comments on the classical approach to infinite-dimensional SDEs. Instead of constructing the process (X,Y) on the space $\widetilde{\Omega}$ as described above, one could also view X as a process on Ω taking values $X(t,k,\omega)_{k>0}$ in some Hilbert space $\mathcal{H}$ of functions (of k) on $(0,\infty)$, and use the theory of Hilbert space-valued SDEs by Da Prato and Zabczyk [19] to obtain existence results for our models. Since our methodology only allows the construction of local implied volatility processes $X(t,k,\omega)$ which are measurable as a function of k, one advantage of the Hilbert-space approach would be the possibility to obtain regularity properties in k via a suitable choice of $\mathcal{H}$. This has been demonstrated for Heath–Jarrow–Morton interest rate models by Filipović [27] (see Sect. 5.1 there), and for term structures of implied volatilities (the case $\mathcal{K} = \{K\}$, $\mathcal{T} = (0,\infty)$) in a recent paper by Brace et al. [10]. In [27], Filipović uses the existence results from Da Prato and Zabczyk [19, Theorems 6.5 and 7.4] for Hilbert space-valued SDEs with Lipschitz coefficients to obtain the existence of HJM models with a specified volatility structure (Theorem 5.2.1, ii, in [27]); the latter is chosen in such a way that the drift coefficients given by the HJM drift restrictions satisfy *global* Lipschitz conditions. For option market models, such a choice is in general not possible because of the complex structure of the drift restrictions, and we can typically only achieve that the drift coefficients are *locally* Lipschitz and of linear growth in the state variables. A general (global) existence result for this type of SDEs is given in Seidler [42, Theorem 1.5, iii]. In the case of option term structure models, Brace et al. [10] deal with the existence problem by a localization argument (see Lemmas 22–25 there) which again allows them to apply the existence results from [19].

We expect that similar arguments as in [42] and [10] will work in our setting as well. Nevertheless, we choose in this paper the alternative approach described in the beginning of this section, where we can also apply a general existence result (Proposition A.3) for SDEs with locally Lipschitz and linearly growing coefficients. This choice is admittedly somewhat arbitrary, and our main reason for making it is that we have the results in [44] easily and readily at our disposal.

5.2 The existence result

We now provide a class of volatility coefficients $\gamma(t), v(t,K)$ for which there exists an arbitrage-free local implied volatility model (4.16), (4.17). Fix $T > 0$ and let $(b_1(t),\dots,b_m(t))$ be a uniformly bounded $\mathbb{R}^m$-valued $\mathbb{F}$-progressively measurable process on Ω. To make things more transparent, we assume that

$\gamma(t) = (\gamma_1(t), 0, \dots, 0)$ (we may always achieve this without loss of generality by an orthogonal transformation of the vector dW_t). We choose the coefficients $\gamma(t)$, $v(t, K) = (v_1(t, K), \dots, v_m(t, K))$ of the functional form

$$\gamma(t, Y) = \big(1 + (T - t)g\big(t, Y(t)\big), 0, \dots, 0\big), \tag{5.1}$$

$$v_j(t, K, X, Y) = f_j\big(t, K, X(t, K)\big)V_j\left(t, K, \int_{K_0}^{K} \frac{dh}{X(t, h)h}, Y(t)\right)\zeta(K)^2 \tag{5.2}$$

for measurable functions $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, $f_j : [0, T] \times (0, \infty)^2 \to \mathbb{R}$ and $V_j : [0, T] \times (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$. With these functions, we define, as in Theorem 4.12,

$$\beta(t, Y) := \frac{1}{2}\frac{Y(t)}{T - t}\big(\gamma_1(t, Y)^2 - 1\big) - \gamma_1(t, Y)b_1(t), \tag{5.3}$$

$$\begin{aligned} u(t, K, X, Y) := \frac{1}{T - t}\Bigg[&\frac{1}{2} - \frac{1}{2}\left|\gamma(t, Y) + \int_{K_0}^{K} \frac{v(t, h, X, Y)}{X(t, h)h}dh\right|^2 \\ &+ \left(Y(t) - \int_{K_0}^{K} \frac{dh}{X(t, h)h}\right) \\ &\times \left(\gamma(t, Y) + \int_{K_0}^{K} \frac{v(t, h, X, Y)}{X(t, h)h}dh\right) \cdot v(t, K, X, Y)\Bigg] \\ &+ \big|v(t, K, X, Y)\big|^2 - v(t, K, X, Y) \cdot b(t). \end{aligned} \tag{5.4}$$

Let $Y_0 \in \mathbb{R}$ and X_0 be a positive measurable function on $(0, \infty)$ with $\int_{K_0}^{K} \frac{dh}{X_0(h)^2} < \infty$ for all $K > 0$. We take $d = 2$ and consider in $\mathcal{S}_c^{p,2}$ the SDE

$$\left.\begin{aligned} dX(t, K) &= u(t, K, X, Y)X(t, K)\,dt + v(t, K, X, Y)X(t, K)\,dW_t, \\ dY(t) &= \beta(t, Y)\,dt + \gamma(t, Y)\,dW_t \end{aligned}\right\} \tag{5.5}$$

with initial condition $X(0, K) = X_0(K)$, $Y(0) = Y_0$. If we have a (unique) solution (X, Y) to (5.5), then Y does not depend on K.

We can now give sufficient conditions for (5.5) to have a unique solution. Recall the definition of the functions ψ and φ in (3.11). In the following, *const* denotes a generic positive constant whose value can change from line to line.

Theorem 5.3 (a) *Let $p > 2$ be sufficiently large and X_0 such that $\varphi(1/X_0(\cdot)) \in L^p(\nu)$. Suppose that γ, v are of the form (5.1), (5.2), f_j is a.e. differentiable in x, and g, V_j and f_j $(j = 1, \dots, m)$ satisfy the Lipschitz conditions*

$$\big|g(t, y) - g(t, y')\big| \le const\,|y - y'|,$$

$$\big|V_1(t, k, w, y) - V_j(t, k, w', y')\big| \le const\,(T - t)\big(|w - w'| + |y - y'|\big),$$

$$\big|V_j(t, k, w, y) - V_j(t, k, w', y')\big| \le const\,\sqrt{T - t}\big(|w - w'| + |y - y'|\big)$$

$$(j = 2, \dots, m),$$

$$\left|x\,\partial_x f_j(t,k,x)\right| \le const \quad (j=1,\dots,m) \tag{5.6}$$

as well as the bounds

$$\left|g(t,y)\right| \le const, \tag{5.7}$$

$$\left|V_1(t,k,w,y)\right| \le \frac{T-t}{1+|y-w|}, \tag{5.8}$$

$$\left|V_j(t,k,w,y)\right| \le \frac{\sqrt{T-t}}{1+|y-w|} \quad (j=2,\dots,m), \tag{5.9}$$

$$\left|f_j(t,k,x)\right| \le const\big(|x| \wedge 1\big) \quad (j=1,\dots,m) \tag{5.10}$$

for all $t \in [0,T]$, $k>0$, $x \ge 0$, $w, w', y, y' \in \mathbb{R}$. *Then* (5.5) *has a unique solution* $(X,Y) \in \mathcal{S}_c^{p,2}$. *$Y$ does not depend on K, we have $X>0$, and $u(t,K,X,Y)$ and $v(t,K,X,Y)$ are uniformly bounded.*

(b) *Moreover, suppose that there exist constants $K_1 \ge 1+K_0$ and $x_0>0$ such that $X_0(k) \le x_0$ for all $k \ge K_1$ and $f_j(t,k,x)=0$ for all $k \ge K_1$ and $x \ge 0$, $j=1,\dots,m$. Then $\sup_{k \ge K_1} X_t(k)$ is uniformly bounded in ω, t, and so the assumption of Proposition* 4.7 *is satisfied.*

It is straightforward to specify examples of functions g, V_j, f_j satisfying the conditions of Theorem 5.3, and we do this below. The above result therefore provides a fairly large class of examples for stochastic local implied volatility models. Note that this stands in contrast to the models of Sect. 3 parametrized by the classical implied volatility; there, a corresponding existence result for the multi-strike case does not seem to be available so far.

Proof of Theorem 5.3 (a) This proof should be read in conjunction with the Appendix. Let $\varphi_1(z) := \varphi'(\psi(z))\psi(z)$, $\varphi_2(z) := \varphi''(\psi(z))\psi(z)^2$, and recall that φ is the inverse of ψ in (3.11). We want to use the transformation $Z=\varphi(\frac{1}{X})$. We consider in $\mathcal{S}_c^{p,2}$ the SDE

$$\left.\begin{aligned} dZ(t,K) &= \bar{u}(t,K,Y,Z)\,dt + \bar{v}(t,K,Y,Z)\,dW_t,\\ dY(t) &= \beta(t,Y)\,dt + \gamma(t,Y)\,dW_t \end{aligned}\right\} \tag{5.11}$$

with initial condition $Z(0,K)=\varphi(\frac{1}{X_0(K)})$, $Y(0)=Y_0$, where

$$\begin{aligned} \bar{u}(t,K,Y,Z) &:= \varphi_1\big(Z(t,K)\big)\left(\left|v\left(t,K,\frac{1}{\psi(Z)},Y\right)\right|^2 - u\left(t,K,\frac{1}{\psi(Z)},Y\right)\right)\\ &\qquad + \varphi_2\big(Z(t,K)\big)\left|v\left(t,K,\frac{1}{\psi(Z)},Y\right)\right|^2,\\ \bar{v}(t,K,Y,Z) &:= -\varphi_1\big(Z(t,K)\big)v\left(t,K,\frac{1}{\psi(Z)},Y\right). \end{aligned}$$

If we have a unique solution (Y, Z) to (5.11), then Itô's lemma readily yields that $(X, Y) = (\frac{1}{\psi(Z)}, Y)$ is the unique solution to (5.5).

We now want to apply Proposition A.3 to (5.11). It follows easily from (5.1), (5.2) that the coefficients in (5.11) are strongly ($\mathcal{S}_c^{p,2}$-)progressively measurable. To check that they are locally Lipschitz on $\mathcal{S}_c^{p,2}$, we use Proposition A.4. It is proved in Lemma 5.4 below that the functions

$$e_1(t, K, Y, Z) := \varphi_1\big(Z(t, K)\big), \tag{5.12}$$

$$g_{1j}(t, K, Y, Z) := f_j\left(t, K, \frac{1}{\psi(Z(t, K))}\right)\varphi_1\big(Z(t, K)\big), \tag{5.13}$$

$$h_{1j}(t, K, Y, Z) := f_j\big(t, K, \frac{1}{\psi(Z(t, K))}\big)^2\varphi_2\big(Z(t, K)\big) \tag{5.14}$$

$(j = 1, \dots, m)$ satisfy (A.2). Next, we introduce the functions

$$\begin{aligned}
g_2(t, K, Y, Z) &:= g\big(t, Y(t)\big),\\
g_3(t, K, Y, Z) &:= \left(Y(t) - \int_{K_0}^{K} \psi\big(Z(t, h)\big)\frac{dh}{h}\right)\zeta(K),\\
g_{4j}(t, K, Y, Z) &:= V_j\left(t, K, \int_{K_0}^{K} \psi\big(Z(t, h)\big)\frac{dh}{h}, Y(t)\right)\zeta(K),\\
g_{5j}(t, K, Y, Z) &:= \int_{K_0}^{K} v_j\left(t, h, \frac{1}{\psi(Z)}, Y\right)\psi\big(Z(t, h)\big)\frac{dh}{h}
\end{aligned} \tag{5.15}$$

and claim that g_2, g_3, $\frac{1}{T-t}g_{41}$, $\frac{1}{T-t}g_{51}$, $\frac{1}{\sqrt{T-t}}g_{4j}$, and $\frac{1}{\sqrt{T-t}}g_{5j}$ $(j \geq 2)$ satisfy the polynomial Lipschitz condition (A.3). This is easily verified for g_2, g_3, $\frac{1}{T-t}g_{41}$, and $\frac{1}{\sqrt{T-t}}g_{4j}$ by using the definition of ζ and the fact that g, V_j, and ψ are Lipschitz, and it is proved in Lemma 5.5 below for $\frac{1}{T-t}g_{51}$ and $\frac{1}{\sqrt{T-t}}g_{5j}$. Now we have, by (5.3), (5.4), and the definitions of $\bar{u}$ and $\bar{v}$ above,

$$\begin{aligned}
\bar{u} = {}& \sum_{j=1}^{m} g_{1j}g_{4j}\,\zeta(K)b_j\\
& - \frac{1}{T-t}\Bigg[-\frac{1}{2}e_1\left(2(T-t)g_2 + (T-t)^2g_2^2 + 2\big(1 + (T-t)g_2\big)g_{51} + \sum_{j=1}^{m} g_{5j}^2\right)\\
& + \left(g_{11}g_{41}\big(1 + (T-t)g_2\big) + \sum_{j=1}^{m} g_{1j}g_{4j}g_{5j}\right)g_3\Bigg]\\
& + \sum_{j=1}^{m} h_{1j}g_{4j}^2\zeta(K)^2,\\
\beta = {}& \frac{1}{2}Y(t)\big(2g_2 + (T-t)g_2^2\big) - \big(1 + (T-t)g_2\big)b_1,
\end{aligned}$$

$$\bar{v}_j = -g_{1j} g_{4j}\, \zeta(K),$$
$$\gamma_1 = 1 + (T-t) g_2,$$

and so Proposition A.4 yields that the coefficients in (5.11) are locally Lipschitz if p is sufficiently large. Finally, we have to check condition (A.1). It is clearly satisfied for β and γ_1, and it follows easily for $\bar{v}_j$ from (5.2) and (5.8)–(5.10). Using Lemma 5.4 and (5.10), we see that

$$|e_1| + |g_{1j}| + |h_{1j}| \le const\big(1 + Z(t,K)\big),$$

and from (5.2), (5.9), (5.10), and $\frac{\zeta(h)}{h} \le \lambda(h)$ we obtain, for $j = 2, \dots, m$,

$$|g_{5j}| \le \int_{K_0}^{K} const\, \frac{1}{\psi(Z(t,h))} \sqrt{T-t}\, \zeta(h)^2 \psi\big(Z(t,h)\big) \frac{dh}{h} \le const\, \zeta^* \sqrt{T-t}$$

and similarly $|g_{51}| \le const\, \zeta^*(T-t)$. Moreover, using (5.8), (5.9), we get $|g_{41} g_3| \le (\zeta^*)^2 (T-t)$ and $|g_{4j} g_3| \le (\zeta^*)^2 \sqrt{T-t}$. Combining these estimates yields (A.1) for $\bar{u}$. Now Proposition A.3 gives us the existence and uniqueness of the solution (Y, Z).

The boundedness of $v(t,K,X,Y)$ is clear, and that of $u(t,K,X,Y)$ follows by writing $u - |v|^2 + v \cdot b$ in a similar form as $\bar{u}$ and then using (5.7)–(5.10) plus the already established boundedness of $\frac{g_{51}}{T-t}$ and $\frac{g_{5j}}{\sqrt{T-t}}$.

(b) By (5.2) we have $v_j(t,K,X,Y) = 0$ for $K \ge K_1$ and $j = 1, \dots, m$, and this implies, for $K \ge K_1$,

$$u(t,K,X,Y) = \frac{1}{2(T-t)} \Bigg[1 - \bigg| 1 + (T-t) g\big(t, Y(t)\big) + \int_{K_0}^{K_1} \frac{v_j(t,h,X,Y)}{X(t,h)h} dh \bigg|^2 - \sum_{j=2}^{m} \bigg| \int_{K_0}^{K_1} \frac{v_j(t,h,X,Y)}{X(t,h)h} dh \bigg|^2 \Bigg].$$

Now using (5.7) and the bounds obtained in the proof of part (a) for g_{5j} defined in (5.15), we find that $|u(t,K,X,Y)| \le const$ for $K \ge K_1$. The assertion follows from $X_t(K) = X_0(K) \exp(\int_0^t u(s,K,X,Y)\, ds)$ for $K \ge K_1$. □

Lemma 5.4 *The functions e_1, g_{1j}, and h_{1j} defined by (5.12)–(5.14) satisfy (A.2).*

Proof We have

$$\frac{d}{dz}\left[f_j\left(t, K, \frac{1}{\psi(z)}\right) \varphi_1(z) \right] = \partial_x f_j\left(t, K, \frac{1}{\psi(z)}\right) \frac{1}{\psi(z)} \frac{-\psi'(z)}{\psi(z)} \varphi_1(z) + f_j\left(t, K, \frac{1}{\psi(z)}\right) \varphi_1'(z),$$

$$\frac{d}{dz}\left[f_j\left(t,K,\frac{1}{\psi(z)}\right)^2\varphi_2(z)\right]=2f_j\left(t,K,\frac{1}{\psi(z)}\right)\partial_x f_j\left(t,K,\frac{1}{\psi(z)}\right)\times\frac{1}{\psi(z)}\frac{-\psi'(z)}{\psi(z)}\varphi_2(z)+f_j\left(t,K,\frac{1}{\psi(z)}\right)^2\varphi_2'(z).$$

Now the functions $\frac{-\psi'(z)}{\psi(z)}\varphi_1(z)$, $\frac{-\psi'(z)}{\psi(z)}\varphi_2(z)$, $\varphi_1'(z)$, and $\varphi_2'(z)$ are bounded on $\mathbb{R}$ (for $z\notin[-a,a]$, this follows by direct computation and for $z\in[-a,a]$ by continuity). Together with (5.6) and (5.10), it follows that the above derivatives are bounded, and therefore the functions are globally Lipschitz in z. This yields (A.2). □

Lemma 5.5 *The functions* $\frac{1}{T-t}g_{51}$ *and* $\frac{1}{\sqrt{T-t}}g_{5j}$ *defined in* (5.15) *satisfy* (A.3).

Proof Define $g_{6j}(t,K,Y,Z):=f_j(t,K,\frac{1}{\psi(Z(t,K))})\psi(Z(t,K))$. Then one shows in the same way as for g_{1j} in Lemma 5.4 that

$$\left|g_{6j}(t,K,Y,Z)-g_{6j}(t,K,Y',Z')\right|\le const\left|Z(t,K)-Z'(t,K)\right|.$$

Since $g_{5j}(t,K,Y,Z)=\int_{K_0}^K g_{6j}(t,h,Y,Z)g_{4j}(t,h,Y,Z)\zeta(h)\frac{dh}{h}$ and $\frac{\zeta(h)}{h}\le\lambda(h)$, we obtain

$$\begin{aligned}&\left|g_{5j}(t,K,Y,Z)-g_{5j}(t,K,Y',Z')\right|\\&\quad\le\int_{K_0}^K\left|g_{6j}(t,h,Y,Z)-g_{6j}(t,h,Y',Z')\right|\left|g_{4j}(t,h,Y,Z)\right|\zeta(h)\frac{dh}{h}\\&\qquad+\int_{K_0}^K\left|g_{6j}(t,h,Y',Z')\right|\left|g_{4j}(t,h,Y,Z)-g_{4j}(t,h,Y',Z')\right|\zeta(h)\frac{dh}{h}\\&\quad\le\sqrt{T-t}\,\zeta^* const\int_0^\infty\left|Z(t,h)-Z'(t,h)\right|\lambda(h)dh\\&\qquad+\sup_{h\in[K_0,K]}\left|g_{4j}(t,h,Y,Z)-g_{4j}(t,h,Y',Z')\right|\\&\qquad\times\left(const\int_0^\infty\left|Z'(t,h)\right|\lambda(h)\,dh+const\right)\end{aligned}$$

for $j\ge 2$, and, for $j=1$, the same holds with $\sqrt{T-t}$ replaced by $T-t$. Now divide this inequality by $\frac{1}{T-t}$ for $j=1$ and $\frac{1}{\sqrt{T-t}}$ for $j\ge 2$. Since $\frac{1}{T-t}g_{41}$ and $\frac{1}{\sqrt{T-t}}g_{4j}$ satisfy (A.3), the assertion follows. □

Example 5.6 Let $g(y)$ be a bounded and Lipschitz function, $x^*>0$ a constant, and $a_j(K)$ bounded functions satisfying $a_j(K)=O(K^2)$ for $K\to 0$ and $a_j(K)=0$ for all sufficiently large K. Define

$$f_j(t,K,x)=\frac{4a_j(K)}{\left(\frac{K}{K_0}\wedge\frac{K_0}{K}\right)^2}\left(|x|\wedge x^*\right),$$

$$V_j(t, K, w, y) = \frac{(T-t)^{r_j}}{1 + |y - w|},$$

where $r_1 = 1$ and $r_j = \frac{1}{2}$ for $j \geq 2$. Clearly, these functions satisfy the conditions of Theorem 5.3. Now let λ be the probability density function on $(0, \infty)$ given by $\lambda(h) = \frac{1}{2K_0} I_{\{h \leq K_0\}} + \frac{K_0}{2h^2} I_{\{h > K_0\}}$. Then we have $\zeta(K) = \frac{1}{2}(\frac{K}{K_0} \wedge \frac{K_0}{K})$ for $K > 0$. So a possible choice for the volatility coefficients of an arbitrage-free local implied volatility model is given by

$$\begin{aligned}\gamma_1(t, Y) &= 1 + (T-t) g\big(Y(t)\big),\\ v_j(t, K, X, Y) &= a_j(K)\big(X(t, K) \wedge x^*\big) \frac{(T-t)^{r_j}}{1 + \big|Y(t) - \int_{K_0}^{K} \frac{dh}{X(t,h)h}\big|}.\end{aligned}$$

This provides a whole family of stochastic dynamic arbitrage-free market models for option prices in the multi-strike case.

Let us conclude this section with a few comments on the conditions in Theorem 5.3. The special choice of $\gamma_1(t, Y(t))$ is natural since it ensures nonexplosion of the drift β in (5.3) near maturity. The Lipschitz conditions as well as (5.7) and (5.10) are essentially technical conditions which guarantee the local Lipschitz continuity of the coefficients in the abstract existence result in Proposition A.3. The appearance of $Y(t) - \int_{K_0}^{K} \frac{dh}{X(t,h)h}$ in the denominator of V_j in (5.9) is also quite natural: It ensures the linear growth of the drift coefficient uX in (5.5). Finally, v_j is chosen proportional to $\sqrt{T-t}$ (and even to $(T-t)$ for $j = 1$) in order to avoid explosion of u in (5.4) near maturity. It is unclear to us whether or not this asymptotic for $t \nearrow T$ can be relaxed.

6 Comments and conclusion

In this paper, we have studied models for a stock S and a set of European call options with one fixed maturity and all strikes $K > 0$. In the traditional martingale approach, option prices are specified as conditional expectations of the payoff under an equivalent martingale measure for S, and the parameters of the *stock model* are usually calibrated to a set of vanilla option prices expressed in terms of implied volatilities. In contrast, we use here *market models* where the stock and option price processes are constructed simultaneously, so that we have at the same time *joint dynamics* for stock and options and *perfect calibration* to the given set of initial vanilla option prices. But because option prices are no longer automatically conditional expectations, the absence of *dynamic* arbitrage now translates into drift conditions on the modeled quantities to ensure the local martingale property of all the tradables' price processes.

A crucial point here is the choice of a good parametrization; this should be done in such a way that the *static* arbitrage restrictions do not already result in a complicated state space for the quantities describing the model. We have argued in Sect. 3.3 that the classical implied volatilities are ill-suited under that aspect and we have proposed the new concept of *local implied volatilities* in Sect. 4 to overcome this difficulty. These new quantities can indeed be used to construct, and prove the existence of, arbitrage-free multi-strike market models with specified volatilities for option prices, as we have shown in Sects. 4.3 and 5.

The setting of the present paper is that of one fixed maturity and all strikes. The symmetric (but slightly simpler) case of one fixed strike and all maturities has been dealt with in our earlier paper [41]. It is a natural and practically important question how these two approaches can be extended to a situation with both multiple strikes and maturities, so that one could attack the problem of arbitrage-free market modeling for the full option surface. From our experience, we expect that this will need more than just combining the two approaches developed so far; we rather believe that another new parametrization idea will be required. It has been suggested in Derman and Kani [23] to model the option price surface via *local volatilities*, and the corresponding arbitrage conditions have recently been analyzed in a rigorous way by Carmona and Nadtochiy [13]. However, the drift conditions in [13] have a rather implicit form that involves functions given not explicitly, but only as solutions to a PDE. Proving the existence of corresponding market models with a given volatility structure is not addressed in these papers and thus remains a (probably difficult) open question. See, however, the work of Wissel [45] for some recent progress.

Our paper is clearly just a first step, and there are many open and important problems. To name but a few, we could mention

– models with multiple strikes *and* maturities: see the discussion above.
– issues of liquidity: which options exactly should be included as tradables in the modeling?
– calibration and implementation: how to solve our SDE systems numerically?
– fine structure: how about recalibration? Is there some Markovian property? (This is probably an area where the approach via Hilbert-space valued SDEs is better suited.)
– applications: how to use the models for hedging? How about the qualitative behavior of the joint dynamics? How do smiles behave and evolve?

The above list is certainly not exhaustive, and one may well feel (like we do) that we have raised more questions than we have given answers. However, all the above issues can only be addressed legitimately once the fundamental questions of parametrization and existence of models have been answered—and this is what we have done here.

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Appendix

In this appendix, we present some further definitions and results from [44] which are used in the proofs of the existence results in Sect. 5. Recall the definitions of

$\widetilde{\Omega}$, $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{G}}$, $\widetilde{\mathbb{G}}$, $\tilde{P}$, $\widetilde{W}$, and $\mathcal{S}_c^p = \mathcal{S}_c^{p,d}$ from Sect. 5.1. Note that tilde quantities here (like $\widetilde{\Omega}$, etc.) correspond to the analogous quantities (like Ω) without tilde in [44], and quantities here without tilde (like Ω, W, etc.) correspond to the analogous quantities (like Ω^1, W^1 etc.) with superscript ^{1} in [44]. In particular, $\widetilde{\omega} = (k, \omega)$ here corresponds to ω in [44].

We repeat some definitions from [44]. The coefficients of our SDEs lie in the following class.

Definition A.1 Let $n \in \mathbb{N}$. A map $f : [0, T] \times \widetilde{\Omega} \times \{X \mid X\ \widetilde{\mathbb{G}}\text{-adapted process}\} \to \mathbb{R}^n$ is called *strongly* ($\mathcal{S}_c^p$-)*progressively measurable* if, for each $X \in \mathcal{S}_c^p$, the map

$$(t, k, \omega) \mapsto f(t, k, \omega, X)$$

is progressively measurable and satisfies, for all $X \in \mathcal{S}_c^p$ and for each $\widetilde{\mathcal{F}}$-measurable stopping time τ,

$$f(t, \cdot, X) I_{\{t \le \tau(\cdot)\}} = f\big(t, \cdot, X^\tau\big) I_{\{t \le \tau(\cdot)\}} \quad \forall t\ \tilde{P}\text{-a.s.}$$

Note that a quantity on $\widetilde{\Omega}$ does not depend on k iff it is $\widetilde{\mathcal{F}}$-measurable. For a process $X \in \mathcal{S}_c^p$, define the process $q(X)$ by

$$q(X)(t) := \left(\int_0^\infty \sup_{0 \le u \le t} \big| X(u, k, \cdot) \big|^p d\nu(k) \right)^{\frac{1}{p}}, \quad t \in [0, T].$$

It is easy to check that $q(X)$ is $\widetilde{\mathcal{F}}$-measurable and $\widetilde{\mathbb{G}}$-adapted, and dominated convergence yields that it is $\tilde{P}$-a.s. continuous in t since $X \in \mathcal{S}_c^p$. Define, for each $X \in \mathcal{S}_c^p$, the sequence of $[0, T] \cup \{\infty\}$-valued stopping times $\tau_N(X)$, $N \in \mathbb{N}$, by

$$\tau_N(X) := \inf\big\{ t \in [0, T] \bigm| q(X)(t) \ge N \big\}$$

with $\inf \emptyset = \infty$. Note that, as a random variable, $\tau_N(X)$ is $\widetilde{\mathcal{F}}$-measurable.

Definition A.2 A strongly progressively measurable function f is called *locally Lipschitz* (*on* $\mathcal{S}_c^p$) if there exist functions C_N with $C_N(t) \xrightarrow{t \to 0} 0$ such that, for all $t \in [0, T]$ and $X, X' \in \mathcal{S}_c^p$, we have

$$\begin{aligned} &\int_0^\infty \left(\int_0^{t \wedge \tau_N(X) \wedge \tau_N(X')} \big| f(u, k, \cdot, X) - f(u, k, \cdot, X') \big|^2 du \right)^{\frac{p}{2}} d\nu(k) \\ &\quad \le C_N(t) \big(q(X - X') \big(t \wedge \tau_N(X) \wedge \tau_N(X') \big) \big)^p. \end{aligned}$$

Proposition A.3 ([44], Theorem 3.1) *Let $d = 2$, $p > 2$, and $(X_0, Y_0) \in L^p(\nu)$. Suppose that β and γ are strongly progressively measurable and locally Lipschitz on $\mathcal{S}_c^{p,2}$. Suppose that $f \in \{\beta, \gamma\}$ satisfy $|f(u, k, \cdot, (0, 0))| \le const$ as well as the growth condition*

$$\mathbb{E}^{\tilde{P}} \left[\int_0^T \big| f\big(u, k, \cdot, (X, Y)\big) \big|^p du \right] \le const\big(1 + \big\|(X, Y)\big\|^p\big) \tag{A.1}$$

for $(X, Y) \in \mathcal{S}_c^{p,2}$. *Then the SDE system*

$$d(X, Y)(t, k, \cdot) = \beta\big(t, k, \cdot, (X, Y)\big)\, dt + \gamma\big(t, k, \cdot, (X, Y)\big)\, dW_t$$

with $(X, Y)(0, k, \cdot) = (X_0(k), Y_0(k))$ *has a unique solution* $(X, Y) \in \mathcal{S}_c^{p,2}$.

The local Lipschitz condition in Definition A.2 is often difficult to verify. We therefore give a criterion which we apply in Sect. 5.2.

Proposition A.4 *Suppose that* $g_1, \ldots, g_n$ *are strongly progressively measurable functions satisfying* $|g_j(t, k, 0)| \le const$ *for* $j = 1, \ldots, n$ *and, for all* $X, X' \in \mathcal{S}_c^p$,

$$\big|g_1(t, k, X) - g_1(t, k, X')\big| \le const\, \big|X(t, k) - X'(t, k)\big|, \tag{A.2}$$

$$\begin{aligned}\big|g_j(t, k, X) - g_j(t, k, X')\big| \le P_j\bigg(\int_0^\infty \big|X(t, h)\big|\, d\nu(h), \int_0^\infty \big|X'(t, h)\big|\, d\nu(h)\bigg)& \\ \times \int_0^\infty \big|X(t, h) - X'(t, h)\big|\, d\nu(h)&\end{aligned} \tag{A.3}$$

for $j = 2, \ldots, n$, *where* P_j *is a polynomial function of degree* $k_j \ge 0$. *If* $p > 2(1 + 2\sum_{j=2}^n k_j)$, *then the product* $g_1 \cdots g_n$ *is locally Lipschitz* (*on* $\mathcal{S}_c^p$).

Proof We use Proposition 3.3 in [44]. Let $\tau := t \wedge \tau_N(X) \wedge \tau_N(X')$ for $t \in [0, T]$. From (A.2) we have

$$\int_0^\infty \int_0^\tau \big|g_1(u, k, X) - g_1(u, k, X')\big|^p\, du\, d\nu(k) \le T\, const\, \big(q(X - X')(t)\big)^p$$

and so we have (3.3) of [44] for g_1. Let $p_j := \frac{p}{2k_j}$; then $\frac{1}{p} + \sum_{j=2}^n \frac{1}{p_j} < \frac{1}{2}$. To show (3.4) of [44] for g_j ($j \ge 2$), note that by Jensen's inequality for the convex function x^{k_j} and the probability measure ν, we have, for some constant $c > 0$,

$$\begin{aligned} B(u) &:= P_j\bigg(\int_0^\infty \big|X(u, h)\big|\, d\nu(h), \int_0^\infty \big|X'(u, h)\big|\, d\nu(h)\bigg) \\ &\le c\bigg(1 + \int_0^\infty \big|X(u, h)\big|^{k_j}\, d\nu(h) + \int_0^\infty \big|X'(u, h)\big|^{k_j}\, d\nu(h)\bigg), \end{aligned} \tag{A.4}$$

$$\begin{aligned} &\int_0^\infty \int_0^\tau \big|X(u, h) - X'(u, h)\big|^{2p_j}\, du\, d\nu(h) \\ &\quad \le \int_0^\infty \Big(T \sup_{0 \le u \le \tau} \big|X(u, h) - X'(u, h)\big|^{2p_j}\Big) d\nu(h) \\ &\quad \le T\Big(\big(q(X - X')(\tau)\big)^p\Big)^{1/k_j}. \end{aligned} \tag{A.5}$$

We now integrate the p_jth power of (A.3) and use Cauchy–Schwarz, then Jensen and (A.4), and then (A.5) and the definition of τ. This yields

$$\begin{aligned}
&\int_0^\tau \left|g_j(u,k,X) - g_j(u,k,X')\right|^{p_j} du \\
&\le \left(\int_0^\tau \left(\int_0^\infty \left|X(u,h) - X'(u,h)\right| d\nu(h)\right)^{2p_j} du\right)^{\frac{1}{2}} \left(\int_0^\tau B(u)^{2p_j} du\right)^{\frac{1}{2}} \\
&\le \left(\int_0^\tau \int_0^\infty \left|X(u,h) - X'(u,h)\right|^{2p_j} d\nu(h)\, du\right)^{\frac{1}{2}} \\
&\quad \times \left(c^{2p_j} 3^{2p_j-1}\right)^{\frac{1}{2}} \left(T + \int_0^\tau \int_0^\infty \left|X(u,h)\right|^{2p_j k_j} d\nu(h)\, du\right. \\
&\quad \left. + \int_0^\tau \int_0^\infty \left|X'(u,h)\right|^{2p_j k_j} d\nu(h)\, du\right)^{\frac{1}{2}} \\
&\le T^{\frac{1}{2}} \left(q(X - X')(\tau)\right)^{\frac{p}{2k_j}} \left(c^{2p_j} 3^{2p_j-1}\right)^{\frac{1}{2}} \left(T + 2TN^p\right)^{\frac{1}{2}},
\end{aligned}$$

and so we have (3.4) of [44] for g_j. □

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