# Mean-variance hedging and mean-variance portfolio selection

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**Abstract:** Suppose discounted asset prices in a financial market are given by a $P$-semimartingale $S$. Mean-variance hedging is the problem of approximating, with minimal mean squared error, a given payoff by the final value of a self-financing trading strategy. Mean-variance portfolio selection consists of finding a self-financing strategy whose final value has maximal mean and minimal variance. In both cases, this leads to projecting a random variable in $L^2(P)$ onto a space of stochastic integrals of $S$, and apart from proving closedness of that space, the main difficulty is to find more explicit descriptions of the optimal integrand. Both problems have a wide range of applications, and many examples and solution techniques can be found in the literature. Nevertheless, challenging open questions still remain.

**Key words:** hedging, portfolio choice, quadratic criterion, variance-optimal martingale measure, mean-variance tradeoff, linear-quadratic stochastic control, backward stochastic differential equations

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In a nutshell, *mean-variance hedging (MVH)* is the problem of approximating, with minimal mean squared error, a given payoff by the final value of a self-financing trading strategy in a financial market. *Mean-variance portfolio selection (MVPS)*, on the other hand, consists of finding a self-financing strategy whose final value has maximal mean and minimal variance.

More precisely, let $S = (S_t)_{0 \le t \le T}$ be an ($\mathbb{R}^d$-valued) stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and think of $S_t$ as discounted time $t$ prices of $d$ underlying risky assets. Assume $S$ is a semimartingale and denote by $\Theta$ a class of ($\mathbb{R}^d$-valued) predictable $S$-integrable processes $\vartheta = (\vartheta_t)_{0 \le t \le T}$ satisfying suitable technical conditions. Together with an initial capital $x$, each $\vartheta$ describes, via its time $t$ holdings $\vartheta_t$ in $S$, a self-financing strategy whose *value* at time $t$ is given by the stochastic integral (compare [stochastic integration])

$$V_t(x, \vartheta) = x + \int_0^t \vartheta_u \, dS_u =: x + G_t(\vartheta). \tag{1}$$

Mean-variance portfolio selection, for some risk aversion parameter $\gamma > 0$, is then to

$$\text{maximise } E[V_T(x, \vartheta)] - \gamma \mathrm{Var}[V_T(x, \vartheta)] \text{ over all } \vartheta \in \Theta, \tag{2}$$

and mean-variance hedging, for a final time $T$ payoff given by a square-integrable $\mathcal{F}_T$-measurable random variable $H$, is to (compare [hedging, general concepts])

$$\text{minimise } E\left[|V_T(x, \vartheta) - H|^2\right] \text{ over all } \vartheta \in \Theta. \tag{3}$$

By writing the objective of (2) as $m(\vartheta) - \gamma E\left[|V_T(x, \vartheta) - m(\vartheta)|^2\right]$ and adding the constraint $m(\vartheta) := E[V_T(x, \vartheta)] = m$, we can solve (2) by first solving (3) for a constant payoff $H \equiv m$ and then optimising over $m$. So we first focus on mean-variance hedging.

**Remark.** A Google Scholar search quickly reveals that the literature on "mean-variance hedging" and "mean-variance portfolio selection" is vast; it cannot be properly surveyed here. Hence we have chosen references partly for historical interest, partly for novelty and partly for other subjective reasons. Any omissions may be blamed on this and lack of space.

In mathematical terms, MVH as in (3) is simply the problem of finding the best approximation in $L^2 = L^2(P)$ of $H$ by an element of $\mathcal{G} := G_T(\Theta)$. Existence (for arbitrary $H$) is thus tantamount to closedness of $\mathcal{G}$ in $L^2$, which depends on the precise choice of $\Theta$; see [50], [44], [18], [16], [24], [15] for results in that direction. Since the optimal approximand is given by the projection in $L^2$ of $H$ onto $\mathcal{G}$, MVH (without constraints on strategies) has the pleasant feature that its solution is linear as a function of $H$. The main challenge, however, is to find more explicit descriptions of the optimal strategy $\tilde{\vartheta}^H$, i.e. the minimiser for (3). The key difficulty there stems from the fact that $S$ is in general a $P$-semimartingale, but not a $P$-martingale.

**Remark.** If $S$ is a $P$-martingale, MVH of $H$ is solved by projecting the $P$-martingale $V^H$ associated to $H$ onto the stable subspace of all stochastic integrals of $S$, and the optimal strategy is the integrand in the Galtchouk-Kunita-Watanabe decomposition of $V^H$ with respect to $S$ under $P$. This is also the (first component of the) strategy which is *risk-minimising* for $H$ in the sense of [22]; see also [11]. However, in this mathematically classical case, MVH is of minor interest for finance since a martingale stock price process has zero excess return.

Historically, mean-variance portfolio selection is much older than mean-variance hedging. It is traditionally credited to Harry Markowitz (1952), although closely related work by Bruno de Finetti (1940) has been discovered recently; see [2] for an overview, and compare also [Markowitz, Harry; biography], [modern portfolio theory]. For the static one-period case where $G_T(\vartheta) = \vartheta^{\mathrm{tr}}(S_T - S_0)$ and $\vartheta$ is a nonrandom vector, [40] and [41] contain a general formulation and [43] an explicit solution; see also [Markowitz efficient frontier]. A multiperiod treatment, whether in discrete or in continuous time, is considerably more delicate; this was already noticed in [45] and will be explained more carefully a bit later.

Mean-variance hedging in the general formulation (3) seems to have been introduced only around 1990. It first appeared in a specific framework in [49] which generalises a particular example from [21], and was subsequently extended to very general settings; see [47], [55] for surveys of the literature up to around 2000. Most of these papers use martingale techniques, and an important quantity in that context is the *variance-optimal martingale measure* $\tilde{P}$, obtained as the solution to the dual problem of minimising over all (signed) local martingale measures $Q$ for $S$ the $L^2(P)$-norm of the density $\frac{dQ}{dP}$ (compare [equivalent martingale measure and ramifications]). It turns out (see [53]) that if one modifies (3) to

$$\text{minimise } E\left[|V_T(x,\vartheta) - H|^2\right] \text{ over all } x \in I\!R \text{ and } \vartheta \in \Theta,$$

the optimal initial capital is given by $\tilde{x} = E_{\tilde{P}}[H]$, and $\tilde{P}$ also plays a key role in finding the optimal strategy $\tilde{\vartheta}^H$. If $S$ is continuous, then $\tilde{P}$ is equivalent to $P$ (compare [equivalence of probability measures]) so that its density process $Z^{\tilde{P}}$ is strictly positive; see [19]. This can then be exploited to give a more explicit description of $\tilde{\vartheta}^H$, either via an elegant change of numeraire (see [24], and compare also [change of numeraire]) or via a change of measure and a recursive formula (see [48]); see also [1] for an overview of partial extensions to discontinuous settings. For general discontinuous $S$, [15] have shown that the optimal strategy can be found like the *locally risk-minimising* strategy (see [55]), provided that one first makes a change from $P$ to a new (their so-called opportunity-neutral) probability measure $P^*$.

One common feature of all the above results is that they require for a more explicit description of $\tilde{\vartheta}^H$ the density process ($Z^{\tilde{P}}$ or $Z^{P^*}$) of some measure, and that this process is very difficult to find in general. Things become much simpler under the (frequently made but restrictive) assumption that $S$ has a *deterministic mean-variance tradeoff* (also called *non-stochastic opportunity set*), because $\tilde{P}$ then coincides with the *minimal martingale measure*

$\widehat{P}$ (compare [minimal martingale measure]) which can always be written down directly from the semimartingale decomposition of $S$; see [52]. The process $S$ typically has a deterministic mean-variance tradeoff if it has independent returns or is a Lévy process (compare [Lévy processes]); this explains why MVH can be used so easily in such settings.

The original MVH problem (3) is a static problem in the sense that one tries at time 0 to find an optimal strategy for the entire interval $[0, T]$. For an intertemporally dynamic formulation, one would at any time $t$

$$\text{minimise } E\left[|V_T(x,\vartheta) - H|^2 \,\middle|\, \mathcal{F}_t\right] \text{ over all } \vartheta \in \Theta_t(\psi), \tag{4}$$

where $\Theta_t(\psi)$ denotes all strategies $\vartheta \in \Theta$ that agree up to time $t$ with a given $\psi \in \Theta$. In view of (1), one recognises in (4) a *linear-quadratic stochastic control (LQSC)* problem, and this point of view allows to exploit additional theory (compare [stochastic control]) and to obtain in some situations more explicit results about the optimal strategy as well. The idea to tackle MVH via LQ control techniques and backward stochastic differential equations (BSDEs; compare [backward stochastic differential equations]) seems to originate with M. Kohlmann and X. Y. Zhou. Together with various coauthors, they developped this approach through several papers in an Itô diffusion setting for $S$; see [31], [61], [29], [37], and [60] for an overview. A key contribution was made a little earlier in [36] in a discrete-time model by embedding the MVPS problem into a class of auxiliary LQSC problems. Extensions beyond the Brownian setting are given in [39], [10] and [38], among others; approaches in discrete time can be found in [51], [25] or [14].

As already said, MVH is very popular and has been used and studied in many examples and contexts. To name but a few, we mention

- stochastic volatility models ([5], [34]; see also [modelling and measuring volatility]);
- insurance and ALM applications ([17], [20], [57]);
- weather derivatives or electricity loads ([12], [46]; see also [weather derivatives], [commodity risk]);
- uncertain horizon models ([42]);
- insider trading ([6], [13], [30]);
- robustness and model uncertainty ([23], [58]; see also [robust portfolio optimization]);
- default risk and credit derivatives ([4], [8], [28]; see also [credit derivatives]).

Perhaps the main difference between mean-variance hedging and mean-variance portfolio selection is that MVPS is not consistent over time, in the following sense. If, in analogy to (4), we consider for each $t$ the problem to

$$\text{maximise } E[V_T(x,\vartheta) \,|\, \mathcal{F}_t] - \gamma \mathrm{Var}[V_T(x,\vartheta) \,|\, \mathcal{F}_t] \text{ over } \vartheta, \tag{5}$$

this is no longer a standard stochastic control problem because of the variance term. In particular, the crucial *dynamic programming* property fails: If $\vartheta^*$ solves (2) on $[0, T]$ and we consider (5) where we optimise over all $\vartheta \in \Theta_t(\vartheta^*)$, i.e. that agree with $\vartheta^*$ up to time $t$, the solution of this conditional problem will over $[t,T]$ differ from $\vartheta^*$ in general. This makes things surprisingly difficult and explains why MVPS in a general multiperiod setting has still not been solved in a satisfactorily explicit manner.

From the purely geometric structure of the problem, one can derive by elementary arguments the optimal final value

$$G_T(\vartheta^*) = \frac{1}{2\gamma}\left(\alpha - \frac{d\tilde{P}}{dP}\right) \tag{6}$$

with $\alpha = E\left[\left(\frac{d\tilde{P}}{dP}\right)^2\right]$; this can be seen from [53], [54] or also be found in [59]. However, (6) mainly shows that finding the optimal strategy $\vartheta^*$ is inextricably linked to a precise knowledge of the variance-optimal martingale measure $\tilde{P}$ which is very difficult to obtain in general. For the case of a *deterministic mean-variance tradeoff* (*non-stochastic opportunity set*), we have already seen that $\tilde{P}$ equals the minimal martingale measure $\widehat{P}$ so that (6) readily gives the solution to the MVPS problem (2) in explicit form. This includes for instance the results by [36] in finite discrete time or by [61] who used BSDE techniques in continuous time. Other work in various settings includes [7], [35] and [56].

One major area of recent developments in MVPS is the inclusion of constraints; see for instance [32], [9], [26], [27], [33]. Another challenging open problem is to find a time-consistent formulation; see [3] for a first attempt.

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