From structural assumptions to a link between assets and interest rates

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Abstract: We derive a link between assets and interest rates in a standard multi-asset diffusion economy from two structural assumptions — one on the volatility and one on the short rate function. Our main result is economically intuitive and testable from data since it only involves empirically observable quantities. A preliminary study illustrates how this could be done.

Key words: assets, interest rates, testable relation, asset index, volatility structure, short rate, homogeneous

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0 Introduction

This paper derives a natural link between assets and interest rates in a broad class of models. Starting from a completely standard multi-asset framework with Markovian diffusion asset prices S, we impose two assumptions of a *structural* nature and show how they combine to yield a close relationship between the forward rate curve and a suitable asset index. All quantities that appear are empirically observable so that our result is testable from available data.

The paper is structured as follows. Section 1 sets up the framework and defines a general asset index as the value process of a self-financing portfolio with unit initial capital. The first structural assumption is imposed on the volatility σ and amounts to the condition that asset prices relative to a suitable (*spherical*) asset index should fluctuate in a (*rigid*) Black-Scholes type fashion. The second structural assumption says that the short rate function $r(\cdot)$ is *homogeneous* of degree 0; this is a mathematical formulation for a natural scale-invariance postulate. Section 2 derives, from this homogeneity alone, a general relation between the drift and volatility of the short rate. Combining the two assumptions yields in Section 3 our main result. For the special case where the short rate volatility is constant, we obtain in particular a distinction into two basic regimes: The instantaneous correlation between the spherical index I and the short rate $r(S)$ is positive or negative, depending on whether the forward rate curve at the short end is downward or upward sloping. A preliminary study in Section 4 illustrates how one could test such results on the basis of empirical data. Section 5 concludes.

1 Basic setup, asset indices, and volatility structures

This section provides the basic framework for our approach. Our ultimate goal is to derive links between the dynamics of assets and interest rates in a general class of models, and we want to achieve this by imposing a pair of simple assumptions. These are of a *structural* nature and in particular aim at obtaining results on quantities which are *empirically observable*. This crucial aspect will come up again at several stages.

We start on a probability space $(\Omega, \mathcal{F}, P)$ with a vector S of n processes $S^1, \dots, S^n$ over a finite time horizon $[0,T]$. These processes model the evolution of our basic *asset prices* and constitute the fundamental given ingredient. We assume in most of the paper that their dynamics are given by the (Markovian) SDEs

$$(1.1) \qquad \frac{dS_t^i}{S_t^i} = dR_t^i = \mu^i(S_t)\,dt + \sum_{j=1}^m \sigma^{ij}(S_t)\,dW_t^j, \qquad S_0^i = x_0^i > 0, \quad i = 1, \dots, n$$

for nice enough real functions $\mu^i(\cdot)$, $\sigma^{ij}(\cdot)$ on $I\!R^n_{++}$. In this section, we could also allow as coefficients general predictable processes μ_t^i, σ_t^{ij}, and so we use here this more compact notation.

To exclude local redundancies between assets, we suppose that

$$(1.2) \qquad \sigma_t = \sigma(S_t) \text{ has full rank} \qquad P\text{-a.s. for every } t \in [0,T].$$

As usual, we write

$$\mu_t^i = r_t + (\sigma\lambda)_t^i = r_t + \sum_{j=1}^m \sigma_t^{ij}\lambda_t^j, \qquad i = 1, \dots, n, \tag{1.3}$$

where $r_t = r(S_t)$ is the *short rate* and $\lambda_t = \lambda(S_t)$ the vector of *market prices of risk* for the assets in S. We later also consider the assumption

$$\mathbf{1} \notin \operatorname{range}(\sigma_t) \qquad P\text{-a.s. for every } t \in [0, T], \tag{1.4}$$

where $\mathbf{1} := (1 \dots 1)^\top \in I\!\!R^n$; this implies in combination with (1.2) that r and λ in (1.3) are unique. Finally, we recall that the pricing kernel of our economy is $1/N$, where N is given by

$$\frac{dN_t}{N_t} = \left(r_t + |\lambda_t|^2\right) dt + \lambda_t^\top dW_t. \tag{1.5}$$

We emphasize that this entire setup is completely standard; see for instance Chapter 1 in [KS] or Chapter 7 in [HK].

Trading in S by self-financing strategies is modelled by pairs (v_0, π), where $v_0 \in (0, \infty)$ is the initial capital and the $I\!\!R^n$-valued process $\pi = (\pi_t)_{0 \le t \le T}$ describes the *fractions* of total wealth held over time in the available assets. More precisely, the (positive) wealth at time t is

$$V_t(v_0, \pi) = v_0\, \mathcal{E}\left(\int \pi^\top dR\right)_t = v_0 \exp\left(\int_0^t \pi_s^\top \sigma_s\, dW_s + \int_0^t \left(\pi_s^\top \mu_s - \tfrac{1}{2}|\pi_s^\top \sigma_s|^2\right) ds\right),$$

and the fraction π_t^i of $V_t(v_0, \pi)$ is currently invested in asset i. Fractions can be negative (we do not exclude short sales), but must sum to 1 so that we have the restriction $\pi_t^\top \mathbf{1} \equiv 1$. This is again the standard setup as in [KS] or [HK].

A general *asset index* or *numeraire* is the value process $I^\pi := V(1, \pi)$ of a strategy $(1, \pi)$ with one unit of initial capital. Our first structural condition will be an assumption on the volatility matrix σ of S which will allow us to construct a particular index with good properties. Before embarking on that, however, we note that the dynamics of any index I^π are

$$\frac{dI_t^\pi}{I_t^\pi} = \pi_t^\top dR_t = \bar{\mu}_t(\pi)\, dt + \bar{\sigma}_t(\pi)^\top dW_t$$

with $\bar{\mu}(\pi) = \pi^\top \mu$ and the $I\!\!R^m$-valued process $\bar{\sigma}(\pi) = \sigma^\top \pi$. If we rewrite this as

$$\frac{dI_t^\pi}{I_t^\pi} = \sum_{i=1}^n \pi_t^i \frac{dS_t^i}{S_t^i}, \tag{1.6}$$

we see that I^π is directly observable from S if the strategy π is, and that the (instantaneous) return of I^π is a generalized convex combination of the returns of the S^i. Because S and I^π are both stochastic exponentials, the I^π-discounted assets $\tilde{S}(\pi) := S/I^\pi$ follow the SDEs

$$\frac{d\tilde{S}_t^i(\pi)}{\tilde{S}_t^i(\pi)} = \tilde{\mu}_t^i(\pi)\, dt + \tilde{\sigma}_t^i(\pi)^\top dW_t$$

with

$$\tilde{\sigma}^{ij}(\pi) := \sigma^{ij} - \bar{\sigma}^j(\pi) \qquad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m$$

and $\tilde{\mu}^i(\pi) := \mu^i - \bar{\mu}(\pi) - \bar{\sigma}(\pi)^\top \tilde{\sigma}^i(\pi)$. Intuitively, $\tilde{S}^i(\pi)$ describes the multiplicative fluctuations of asset S^i around the index I^π, and $\tilde{\sigma}(\pi)$ is the matrix of *intrinsic volatilities* with respect to I^π. Like the exchange prices in [P00], the $\tilde{S}^i(\pi)$ are ratios of two Itô processes and thus have a specific drift and volatility structure.

A first well-known choice of a particular index is the *numeraire portfolio* $N^* = I^{\pi^*}$, defined by the property that all N^*-discounted prices $\tilde{S}^i(\pi^*) = S^i/N^*$ become local martingales under the original measure P. Thanks to (1.2) and (1.3), N^* exists, and it is also known to have good properties like for instance growth-optimality; see [Be] for a theoretical treatment and [P04, P05] for application-oriented aspects. However, N^* has from our perspective the drawback that it is not genuinely observable; its construction requires the knowledge of the asset drifts μ^i which (in contrast to the volatility matrix σ) cannot be recovered from a single trajectory $S_{\boldsymbol{\cdot}}(\omega)$ of asset price observations. This motivates our introduction and study of a different class of indices.

Definition. The volatility structure σ of S is called *spherical* if there exists an index I^π such that all n corresponding intrinsic volatility vectors $\tilde{\sigma}^i(\pi) = \sigma^i - \bar{\sigma}(\pi)$ are of equal magnitude, i.e., $|\sigma^i_t - \bar{\sigma}_t(\pi)| = \text{const.}(t, \omega)$ P-a.s. for every $t \in [0, T]$ and $i = 1, \dots, n$. Any such index I^π is called a *spherical index* (for the volatility structure σ).

Put differently, σ is spherical if all its volatility vectors σ^i lie on a sphere whose center has the form $\bar{\sigma}(\pi) = \sigma^\top \pi$ for some π with $\pi^\top \mathbf{1} \equiv 1$. This formulation is more convenient to work with and equivalent to the above definition. In fact, due to (1.1) and (1.3), the drift $\bar{\mu}(\pi)$ of any spherical index I^π is uniquely determined from $\bar{\sigma}(\pi)$, since (1.3) gives $\mu = r\mathbf{1} + \sigma\lambda$ and so

$$\bar{\mu}(\pi) = \mu^\top \pi = r\pi^\top \mathbf{1} + \lambda^\top \sigma^\top \pi = r + \lambda^\top \bar{\sigma}(\pi).$$

Hence a spherical index I^π is unique as soon as its volatility vector $\bar{\sigma}(\pi)$ is unique.

The importance of a spherical index I^π is that it gives a numeraire in which relative asset prices $\tilde{S}(\pi) = S/I^\pi$ have a simple volatility structure: the intrinsic volatilities $\tilde{\sigma}^i(\pi) = \sigma^i - \bar{\sigma}(\pi)$ always lie on a sphere. Note that like $\pi = \big(\pi_t(\omega)\big)$, this sphere can be random and time-dependent. Things become even simpler if $\tilde{\sigma}^i(\pi)$ does not depend on ω and t, because we then have a multidimensional Black-Scholes type fluctuation around the reference level I^π.

Our first result shows that existence of a spherical index is not a restrictive condition. In fact, if the number n of assets is fixed, we can always ensure existence of a spherical index by increasing the number m of driving factors. This is especially useful if n is small, e.g., if we have a situation with 3 or 4 representative assets that each summarize one market segment.

Proposition 1 *Assume (1.1) – (1.3). Then a spherical index is always unique. Recall that $m = \dim W$, $n = \dim S$ and assume in addition*

1) if $m > n - 1$: nothing extra.

2) if $m = n-1$: that (1.4) holds.

3) if $m < n-1$: that σ is spherical.

Then there exists a (unique) spherical index.

Proof. a) Existence follows if σ is spherical. This is true in case 3) by assumption and clear in case 1) because due to (1.2), the n points $\sigma^1, \dots, \sigma^n$ in $\mathbb{R}^m$ always lie on a sphere in $\mathbb{R}^m$ if $m \geq n$. For case 2), we have to use (1.4). Generally, $n = m+1$ vectors in $\mathbb{R}^m$ lie on a sphere if and only if they are not all in some hyperplane of dimension $\leq m-1$. But the latter cannot happen for $\sigma^1, \dots, \sigma^n$, because (1.4) excludes the case where the vectors lie in a hyperplane not containing the origin and (1.2) the case of a hyperplane through the origin.

b) If we have two spherical indices I^{π_1}, I^{π_2}, then $|\sigma^i - \bar{\sigma}(\pi_k)| = |\sigma^1 - \bar{\sigma}(\pi_k)|$ for all i and thus

$$2(\sigma^i - \sigma^1)^\top \bar{\sigma}(\pi_k) = |\sigma^i|^2 - |\sigma^1|^2 \qquad \text{for } i = 2, \dots, n \text{ and } k = 1, 2. \tag{1.7}$$

Hence $\bar{\sigma}(\pi_1) - \bar{\sigma}(\pi_2)$ is orthogonal to $\sigma^i - \sigma^1$ for $i = 2, \dots, n$. But we also know that $\bar{\sigma}(\pi_k) = \sigma^\top \pi_k$ and $\pi_k^\top \mathbf{1} \equiv 1$ and therefore

$$\bar{\sigma}(\pi_k) = (\sigma^\top - \sigma^1 \mathbf{1}^\top)\, \pi_k + \sigma^1 \mathbf{1}^\top \pi_k = \sigma^1 + (\sigma - \sigma^1 \mathbf{1})^\top \pi_k.$$

Hence $\bar{\sigma}(\pi_1) - \bar{\sigma}(\pi_2) = (\sigma - \sigma^1 \mathbf{1})^\top (\pi_1 - \pi_2)$ is also in the span of the vectors $\sigma^i - \sigma^1$, $i = 2, \dots, n$, and so we must have $\bar{\sigma}(\pi_1) - \bar{\sigma}(\pi_2) = 0$. Uniqueness follows because $\bar{\sigma}(\pi)$ determines $\bar{\mu}(\pi)$. ∎

To construct a spherical index I^π, we need a generating strategy π. Although π need not be unique, I^π always is so that the choice of π does not matter. But later, we need some π explicitly to generate I^π from S via (1.6). Due to (1.7), π is a solution of the equations

$$|\sigma^i|^2 - |\sigma^1|^2 = 2\pi^\top \sigma(\sigma^i - \sigma^1) = 2 \sum_{\ell=1}^n \pi^\ell \big((\sigma^\ell)^\top \sigma^i - (\sigma^\ell)^\top \sigma^1 \big) \qquad \text{for } i = 2, \dots, n \tag{1.8}$$

with the constraint that $\pi^\top \mathbf{1} \equiv 1$. Note that (1.8) only involves quantities that are ω-wise computable from the asset price data $S_\cdot(\omega)$ since we need the volatilities $|\sigma_t^i|^2$ (only the lengths, not the entire vectors) and the instantaneous return covariances

$$(\sigma_t^\ell)^\top \sigma_t^i = \frac{d}{dt} \left\langle \int \frac{dS^\ell}{S^\ell}, \int \frac{dS^i}{S^i} \right\rangle_t . \tag{1.9}$$

As these can all be estimated from asset price data, I^π is always empirically observable from S. Section 4 explains in more detail in an example how this estimation can be done.

While assuming the existence of a spherical index is not restrictive, the next notion is a bit more special. To motivate the underlying idea, consider generalizing the multidimensional Black-Scholes model which is obtained from (1.1) under the assumption of *constant* drift μ and volatility σ. The idea is to relax this by assuming only that *relative/intrinsic* values $\tilde{S}^i$ behave in a Black-Scholes type fashion. Hence not the absolute fluctuations of S, but the relative fluctuations of $\tilde{S}$ around an index I^π are assumed to have constant volatility. We shall see below that this is less dependent on the choice of I^π than appears at first sight.

Let us now make these ideas more precise.

Definition. A volatility structure σ is called *rigid* if there exists an $I\!R^m$-valued predictable process σ_0 such that $\sigma^i - \sigma_0$ is constant in ω, t for $i = 1, \dots, n$.

Lemma 2 *A volatility structure σ is rigid if and only if for all constant vectors $b \in I\!R^n$ with $b^\top \mathbf{1} = 1$, the difference $\sigma^i - \sigma_{(b)}$ is constant in ω, t for $i = 1, \dots, n$, where $\sigma_{(b)} := \sum_{\ell=1}^n b^\ell \sigma^\ell = \sigma^\top b$.*

Proof. For any $b \in I\!R^n$ with $b^\top \mathbf{1} = 1$ and any process σ_0, we have

$$\sigma^i - \sigma_{(b)} = \sigma^i - \sigma_0 + \sigma_0 - \sigma_{(b)} = \sigma^i - \sigma_0 - \sum_{\ell=1}^n b^\ell (\sigma^\ell - \sigma_0).$$

This shows the "only if" part, and the "if" part is obvious if we take as σ_0 any $\sigma_{(b)}$. ■

An immediate consequence of Lemma 2 is the promised assertion that rigidity does not depend very much on the choice of the index. More precisely, we have

Corollary 3 *If the relative prices S^i/S^k have constant volatility vectors for $i = 1, \dots, n$ and for at least one asset S^k, then all relative price processes S^i/S^j have constant volatility vectors for $i, j = 1, \dots, n$, and then σ is rigid. Hence:*

1) *A rigid volatility structure may be viewed as a multivariate Black-Scholes volatility structure for relative prices.*

2) *The structural property of being rigid does not depend on the choice of discounting index I^b as long as b is constant.*

Theorem 4 *Assume (1.1) – (1.3). If σ is rigid and spherical, the unique spherical index can be generated by a constant strategy $\pi \in I\!R^n$ with $\pi^\top \mathbf{1} = 1$. The corresponding relative prices $\tilde{S}^i(\pi) = S^i/I^\pi$ then have constant volatility vectors $\tilde{\sigma}^i(\pi) = \sigma^i - \bar{\sigma}(\pi)$ whose length is the same for all i.*

Proof. Because σ is spherical, the sphere center $\bar{\sigma}$ exists, satisfies $|\sigma^i - \bar{\sigma}| = |\sigma^1 - \bar{\sigma}|$ for $i = 2, \dots, n$ and may be written as $\bar{\sigma} = \sum_{\ell=1}^n \pi^\ell \sigma^\ell$ with a process π satisfying $\pi^\top \mathbf{1} \equiv 1$. Hence

$$0 = |\sigma^i - \bar{\sigma}|^2 - |\sigma^1 - \bar{\sigma}|^2 = (\sigma^i + \sigma^1 - 2\bar{\sigma})^\top (\sigma^i - \sigma^1), \qquad i = 2, \dots, n$$

and so

$$\sum_{\ell=1}^n \pi^\ell (\sigma^i - \sigma^\ell + \sigma^1 - \sigma^\ell)^\top (\sigma^i - \sigma^1) = 0, \qquad i = 2, \dots, n.$$

This is a system of linear equations for π whose coefficients are constant in ω, t because of rigidity and Lemma 2. Hence the solution π (which exists since σ is spherical) may also taken to be constant. The rest follows from Corollary 3. ∎

Our first structural assumption is that σ is rigid and spherical. Thanks to Theorem 4, this later gives major simplifications in a number of formulae by exploiting the fact, both that $|\tilde{\sigma}^i(\pi)|$ does not depend on i (σ is spherical), and that the strategy π generating the spherical index I^π is constant (σ is rigid).

2 Homogeneity: Motivation and consequences

This section introduces our second basic assumption. Before giving a mathematical formulation, we motivate the underlying economic idea by the following postulate of *scale-invariance*: *"The real state of an economy is not affected by a simultaneous scaling of prices for tradable goods. Put differently, real or intrinsic values of all products in an economy are determined only by the totality of relative prices of goods."* A similar idea appears in Section 7.3.1 of [HK] who say that the so-called natural filtration generated by all relative asset prices is arguably the most fundamental one since it contains only information intrinsic to the economy.

Our economy is modelled by the diffusion (1.1) with (inverse) pricing kernel (1.5). In this setup, we formalize the above economic principle mathematically by the assumption that

$$r(\cdot) \text{ is homogenous of degree } 0, \tag{2.1}$$

i.e., $r(\gamma x) = r(x)$ for all $\gamma > 0$ and $x \in \mathbb{R}^n_{++}$. To explain why this indeed captures scale-invariance, let us first consider a complete market where all prices and in particular the term structure of interest rates are determined by S. Then the short rate r and the market price of risk λ are both uniquely given in terms of μ and σ via (1.3); see Corollary 4 of [RSS] for details. To exploit now scale-invariance, note that a simultaneous scaling of all asset prices should not affect the return dynamics R^i since all prices are relative. Hence μ and σ should be homogeneous of degree 0, and this entails the same for r and λ. For a second, alternative motivation, we could start with the dynamics (1.5) of the (inverse) pricing kernel N. If we now scale all prices by a factor $\gamma > 0$, the new relevant asset prices are γS instead of S, and these (instead of the obsolete values S) should be plugged as arguments into $r(\cdot)$ and $\lambda(\cdot)$. But since scaling prices does not change anything economically, the resulting dynamics of N should remain unchanged, and thus the coefficients r (and λ) should be homogeneous of degree 0.

Remark. The idea of scale-invariance also appears in [HN1, HN2], and even the condition that μ and σ should be homogeneous of degree 0 can be found there. The main thrust of [HN1, HN2] is that "any payoff function should be representable by a homogeneous function of degree one in tradables", and this is then exploited to give alternative derivations for a number of well-known option pricing results. The same idea of using homogeneity already appears earlier in [Ja] who even proves that homogeneous payoffs can always be hedged under more general conditions. Our thrust here goes in a different direction, and in contrast to [HN1, HN2], we clearly distinguish between economic intuition and mathematical derivation. $\diamond$

After σ being rigid and spherical, homogeneity of r is our second structural assumption. We combine the two conditions in the next section to obtain our main result, but focus first on the consequences we can obtain from (2.1) alone. We start with a simple analytic lemma.

Lemma 5 *If $h : \mathbb{R}^n_{++} \to \mathbb{R}$ is C^2 and homogeneous of degree 0, then*

$$\sum_{i=1}^n x^i \frac{\partial h}{\partial x^i} \equiv 0, \qquad \frac{\partial h}{\partial x^k} + \sum_{i=1}^n x^i \frac{\partial^2 h}{\partial x^i \partial x^k} \equiv 0, \qquad \sum_{i,k=1}^n x^i x^k \frac{\partial^2 h}{\partial x^i \partial x^k} \equiv 0.$$

Proof. By homogeneity, $\gamma \mapsto h(\gamma x)$ is constant on $(0,\infty)$ for each $x \in \mathbb{R}^n_{++}$. Differentiate to get the first result, differentiate that with respect to x^k to get the second one, and multiply by x^k and sum over k to get the third result by using the first one. $\blacksquare$

If Q is a risk-neutral measure for the assets S and the bank account $B := \exp\big(\int r(S_u)\,du\big)$, then S/B is a local Q-martingale and $\hat{W} = W + \int \lambda(S_u)\,du$ is a standard Brownian motion under Q. Combining this with (1.3), Itô's formula and the first property in Lemma 5 yields

$$\begin{aligned} dr(S_t) &= \sum_{i=1}^n \frac{\partial r}{\partial x^i} \mu^i S^i_t\,dt + \frac{1}{2}\sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} S^i_t S^k_t (\sigma\sigma^\top)^{ik}\,dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} S^i_t (\sigma\,dW_t)^i \\ &= \frac{1}{2}\sum_{i,k=1}^n \frac{\partial^2 r}{\partial x^i \partial x^k} S^i_t S^k_t (\sigma\sigma^\top)^{ik}\,dt + \sum_{i=1}^n \frac{\partial r}{\partial x^i} S^i_t (\sigma\,d\hat{W}_t)^i \\ &= \hat{c}(S_t)\,dt + b(S_t)^\top d\hat{W}_t \qquad (2.2) \\ &= c(S_t)\,dt + b(S_t)^\top dW_t \end{aligned}$$

for the dynamics of $r(S)$, where we have dropped the argument S_t in most functions and set

$$b(x) := \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x)\sigma^i(x), \tag{2.3}$$

$$\hat{c}(x) := c(x) - (b^\top\lambda)(x) := \frac{1}{2}\sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(x)(\sigma\sigma^\top)^{ik}(x). \tag{2.4}$$

We next fix an auxiliary function $\sigma_{\text{ref}} : \mathbb{R}^n \to \mathbb{R}^m$ and set $\tilde{\sigma}^{ij} := \sigma^{ij} - \sigma^j_{\text{ref}}$ for $i = 1,\dots,n$ and $j = 1,\dots,m$. Then we plug $\sigma^i = \tilde{\sigma}^i + \sigma_{\text{ref}}$ into (2.3) and use Lemma 5 and the $\mathbb{R}^m$-valued functions $\tilde{\sigma}^i := (\tilde{\sigma}^{ij})_{j=1,\dots,m}$ to obtain

$$b(x) = \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}(x)\tilde{\sigma}^i(x). \tag{2.5}$$

Differentiating with respect to x^k, multiplying by $x^k(\tilde{\sigma}^k)^\top$ and summing over k gives

$$\sum_{k=1}^n x^k(\tilde{\sigma}^k)^\top \frac{\partial b}{\partial x^k} = \sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(\tilde{\sigma}\tilde{\sigma}^\top)^{ik} + \sum_{i=1}^n x^i \frac{\partial r}{\partial x^i}|\tilde{\sigma}^i|^2 + \sum_{i,k=1}^n x^i x^k \frac{\partial r}{\partial x^i}(\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k}. \tag{2.6}$$

On the other hand, we can also plug $\sigma^i = \tilde{\sigma}^i + \sigma_{\mathrm{ref}}$ into (2.4) and use the first property in Lemma 5, then the second one and then (2.5) to get

$$\hat{c} = c - b^\top \lambda = -b^\top \sigma_{\mathrm{ref}} + \frac{1}{2}\sum_{i,k=1}^n x^i x^k \frac{\partial^2 r}{\partial x^i \partial x^k}(\tilde{\sigma}\tilde{\sigma}^\top)^{ik}. \tag{2.7}$$

Finally, replacing the second derivatives in (2.7) via (2.6) and using the first property in Lemma 5 leads to

Proposition 6 *Assume (1.1) – (1.3). If r satisfies (2.1) and is in C^2, then (dropping all arguments S_t) the coefficients $\hat{c}$, c and b in the short rate dynamics (2.2) are related by*

$$\begin{aligned} \hat{c} &= c - b^\top \lambda \\ &= -b^\top \sigma_{\mathrm{ref}} + \frac{1}{2}\sum_{i=1}^n S_t^i (\tilde{\sigma}^i)^\top \frac{\partial b}{\partial x^i} - \frac{1}{2}\sum_{i,k=1}^n S_t^i S_t^k \frac{\partial r}{\partial x^i}(\tilde{\sigma}^k)^\top \frac{\partial \tilde{\sigma}^i}{\partial x^k} - \frac{1}{2}\sum_{i=1}^n S_t^i \frac{\partial r}{\partial x^i}\left(|\tilde{\sigma}^i|^2 - \tilde{\sigma}^2_{\mathrm{av}}\right), \end{aligned} \tag{2.8}$$

where σ_{ref} is any reference function and $\tilde{\sigma}^2_{\mathrm{av}} := \frac{1}{n}\sum_{i=1}^n |\tilde{\sigma}^i|^2 = \frac{1}{n}\sum_{i=1}^n |\sigma^i - \sigma_{\mathrm{ref}}|^2$.

To clarify matters, let us emphasize that Proposition 6 is not about the construction of possible short rate models. We have started in Section 1 from a standard Markovian diffusion framework for asset prices and have then derived a *general result* on the structure of the associated short rate dynamics. This derivation uses solely the structural assumption that the short rate function $r(\cdot)$ is (C^2 and) homogeneous of degree 0.

3 The main result: A link between index, short rate, and slope of the yield curve

In this section, we combine our two structural assumptions on r and σ to derive our main result. The plan for this is as follows. Homogeneity of $r(\cdot)$ on its own gives in Proposition 6 for the coefficients in the short rate dynamics (2.2) a relation which depends on an arbitrary reference function σ_{ref}. If now in addition σ is rigid and spherical, a good choice of σ_{ref} considerably simplifies that relation; see Proposition 7 below. In a second step, this is then transformed into a result with a clear economic interpretation.

We start with the first step. Throughout this section, we assume that $r(\cdot)$ is like $\mu(\cdot)$ and $\sigma(\cdot)$ sufficiently nice. If σ is spherical with spherical index I^π, (1.8) makes it clear that $\pi_t(\omega)$

is like $\sigma\big(S_t(\omega)\big)$ a function of $S_t(\omega)$. The same is then true for the process $\bar\sigma(\pi) = \sigma^\top\pi$, and we write $\bar\sigma_\pi(\cdot)$ for the corresponding function $\sigma^\top(\cdot)\,\pi(\cdot)$. So if we choose for σ_{ref} the volatility function $\bar\sigma_\pi$ of I^π, the vectors $\tilde\sigma^i_\pi = \sigma^i - \bar\sigma_\pi$ all have the same length because σ is spherical; hence we get $\tilde\sigma^2_{\mathrm{av}} = |\tilde\sigma^i|^2$ for all i, and the last term in (2.8) vanishes. If σ is also rigid, $\pi(\cdot)$ can by Theorem 4 be chosen constant in x, and $\tilde\sigma^i_\pi$ then also becomes constant in x due to Lemma 2. So the double sum in (2.8) vanishes as well and we get (dropping all arguments S_t)

Proposition 7 *Assume (1.1) – (1.3) and (2.1). If σ is rigid and spherical with spherical index I^π, the coefficients $\hat c$, c and b in the short rate dynamics (2.2) are related by*

$$\hat c = -b^\top\bar\sigma_\pi + \frac12\sum_{i=1}^n S^i_t(\tilde\sigma^i_\pi)^\top\frac{\partial b}{\partial x^i} = -\frac{d}{dt}\left\langle r(S), \int\frac{dI^\pi}{I^\pi}\right\rangle_t + \frac12\sum_{i=1}^n S^i_t(\tilde\sigma^i_\pi)^\top\frac{\partial b}{\partial x^i}. \tag{3.1}$$

Proposition 7 is only an auxiliary intermediate result. It relates the risk-neutral drift $\hat c(S)$ of the short rate $r(S)$ to the spherical index I^π, the constant intrinsic volatility vectors $\tilde\sigma^i_\pi$, the assets S and the volatility vector $b(S)$ of the short rate. To transform this into our main result, we first recall the well-known fact that in any term structure model with nice coefficients, the risk-neutral short rate drift is equal to the slope of the forward rate curve at the short end, i.e.,

$$\hat c_{t_0} = \frac{\partial}{\partial T}f_{t_0,T}\bigg|_{T=t_0}. \tag{3.2}$$

A proof is given in Appendix A. In addition, we need a minor extra assumption because the vector $b(S)$ of instantaneous volatilities is (as a vector) not observable. Hence we assume that

$$b(x) = b_0\big(r(x)\big) \qquad \text{for some nice function } b_0 : I\!R \to I\!R^n. \tag{3.3}$$

One possible justification is that this goes towards a Markovian short rate model which is a popular assumption in the literature. We provide below an alternative characterization of (3.3) showing that this is a condition on the structure of the homogeneous function $r(\cdot)$.

Remark. Models of rigid and spherical Markovian asset markets do exist, even if we add the condition (3.3). See Appendix B. $\diamond$

Theorem 8 *Assume (1.1) – (1.3), (2.1) and that σ is rigid and spherical with spherical index I^π. If (3.3) holds and the term structure has nice coefficients, then*

$$\begin{aligned}\frac{\partial}{\partial T}f_{t_0,T}\bigg|_{T=t_0} &= -b_0\big(r(S_{t_0})\big)^\top\bar\sigma_\pi(S_{t_0}) + \frac12\big(|b_0||b_0|'\big)\big(r(S_{t_0})\big)\\ &= -\frac{d}{dt}\left\langle r(S), \int\frac{dI^\pi}{I^\pi}\right\rangle_t\bigg|_{t=t_0} + \frac12\big(|b_0||b_0|'\big)\big(r(S_{t_0})\big) \qquad \textit{for each } t_0\in[0,T].\end{aligned} \tag{3.4}$$

Proof. Thanks to our preparations, this is straightforward. Since (3.3) and (2.5) give $\sum_{i=1}^n x^i\tilde\sigma^i_\pi\frac{\partial b}{\partial x^i}(x) = \Big(b_0'\big(r(x)\big)\Big)^\top b_0\big(r(x)\big)$ and because $(b_0')^\top b_0 = \frac12\frac{d}{dr}b_0^\top b_0 = \frac12\frac{d}{dr}|b_0|^2 = |b_0||b_0|'$, the assertion follows from (3.1) and (3.2). ∎

Theorem 8 is our main result on the link between interest rates and the spherical index I^π under our two structural assumptions. An economic interpretation is as follows. Think of a fixed short rate level and consider the effect of a change in expected interest rates in the near future. Because the last term in (3.4) is then constant, we see that *higher expectations about future interest rates* (in the form of an increased slope of the initial forward rate curve) *go with a decrease of correlation between the short rate and the spherical index, and vice versa.*

The central relation (3.4) is (almost) observable and *testable* in the sense that (almost) all its ingredients can be computed ω-wise from available data. For the left-hand side, we only need the initial forward rate curve $T \mapsto f_{t_0,T}$ near t_0. The first term on the right-hand side can be written as (dropping the argument S_{t_0})

$$-(b_0 \circ r)^\top \bar{\sigma}_\pi = -|b_0 \circ r|\, |\bar{\sigma}_\pi|\, \rho_{I^\pi,r},$$

where

$$\rho_{I^\pi,r} := \frac{(b_0 \circ r)^\top \bar{\sigma}_\pi}{|b_0 \circ r|\, |\bar{\sigma}_\pi|} = \frac{\frac{d}{dt}\left\langle r(S), \int \frac{dI^\pi}{I^\pi}\right\rangle_t}{\sqrt{\left(\frac{d}{dt}\langle r(S)\rangle_t\right)\left(\frac{d}{dt}\left\langle \int \frac{dI^\pi}{I^\pi}\right\rangle_t\right)}}\Bigg|_{t=t_0} \tag{3.5}$$

is the instantaneous correlation at time t_0 between the short rate $r(S)$ and the return of the spherical index I^π. Since both I^π and $r(S)$ are observable, so are $\rho_{I^\pi,r}$ and the volatilities $|b_0 \circ r|$ of r and $|\bar{\sigma}_\pi|$ of I^π. The final term in (3.4) becomes observable under an auxiliary parametric assumption on b_0; for instance, we could try $|b_0(r)| = \beta r^{1/2}$ if we believe in a CIR-like model. Alternatively, we could use (3.4) to estimate the parameters in a specific model for b_0.

The simplest case of (3.3) occurs if $b(x)$ is a constant vector b_*; this corresponds to a "semi-Vasiček" type model for the short rate with constant volatility $|b_*|$. The last term in (3.4) then vanishes and we are left with the simplified relation

$$\frac{\partial}{\partial T} f_{t_0,T}\bigg|_{T=t_0} = -\frac{d}{dt}\left\langle r(S), \int \frac{dI^\pi}{I^\pi}\right\rangle_t\bigg|_{t=t_0}. \tag{3.6}$$

This also has a very appealing and plausible economic interpretation: *If the forward rate curve is upward (downward) sloping at the short end, the short rate is negatively (positively) correlated with the spherical index I^π.* Section 4 shows some results from a simple empirical study of (3.6).

We conclude this section with the promised characterization of condition (3.3). The key point is that this can also be viewed as a structural assumption on the short rate function $r(\cdot)$, in line with our overall approach.

Proposition 9 *If σ is rigid and spherical, a sufficient condition for*

$$b(x) = b_0\big(r(x)\big) \neq 0 \qquad \text{for all } x \in \mathbb{R}^n_{++} \tag{3.7}$$

is that

$$r(x) = \varphi\big(J_a(x)\big) \qquad \text{for all } x \in \mathbb{R}^n_{++} \tag{3.8}$$

for some a and some strictly monotone C^1 function $\varphi : [0,\infty) \to I\!R$, where $J_a : I\!R^n_{++} \to I\!R$ is the homogeneous function

$$J_a(x) := \prod_{i=1}^{n} (x^i)^{a^i} \qquad \textit{with } a \in I\!R^n \setminus \{0\} \textit{ satisfying } \sum_{i=1}^{n} a^i = 0.$$

Conversely, (3.8) is also necessary for (3.7) if we have (1.2) and either $m \geq n$ or the combination of $m = n-1$ with (1.4) in the form

$$\mathbf{1} \notin \mathrm{range}\big(\sigma(x)\big) \qquad \textit{for every } x \in I\!R^n_{++}. \tag{3.9}$$

Proof. The sufficiency part is easy. Denote by φ^{inv} the inverse function of φ, differentiate (3.8) and use (2.5) to get

$$b(x) = J_a(x)\varphi'\big(J_a(x)\big)\sum_{i=1}^{n} a^i\tilde{\sigma}^i_\pi = \varphi^{\mathrm{inv}}\big(r(x)\big)\varphi'\Big(\varphi^{\mathrm{inv}}\big(r(x)\big)\Big)\sum_{i=1}^{n} a^i\tilde{\sigma}^i_\pi =: b_0\big(r(x)\big). \tag{3.10}$$

Note that $\bar{v}_a := \sum_{i=1}^{n} a^i\tilde{\sigma}^i_\pi$ is a constant vector since σ is rigid and spherical. The necessity part is more involved; its proof can be found in Appendix C. ■

The proof of Proposition 9 shows in particular that $b(x)$ is a scalar $r(x)$-dependent multiple of a constant vector $\bar{v}_a$ and gives with (3.10) an expression for the function b_0 in terms of φ. Simple examples are $\varphi(z) = \log z$ which leads to $b_0(r) = b_*$ for a constant vector b_*, or $\varphi(z) = (\log z)^{\frac{1}{1-\beta}}$ which gives $b_0(r) = Cr^\beta$ with another constant vector C. Hence our setup contains in particular a rich class of term structure models, and these have a Markovian short rate as soon as (3.7) holds and the projection of $\bar{\sigma}_\pi(x)$ on $\bar{v}_a$ is a function of $r(x)$ only; see (3.4).

Remark. It may seem strange that we impose conditions on the short rate r without actually postulating or studying a model for r. To explain this, let us recall that our basic ingredients are really the drift μ and volatility σ of our asset prices S. In view of (1.3), it then remains to specify either r or the market price of risk λ, and assumptions on r should thus rather be viewed as implicit assumptions on the basic objects μ, σ and λ. ◇

4 A first empirical study

This section presents a first empirical study for the simplified relation (3.6) deduced from Theorem 8. We indicate how one could test (3.6) on the basis of available market data. But this is only preliminary work, and a detailed statistical analysis still remains to be done.

4.1 Estimation of the required quantities

Our data come from the Euro asset market, but we do not consider all stocks. We take the German index DAX, the French index CAC and the Dutch index AEX as representative price processes and regard these three stock indices as our basic assets. We have daily closing price data for them and also data of riskless Euro yields for 1, 2 and 3 months time to maturity.

In order to test (3.6), we construct for the above data the spherical index, obtain the correlation between this index and the short rate, and determine the initial slope of the forward yield curve. Since Theorem 8 is a local result, we carry out our estimations over rather short time periods of four months. We outline the procedure in three steps below.

Step 1: Estimation of the spherical index. We first estimate the covariance matrix of S. We observe the asset prices S_k^j, $j = 1, \dots, n$ (with $n = 3$ in our example) at $N+1$ (daily) dates T_k, $k = 1, \dots, N+1$. In view of (1.9), we estimate the local return covariance rates by

$$\widehat{(\sigma^\ell)^\top \sigma^i} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{T_{k+1} - T_k} \frac{S_{k+1}^\ell - S_k^\ell}{S_k^\ell} \frac{S_{k+1}^i - S_k^i}{S_k^i}. \tag{4.1}$$

We solve the system (1.8) of linear equations by replacing $(\sigma^\ell)^\top \sigma^i$ with the estimates from (4.1) and obtain the estimated weights $\hat{\pi}^\ell$ of the spherical index. The estimate for the squared volatility norm of the spherical index is then computed via

$$\widehat{|\bar{\sigma}_\pi|}^2 = \sum_{i,\ell=1}^{n} \hat{\pi}^i \hat{\pi}^\ell \widehat{(\sigma^\ell)^\top \sigma^i}.$$

After computing the index weight estimates, we construct from the asset returns via (1.6) a time series of estimates $\widehat{I_k^\pi}$ of the spherical index . The result is shown in Figure 3 below.

Step 2: Covariance and correlation between short rate and spherical index. In view of the available data, we use the 1 month spot yield as approximation for the short rate. Denote by (r_k) a time series of this process observed at the dates T_k and let b_* be the (constant) volatility vector of the short rate. The local covariance between the short rate and the spherical index I^π is next estimated by

$$\widehat{b_*^\top \bar{\sigma}_\pi} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{T_{k+1} - T_k} (r_{k+1} - r_k) \frac{\widehat{I_{k+1}^\pi} - \widehat{I_k^\pi}}{\widehat{I_k^\pi}}.$$

After computing an estimate for the squared short rate volatility norm by

$$\widehat{|b_*|}^2 = \frac{1}{N} \sum_{k=1}^{N} \frac{(r_{k+1} - r_k)^2}{T_{k+1} - T_k},$$

we may estimate the instantaneous correlation between I^π and r due to (3.5) by

$$\widehat{\rho}_{I^\pi, r} = \frac{\widehat{b_*^\top \bar{\sigma}_\pi}}{\widehat{|b_*|}\, \widehat{|\bar{\sigma}_\pi|}}.$$

Step 3: Estimation of the initial forward yield slope. To estimate at date t_0 the slope of the instantaneous forward yield curve at the short end, we use riskless continuously compounded bond yields $R_{t_0,T}$. These are connected with the zero coupon bond prices $B_{t_0,T}$ via

$$R_{t_0,T} = -\frac{\log B_{t_0,T}}{T - t_0}.$$

We first estimate the value of $\lim_{T \searrow t_0} \frac{\partial R_{t_0,T}}{\partial T}$ by the slope of the regression line through the three points given by the 1, 2 and 3 month yields; standard calculus then gives

$$\left.\frac{\partial f_{t_0,T}}{\partial T}\right|_{T=t_0} = -\lim_{T \searrow t_0} \frac{\partial^2 \log B_{t_0,T}}{\partial T^2} = 2 \lim_{T \searrow t_0} \frac{\partial R_{t_0,T}}{\partial T}.$$

Remark. As already mentioned, (3.6) is a local result that involves instantaneous volatilities and correlations. Steps 1 and 2 use time averages from time series to generate estimates for these local quantities, and this works well only if the time series under consideration come from stationary processes. The same comment applies to Step 3 where we estimate the desired slope by averaging (over all observation dates) the slopes obtained at each date. Once the stationarity assumption is not met, the reliability of such crude estimates is drastically reduced and more sophisticated methods are called for. In the same vein, one might also look for and use other observable financial products for obtaining estimates of the quantities we need. ◇

4.2 Empirical results

On the basis of the previous description, we performed an analysis of the data in the periods Jan – Apr 2001 and Jan – Apr 2002. During the period in 2001, the yield curve at the short end was downward sloping; Figures 1 and 2 show the yield curves of the first and the last day of this period.

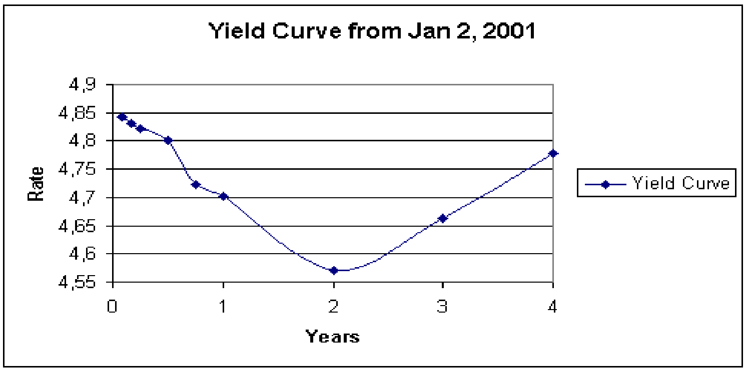

Figure 1: Euro yield curve on Jan 2, 2001. Estimated initial slope is ≈ -0.0012.

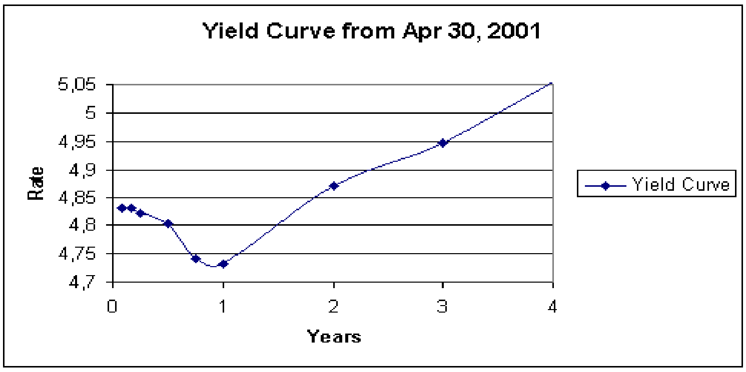

Figure 2: Euro yield curve on Apr 30, 2001. Estimated initial slope is ≈ -0.0006.

From Step 1, we obtain estimates for the instantaneous covariances of the three assets, the spherical index weights and the spherical index volatility. The assets and the corresponding spherical index estimates, both scaled by their initial values, are shown in Figure 3.

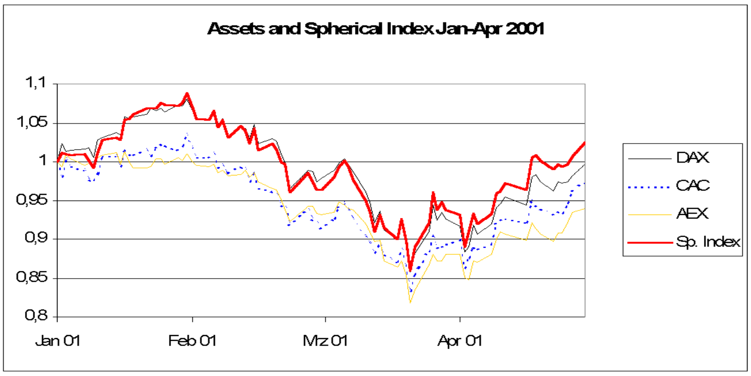

Figure 3: Estimated weights of spherical index are $\hat{\pi}_{\text{DAX}} = 0.8908$, $\hat{\pi}_{\text{CAC}} = 0.6554$ and $\hat{\pi}_{\text{AEX}} = -0.5462$.

For the period Jan – Apr 2001 we then obtain the following estimates:

Short rate volatility $\widehat{\lvert b_* \rvert}$:	0.0061
Spherical index volatility $\widehat{\lvert \bar{\sigma}_\pi \rvert}$:	0.3097
Correlation $\widehat{\rho}_{I^\pi, r}$ between index and short rate:	0.2169
Average initial slope of yield curve $T \mapsto R_{t_0,T}$:	-0.0040
Average initial slope of forward rate curve $T \mapsto f_{t_0,T}$: $2 \times -0.0040 =$	-0.0080

According to (3.6), the product of the first three numbers, $0.0061 \times 0.3097 \times 0.2169 \approx 0.0004$, should be equal to minus the last one, 0.0080. So we might conclude that with respect to sign and order of magnitude, the empirical results for this example are roughly consistent with (3.6). But we repeat that the main point of this section is to illustrate the basic approach, and a proper test of (3.6) by appropriate econometric methods is left for future research.

During the period in 2002, the situation on the interest rate market was quite different from 2001. As we see in Figures 4 and 5, the initial slope of the yield curve was changing from negative to positive. This indicates that an assumption of stationarity is here probably violated. Indeed, a similar test of (3.6) in this case did not yield conclusive results.

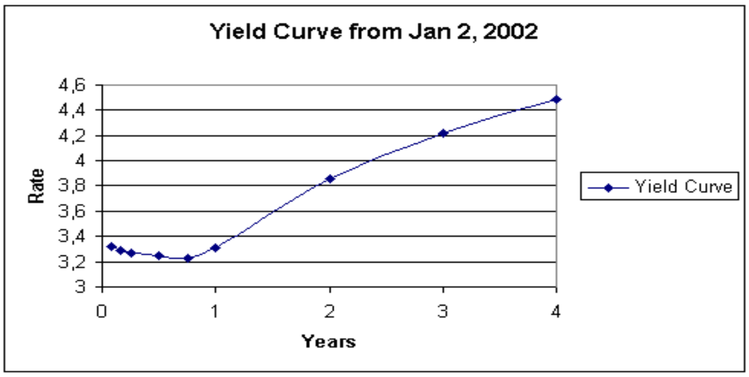

Figure 4: Euro yield curve on Jan 2, 2002. Estimated initial slope is ≈ -0.0025.

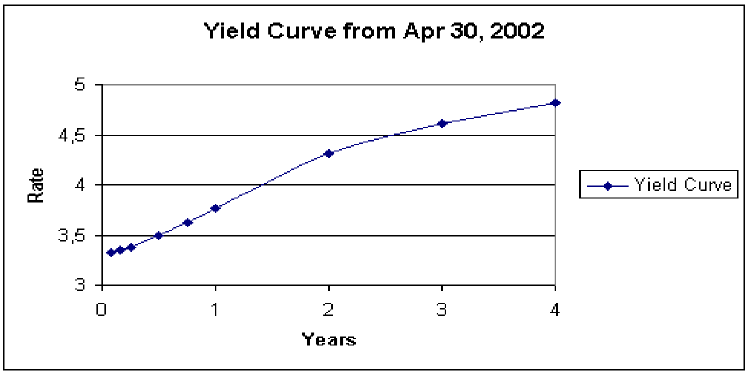

Figure 5: Euro yield curve on Apr 30, 2002. Estimated initial slope is $\approx +0.0029$.

5 Conclusion

We have presented an economically intuitive and empirically testable link between assets and interest rates in a general Markovian diffusion framework. This is derived from two structural assumptions on the coefficients of the model: The volatility structure is rigid and spherical, and the short rate function is homogeneous of degree 0. These mathematical assumptions are motivated via a scale invariance postulate in an intrinsic Black-Scholes economy. A preliminary empirical analysis has also indicated how our result could be tested.

Appendix A: Proof of (3.2)

In this appendix, we prove the representation

$$\hat{c}_{t_0} = \frac{\partial}{\partial T} f_{t_0,T}\Big|_{T=t_0} \tag{3.2}$$

for the risk-neutral short rate drift $\hat{c}$. We start with the risk-neutral forward rate dynamics

$$df_{t,T} = \gamma_{t,T}\, dt + \delta_{t,T}^{\top}\, d\hat{W}_t$$

to obtain for $T = t > t_0$

$$r_t = f_{t,t} = f_{t_0,t} + \int_{t_0}^{t} \gamma_{u,t}\, du + \int_{t_0}^{t} \delta_{u,t}^{\top}\, d\hat{W}_u.$$

Writing a dot $\dot{}$ for partial derivatives with respect to the second argument, we get

$$dr_t = \dot{f}_{t_0,t}\, dt + \gamma_{t,t}\, dt + \Big(\int_{t_0}^{t} \dot{\gamma}_{u,t}\, du\Big)\, dt + \delta_{t,t}^{\top}\, d\hat{W}_t + \Big(\int_{t_0}^{t} \dot{\delta}_{u,t}^{\top}\, d\hat{W}_u\Big)\, dt \tag{A.1}$$

because all quantities are sufficiently nice. Now the HJM drift condition (see for instance [Bj], Proposition 23.2) says that

$$\gamma_{u,t} = \delta_{u,t}^{\top} \int_{u}^{t} \delta_{u,s}\, ds$$

since γ is the risk-neutral forward rate drift. Plugging this into (A.1) yields

$$dr_t = \dot{f}_{t_0,t}\, dt + \Big(\int_{t_0}^{t} \dot{\delta}_{u,t}^{\top} \int_{u}^{t} \delta_{u,s}\, ds\, du + \int_{t_0}^{t} |\delta_{u,t}|^2\, du + \int_{t_0}^{t} \dot{\delta}_{u,t}^{\top}\, d\hat{W}_u\Big)\, dt + \delta_{t,t}^{\top}\, d\hat{W}_t.$$

Letting $t \searrow t_0$, we obtain (3.2). ■

Appendix B: Existence of models

We show here that many rigid and spherical asset markets with a short rate volatility of the form $b(x) = b_0\big(r(x)\big)$ exist. To that end, we first choose a homogeneous, rigid and spherical asset volatility structure with (1.2) and set $r(x) := \varphi\big(J_a(x)\big)$ as in (3.8). Then $r(x)$ is also homogeneous, and Proposition 9 shows that we have $b(x) = b_0\big(r(x)\big)$. If we then choose some homogeneous function $\lambda(x)$, we obtain a homogeneous $\mu(x)$ from the drift condition (1.3). This construction also illustrates that we usually have enough freedom in the choice of our parameters to produce a model with (for instance) a desired short rate process as output. ■

Appendix C: Proof of Proposition 9

In this appendix, we prove that (3.8) is necessary for (3.7).

Step 1: For each $x \in I\!R^n_{++}$, $x^j \frac{\partial r}{\partial x^j}(x)$ is uniquely determined by $b(x) = b_0\big(r(x)\big)$ so that

$$x^j \frac{\partial r}{\partial x^j}(x) = f_j\big(r(x)\big) \tag{C.1}$$

for functions $f_1, \ldots, f_n : I\!R \to I\!R$. To see this, suppose that we have two representations

$$b(x) = \sum_{i=1}^{n} x^i \frac{\partial r}{\partial x^i}(x)\sigma^i(x) = \sum_{i=1}^{n} x^i \frac{\partial \bar{r}}{\partial x^i}(x)\sigma^i(x)$$

of the form (2.3) with homogeneous functions r and $\bar{r}$. Then we get for $h := r - \bar{r}$ that

$$\sum_{i=1}^{n} x^i \frac{\partial h}{\partial x^i}(x)\sigma^i(x) = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} x^i \frac{\partial h}{\partial x^i}(x) = 0, \tag{C.2}$$

the latter by Lemma 5. Now if $m \geq n$, the full rank condition (1.2) yields $\mathrm{range}\big(\sigma(x)\big) = I\!R^n$ and therefore $x^i \frac{\partial h}{\partial x^i}(x) = 0$ for $i = 1, \ldots, n$ due to (C.2). If $m = n - 1$, combining (1.2) with (3.9) implies that $\mathrm{span}\Big(\mathbf{1}, \mathrm{range}\big(\sigma(x)\big)\Big) = I\!R^n$ and we see again from (C.2) that $x^i \frac{\partial h}{\partial x^i}(x) = 0$. This proves the assertion.

Step 2: Because of (C.1) and

$$\sum_{i=1}^{n} x^i \frac{\partial r}{\partial x^i}(x)\tilde{\sigma}^i_\pi = b(x) = b_0\big(r(x)\big) \neq 0,$$

there exists for any $x_0 \in I\!R^n_{++}$ an open ball U_0 around x_0 and an index j_0 such that

$$x^{j_0} \frac{\partial r}{\partial x^{j_0}}(x) = f_{j_0}\big(r(x)\big) \neq 0 \qquad \text{for all } x \in U_0.$$

This implies the existence of constants a^i with $a^{j_0} = 1$ and $\sum\limits_{i=1}^{n} a^i = 0$ such that

$$f_i \circ r = a^i f_{j_0} \circ r \qquad \text{on } U_0 \text{ for all } i. \tag{C.3}$$

To prove this, we may assume that $j_0 = 1$ so that $f_1 \circ r \neq 0$ on U_0. For $i \neq j$, (C.1) implies

$$\frac{\partial^2 r}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^i} \frac{f_j}{x^j} = \frac{1}{x^j} f_j' \frac{\partial r}{\partial x^i} = \frac{f_j' f_i}{x^i x^j} = \frac{f_i' f_j}{x^i x^j},$$

hence

$$f_i'\big(r(x)\big) = \frac{f_1'\big(r(x)\big)}{f_1\big(r(x)\big)} f_i\big(r(x)\big) \qquad \text{on } U_0.$$

This ODE has for each i the unique solution $f_i\big(r(x)\big) = a^i f_1\big(r(x)\big)$ for a constant a^i, and homogeneity of r enforces by Lemma 5 that $\sum_{i=1}^n a^i = 0$. This proves (C.3).

Step 3: A priori, (C.3) holds only on U_0, and the constants a^i could depend on x_0. We claim that the a^i are global constants and that (C.3) holds on all of $I\!R^n_{++}$. Since $I\!R^n_{++}$ is σ-compact, both assertions follow once we prove the following result: If we have two representations

$$f_i \circ r = a^i_\ell \, f_{j_\ell} \circ r \quad \text{with } f_{j_\ell} \circ r \neq 0 \qquad \text{on } U_\ell, \text{ for } \ell = 1, 2$$

with open balls U_1, U_2 such that $U_1 \cap U_2 \neq \emptyset$, then

$$f_i \circ r = a^i_1 \, f_{j_1} \circ r = a^i_2 \, f_{j_2} \circ r \qquad \text{holds on } U_1 \cup U_2. \tag{C.4}$$

To see this, use the shorthand $g := f \circ r$ and note that we have $g_i = a^i_1 \, g_{j_1} = a^i_2 \, g_{j_2}$ on $U_1 \cap U_2$; hence $a^{j_2}_1 g_{j_1} = g_{j_2} \neq 0$ on $U_1 \cap U_2$, so $a^{j_2}_1 \neq 0$, and then it follows that $a^i_1 \, g_{j_1} = a^i_2 \, a^{j_2}_1 g_{j_1}$ on $U_1 \cap U_2$ so that $a^i_1 = a^i_2 \, a^{j_2}_1$. Using this, we have on U_1 that

$$g_i = a^i_1 \, g_{j_1} = a^i_2 \, a^{j_2}_1 g_{j_1} = a^i_2 \, g_{j_2},$$

and so (C.4) follows.

Step 4: Now define the function $\tilde{r} : I\!R^n_{++} \to I\!R$ by $\tilde{r}(y) := r(\exp[y]) = r(x)$ with $x = \exp[y] := \big(\exp(y_1), \ldots, \exp(y_n)\big)^\top$. Then we have from (C.3)

$$\frac{\partial \tilde{r}}{\partial y^j}(y) = x^j \frac{\partial r}{\partial x^j}(x) = a^j f_{j_0}\big(r(x)\big) = a^j f_{j_0}\big(\tilde{r}(y)\big). \tag{C.5}$$

Choose $H : I\!R \to I\!R$ with $H' = 1/f_{j_0}$ and integrate (C.5) with respect to y^j for a fixed j to get

$$H\big(\tilde{r}(y)\big) = C_j(y) + a^j y^j, \tag{C.6}$$

where $C_j(y)$ does not depend on y^j. Now differentiate (C.6) with respect to y^i for $i \neq j$ and use (C.5) to get $\frac{\partial C_j(y)}{\partial y^i} = a^i$. This yields $C_j(y) = a^i y^i + C_{i,j}(y)$, where $C_{i,j}(y)$ now depends neither on y^i nor on y^j, and

$$H\big(\tilde{r}(y)\big) = C_{i,j}(y) + a^i y^i + a^j y^j.$$

Iterating this argument finally gives

$$H\big(\tilde{r}(y)\big) = C + \sum_{i=1}^n a^i y^i = C + \sum_{i=1}^n a^i \log x^i = C + \log J_a(x)$$

with a constant C, and inverting H yields

$$r(x) = \tilde{r}(y) = H^{\text{inv}}\big(C + \log J_a(x)\big)$$

which is of the form (3.8). ∎

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