# Implied Savings Accounts are Unique

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**Abstract:** An implied savings account for a given term structure model is a strictly positive predictable increasing process $A$ such that zero coupon bond prices are given by $B(t,T) = E^Q\left[\frac{A_t}{A_T}\middle|\mathcal{F}_t\right]$ for some $Q$ equivalent to the original probability measure. We prove that if $(A', Q')$ is another pair with the same properties, then $A$ and $A'$ are indistinguishable. This extends a result given by Musiela/ Rutkowski (1997a) who considered the case of a Brownian filtration, and fills a gap in their arguments.

**Key words:** term structure models, implied savings account, Doob-Meyer decomposition, semimartingales, multiplicative decomposition

**1991 Mathematics Subject Classification:** 60G07, 90A09

**JEL Classification Numbers:** G10

*(Finance & Stochastics 4 (2000), 431–442)*

*This version: 12.11.1999*

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## 0. Introduction

This paper resolves a problem raised by the theory of interest rates for general term structure models. Fix a time horizon $T'$ and denote for each $T \le T'$ by $B(t,T)$ the price at time $t \le T$ of a zero coupon bond with maturity $T$. Following Musiela/Rutkowski (1997a,b), we call a strictly positive predictable process $A$ of finite variation an *implied savings account with respect to* $Q$ for the term structure model $B$ if we have

$$B(t,T) = E^Q\left[\frac{A_t}{A_T}\bigg|\mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T \le T' \tag{0.1}$$

for some $Q$ equivalent to the original probability measure $P$. If $A$ is absolutely continuous and of the form $A = \exp\left(\int r_u\,du\right)$, we recover the familiar formula

$$B(t,T) = E^Q\left[\exp\left(-\int_t^T r_u\,du\right)\bigg|\mathcal{F}_t\right]$$

so that an implied savings account is a generalization of the classical savings account to a situation where there is possibly no short rate $r$. This happens for instance in equilibrium models with agents having finite marginal utility from consumption at the origin; see Karatzas/Lehoczky/Shreve (1991).

The problem we solve here is the uniqueness question for an implied savings account: Can there be another pair $(A', Q')$ generating $B$ in the sense that we also have

$$B(t,T) = E^{Q'}\left[\frac{A'_t}{A'_T}\bigg|\mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T \le T' \ ? \tag{0.2}$$

This question first appeared in Rutkowski (1996) in the particular context of a HJM model. We prove here that the answer quite generally is negative: If either $A, A'$ are of class (D) under $Q, Q'$ respectively or if (0.1) and (0.2) hold with stopping times $\sigma \le \tau$ replacing the deterministic times $t \le T$, then $A$ and $A'$ are indistinguishable. This was already asserted in Musiela/Rutkowski (1997a,b), but only for the case where the underlying filtration is generated by a Brownian motion. Moreover, the arguments given in these references are not clear and even contain gaps in some places. We provide here a complete rigorous proof that works in full generality.

For the special case where $Q$ and $Q'$ coincide, our two main theorems reduce to the well-known uniqueness results for the multiplicative decomposition of supermartingales and for the Doob-Meyer decomposition. In the multiplicative case, this was already pointed out in Musiela/Rutkowski (1997a). It may be interesting to note that our method of proof takes up an old argument due to Rao (1969).

The paper is structured as follows. Section 1 contains the main results. We present both the above multiplicative and a similar additive uniqueness theorem and explain how these can be viewed as generalizing the uniqueness of the Doob-Meyer decomposition. Section 2 gives an approximation result for processes of finite variation that is of independent interest. Sections 3 and 4 contain the proofs of the additive and multiplicative uniqueness theorems and section 5 concludes with some comments.

## 1. Main results

Let $\big(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0 \le t \le T'}\big)$ be a filtered probability space satisfying the usual conditions and $T' \in (0, \infty]$ a fixed time horizon. If $Q$ is any probability measure, we denote by $\mathcal{M}^{+}_{1,\mathrm{loc}}(Q)$ the set of strictly positive local $Q$-martingales $M$ with $M_0 = 1$ and by $\mathcal{A}^{+}_{1}$ the set of strictly positive predictable RCLL processes $A$ of finite variation with $A_0 = 1$. For other notations from the general theory of processes, see Dellacherie/Meyer (1982) or Protter (1990).

Let us first recall a well-known result. If $A$ is an integrable increasing predictable process, $A$ generates a nonnegative supermartingale $Z$ by

$$Z_t := E\left[A_{T'} - A_t | \mathcal{F}_t\right] \qquad , \qquad 0 \le t \le T'.$$

If $A$ has no jump at $T'$, the nonnegative supermartingale $Z$ is even a potential on $[0, T']$, i.e., $\lim\limits_{t \nearrow T'} E[Z_t] = 0$. In this case, $Z$ is called the *potential generated by* $A$. Conversely, the potential $Z$ determines $A$ in the sense that if both $A$ and $A'$ generate the same potential $Z$, then $A$ and $A'$ coincide. This well-known fact is slightly generalized in the following result which is an immediate consequence of the uniqueness of the Doob-Meyer decomposition.

**Proposition 1.** *Let $A, A'$ be predictable integrable processes of finite variation with $A_0 = A'_0 = 1$. If*

$$E[A_{T'} - A_t|\mathcal{F}_t] = E[A'_{T'} - A'_t|\mathcal{F}_t] \qquad , \qquad 0 \le t \le T', \tag{1.1}$$

*then $A$ and $A'$ are indistinguishable.*

**Proof.** By (1.1),

$$E[A_{T'}|\mathcal{F}_t] - A_t = E[A'_{T'}|\mathcal{F}_t] - A'_t \qquad , \qquad 0 \le t \le T'.$$

Since $A, A'$ are predictable and of finite variation, the assertion follows from the uniqueness of the Doob-Meyer decomposition.

**q.e.d.**

Proposition 1 can be viewed as a uniqueness result for the *additive* decomposition of special semimartingales. To formulate a *multiplicative* version, we recall the *multiplicative decomposition* for strictly positive special semimartingales. The following well-known result can be found in Jacod (1979), Propositions 6.19 and 6.20.

**Proposition 2 (Multiplicative decomposition of semimartingales)**

*Let $Q$ be a probability measure equivalent to $P$. Any strictly positive special $Q$-semimartingale $X$ satisfying $X_- > 0$ and $X_0 = 1$ admits a unique multiplicative decomposition*

$$X = MA,$$

*where $M \in \mathcal{M}^{+}_{1,\mathrm{loc}}(Q)$ and $A \in \mathcal{A}^{+}_{1}$. Uniqueness means that if we have two such decompositions $X = MA = M'A'$, then $M$ and $M'$ as well as $A$ and $A'$ are indistinguishable.*

The uniqueness statement in the above result leads immediately to a multiplicative version of Proposition 1.

**Proposition 3.** *Let $A, A' \in \mathcal{A}_1^+$ be integrable processes with $A_- > 0$ or $A'_- > 0$. If*

$$(1.2) \qquad E\left[\frac{A_{T'}}{A_t}\bigg|\mathcal{F}_t\right] = E\left[\frac{A'_{T'}}{A'_t}\bigg|\mathcal{F}_t\right] \quad , \quad 0 \le t \le T',$$

*then $A$ and $A'$ are indistinguishable.*

**Proof.** Assume that $A_- > 0$ and let

$$X_t := \frac{1}{A_t} E\left[A_{T'}|\mathcal{F}_t\right] = \frac{1}{A'_t} E\left[A'_{T'}|\mathcal{F}_t\right] \quad , \quad 0 \le t \le T'.$$

As a predictable RCLL process, $1/A$ is locally bounded by VII.32 of Dellacherie/Meyer (1982); this uses that $A_- > 0$. Again by VII.32 of Dellacherie/Meyer (1982), $X$ is hence a special semimartingale. Moreover, $X > 0$ and since $A_- > 0$, we also have $X_- > 0$. The uniqueness result in Proposition 2 then yields $\frac{1}{A} = \frac{1}{A'}$ and hence $A$ and $A'$ coincide.

**q.e.d.**

The main results of this paper are generalizations of Proposition 1 and Proposition 3 to the case where the expectations in (1.2) and (1.1) are taken under two *different* equivalent measures $Q$ and $Q'$. As explained above, this is directly motivated by a question from interest rate theory. The precise results are as follows.

**Theorem 4.** *Let $A, A' \in \mathcal{A}_1^+$ and let $Q, Q'$ be equivalent probability measures. If*

$$(1.3) \qquad E^Q\left[\frac{A_\tau}{A_\sigma}\bigg|\mathcal{F}_\sigma\right] = E^{Q'}\left[\frac{A'_\tau}{A'_\sigma}\bigg|\mathcal{F}_\sigma\right] \quad \textit{for all stopping times } 0 \le \sigma \le \tau \le T'$$

*or if $A, A'$ are of class (D) under $Q, Q'$ respectively and*

$$(1.4) \qquad E^Q\left[\frac{A_T}{A_t}\bigg|\mathcal{F}_t\right] = E^{Q'}\left[\frac{A'_T}{A'_t}\bigg|\mathcal{F}_t\right] \quad , \quad 0 \le t \le T \le T',$$

*then $A$ and $A'$ are indistinguishable.*

**Theorem 5.** *Let $A, A'$ be predictable RCLL processes of finite variation with $A_0 = A'_0 = 1$ and let $Q, Q'$ be equivalent probability measures. If*

$$(1.5) \qquad E^Q[A_\tau - A_\sigma|\mathcal{F}_\sigma] = E^{Q'}[A'_\tau - A'_\sigma|\mathcal{F}_\sigma]$$

*for all stopping times $0 \le \sigma \le \tau \le T'$ for which the conditional expectations in (1.5) are well-defined, or if $A, A'$ are of class (D) under $Q, Q'$ respectively and*

$$(1.6) \qquad E^Q[A_T - A_t|\mathcal{F}_t] = E^{Q'}[A'_T - A'_t|\mathcal{F}_t] \quad , \quad 0 \le t \le T \le T',$$

*then $A$ and $A'$ are indistinguishable.*

For comments and relations to the literature, we refer to section 5.

## 2. An auxiliary convergence result

The key result for our proofs of Theorem 5 and Theorem 4 is the following convergence result of independent interest. For its formulation, we first have to introduce some notation.

**Definition.** Let $s, t \in \mathbb{R}$ with $s \leq t$. A *partition* of $[s,t]$ is a finite family $\pi = \{t_0, t_1, \ldots, t_k\}$ with $s = t_0 \leq t_1 \leq \ldots \leq t_k \leq t$. A sequence $(\pi_n)_{n\in\mathbb{N}} = (\{t_0^n, t_1^n, \ldots, t_{k_n}^n\})_{n\in\mathbb{N}}$ of partitions of $[s,t]$ *tends to the identity* if $\lim\limits_{n\to\infty} t_{k_n}^n = t$ and $\lim\limits_{n\to\infty} \max\limits_{i=1,\ldots,k_n} (t_i^n - t_{i-1}^n) = 0$. A sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions is called *increasing* if $\pi_n \subseteq \pi_{n+1}$ for each $n$.

**Definition.** Let $\varrho \leq \tau \leq T'$ be stopping times. A *random partition of* $[\![\varrho, \tau]\!]$ is a finite family $\Pi = \{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ of stopping times with $\varrho = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_k \leq \tau$. A sequence $(\Pi_n)_{n\in\mathbb{N}} = (\{\sigma_0^n, \sigma_1^n, \ldots, \sigma_{k_n}^n\})_{n\in\mathbb{N}}$ of random partitions of $[\![\varrho, \tau]\!]$ *tends to the identity* if $\lim\limits_{n\to\infty} \sigma_{k_n}^n = \tau$ $P$-a.s. and $\lim\limits_{n\to\infty} \max\limits_{i=1,\ldots,k_n} (\sigma_i^n - \sigma_{i-1}^n) = 0$ $P$-a.s. A sequence $(\Pi_n)_{n\in\mathbb{N}}$ of random partitions is called *increasing* if $\Pi_n \subseteq \Pi_{n+1}$ for each $n$.

**Proposition 6.** *Suppose that the RCLL process $A$ of finite variation is of class (D) under $P$ and that $G$ is a bounded adapted RCLL process. Let $(\Pi_n)_{n\in\mathbb{N}}$ be an arbitrary increasing sequence of random partitions of $[\![0, T']\!]$ tending to the identity. If $\tau \leq T'$ is any stopping time such that $A^\tau$ is of $P$-integrable variation, then*

$$\lim_{n\to\infty} Y_n := \lim_{n\to\infty} \sum_{\substack{\sigma_i^n, \sigma_{i+1}^n \in \Pi_n \\ \sigma_i^n \leq \tau}} E\left[ G_{\sigma_i^n} \left( A_{\sigma_{i+1}^n} - A_{\sigma_i^n} \right) \middle| \mathcal{F}_{\sigma_i^n} \right] = \int_0^\tau G_{s-}\, dA_s^p \tag{2.1}$$

*weakly in $L^1(P)$, where $A^p$ denotes the dual predictable projection of $A$ under $P$ (which exists at least on $[\![0, \tau]\!]$).*

**Proof.** Fix $M \in L^\infty(\Omega, \mathcal{F}_{T'}, P)$ and an RCLL version of the martingale $M_t := E[M|\mathcal{F}_t]$, $0 \leq t \leq T'$. Fix an increasing sequence $(\Pi_n)_{n\in\mathbb{N}}$ of random partitions of $[\![0, T']\!]$ tending to the identity and set $\tau^n := \sigma_{i+1}^n$ on the set $\{\sigma_i^n \leq \tau < \sigma_{i+1}^n\}$ and $\tau^n := T'$ on $\{\tau = T'\}$. Because $\{\sigma_i^n \leq \tau\}$ is in $\mathcal{F}_{\sigma_i^n}$, conditioning on $\mathcal{F}_{\sigma_i^n}$ yields

$$\begin{aligned}
E[MY_n] &= E\left[ \sum_{\sigma_i^n, \sigma_{i+1}^n \in \Pi_n} M E\left[ I_{\{\sigma_i^n \leq \tau\}} G_{\sigma_i^n} \left( A_{\sigma_{i+1}^n} - A_{\sigma_i^n} \right) \middle| \mathcal{F}_{\sigma_i^n} \right] \right] \\
&= E\left[ \sum_{\sigma_i^n, \sigma_{i+1}^n \in \Pi_n} I_{\{\sigma_i^n \leq \tau\}} M_{\sigma_i^n} G_{\sigma_i^n} \left( A_{\sigma_{i+1}^n} - A_{\sigma_i^n} \right) \right] \\
&= E\Bigg[ \sum_{\sigma_i^n, \sigma_{i+1}^n \in \Pi_n} I_{\{\sigma_i^n \leq \tau\}} M_{\sigma_i^n} G_{\sigma_i^n} \left( A^\tau_{\sigma_{i+1}^n} - A^\tau_{\sigma_i^n} \right) \\
&\qquad + \sum_{\sigma_i^n, \sigma_{i+1}^n \in \Pi_n} I_{\{\sigma_i^n \leq \tau < \sigma_{i+1}^n\}} M_{\sigma_i^n} G_{\sigma_i^n} \left( A_{\sigma_{i+1}^n} - A_\tau \right) \Bigg].
\end{aligned} \tag{2.2}$$

Since $M$ and $G$ are bounded, the definition of $\tau^n$ yields

$$\left| E\left[ \sum_{\sigma_i^n,\sigma_{i+1}^n\in\Pi_n} I_{\{\sigma_i^n\le\tau<\sigma_{i+1}^n\}} M_{\sigma_i^n} G_{\sigma_i^n}\left(A_{\sigma_{i+1}^n} - A_\tau\right)\right]\right| \le \text{const.}\, E\big[\,|A_{\tau^n} - A_\tau|\,\big] \longrightarrow 0$$

as $n$ tends to infinity because $A$ is an RCLL process of class (D) and $\tau^n$ decreases to $\tau$. Hence (2.2) implies that

$$\lim_{n\to\infty} E\left[MY_n\right] = \lim_{n\to\infty} E\left[ \sum_{\sigma_i^n,\sigma_{i+1}^n\in\Pi_n} M_{\sigma_i^n} G_{\sigma_i^n}\left(A^\tau_{\sigma_{i+1}^n} - A^\tau_{\sigma_i^n}\right)\right]. \tag{2.3}$$

By Proposition II.21 of Protter (1990), we have

$$\lim_{n\to\infty} \sum_{\sigma_i^n,\sigma_{i+1}^n\in\Pi_n} M_{\sigma_i^n} G_{\sigma_i^n}\left(A^\tau_{\sigma_{i+1}^n} - A^\tau_{\sigma_i^n}\right) = \int_0^{T'} M_{s-}G_{s-}\, dA^\tau_s = \int_0^\tau M_{s-}G_{s-}\, dA_s$$

in probability. Since $M$ and $G$ are bounded and $A^\tau$ is of $P$-integrable variation, we therefore obtain by the generalized Lebesgue convergence theorem

$$\lim_{n\to\infty} E\left[ \sum_{\sigma_i^n,\sigma_{i+1}^n\in\Pi_n} M_{\sigma_i^n} G_{\sigma_i^n}\left(A^\tau_{\sigma_{i+1}^n} - A^\tau_{\sigma_i^n}\right)\right] = E\left[\int_0^\tau M_{s-}G_{s-}\, dA_s\right] = E\left[\int_0^\tau M_{s-}G_{s-}\, dA^p_s\right]$$

because $\int M_- G_-\, d(A - A^p)^\tau$ is a $P$-martingale. Combining this with (2.3) and using first Proposition VI.61 of Dellacherie/Meyer (1982) and then the optional stopping theorem yields

$$\lim_{n\to\infty} E\left[MY_n\right] = E\left[\int_0^\tau M_{s-}G_{s-}\, dA^p_s\right] = E\left[M_\tau \int_0^\tau G_{s-}\, dA^p_s\right] = E\left[M \int_0^\tau G_{s-}\, dA^p_s\right].$$

This completes the proof.

**q.e.d.**

**Remarks. 1)** Proposition 6 is a variation of well-known results; see for instance Doléans (1967), VII.21 of Dellacherie/Meyer (1982), Lemma 2.3 of Musiela/Rutkowski (1997a) or Lemma 2.14 of Jacod (1984). But all these results are formulated for the case where one can take $\tau = T'$ and the last assumes in addition that $A$ is continuous.

**2)** If $A$ in Proposition 6 is predictable, we have of course $A^p = A$. Our subsequent arguments only need this special case.

## 3. Proof of Theorem 5

The idea for the proof is rather simple. We use Proposition 6 to approximate both $A$ and $A'$ by sums of the conditional expectations of their increments. By assumption, the respective summands in the two sums always agree and hence so must the limits. Note that since $Q$ and $Q'$ are equivalent, we have $L^\infty(Q) = L^\infty(Q')$ and can write $L^\infty$ for short.

1) Suppose first that (1.5) holds. By VII.32 of Dellacherie/Meyer (1982), there exists an increasing sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ converging stationarily to $T'$ such that $\int_0^{\tau_m} |dA_u|+|dA'_u| \in L^\infty$. In particular, $A^{\tau_m}$ and $(A')^{\tau_m}$ are of class (D) under $Q, Q'$ respectively. Fix $t \in [0,T']$ and an increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,T']$ tending to the identity. By Proposition 6 with $P = Q$, $\tau = t\wedge\tau_m$, $G \equiv 1$ and $A^{\tau_m}$ instead of $A$,

$$
\begin{aligned}
(3.1) \qquad A_{t\wedge\tau_m} - A_0 &= \lim_{n\to\infty} \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t\wedge\tau_m}} E^Q\left[ A^{\tau_m}_{t^n_{i+1}} - A^{\tau_m}_{t^n_i} \,\middle|\, \mathcal{F}_{t^n_i} \right] \\
&= \lim_{n\to\infty} \sum_{t_i^n, t_{i+1}^n \in \pi_n} I_{\{t_i^n \le t\wedge\tau_m\}} E^Q\left[ A_{t^n_{i+1}\wedge\tau_m} - A_{t^n_i\wedge\tau_m} \,\middle|\, \mathcal{F}_{t^n_i\wedge\tau_m} \right],
\end{aligned}
$$

where the last equality uses the well-known fact that

$$
(3.2) \qquad E[X|\mathcal{F}_u] = E[X|\mathcal{F}_{u\wedge\tau_m}] \qquad \text{on } \{u \le \tau_m\}
$$

for any $X \in L^1(Q)$ and any $u \in [0,T']$. The same arguments for $Q', A'$ instead of $Q, A$ yield

$$
(3.3) \qquad A'_{t\wedge\tau_m} - A'_0 = \lim_{n\to\infty} \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t\wedge\tau_m}} E^{Q'}\left[ A'_{t^n_{i+1}\wedge\tau_m} - A'_{t^n_i\wedge\tau_m} \,\middle|\, \mathcal{F}_{t^n_i\wedge\tau_m} \right].
$$

But by (1.6) the sequences on the right-hand sides of (3.1) and (3.3) coincide. Thus we have a sequence converging at the same time to $A_{t\wedge\tau_m} - A_0$ weakly in $L^1(Q)$ and to $A'_{t\wedge\tau_m} - A'_0$ weakly in $L^1(Q')$, where $Q$ is equivalent to $Q'$. Hence Lemma 7 below tells us that

$$
A_{t\wedge\tau_m} - A_0 = A'_{t\wedge\tau_m} - A'_0 \qquad , \qquad 0 \le t \le T'
$$

and so $A$ and $A'$ coincide on $[\![0,\tau_m]\!]$. Letting $m$ tend to infinity shows $A = A'$.

2) Suppose now that $A, A'$ are of class (D) under $Q, Q'$ respectively and that (1.6) holds. Fix again an increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,T']$ tending to the identity. Applying Proposition 6 with $G \equiv 1$ and a sequence $(\tau_m)$ of stopping times as in part 1) yields for all $t \in [0,T']$ and $m \in \mathbb{N}$

$$
(3.4) \qquad A_{t\wedge\tau_m} - A_0 = \lim_{n\to\infty} \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t\wedge\tau_m}} E^Q\big[ A_{t^n_{i+1}} - A_{t^n_i} \big| \mathcal{F}_{t^n_i} \big] \qquad \text{weakly in } L^1(Q)
$$

and

$$
(3.5) \qquad A'_{t\wedge\tau_m} - A'_0 = \lim_{n\to\infty} \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t\wedge\tau_m}} E^{Q'}\big[ A'_{t^n_{i+1}} - A'_{t^n_i} \big| \mathcal{F}_{t^n_i} \big] \qquad \text{weakly in } L^1(Q').
$$

Because of (1.6), the sequences on the right-hand sides of (3.4) and (3.5) coincide and so the same arguments as in part 1) show that $A^{\tau_m}$ and $(A')^{\tau_m}$ must also coincide. Letting $m$ tend to infinity completes the proof.

**q.e.d.**

In the above argument, we have used the following well-known result.

**Lemma 7.** *If a sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly in $L^1(Q)$ to $u$ and weakly in $L^1(Q')$ to $u'$ where $Q'$ is equivalent to $Q$, then $u = u'$ $Q$-a.s. and $Q'$-a.s.*

**Proof.** Let $Z := \frac{dQ'}{dQ}$ and fix $\varphi \in L^\infty(Q)$ and $a \geq 0$. By definition of the weak convergence $\sigma(L^1, L^\infty)$, we have

$$E^{Q'}\left[u'\varphi I_{\{Z\leq a\}}\right] = \lim_{n\to\infty} E^{Q'}\left[u_n\varphi I_{\{Z\leq a\}}\right] = \lim_{n\to\infty} E^{Q}\left[u_n\varphi Z I_{\{Z\leq a\}}\right] = E^{Q}\left[u\varphi Z I_{\{Z\leq a\}}\right]$$

because $ZI_{\{Z\leq a\}}$ is bounded. Thus we have

$$E^{Q}\left[u\varphi Z I_{\{Z\leq a\}}\right] = E^{Q}\left[u'\varphi Z I_{\{Z\leq a\}}\right] \qquad \text{for all } \varphi \in L^\infty(Q) \text{ and all } a \geq 0$$

and this implies the assertion.

**q.e.d.**

## 4. Proof of Theorem 4

The basic idea for this proof is again to use approximations via Proposition 6, but the complete argument requires some care. We first show that

$$A'_-\,dA = A_-\,dA' \tag{4.1}$$

which implies that $A$ and $A'$ must coincide at least until either $A_-$ or $A'_-$ hits zero. We then prove that neither does.

As in the last proof, VII.32 of Dellacherie/Meyer (1982) gives us an increasing sequence of stopping times $(\tau_m)_{m\in\mathbb{N}}$ converging stationarily to $T'$ such that $\int_0^{\tau_m} |dA_u| + |dA'_u| \in L^\infty$. Fix an increasing sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,T']$ tending to the identity.

1) Suppose first that (1.3) holds. By Proposition 6 with $P = Q$, $\tau = t\wedge\tau_m$, $G = (A')^{\tau_m}_-$ and $A^{\tau_m}$ instead of $A$,

$$\lim_{n\to\infty} U_n(t) = \int_0^{t\wedge\tau_m} (A')^{\tau_m}_{s-}\,dA^{\tau_m}_s = \int_0^{t\wedge\tau_m} A'_{s-}\,dA_s \qquad \text{weakly in } L^1(Q) \tag{4.2}$$

for each $t$, where

$$\begin{aligned} U_n(t) &:= \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \leq t\wedge\tau_m}} E^Q\left[(A')^{\tau_m}_{t_i^n}\left(A^{\tau_m}_{t_{i+1}^n} - A^{\tau_m}_{t_i^n}\right)\Big|\mathcal{F}_{t_i^n}\right] \\ &= \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \leq t\wedge\tau_m}} E^Q\left[A'_{t_i^n\wedge\tau_m}\left(A_{t_{i+1}^n\wedge\tau_m} - A_{t_i^n\wedge\tau_m}\right)\Big|\mathcal{F}_{t_i^n\wedge\tau_m}\right]; \end{aligned}$$

the last equality is due to (3.2). Reversing the roles of $A, Q$ and $A', Q'$ shows in the same way that we also have

$$\lim_{n\to\infty} U'_n(t) = \int_0^{t\wedge\tau_m} A_{s-}\,dA'_s \qquad \text{weakly in } L^1(Q') \tag{4.3}$$

for each $t$, where

$$U'_n(t) := \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t \wedge \tau_m}} E^{Q'} \left[ A_{t_i^n \wedge \tau_m} \left( A'_{t_{i+1}^n \wedge \tau_m} - A'_{t_i^n \wedge \tau_m} \right) \Big| \mathcal{F}_{t_i^n \wedge \tau_m} \right].$$

But due to (1.3), we have $U_n(t) = U'_n(t)$ a.s. for all $n, t$ and so (4.2) and (4.3) show that we have one sequence of random variables converging at the same time weakly in $L^1(Q)$ and weakly in $L^1(Q')$, where $Q$ is equivalent to $Q'$. This implies that the limits must coincide a.s. and so we obtain

$$\int_0^{t\wedge\tau_m} A'_{s-}\, dA_s = \int_0^{t\wedge\tau_m} A_{s-}\, dA'_s \qquad \text{a.s. for all } t, m.$$

Letting $m$ tend to infinity yields $A'_-\, dA = A_-\, dA'$.

2) Suppose now that $A, A'$ are of class (D) under $Q, Q'$ respectively and that (1.4) holds. Then Proposition 6 with $P = Q$, $\tau = t \wedge \tau_m$ and $G = A'_-$ yields

$$\lim_{n\to\infty} V_n(t) = \int_0^{t\wedge\tau_m} A'_{s-}\, dA_s \qquad \text{weakly in } L^1(Q)$$

for each $t$, where

$$V_n(t) := \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t \wedge \tau_m}} E^{Q} \left[ A'_{t_i^n} \left( A_{t_{i+1}^n} - A_{t_i^n} \right) \Big| \mathcal{F}_{t_i^n} \right].$$

Again reversing the roles of $A, Q$ and $A', Q'$ also gives

$$\lim_{n\to\infty} V'_n(t) = \int_0^{t\wedge\tau_m} A_{s-}\, dA'_s \qquad \text{weakly in } L^1(Q')$$

for each $t$, where

$$V'_n(t) := \sum_{\substack{t_i^n, t_{i+1}^n \in \pi_n \\ t_i^n \le t \wedge \tau_m}} E^{Q'} \left[ A_{t_i^n} \left( A'_{t_{i+1}^n} - A'_{t_i^n} \right) \Big| \mathcal{F}_{t_i^n} \right].$$

But $V_n(t) = V'_n(t)$ a.s. for all $n, t$ by (1.4) and so the same arguments as in part 1) show that $A'_-\, dA = A_-\, dA'$.

3) Now let $\varrho < T'$ be a random variable such that $A_\varrho = A'_\varrho$. Because $A$ and $A'$ are both right-continuous and strictly positive, there exists a random variable $\sigma > \varrho$ such that $A_- > 0$ and $A'_- > 0$ on $]\!]\varrho, \sigma]\!]$ and so (4.1) is on $]\!]\varrho, \sigma]\!]$ equivalent to

$$\frac{dA}{A_-} = \frac{dA'}{A'_-} \qquad , \qquad A_\varrho = A'_\varrho.$$

By the uniqueness of the stochastic exponential, $A$ and $A'$ therefore coincide on $[\![\varrho, \sigma]\!]$. So if we define

$$\tau := \inf\left\{ t \in [0, T'] \,\middle|\, A_t \neq A'_t \right\} \wedge T',$$

the above argument shows that

$$A_\tau \neq A'_\tau \qquad \text{if } \tau < T'.$$

4) To complete the proof, we now show that $A_\tau = A'_\tau$. This would be obvious if $A$ and $A'$ were left-continuous, but since they are only predictable, we have to use (1.3) or (1.4). We denote by $\Delta A_t := A_t - A_{t-}$, $0 \le t \le T'$, the process of the jumps of $A$. The stopping time $\tau$ is in general not predictable, but by Theorem B in the complements to chapter IV in Dellacherie/Meyer (1982), there exists a sequence $(\sigma_n)_{n\in I\!N}$ of predictable stopping times such that

$$\big\{(\omega,t)\,\big|\,\Delta A_t(\omega) \neq 0 \text{ or } \Delta A'_t(\omega) \neq 0\big\} = \bigcup_{n\in I\!N} [\![\sigma_n]\!].$$

Since each $\sigma_n$ is a predictable stopping time, there exists for every $n$ a strictly increasing sequence of stopping times $(\sigma_n^p)_{p\in I\!N}$ tending to $\sigma_n$.

If (1.3) holds, we have

$$E^Q\left[A_{\sigma_n}\big|\mathcal{F}_{\sigma_n^p}\right] = E^{Q'}\left[A'_{\sigma_n}\big|\mathcal{F}_{\sigma_n^p}\right] \qquad \text{on the set } C_n^p := \{A = A' \text{ on } [\![0,\sigma_n^p]\!]\}.$$

Letting $p$ tend to infinity and using that $A, A'$ are predictable yields

$$\text{(4.4)} \qquad A_{\sigma_n} = E^Q\left[A_{\sigma_n}\big|\mathcal{F}_{\sigma_n-}\right] = E^{Q'}\left[A'_{\sigma_n}\big|\mathcal{F}_{\sigma_n-}\right] = A'_{\sigma_n} \qquad \text{on the set } C_n := \bigcap_{p\in I\!N} C_n^p.$$

By the definition of $\tau$, we have $A = A'$ on $[\![0,\tau[\![$ and therefore $A_{\tau-} = A'_{\tau-}$. This implies that

$$\{A_\tau \neq A'_\tau\} \subseteq \{\Delta A_\tau \neq 0 \text{ or } \Delta A'_\tau \neq 0\} \subseteq \bigcup_{n\in I\!N}\{\tau = \sigma_n\}$$

since the sequence $(\sigma_n)$ exhausts the jumps of $A$ and $A'$. So for any $\omega \in \{A_\tau \neq A'_\tau\}$ we have $\tau(\omega) = \sigma_n(\omega)$ for some $n$ and therefore $A_t(\omega) = A'_t(\omega)$ for any $t \le \sigma_n^p(\omega)$ and all $p$, because $\big(\sigma_n^p(\omega)\big)_{p\in I\!N}$ increases strictly to $\sigma_n(\omega) = \tau(\omega)$ and $A = A'$ on $[\![0,\tau[\![$. Hence $\omega \in C_n$ for the above $n$ and therefore $A_\tau(\omega) = A_{\sigma_n}(\omega) = A'_{\sigma_n}(\omega) = A'_\tau(\omega)$ by (4.4), contradicting the fact that $\omega \in \{A_\tau \neq A'_\tau\}$. Thus we conclude that $A_\tau = A'_\tau$.

Suppose now that $A, A'$ are of class (D) under $Q, Q'$ respectively and that (1.4) holds. Fix $t \in [0,T']$ and use (1.4) to obtain

$$\varphi(u) := \frac{E^Q[A_t|\mathcal{F}_{u\wedge t}]}{A_{u\wedge t}} = \frac{E^{Q'}[A'_t|\mathcal{F}_{u\wedge t}]}{A'_{u\wedge t}} =: \varphi'(u) \qquad , \qquad 0 \le u \le T'.$$

Applying the optional stopping theorem, we get $\varphi(\sigma_n^p) = \varphi'(\sigma_n^p)$ for all $n$ and $p$. Letting $p$ tend to infinity then yields

$$\text{(4.5)} \qquad E^Q\left[A_t\big|\mathcal{F}_{\sigma_n-}\right] = E^{Q'}\left[A'_t\big|\mathcal{F}_{\sigma_n-}\right] \qquad \text{on } C_n \cap \{t \ge \sigma_n\}.$$

For all $k \in I\!N$, we set $\tau_n^m := \frac{k+1}{2^m}$ if $\frac{k}{2^m} \le \sigma_n < \frac{k+1}{2^m}$. Then (4.5) implies

$$E^Q\left[A_{\tau_n^m}\big|\mathcal{F}_{\sigma_n-}\right] = E^{Q'}\left[A'_{\tau_n^m}\big|\mathcal{F}_{\sigma_n-}\right] \qquad \text{on } C_n$$

by the definition of $\tau_n^m$. Letting $m$ tend to infinity and using that $A, A'$ are right-continuous and of class (D) under $Q, Q'$ respectively, we again obtain (4.4) and hence $A_\tau = A'_\tau$ by the same argument as above. This completes the proof of Theorem 4.

**q.e.d.**

## 5. Comments and conclusion

The main contribution of this paper is a complete and general proof for the uniqueness of an implied savings account under clearly specified (and very weak) assumptions. Proposition 6 and in particular Theorem 4 improve Lemma 2.3 and Proposition 2.3 in Musiela/Rutkowski (1997a) in two directions. They are more general because we work with an arbitrary filtration which need not be generated by a Brownian motion. They are also more precise because Musiela/Rutkowski (1997a) apply their Lemma 2.3 in a situation where its assumptions are not satisfied. By exploiting the symmetry between $Q$ and $Q'$, we have in addition simplified the structure of the argument originally proposed by Musiela/Rutkowski (1997a).

The generalization in Theorem 4 of the uniqueness of the multiplicative decomposition yields as direct application that implied savings accounts are unique. We have at present no direct use for the additive counterpart in Theorem 5, but we hope to see this appear in a problem from finance or elsewhere. We remark that the crucial convergence result in Proposition 6 has also been used by Döberlein/Schweizer (1999) to replicate a continuous implied savings account by a roll-over strategy in just maturing bonds.

Finally, let us briefly comment on the assumptions in Theorem 4. Of course, (1.4) is just the condition that $(A, Q)$ and $(A', Q')$ generate via (0.1) the same term structure model. The class (D) assumption imposes a weak integrability property that allows us to interchange limits and expectations. Condition (1.3) on the other hand is more intuitive. If we compare it to (0.1), it tells us that we should also consider bonds with random maturities $\tau$ which are stopping times and that the prices of such bonds should also be generated by both $(A, Q)$ and $(A', Q')$.

**Acknowledgements.** We thank the co-editor and an anonymous referee for providing additional references and Walter Schachermayer and Frank Oertel for stimulating discussions. The research of this paper was partially supported by the INTAS-RFBR Grant 95-0061. Financial support by the Deutsche Forschungsgemeinschaft through Graduiertenkolleg "Stochastische Prozesse und probabilistische Analysis" at the Technical University of Berlin is gratefully acknowledged.

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