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Mean-Variance Hedging for General Claims

by

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Abstract: We consider a hedger with a mean-variance objective who faces a random loss at a fixed time. The size of this loss depends quite generally on two correlated asset prices, while only one of them is available for hedging purposes. We present a simple solution of this hedging problem by introducing the intrinsic value process of a contingent claim.

Key words: hedging, mean-variance criterion, continuous trading, option valuation, contingent claims, equivalent martingale measures.

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0. Introduction

In this paper, we solve the continuous-time hedging problem with a mean-variance objective for general contingent claims. A special case of this problem was treated by Duffie/Richardson (1991) and provided the motivation for this work.

There are two assets whose prices are both modelled by exponential Brownian motions with time-dependent random coefficients. The rates of return between assets are correlated. At a fixed time, the hedger faces a random loss which depends in full generality on the entire evolution of both asset prices. For the purpose of hedging against this risk, however, only one asset is available. This implies that markets are incomplete and contingent claims cannot be replicated by trading. The goal of the hedger is to minimize his total expected quadratic costs, or equivalently to maximize his expected utility from terminal wealth for a quadratic utility function. A precise statement is given in Section 1, and the solution is presented in Section 3. We remark that the same arguments would also work for any number N of driving assets with n hedging assets, where $1 \leq n \leq N$.

Our approach to this problem follows the method of Duffie/Richardson (1991): we show that the inner product associated with the normal equations for orthogonal projection is defined by an ordinary differential equation in time with an explicit solution. This is done by choosing a suitable tracking process for the contingent claim under consideration. The essential difference to the above paper lies in two points: we are able to solve the hedging problem for a general contingent claim, and we do not have to conjecture the solution from discrete-time reasoning. In fact, our approach shows that the natural choice for the optimal tracking process is provided by the *intrinsic value process* associated to the given contingent claim. This process is defined in terms of the *minimal equivalent martingale measure* for that asset price which is used for hedging. Both of these concepts are explained in more detail in Section 2. We conclude the paper in Section 4 with a class of examples where explicit formulas can be derived, including as a special case the result of Duffie/Richardson (1991).

1. The problem

In this section, we formulate the general hedging problem and recall the basic approach to solving it. For ease of reference, we use the same notations as in Duffie/Richardson (1991), subsequently abbreviated as D/R. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a two-dimensional Brownian motion (B, ε) and $I\!F = (\mathcal{F}_t)_{0 \leq t \leq T}$ the augmentation of the filtration generated by (B, ε), where $T > 0$ is a fixed time horizon. Let (μ_t), (m_t), (σ_t), (v_t) and (ϱ_t) be bounded adapted processes and assume that (v_t) is bounded away from 0 (uniformly in ω), $|\varrho_t| \leq 1$

for all t and

$$\left(\frac{m_t}{v_t}\right)_{0 \le t \le T} \quad \text{is a deterministic function.} \tag{1.1}$$

Now define a Brownian motion ξ by setting

$$\xi_t = \int_0^t \varrho_u \, dB_u + \int_0^t \sqrt{1-\varrho_u^2} \, d\varepsilon_u \quad , \qquad 0 \le t \le T. \tag{1.2}$$

The asset price processes S and F are then described by the stochastic differential equations

$$\begin{aligned} dS_t &= \mu_t S_t \, dt + \sigma_t S_t \, dB_t \quad , \quad S_0 > 0 \\ dF_t &= m_t F_t \, dt + v_t F_t \, d\xi_t \quad , \quad F_0 > 0. \end{aligned} \tag{1.3}$$

Since trading in F is possible, F may be used for hedging purposes. A *hedging strategy* ϑ is an $I\!\!F$-predictable process satisfying

$$E\left[\int_0^T \vartheta_u^2 F_u^2 \, du\right] < \infty.$$

Its associated cumulative gains process $G(\vartheta)$ is given by the stochastic integral

$$G_t(\vartheta) = \int_0^t \vartheta_u \, dF_u \quad , \qquad 0 \le t \le T.$$

Θ denotes the set of all hedging strategies.

A *contingent claim* Π is an F_T-measurable random variable satisfying

$$\Pi \in \mathcal{L}^p(P) \qquad \text{for some } p > 2. \tag{1.4}$$

The hedging problem is then

$$\min_{\vartheta \in \Theta} E\left[\left(\Pi + L - G_T(\vartheta)\right)^2\right], \tag{1.5}$$

where $L > 0$ is a given constant.

Interpretation. Π can be viewed as a random loss suffered by the hedger at time T. The constant L plays the role of an initial cost, and $\Pi + L - G_T(\vartheta)$ describes the total loss or costs incurred at time T. Thus, the hedger's objective is to minimize his expected quadratic costs.

Remarks. 1) If $\Pi \leq 0$, then $-\Pi$ can also be interpreted as a payoff received by the hedger at time T. His total final wealth, including gains from trade, is then given by $-\Pi + G_T(\vartheta)$. (Of course, this interpretation still holds for a general Π if one is willing to receive possibly negative payoffs.) Rewriting (1.5) as

$$\min_{\vartheta \in \Theta} E\left[\big(-\Pi + G_T(\vartheta) - L\big)^2\right]$$

then yields the interpretation of minimizing the expected quadratic deviation of the terminal wealth from a fixed target level L. This is the question addressed in D/R for the special case $\Pi = -kS_T$.

2) If we set $u(x) = x - cx^2$ and $c = \frac{1}{2L}$, the hedging problem (1.5) can equivalently be written as

$$\max_{\vartheta \in \Theta} E\left[u\big(G_T(\vartheta) - \Pi\big)\right], \tag{1.6}$$

i.e., as maximization of expected utility from terminal wealth for the quadratic utility function u. We refer to D/R for other related optimization problems.

3) There are several well-known objections to the preceding criteria (e.g., increasing absolute risk aversion, ranges of nonmonotonicity and possibility of negative wealth). Nevertheless, they have been widely used in practice and can often be helpful as a first approximation.

Let us now recall the basic approach to solving (1.5). The same Hilbert space projection argument as in D/R shows that a hedging strategy ϑ^* is optimal if and only if it satisfies

$$E\left[\big(\Pi + L - G_T(\vartheta^*)\big)\cdot G_T(\vartheta)\right] = 0 \tag{1.7}$$

for all $\vartheta \in \Theta$. To obtain (1.7), one defines the function

$$H(t) := E\left[\big(Z_t + L - G_t(\vartheta^*)\big)\cdot G_t(\vartheta)\right] \qquad , \qquad 0 \leq t \leq T$$

for a suitable *tracking process* Z with $Z_T = \Pi$. Using (1.1), one derives a differential equation for H whose unique solution is $H(t) \equiv 0$, thus proving in particular (1.7). The crucial problem is then the choice of ϑ^* and, even more importantly, the tracking process Z. In D/R, the solution for their particular contingent claim was conjectured from discrete-time arguments. We shall see in Section 3 that the simplest choice for Z is quite generally given by the intrinsic value process $\widehat{V}$ of Π, and this will also yield a simple description and intuitive interpretation of the optimal strategy ϑ^*.

2. Claims and their intrinsic values

In this section, we recall from Föllmer/Schweizer (1991) and Schweizer (1991) the notions of a minimal martingale measure and of an intrinsic value process. Since this is of interest in itself, we begin with a few words of motivation.

In a general formulation of option trading problems, one usually starts out with a price process X and an option or contingent claim H. Two key issues are then the valuation and the hedging of H by means of a suitable trading strategy based on X. In this context, an important role is played by the set $\mathbb{P}$ of equivalent martingale measures for X. The elements of $\mathbb{P}$ are probability measures Q which are equivalent to P (i.e., have the same null sets as P) and under which the price process X becomes a martingale. By no-arbitrage-type arguments, the value process V of a claim H must have the form $V = E_Q[H|\mathbb{F}]$ for some $Q \in \mathbb{P}$. If $\mathbb{P}$ is a singleton, this obviously leads to a unique solution of the valuation problem. In general, however, $\mathbb{P}$ has many elements, all of which give rise to a potential value process. But it turns out that in many cases, there is a unique *minimal equivalent martingale measure* $\widehat{P} \in \mathbb{P}$. This concept was introduced in Föllmer/Schweizer (1991) and used in Schweizer (1991) to obtain hedging strategies which are optimal in a certain sense (different from the one used here). Furthermore, it is shown in Schweizer (1991) that the induced valuation process $\widehat{V} = E_{\widehat{P}}[H|\mathbb{F}]$ can be viewed as the *intrinsic value process* of the claim H and that it corresponds to a risk-neutral approach to option valuation.

In our present hedging problem, the price process X to be used for hedging is given by F and the contingent claim H by Π. Clearly, every valuation process $V := E_Q[\Pi|\mathbb{F}]$ satisfies $V_T = \Pi$ and thus could in principle serve as a tracking process Z. But as we shall see in the next section, the choice of the intrinsic value process $\widehat{V}$ allows us to give a very simple solution for (1.5).

Let us now turn to the actual construction of the minimal martingale measure $\widehat{P}$ for F. For this purpose, we write the P-semimartingale F in its canonical decomposition

$$F = F_0 + M + A$$

with

$$\begin{aligned} dM_t &= v_t F_t \, d\xi_t, \\ dA_t &= m_t F_t \, dt. \end{aligned}$$

This implies $d\langle M\rangle_t = v_t^2 F_t^2 \, dt$ and therefore $dA_t = \alpha_t \, d\langle M\rangle_t$, with

$$\alpha_t = \frac{m_t}{v_t^2 F_t}.$$

The minimal martingale measure $\widehat{P}$ for F is then defined by

$$\text{(2.1)} \qquad \frac{d\widehat{P}}{dP} := U_T := \mathcal{E}\left(-\int \alpha\, dM\right)_T$$
$$= \exp\left(-\int_0^T \alpha_u\, dM_u - \frac{1}{2}\cdot\int_0^T \alpha_u^2\, d\langle M\rangle_u\right)$$
$$= \exp\left(-\int_0^T \frac{m_u}{v_u}\, d\xi_u - \frac{1}{2}\cdot\int_0^T \frac{m_u^2}{v_u^2}\, du\right);$$

see Föllmer/Schweizer (1991). For future reference, we note that

$$\text{(2.2)} \qquad U_T \in \bigcap_{p\geq 1} \mathcal{L}^p(P) \qquad \text{and} \qquad U_T^{-1} \in \bigcap_{p\geq 1} \mathcal{L}^p(\widehat{P});$$

this follows immediately from the fact that $\left(\frac{m_t}{v_t}\right)$ is bounded.

Let us now define the square integrable P-martingale N by

$$N_t := \int_0^t \sqrt{1-\varrho_u^2}\, dB_u - \int_0^t \varrho_u\, d\xi_u \qquad , \qquad 0 \leq t \leq T.$$

Using Girsanov's theorem and Itô's representation theorem, one can easily verify the following facts:

(i) Both F and N are square-integrable martingales under $\widehat{P}$.

(ii) M and N are orthogonal under P, and every $B \in \mathcal{L}^2(\mathcal{F}_T, P)$ with zero expectation is the sum of two stochastic integrals with respect to M and N, respectively.

(iii) F and N are orthogonal under $\widehat{P}$, and every $B \in \mathcal{L}^2(\mathcal{F}_T, \widehat{P})$ with zero expectation is the sum of two stochastic integrals with respect to F and N, respectively.

We remark that (iii) is straightforward since, due to (1.1), the change of measure from P to $\widehat{P}$ only involves a deterministic change of drift.

Lemma. *Suppose* Π *is a contingent claim satisfying (1.4). Then* Π *admits a decomposition*

$$\text{(2.3)} \qquad \Pi = \widehat{\Pi}_0 + \int_0^T \widehat{\vartheta}_u\, dF_u + \int_0^T \nu_u\, dN_u \qquad P-a.s.,$$

where $\widehat{\vartheta}$ is a hedging strategy and ν satisfies $E\left[\int\limits_0^T \nu_u^2\,du\right] < \infty$. Furthermore, the intrinsic value process $\widehat{V}$ associated to Π is given by

$$(2.4)\qquad \widehat{V}_t = E_{\widehat{P}}[\Pi|\mathcal{F}_t] = \widehat{\Pi}_0 + \int\limits_0^t \widehat{\vartheta}_u\,dF_u + \int\limits_0^t \nu_u\,dN_u \qquad , \qquad 0 \le t \le T$$

and satisfies $\sup\limits_{0\le t\le T} |\widehat{V}_t| \in \mathcal{L}^2(P)$.

Proof. This follows immediately from (i) – (iii), (1.4) and (2.2).

q.e.d.

3. The solution

In this section, we present the general solution to the hedging problem (1.5). Since the method of proof is the same as in D/R, we shall confine ourselves to a brief outline of the required steps.

Theorem. *Let Π be a contingent claim satisfying (1.4) and denote by $\widehat{V} = E_{\widehat{P}}[\Pi|\mathbb{F}]$ its associated intrinsic value process. Let G^* be the solution of the stochastic differential equation*

$$(3.1)\qquad dG_t^* = \Phi(G_t^*)\,dF_t \quad , \quad G_0^* = 0,$$

where

$$(3.2)\qquad \Phi(G_t^*) = \widehat{\vartheta}_t + \frac{m_t}{v_t^2 F_t}\cdot(\widehat{V}_t + L - G_t^*)$$

and $\widehat{\vartheta}$ is taken from (2.3). Then the hedging strategy $\vartheta_t^* := \Phi(G_t^*)$ solves (1.5).

Proof. 1) Using (2.4), (1.4) and (2.2), one can show as in Protter (1990) that (3.1) has a unique solution satisfying $\sup\limits_{0\le t\le T} E\left[(G_t^*)^2\right] < \infty$. Combining this with (2.4) then implies that ϑ^* is indeed a hedging strategy with gains process $G(\vartheta^*) = G^*$.

2) Fix any hedging strategy ϑ and define the function

$$H(t) := E\left[\left(\widehat{V}_t + L - G_t^*\right)\cdot G_t(\vartheta)\right] \qquad , \qquad 0 \le t \le T.$$

If we can show that $H(t) \equiv 0$, then $H(T) = 0$ will imply the optimality of ϑ^* by (1.7). For this purpose, we apply the product rule to $(\widehat{V} + L - G^*)\cdot G(\vartheta)$, note that all martingale terms have expectation 0 and use Fubini's theorem to obtain

$$H(t) = -\int\limits_0^t E\left[\frac{m_u^2}{v_u^2}\cdot\left(\widehat{V}_u + L - G_u^*\right)\cdot G_u(\vartheta)\right]\,du.$$

Since $\left(\frac{m_t}{v_t}\right)$ is deterministic by (1.1), this implies

$$H(t) = -\int_0^t \frac{m_u^2}{v_u^2} \cdot H(u)\,du$$

and therefore

$$\frac{d}{dt}H(t) = -\frac{m_t^2}{v_t^2} \cdot H(t).$$

But $G_0(\vartheta) = 0$ implies $H(0) = 0$, and thus we must have $H(t) \equiv 0$.

q.e.d.

Remarks. 1) The preceding proof relies in a crucial way on the assumption that $\left(\frac{m_t}{v_t}\right)$ is deterministic. Clearly, a stochastic mean-variance tradeoff for the hedging instrument F would be more realistic. It would be interesting to see a solution of (1.5) in this general case.

2) Recall that the hedging problem (1.5) can be rewritten as a problem (1.6) of maximizing expected utility from terminal wealth. The corresponding quadratic utility function $u(x) = x - \frac{1}{2L}x^2$ has absolute risk aversion

$$R_a(x) = \frac{1}{L-x}.$$

The optimal strategy ϑ^* in (3.2) can therefore be written as

$$\vartheta_t^* = \widehat{\vartheta}_t + \frac{1}{R_a(G_t^* - \widehat{V}_t)} \cdot \frac{m_t}{v_t^2 F_t}. \tag{3.3}$$

But the infinitesimal conditional mean and variance of the hedging instrument F are given by

$$E[dF_t|\mathcal{F}_t] = m_t F_t\,dt,$$
$$\mathrm{Var}[dF_t|\mathcal{F}_t] = v_t^2 F_t^2\,dt.$$

Thus (3.3) shows that ϑ^* decomposes into a pure hedging demand $\widehat{\vartheta}$ and a second component representing a demand for mean-variance purposes. Such a decomposition was already obtained by Merton (1973) for general utility functions u, but under the assumption of a Markovian structure for the underlying assets.

4. Some explicit results

In this section, we give explicit expressions for $\widehat{V}$ and $\widehat{\vartheta}$ in the case where all coefficients are deterministic functions and the claim Π only depends on the terminal values F_T and S_T of the asset prices. In particular, we recover as a special case the result obtained by D/R.

Suppose in addition to our standing assumptions that the coefficients (μ_t), (m_t), (σ_t), (v_t) and (ϱ_t) are all deterministic. Define

$$\gamma_t := \frac{\sigma_t m_t \varrho_t}{v_t} - \mu_t \qquad , \qquad 0 \le t \le T$$

and

$$Y_t := S_t \cdot \exp\left(- \int_t^T \gamma_u \, du \right) \qquad , \qquad 0 \le t \le T,$$

so that $Y_T = S_T$. Then it is easy to see from (1.3) that (F, Y) is a two-dimensional Markov process. If the claim Π has the form

$$\Pi = g(F_T, S_T) \tag{4.1}$$

for a function g satisfying some growth conditions, the corresponding intrinsic value process $\widehat{V}$ is therefore given by

$$\widehat{V}_t = E_{\widehat{P}}[\Pi | \mathcal{F}_t] = f(F_t, Y_t, t) \tag{4.2}$$

with

$$f(x, y, t) = E\left[g\left(x \cdot \exp\left(W_1 - \frac{1}{2}\mathrm{Var}[W_1] \right), y \cdot \exp\left(W_2 - \frac{1}{2}\mathrm{Var}[W_2] \right) \right) \right], \tag{4.3}$$

where W_1 and W_2 are jointly normally distributed with variances $\int_t^T v_u^2 \, du$ and $\int_t^T \sigma_u^2 \, du$, respectively, and covariance $\int_t^T v_u \sigma_u \varrho_u \, du$. To obtain the decomposition (2.3) in terms of f, we first apply Itô's lemma:

$$d\widehat{V}_t = f_x \, dF_t + f_y \, dY_t + \text{terms of finite variation.}$$

But since we know from (2.4) that

$$d\widehat{V}_t = \widehat{\vartheta}_t \, dF_t + \nu_t \, dN_t$$

is a continuous martingale under $\widehat{P}$, all finite variation terms must vanish, and it only remains to express the martingale part (under $\widehat{P}$) of Y in terms of F and N. After some calculations, this leads to

$$\begin{aligned}\widehat{\vartheta}_t &= f_x(F_t, Y_t, t) + f_y(F_t, Y_t, t)\cdot Y_t\cdot \frac{\sigma_t \varrho_t}{v_t F_t}, \\ \nu_t &= f_y(F_t, Y_t, t)\cdot Y_t\cdot \sigma_t\cdot \sqrt{1-\varrho_t^2}.\end{aligned} \tag{4.4}$$

By solving (3.1) for G^*, we can therefore obtain the optimal hedging strategy ϑ^* from (3.2).

Example. Let us take the claim $\Pi = -kS_T$, $k > 0$, considered by D/R. Then

$$f(x,y,t) = E\left[-k\cdot y\cdot e^{W_2 - \frac{1}{2}\mathrm{Var}[W_2]}\right] = -k\cdot y$$

implies $f_x = 0$, $f_y = -k$ and thus

$$\widehat{\vartheta}_t = -k\cdot Y_t\cdot \frac{\sigma_t \varrho_t}{v_t F_t}.$$

The intrinsic value process is

$$\widehat{V}_t = -k\cdot Y_t = -k\cdot S_t\cdot \exp\left(-\int_t^T \gamma_u\, du\right), \tag{4.5}$$

and the optimal strategy is given by

$$\vartheta_t^* = \frac{1}{F_t}\cdot\left(\frac{m_t}{v_t^2}\cdot(L + \widehat{V}_t - G_t^*) + \frac{\sigma_t \varrho_t}{v_t}\cdot \widehat{V}_t\right), \tag{4.6}$$

where G^* solves

$$dG_t^* = \frac{1}{F_t}\cdot\left(\frac{m_t}{v_t^2}\cdot(L + \widehat{V}_t - G_t^*) + \frac{\sigma_t \varrho_t}{v_t}\cdot \widehat{V}_t\right) dF_t \quad , \quad G_0^* = 0.$$

This is exactly the solution obtained by D/R. In particular, their tracking process Z coincides (up to a change of sign) with the intrinsic value process $\widehat{V}$ of Π.

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References

D. Duffie and H. R. Richardson (1991), "Mean-Variance Hedging in Continuous-Time", *Annals of Applied Probability 1*, 1–15

H. Föllmer and M. Schweizer (1991), "Hedging of Contingent Claims under Incomplete Information", *in: M. H. A. Davis and R. J. Elliott (eds.), "Applied Stochastic Analysis", Stochastics Monographs, Vol. 5, Gordon and Breach, London/New York*, 389–414

R. C. Merton (1973), "An Intertemporal Capital Asset Pricing Model", *Econometrica 41*, 867–887

P. Protter (1990), "Stochastic Integration and Differential Equations. A New Approach", *Springer, New York*

M. Schweizer (1991), "Option Hedging for Semimartingales", *Stochastic Processes and their Applications 37*, 339–363

M. Schweizer (1990), "Hedging and the CAPM", *unpublished manuscript*