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by

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On the Minimal Martingale Measure and the Föllmer-Schweizer Decomposition

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Abstract: We provide three characterizations of the minimal martingale measure $\widehat{P}$ associated to a given d-dimensional semimartingale X. In each case, $\widehat{P}$ is shown to be the unique solution of an optimization problem where one minimizes a certain functional over a suitable class of signed local martingale measures for X. Furthermore, we extend a result of Ansel and Stricker on the Föllmer-Schweizer decomposition to the case where X is continuous, but multidimensional.

Key words: minimal signed martingale measure, Föllmer-Schweizer decomposition, martingale densities, structure condition, semimartingales

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0. Introduction

Suppose $X = (X_t)_{0 \leq t \leq T}$ is an $\mathbb{R}^d$-valued adapted RCLL process. An important notion in financial mathematics is the concept of an equivalent martingale measure for X, i.e., a probability measure P^* equivalent to the original measure P such that X is a martingale under P^*. If such a P^* exists, then its density process Z^* with respect to P is a strict *martingale density* for X: Z^* is strictly positive, and both Z^* and $Z^* X$ are local martingales under P. Under some very weak integrability assumptions, one can show in turn that the existence of such a strict martingale density already implies a certain structure for X. In fact, X must be a *semimartingale* under P of the form (we take here $d = 1$ for notational simplicity)

$$X = X_0 + M + \int \alpha \, d\langle M \rangle$$

for some predictable process α. Moreover, every locally square-integrable martingale density Z can then be obtained as solution of a stochastic differential equation,

$$Z_t = 1 - \int_0^t Z_{s-} \alpha_s \, dM_s + R_t \qquad , \qquad 0 \leq t \leq T, \tag{0.1}$$

for some $R \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ strongly orthogonal to M. The multidimensional versions of these results are formulated and proved in section 1; they generalize previous work by Ansel/Stricker [1,2] and Schweizer [15].

In section 2, we provide three characterizations of the *minimal martingale measure*. This is the (possibly signed) measure $\widehat{P}$ associated to the minimal martingale density $\widehat{Z} := \mathcal{E}\left(-\int \alpha \, dM\right)$ corresponding to $R \equiv 0$ in (0.1). If X is continuous, we first characterize $\widehat{P}$ among all local martingale measures for X as the unique solution of a minimization problem involving the relative entropy with respect to P; this slightly extends a previous result of Föllmer/Schweizer [10]. Next we show that $\widehat{P}$ also minimizes

$$D(Q, P) := \left\| \frac{dQ}{dP} - 1 \right\|_{\mathcal{L}^2(P)} = \sqrt{\mathrm{Var}\left[\frac{dQ}{dP}\right]}$$

over all equivalent local martingale measures Q for X if, in addition, the random variable $\int_0^T \alpha_s^2 \, d\langle M \rangle_s$ is deterministic. By a completely different argument, we then prove that this characterization still holds if X is possibly discontinuous, provided that we minimize over all signed local martingale measures for X and that the entire process $\int \alpha^2 \, d\langle M \rangle$ is deterministic. These results give further support for the terminology "minimal martingale measure" used for $\widehat{P}$; they are of course stated and proved for $d \geq 1$.

In section 3, we study the *Föllmer-Schweizer decomposition* in the multidimensional case. Extending a result of Ansel/Stricker [1] to the case $d \geq 1$, we obtain a necessary and sufficient condition for the existence of such a decomposition if X is continuous. Furthermore, we also show how to deduce more specific integrability properties for the various terms in this decomposition from assumptions on α and the random variable H to be decomposed.

1. Martingale densities and the structure of X

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness, where $T > 0$ is a fixed and finite time horizon. All stochastic processes will be defined for $t \in [0, T]$. Let X be an $I\!\!F$-adapted $I\!\!R^d$-valued process such that X^i is right-continuous with left limits (RCLL for short) for $i = 1, \dots, d$. We recall from Schweizer [15] the following

Definition. A real-valued process Z is called a *martingale density* for X if Z is a local P-martingale with $Z_0 = 1$ P-a.s. and such that the product XZ is a local P-martingale. We can and shall always choose an RCLL version of Z. If Z is in addition strictly positive, Z is called a *strict martingale density* for X.

As explained in Schweizer [15], the concept of a strict martingale density for X generalizes the notion of an equivalent martingale measure for X. The existence of one or the other is closely related to a condition of absence of arbitrage for X and has therefore a very appealing economic interpretation; see for instance Delbaen/Schachermayer [6] for recent results and a comprehensive list of references. Note that any $I\!\!F$-adapted process X admitting a strict martingale density Z is necessarily a P-semimartingale. In fact, $\frac{1}{Z}$ is a P-semimartingale by Itô's formula, since Z is strictly positive, and X is the product of $\frac{1}{Z}$ and the local P-martingale XZ.

Our first result clarifies the structure of processes admitting a strict martingale density. For unexplained notations and terminology from martingale theory, we refer to Jacod [12] and Dellacherie/Meyer [7]. In order to abbreviate future statements, we say that X *satisfies the structure condition (SC)* if X is a special P-semimartingale with canonical decomposition

$$X = X_0 + M + A$$

which satisfies

$$M \in \mathcal{M}^2_{0,\mathrm{loc}}(P) \tag{1.1}$$

and

$$A^i \ll \langle M^i \rangle \qquad \text{with predictable density } \alpha^i \tag{1.2}$$

for $i = 1, \dots, d$, and if there exists a predictable process $\widehat{\lambda} \in L^2_{\mathrm{loc}}(M)$ with

$$\sigma_t \widehat{\lambda}_t = \gamma_t \qquad P\text{-a.s. for } t \in [0, T]. \tag{1.3}$$

The predictable processes σ and γ in (1.3) are defined by

$$\gamma^i_t := \alpha^i_t \sigma^{ii}_t \qquad \text{for } i = 1, \dots, d$$

and

$$\sigma^{ij}_t := \frac{d\langle M^i, M^j \rangle_t}{dB_t} \qquad \text{for } i, j = 1, \dots, d,$$

where B is a fixed increasing predictable RCLL process null at 0 such that $\langle M^i \rangle \ll B$ for each i. Such a process always exists, and it is easy to check that the stochastic integral $\int \widehat{\lambda}\, dM$ does not depend on the choice of $\widehat{\lambda}$ satisfying (1.3); see Jacod [12].

Theorem 1. *Suppose that X admits a strict martingale density Z^* and that either*

$$X \text{ is continuous} \tag{1.4}$$

or

$$X \text{ is a special semimartingale satisfying (1.1)} \tag{1.5a}$$

and

$$Z^* \in \mathcal{M}^2_{\mathrm{loc}}(P). \tag{1.5b}$$

Then X satisfies the structure condition (SC), and

$$\alpha^i \in L^2_{\mathrm{loc}}(M^i) \qquad \text{for } i = 1, \dots, d. \tag{1.6}$$

Furthermore, Z^* can be written as

$$Z^* = \mathcal{E}\left(- \int \widehat{\lambda}\, dM + L \right), \tag{1.7}$$

where $L \in \mathcal{M}_{0,\mathrm{loc}}(P)$ is strongly orthogonal to M^i for each i. If (1.5) holds, then we have $L \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$; if (1.4) holds, (1.7) can be simplified to

$$Z^* = \mathcal{E}\left(- \int \widehat{\lambda}\, dM \right) \mathcal{E}(L). \tag{1.8}$$

Proof. 1) Choose $N^1, \dots, N^d \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ pairwise strongly orthogonal such that each M^i is in the stable subspace of $\mathcal{M}^2_0(P)$ generated by $N^1, \dots, N^d$. Each M^i can then be written as

$$M^i = \sum_{j=1}^d \int \varrho^{ij}\, dN^j$$

for some predictable $d \times d$ matrix-valued process ϱ with $\varrho^{ij} \in L^2_{\mathrm{loc}}(N^j)$ for all i, j. If X (hence also M) is continuous, N can also be taken continuous. Choose an increasing predictable RCLL process B null at 0 with $\langle N^i \rangle \ll B$ for each i and set

$$\zeta^i_t := \frac{d\langle N^i \rangle_t}{dB_t} \qquad \text{for } i = 1, \dots, d.$$

By replacing N^i with $\int I_{\{\zeta^i \neq 0\}} \frac{1}{\sqrt{\zeta^i}}\, dN^i$, we may and shall assume without loss of generality that $\zeta^i_t \in \{0, 1\}$ for all i, t. Since

$$\int I_{\{\zeta^j = 0\}}\, d\langle N^j \rangle = \int \zeta^j I_{\{\zeta^j = 0\}}\, dB = 0, \tag{1.9}$$

we may and shall also assume that $\varrho^{ij}_t = 0$ on the set $\{\zeta^j_t = 0\}$ for all i, j, t. This implies that

$$\varrho^{ij}_t \zeta^j_t = \varrho^{ij}_t \qquad \text{for all } i, j, t \tag{1.10}$$

and therefore

$$\langle M^i, M^j \rangle = \sum_{k=1}^d \int \varrho^{ik} \varrho^{jk} \, d\langle N^k \rangle = \sum_{k=1}^d \int \varrho^{ik} \varrho^{jk} \zeta^k \, dB = \int (\varrho \varrho^{\mathrm{tr}})^{ij} dB$$

by the pairwise strong orthogonality of the components of N and (1.10). Hence we conclude that

$$\sigma_t = \varrho_t \varrho_t^{\mathrm{tr}} \qquad P\text{-a.s. for } t \in [0,T]. \tag{1.11}$$

2) Define $U \in \mathcal{M}_{0,\mathrm{loc}}(P)$ by $U := \int \frac{1}{Z^*_-} \, dZ^*$; this is well-defined since Z^* is strictly positive. Decompose U as $U = U^1 + U^2$ with $U^1 \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ and $U^2 \in \mathcal{M}_{0,\mathrm{loc}}(P)$ such that U^2 is strongly orthogonal to each N^i. In fact, we can choose $U^2 \equiv 0$ if (1.5b) holds, and under (1.4), we can take $U^1 = U^c$ and $U^2 = U^d$ as the continuous and purely discontinuous martingale parts of U, respectively. By the Galtchouk-Kunita-Watanabe decomposition theorem, U^1 can be written as

$$U^1 = -\sum_{j=1}^d \int \psi^j \, dN^j + R, \tag{1.12}$$

where $\psi^j \in L^2_{\mathrm{loc}}(N^j)$ and $R \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ is strongly orthogonal to N^j for each j. Furthermore, (1.9) implies that we can choose ψ such that

$$\psi^j_t = 0 \qquad \text{on the set } \{\zeta^j_t = 0\} \tag{1.13}$$

for all j, t. Applying the product rule to X^i and Z^* now yields

$$d(Z^* X^i) = \big(X^i_- \, dZ^* + Z^*_- \, dM^i + d[Z^*, A^i]\big) + Z^*_- \, dA^i + d[Z^*, M^i]. \tag{1.14}$$

Since Z^* is a strict martingale density for X, the left-hand side is (the differential of) a local P-martingale, and by Yoeurp's lemma, so is the term in brackets on the right-hand side. Furthermore, $Z^* = \mathcal{E}(U)$ implies that

$$[Z^*, M^i] = \int Z^*_- \, d[U, M^i],$$

and

$$\begin{aligned}
[U, M^i] &= \left[-\sum_{j=1}^d \int \psi^j \, dN^j + R + U^2, \sum_{k=1}^d \int \varrho^{ik} \, dN^k \right] \\
&= -\sum_{j=1}^d \int \psi^j \varrho^{ij} \, d\langle N^j \rangle + \left[R + U^2, \sum_{k=1}^d \int \varrho^{ik} \, dN^k \right] \\
&= -\sum_{j=1}^d \int \varrho^{ij} \psi^j \, dB + \left[R + U^2, \sum_{k=1}^d \int \varrho^{ik} \, dN^k \right]
\end{aligned}$$

by the pairwise strong orthogonality of the components of N and (1.10). Since the last term on the right-hand side is also a local P-martingale by the strong orthogonality of R and U^2 to each N^k, we conclude from (1.14) that

$$A^i = \int (\varrho \psi)^i \, dB \qquad \text{for } i = 1, \ldots, d, \tag{1.15}$$

since A is predictable and Z^*_- is strictly positive.

3) Now denote by $\widehat{\psi}$ the projection of ψ on $(\operatorname{Ker}\varrho)^\perp = \operatorname{Im}\varrho^{\mathrm{tr}}$ so that

$$\psi = \widehat{\psi} + \nu = \varrho^{\mathrm{tr}}\widehat{\lambda} + \nu \tag{1.16}$$

for some predictable processes $\widehat{\lambda}$, ν with $\varrho\nu = 0$. From (1.15) and (1.11), we then obtain

$$A^i = \int (\sigma\widehat{\lambda})^i \, dB \qquad \text{for } i = 1, \dots, d, \tag{1.17}$$

and since $\sigma_t^{ij} = 0$ on the set $\{\sigma_t^{ii} = 0\}$ by the Kunita-Watanabe inequality, we conclude that $A^i \ll \langle M^i \rangle$ with density

$$\alpha^i = \frac{(\sigma\widehat{\lambda})^i}{\sigma^{ii}}$$

and that $\widehat{\lambda}$ satisfies (1.3). Furthermore, we have

$$\begin{aligned}
\int (\alpha^i)^2 d\langle M^i \rangle &= \int \frac{1}{\sigma^{ii}} \left((\varrho\psi)^i \right)^2 dB \\
&\le \int \frac{1}{\sigma^{ii}} \sum_{j=1}^d (\varrho^{ij})^2 \sum_{j=1}^d (\psi^j)^2 \, dB \\
&= \sum_{j=1}^d \int (\psi^j)^2 \, dB \\
&= \sum_{j=1}^d \int (\psi^j)^2 \, d\langle N^j \rangle
\end{aligned}$$

by (1.15) and (1.17), the Cauchy-Schwarz inequality, (1.11) and (1.13). Because each ψ^j is in $L^2_{\mathrm{loc}}(N^j)$, this yields $\alpha^i \in L^2_{\mathrm{loc}}(M^i)$ for each i, hence (1.6). Similarly, (1.11), (1.16) and (1.13) imply that

$$\int \widehat{\lambda}^{\mathrm{tr}} \sigma \widehat{\lambda} \, dB = \int \left\| \varrho^{\mathrm{tr}} \widehat{\lambda} \right\|^2 dB \le \int \|\psi\|^2 \, dB = \sum_{j=1}^d \int (\psi^j)^2 \, d\langle N^j \rangle$$

and therefore $\widehat{\lambda} \in L^2_{\mathrm{loc}}(M)$ by (4.34) of Jacod [12]. In particular, the process $\int \widehat{\lambda} \, dM \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ is well-defined, and

$$\left\langle Y, \int \widehat{\lambda} \, dM \right\rangle = \sum_{i=1}^d \int \widehat{\lambda}^i \, d\langle Y, M^i \rangle = \sum_{i=1}^d \sum_{j=1}^d \int \widehat{\lambda}^i \varrho^{ij} \, d\langle Y, N^j \rangle = \left\langle Y, \sum_{j=1}^d \int (\varrho^{\mathrm{tr}} \widehat{\lambda})^j \, dN^j \right\rangle$$

for every $Y \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ shows that

$$\int \widehat{\lambda} \, dM = \sum_{j=1}^d \int (\varrho^{\mathrm{tr}} \widehat{\lambda})^j \, dN^j.$$

Hence $Z^* = \mathcal{E}(U)$ with

$$U = -\int \widehat{\lambda}\,dM + R + U^2 - \sum_{j=1}^{d} \int \nu^j\,dN^j$$

by (1.12) and (1.16), and since $\varrho\nu = 0$, $L := R + U^2 - \sum\limits_{j=1}^{d} \int \nu^j\,dN^j$ is strongly orthogonal to N^k, hence also to M^k, for each k. By (1.16), $\nu^j \in L^2_{\mathrm{loc}}(N^j)$ for each j, and so $L \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ under (1.5b). Finally, (1.4) implies that

$$\left[L, \int \widehat{\lambda}\,dM\right] = \left\langle L, \int \widehat{\lambda}\,dM \right\rangle = 0,$$

since M is continuous and L is strongly orthogonal to each M^k; thus (1.7) implies (1.8) by Proposition (6.4) of Jacod [12].

q.e.d.

Remark. Theorem 1 is at the same time a unification and a slight generalization of previous results. For the case where X is continuous and admits an equivalent martingale measure, the theorem was proved by Ansel/Stricker [1,2]; the scheme of the preceding proof is essentially due to them. The extension to general X admitting a locally square-integrable strict martingale density was obtained in Schweizer [15] under an invertibility assumption on the process σ. For related results with $d = 1$, see also Christopeit/Musiela [5].

The next result is a sort of converse to Theorem 1; it provides in addition a characterization of all martingale densities Z in $\mathcal{M}^2_{\mathrm{loc}}(P)$ if X satisfies the structure condition (SC). Note that (1.20) below is more general than (1.7) since Z may vanish or even become negative.

Proposition 2. *Suppose that X satisfies the structure condition (SC). Then*

$$\widehat{Z} := \mathcal{E}\left(-\int \widehat{\lambda}\,dM\right) \tag{1.18}$$

is a martingale density for X; $\widehat{Z}$ is a strict martingale density for X if and only if

$$\widehat{\lambda}^{\mathrm{tr}}_t \Delta M_t < 1 \qquad P\text{-a.s. for } t \in [0,T]. \tag{1.19}$$

More generally, $Z \in \mathcal{M}^2_{\mathrm{loc}}(P)$ is a martingale density for X if and only if Z satisfies the stochastic differential equation

$$Z_t = 1 - \int_0^t Z_{s-}\widehat{\lambda}_s\,dM_s + R_t \qquad , \qquad 0 \le t \le T \tag{1.20}$$

for some $R \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ strongly orthogonal to M^i for each i.

Proof. For $d = 1$, the second assertion is due to Yoeurp/Yor [17], Théorème 2.1; for $d \ge 1$, the "only if" part can be proved exactly as in Proposition 5 of Schweizer [15] since $\widehat{\lambda}$ exists and satisfies (1.3). Conversely, if $Z \in \mathcal{M}^2_{\mathrm{loc}}(P)$ satisfies (1.20), the product rule yields

$$d(ZX^i) = \Big(X^i_-\,dZ + Z_-\,dM^i + d[Z,A^i] + d[Z,M^i] - d\langle Z,M^i\rangle\Big) + Z_-\,dA^i + d\langle Z,M^i\rangle.$$

By Yoeurp's lemma, the term in brackets is (the differential of) a local P-martingale, and by (1.2), (1.20) and (1.3), we have

$$Z_- \, dA^i + d\langle Z, M^i \rangle = Z_- \gamma^i \, dB - Z_- (\sigma \widehat{\lambda})^i \, dB = 0.$$

Thus Z is a martingale density which proves the "if" part. Finally, (1.18) follows from (1.20) for $R \equiv 0$.

q.e.d.

Corollary 3. *A continuous $I\!\!F$-adapted process X admits a strict martingale density if and only if it satisfies the structure condition (SC).*

Proof. Since continuity of X implies (1.19), this follows immediately from Theorem 1 and Proposition 2.

q.e.d.

2. Characterizations of the minimal martingale measure

Let X be an $I\!\!F$-adapted $I\!\!R^d$-valued RCLL process. If Z is any martingale density for X, we can define a signed measure $Q \ll P$ on $(\Omega, \mathcal{F})$ by setting

$$\frac{dQ}{dP} := Z_T.$$

If Z is not only a local P-martingale, but a P-martingale, then $E[Z_T] = 1$ and Q is a signed local martingale measure for X in the sense of the following

Definition. A signed measure Q on $(\Omega, \mathcal{F})$ is called a *signed local martingale measure* for X if $Q[\Omega] = 1$, $Q \ll P$ on $\mathcal{F}_T$, $Q = P$ on $\mathcal{F}_0$ and X is a local Q-martingale in the sense that (an RCLL version of) the density process $\left(\frac{dQ}{dP} \big|_{\mathcal{F}_t} \right)_{0 \le t \le T}$ is a martingale density for X. Q is called a *local martingale measure* for X if in addition, Q is a measure, i.e., nonnegative, and an *equivalent local martingale measure* for X if in addition, $Q \approx P$ on $\mathcal{F}_T$.

Definition. Suppose that X satisfies the structure condition (SC). The increasing predictable process $\widehat{K}$ defined by

$$\widehat{K}_t := \int_0^t \widehat{\lambda}_s^{\mathrm{tr}} \, dA_s = \int_0^t \widehat{\lambda}_s^{\mathrm{tr}} \sigma_s \widehat{\lambda}_s \, dB_s = \left\langle \int \widehat{\lambda} \, dM \right\rangle_t \tag{2.1}$$

is called the *mean-variance tradeoff process* of X; we always choose an RCLL version. If $\widehat{Z} = \mathcal{E}\left(- \int \widehat{\lambda} \, dM \right)$ is a martingale, the signed local martingale measure $\widehat{P}$ with density $\widehat{Z}_T$ with respect to P is called the *minimal signed local martingale measure* for X.

It is clear from (1.20) that $\widehat{Z}$ is in a sense the simplest martingale density for X. Originally, however, the expression "minimal" was motivated in a different way when $\widehat{P}$ was first introduced in Föllmer/Schweizer [10]. They studied the case where X is continuous and $\widehat{Z}$ is square-integrable; for more general situations and various properties of $\widehat{P}$, see also Ansel/Stricker [1,2], El Karoui/Quenez [8], Hofmann/Platen/Schweizer [11] and the references contained in these papers. Our goal in this section is to give three characterizations of $\widehat{P}$ by proving certain minimality properties within a suitable class of signed local martingale measures for X.

A first characterization in terms of a relative entropy can be obtained if X is continuous; Theorem 5 below is a slight refinement of the basic result due to Föllmer/Schweizer [10]. If Q and P are probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra, the relative entropy on $\mathcal{G}$ is defined by

$$H_{\mathcal{G}}(Q|P) := \begin{cases} E_Q\left[\log \left.\dfrac{dQ}{dP}\right|_{\mathcal{G}}\right] & \text{, if } Q \ll P \text{ on } \mathcal{G} \\ +\infty & \text{, otherwise.} \end{cases}$$

Recall that $H_{\mathcal{G}}(Q|P)$ is always nonnegative, increasing in $\mathcal{G}$, and that $H(Q|P) := H_{\mathcal{F}}(Q|P)$ is 0 if and only if $Q = P$.

Lemma 4. *Suppose that X is a continuous $\mathbb{F}$-adapted process admitting a strict martingale density and that $E[\widehat{Z}_T] = 1$. If Q is any local martingale measure for X with*

$$H(Q|P) < \infty, \tag{2.2}$$

then

$$E_Q[\log \widehat{Z}_T] = \frac{1}{2} E_Q[\widehat{K}_T] < \infty \tag{2.3}$$

and

$$H(Q|\widehat{P}) = H(Q|P) - \frac{1}{2} E_Q[\widehat{K}_T]. \tag{2.4}$$

Proof. Since X is continuous, $\widehat{Z}$ is a strictly positive local martingale and therefore a martingale because of $E[\widehat{Z}_T] = 1$. Thus $\widehat{P} \approx P$ and so (2.2) implies that $Q \ll \widehat{P}$ and

$$\frac{dQ}{dP} = \widehat{Z}_T \frac{dQ}{d\widehat{P}}. \tag{2.5}$$

Moreover, the stochastic integral $\int \widehat{\lambda}\, dX$ is well-defined under Q, the same as under P and a local Q-martingale; see Propriété f) of Chou/Meyer/Stricker [4] and Proposition 1 of Emery [9], respectively. If $(T_n)_{n\in\mathbb{N}}$ is a localizing sequence for $\int \widehat{\lambda}\, dX$ under Q, (2.2) yields

$$\sup_{n\in\mathbb{N}} H_{\mathcal{F}_{T_n}}(Q|P) \le H(Q|P) < \infty$$

and therefore

$$\sup_{n\in\mathbb{N}} \left| \log \left.\frac{dQ}{dP}\right|_{\mathcal{F}_{T_n}} \right| \in \mathcal{L}^1(Q) \tag{2.6}$$

by Lemma 2 of Barron [3]. Furthermore, (2.1) implies that

$$\log \widehat{Z}_{T_n} = -\int_0^{T_n} \widehat{\lambda}_s \, dM_s - \frac{1}{2}\widehat{K}_{T_n} = -\int_0^{T_n} \widehat{\lambda}_s \, dX_s + \frac{1}{2}\widehat{K}_{T_n}$$

and therefore

$$E_Q\left[\log \widehat{Z}_{T_n}\right] = \frac{1}{2}E_Q\left[\widehat{K}_{T_n}\right] \geq 0, \tag{2.7}$$

hence

$$\sup_{n\in\mathbb{N}} H_{\mathcal{F}_{T_n}}(Q|\widehat{P}) \leq \sup_{n\in\mathbb{N}} H_{\mathcal{F}_{T_n}}(Q|P) \leq H(Q|P) < \infty$$

by (2.5) and (2.2) and thus

$$\sup_{n\in\mathbb{N}} \left| \log \frac{dQ}{d\widehat{P}}\bigg|_{\mathcal{F}_{T_n}} \right| \in \mathcal{L}^1(Q)$$

again by Lemma 2 of Barron [3]. Combining this with (2.6) and (2.5) shows that

$$\sup_{n\in\mathbb{N}} \left|\log \widehat{Z}_{T_n}\right| \in \mathcal{L}^1(Q),$$

and passing to the limit in (2.7) yields by continuity of $\widehat{Z}$ and $\widehat{K}$ the equality in (2.3). Since

$$\frac{1}{2}E_Q\left[\widehat{K}_{T_n}\right] = H_{\mathcal{F}_{T_n}}(Q|P) - H_{\mathcal{F}_{T_n}}(Q|\widehat{P}) \leq H(Q|P)$$

for all n by (2.7) and (2.5), we obtain (2.3) by monotone integration. Finally, (2.4) follows from (2.5) and (2.3).

q.e.d.

Theorem 5. *Suppose that X is a continuous $\mathbb{F}$-adapted process admitting a strict martingale density and that $E[\widehat{Z}_T] = 1$. If $H(\widehat{P}|P) < \infty$, then $\widehat{P}$ is the unique solution of*

(2.8) *Minimize $H(Q|P) - \frac{1}{2}E_Q[\widehat{K}_T]$ over all local martingale measures Q for X satisfying the "finite energy condition" $E_Q[\widehat{K}_T] < \infty$.*

Proof. Due to Lemma 4, $\widehat{P}$ satisfies the condition in (2.8). If $H(Q|P) = \infty$, there is nothing to prove; otherwise, Lemma 4 implies that

$$H(Q|P) - \frac{1}{2}E_Q[\widehat{K}_T] = H(Q|\widehat{P}) \geq 0$$

by (2.4), with equality if and only if $Q = \widehat{P}$.

q.e.d.

Corollary 6. *Suppose that X is a continuous $\mathbb{F}$-adapted process admitting a strict martingale density. If $\widehat{K}_T$ is bounded, then $\widehat{P}$ is the unique solution of*

Minimize $H(Q|P) - \frac{1}{2}E_Q[\widehat{K}_T]$ over all local martingale measures Q for X.

In particular, if $\widehat{K}_T$ is deterministic, then $\widehat{P}$ minimizes the relative entropy $H(Q|P)$ over all local martingale measures Q for X.

Proof. If $\widehat{K}$ is bounded, (2.1) implies by the continuity of M that $\widehat{Z}_T$ is in $\mathcal{L}^p(P)$ for every $p < \infty$ and in particular $H(\widehat{P}|P) < \infty$. Hence the assertion follows from Theorem 5.

q.e.d.

If we measure the distance from a given probability measure by the relative entropy, Corollary 6 shows that within the class of all local martingale measures for X, $\widehat{P}$ is closest to the original measure P if X is continuous and $\widehat{K}_T$ is deterministic. Our second characterization of $\widehat{P}$ gives a similar result if we replace the relative entropy by the χ^2-distance

$$D(Q,P) := \left\| \frac{dQ}{dP} - 1 \right\|_{\mathcal{L}^2(P)} = \sqrt{\operatorname{Var}\left[\frac{dQ}{dP}\right]} .$$

In the following, all expectations without subscript are with respect to P.

Theorem 7. *Suppose that X is a continuous $\mathbb{F}$-adapted process admitting a strict martingale density. If $\widehat{K}_T$ is deterministic, then $\widehat{P}$ is the unique solution of*

(2.9) *Minimize $D(Q,P)$ over all equivalent local martingale measures Q for X with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$.*

Proof. Since X is continuous, (2.1) implies that

$$\widehat{Z} = \exp\left(-\int \widehat{\lambda}\, dX + \frac{1}{2}\widehat{K}\right) = \mathcal{E}\left(-\int \widehat{\lambda}\, dX\right) e^{\widehat{K}} \tag{2.10}$$

and

$$\frac{1}{\widehat{Z}} = \mathcal{E}\left(\int \widehat{\lambda}\, dX\right). \tag{2.11}$$

Since $\widehat{K}_T$ is deterministic, hence bounded, we deduce first that

$$\widehat{Z} \in \mathcal{M}^r(P) \qquad \text{for every } r \geq 1; \tag{2.12}$$

hence $\widehat{P}$ is well-defined and satisfies the condition in (2.9). As in the proof of Lemma 4, $\int \widehat{\lambda}\, dX$ is a continuous local $\widehat{P}$-martingale, and the boundedness of $\widehat{K} = \left\langle \int \widehat{\lambda}\, dX \right\rangle$ implies that

$$\mathcal{E}\left(-\int \widehat{\lambda}\, dX\right) \in \mathcal{M}^r(\widehat{P}) \qquad \text{for every } r \geq 1 \tag{2.13}$$

and

$$\frac{1}{\widehat{Z}} \in \mathcal{M}^r(\widehat{P}) \qquad \text{for every } r \geq 1. \tag{2.14}$$

If Q is any signed local martingale measure for X satisfying the integrability condition in (2.9), so is $R := 2Q - \widehat{P}$, and $Q = \widehat{P} + \frac{1}{2}(R - \widehat{P})$ yields

$$D^2(Q,P) = D^2(\widehat{P},P) + \frac{1}{4}E\left[\left(\frac{dR}{dP} - \frac{d\widehat{P}}{dP}\right)^2\right] + E\left[\left(\frac{dR}{dP} - \frac{d\widehat{P}}{dP}\right)\left(\frac{d\widehat{P}}{dP} - 1\right)\right].$$

But the last term equals $E_R[\widehat{Z}_T] - E_{\widehat{P}}[\widehat{Z}_T]$, and thus it only remains to show that $E_R[\widehat{Z}_T]$ is constant over all signed local martingale measures R for X satisfying the integrability condition in (2.9). If Z is the density process of any such R, then ZX is a local P-martingale, hence so is the product of Z and $\int \widehat{\lambda}\, dX$, and we conclude that

$$Z\mathcal{E}\left(-\int \widehat{\lambda}\, dX\right) \in \mathcal{M}_{\text{loc}}(P).$$

But $Z \in \mathcal{M}^2(P)$ by (2.9) and

$$E\left[\sup_{0\le t\le T}\left|\mathcal{E}\left(-\int \widehat{\lambda}\, dX\right)_t\right|^2\right] = \widehat{E}\left[\frac{1}{\widehat{Z}_T}\sup_{0\le t\le T}\left|\mathcal{E}\left(-\int \widehat{\lambda}\, dX\right)_t\right|^2\right] < \infty$$

by (2.14) and (2.13) and so $Z\mathcal{E}\left(-\int \widehat{\lambda}\, dX\right)$ is a true P-martingale. Since $\widehat{K}_T$ is deterministic, we conclude from (2.10) that

$$E_R[\widehat{Z}_T] = e^{\widehat{K}_T}$$

for every signed local martingale measure R satisfying the integrability condition in (2.9), and this completes the proof.

q.e.d.

For a general, not necessarily continuous process X, we have a third characterization of $\widehat{P}$ under the stronger assumption that the entire process $\widehat{K}$ is deterministic. Although Theorem 8 looks quite similar to Theorem 7, we believe it is worth stating separately, because its proof is entirely different from the preceding one.

Theorem 8. *Suppose that X satisfies the structure condition (SC). If the mean-variance tradeoff process $\widehat{K}$ of X is deterministic, then $\widehat{P}$ is the unique solution of*

(2.15) *Minimize $D(Q,P)$ over all signed local martingale measures Q for X with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$.*

Proof. Since $\widehat{K}$ is deterministic, hence bounded, $\widehat{Z} \in \mathcal{M}^2(P)$ by Théorème II.2 of Lepingle/Mémin [14], and so $\widehat{P}$ satisfies the conditions in (2.15). Now fix any Q as in (2.15) and denote by Z its density process with respect to P. Then $Z \in \mathcal{M}^2(P)$ is a martingale density for X and therefore

$$\langle Z\rangle_t = \int_0^t Z_{s-}^2\, d\widehat{K}_s + \langle R\rangle_t \qquad , \qquad 0 \le t \le T$$

for some $R \in \mathcal{M}^2_{0,\mathrm{loc}}(P)$ by (1.20) and (2.1). Since $Z^2 - \langle Z \rangle$ is a P-martingale whose initial value is $Z_0^2 = 1$ because of $Q = P$ on $\mathcal{F}_0$, we thus have

$$E[Z_t^2] - 1 = E[\langle Z \rangle_t] = \int_0^t E[Z_{s-}^2] \, d\widehat{K}_s + E[\langle R \rangle_t] \tag{2.16}$$

for all $t \in [0,T]$, where the last step uses Fubini's theorem and the fact that $\widehat{K}$ is deterministic. If we now define the functions h and g on $[0,T]$ by

$$h(t) := E[Z_t^2] \qquad , \qquad 0 \le t \le T$$

and

$$g(t) := 1 + E[\langle R \rangle_t] \qquad , \qquad 0 \le t \le T,$$

then $Z \in \mathcal{M}^2(P)$ implies that

$$h(s-) = E[Z_{s-}^2].$$

Thus (2.16) shows that h satisfies the equation

$$h(t) = g(t) + \int_0^t h(s-) \, d\widehat{K}_s \qquad , \qquad 0 \le t \le T,$$

and by Théorème (6.8) of Jacod [12], h is therefore given by

$$h(t) = \mathcal{E}(\widehat{K})_t + \int_0^t \frac{\mathcal{E}(\widehat{K})_t}{\mathcal{E}(\widehat{K})_s} \, dg(s) \qquad , \qquad 0 \le t \le T, \tag{2.17}$$

since $\widehat{K}$ is increasing, hence $\Delta \widehat{K} > -1$. But since $\widehat{K}$ and g are both increasing and nonnegative, we obtain

$$E\left[\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_t} - 1\right)^2\right] = E[Z_t^2] - 1 \ge \mathcal{E}(\widehat{K})_t - 1 = E\left[\left(\frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t} - 1\right)^2\right],$$

where the last equality follows from (2.17) with $g \equiv 1$ which corresponds to $R \equiv 0$, i.e., $Q = \widehat{P}$; the inequality is strict unless $R \equiv 0$ P-a.s., i.e., $Q = \widehat{P}$, and this proves the assertion.

q.e.d.

Remark. Actually, the preceding argument shows that $\widehat{P}$ even minimizes $D\left(Q\big|_{\mathcal{F}_t}, P\big|_{\mathcal{F}_t}\right)$ for each $t \in [0,T]$ over all signed local martingale measures Q for X such that $\frac{dQ}{dP} \in \mathcal{L}^2(P)$; the assumption that $\widehat{K}$ is deterministic seems therefore stronger than really necessary to obtain Theorem 8.

3. On the Föllmer-Schweizer decomposition

Let X be a continuous $I\!\!F$-adapted $I\!\!R^d$-valued process admitting a strict martingale density and denote by $\widehat{Z} = \mathcal{E}\left(-\int \widehat{\lambda}\, dM\right)$ the minimal martingale density for X. We recall from Ansel/Stricker [1] the following

Definition. An $\mathcal{F}_T$-measurable random variable H is said to admit a *generalized Föllmer-Schweizer decomposition* if there exist a constant H_0, a predictable X-integrable process ξ^H and a local P-martingale L^H strongly orthogonal to M^i for each i such that H can be written as

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H \qquad P\text{-a.s.} \tag{3.1}$$

and such that the process $\widehat{Z}\widehat{V}$ is a P-martingale, where

$$\widehat{V} := H_0 + \int \xi^H dX + L^H. \tag{3.2}$$

Recall from Jacod [13] and Chou/Meyer/Stricker [4] that a (possibly not locally bounded) predictable process ξ is called X*-integrable* with respect to the semimartingale X if the sequence $Y^n = \int \xi I_{\{|\xi|\leq n\}}\, dX$ converges to a semimartingale Y in the semimartingale topology; the limit Y is then denoted by $\int \xi\, dX$ and called the stochastic integral of ξ with respect to X. We do not explain here how the semimartingale topology is defined; we only remark that those results of Chou/Meyer/Stricker [4] that we use below extend in a straightforward fashion from their situation of a real-valued X to the case where X takes values in $I\!\!R^d$. As a matter of fact, the definition of a generalized Föllmer-Schweizer decomposition given in Ansel/Stricker [1] is slightly different. They assume $\mathcal{F}_0$ to be trivial and allow X to be possibly discontinuous. However, the proof of Theorem 9 below shows that L^H is null at 0 if $\mathcal{F}_0$ is trivial, and thus it follows from their Remarque (ii) that the two definitions agree for X continuous and $\mathcal{F}_0$ trivial.

The main result of this section is a necessary and sufficient condition for H to admit a generalized Föllmer-Schweizer decomposition. For the case $d = 1$, i.e., if X is real-valued, this is due to Ansel/Stricker [1]; we shall comment below on the difficulty in the multidimensional case. Before we state our theorem, we introduce some notation. For any stochastic process Y and any stopping time S, we denote by

$$Y^S = (Y_t^S)_{0\leq t\leq T} := (Y_{t\wedge S})_{0\leq t\leq T}$$

the process Y stopped at S. Furthermore, $\langle M^i\rangle^{\mathrm{qv}}$ denotes the pathwise quadratic variation of M^i along a fixed sequence $(\tau_n)_{n\in I\!\!N}$ of partitions of $[0,T]$ whose mesh size $|\tau_n| := \max\limits_{t_\ell, t_{\ell+1}\in\tau_n} |t_{\ell+1} - t_\ell|$ tends to 0 as $n \to \infty$. Then

$$\langle M^i\rangle^{\mathrm{qv}} = \langle M^i\rangle^P = [M^i] \qquad P\text{-a.s. for } i = 1,\dots,d. \tag{3.3}$$

Recall that $\langle M^i\rangle^P$ is the sharp bracket process associated to M^i with respect to P; the notational distinction between $\langle M^i\rangle^{\mathrm{qv}}$ and $\langle M^i\rangle^P$ is made to clarify which definition of $\langle M^i\rangle$ is used.

Theorem 9. *Suppose that X is a continuous $I\!\!F$-adapted process admitting a strict martingale density. An $\mathcal{F}_T$-measurable random variable H admits a generalized Föllmer-Schweizer decomposition if and only if H satisfies*

$$H\widehat{Z}_T \in \mathcal{L}^1(P). \tag{3.4}$$

Proof. 1) Define the process N by

$$N_t := \frac{1}{\widehat{Z}_t} E[H\widehat{Z}_T|\mathcal{F}_t] \tag{3.5}$$

so that $N\widehat{Z}$ is a P-martingale by (3.4). Choose a localizing sequence $(T_m)_{m\in I\!\!N}$ for the local P-martingales $\widehat{Z}$ and $X\widehat{Z}$, and define for each $m \in I\!\!N$ the probability measure $\widehat{P}^m$ on $(\Omega, \mathcal{F})$ by

$$d\widehat{P}^m := \widehat{Z}_T^{T_m}\, dP = \widehat{Z}_{T_m}\, dP.$$

Since $\widehat{Z}$ is strictly positive, $\widehat{P}^m$ is equivalent to P, and

$$N^{T_m}\widehat{Z}^{T_m} = (N\widehat{Z})^{T_m}$$

is a P-martingale by the stopping theorem; hence N^{T_m} is a $\widehat{P}^m$-martingale for each m, and so is X^{T_m} by the same argument. Now fix $m \in I\!\!N$ and write

$$N^{T_m} = N_0 + (N^{T_m})^c + (N^{T_m})^d \tag{3.6}$$

for the decomposition of N^{T_m} with respect to $\widehat{P}^m$ into a continuous and a purely discontinuous local $\widehat{P}^m$-martingale. Since both $(N^{T_m})^c$ and X^{T_m} are continuous local $\widehat{P}^m$-martingales, the Galtchouk-Kunita-Watanabe decomposition theorem implies that

$$(N^{T_m})^c = \int \xi^m\, dX^{T_m} + L^m \tag{3.7}$$

for a unique predictable process $\xi^m \in L^2_{\text{loc}}(X^{T_m}, \widehat{P}^m)$ and a unique continuous L^m in $\mathcal{M}^2_{0,\text{loc}}(\widehat{P}^m)$ strongly $\widehat{P}^m$-orthogonal to each $(X^{T_m})^i$. In particular, ξ^m is X^{T_m}-integrable with respect to $\widehat{P}^m$ by Propriété c) of Chou/Meyer/Stricker [4]; hence Propriété f) of Chou/Meyer/Stricker [4] implies that ξ^m is also X^{T_m}-integrable with respect to $P \approx \widehat{P}^m$. Furthermore, (3.3) yields by polarization

$$\begin{aligned}
\big[(X^{T_m})^i, L^m + (N^{T_m})^d\big] &= \big\langle (X^{T_m})^i, L^m + (N^{T_m})^d\big\rangle^{\text{qv}} \\
&= \big\langle (X^{T_m})^i, L^m + (N^{T_m})^d\big\rangle^{\widehat{P}^m} \\
&= 0 \qquad P\text{-a.s. for } i = 1, \ldots, d,
\end{aligned} \tag{3.8}$$

since L^m and $(N^{T_m})^d$ are strongly $\widehat{P}^m$-orthogonal to $(X^{T_m})^i$, and $P \approx \widehat{P}^m$.

2) Now define processes ξ^H and L^H by setting

$$\xi^H := \xi^m \qquad \text{on } [\![0, T_m]\!] \tag{3.9}$$

and

$$L^H := N_0 - E[N_0] + L^m + (N^{T_m})^d \qquad \text{on } [\![0, T_m]\!]. \tag{3.10}$$

The first problem is then to show that these definitions make sense: since ξ^m, L^m and $(N^{T_m})^d$ are obtained by decomposing with respect to different measures $\widehat{P}^m$ for different m, it is not clear a priori that they are consistent in the sense that $\xi^{m+1} = \xi^m$ on $[\![0, T_m]\!]$ and so on. Consider first (3.6). We want to show that

$$\left((N^{T_{m+1}})^{x,\widehat{P}^{m+1}}\right)^{T_m} = (N^{T_m})^{x,\widehat{P}^m} \qquad \text{for } x \in \{c, d\}, \tag{3.11}$$

where the superscripts indicate the measures with respect to which the decomposition (3.6) is taken. Now first of all, $(N^{T_{m+1}})^{c,\widehat{P}^{m+1}} \in \mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^{m+1})$, so $(N^{T_{m+1}})^{c,\widehat{P}^{m+1}} \widehat{Z}^{T_{m+1}} \in \mathcal{M}^c_{\mathrm{loc}}(P)$, hence by stopping $\left((N^{T_{m+1}})^{c,\widehat{P}^{m+1}}\right)^{T_m}$ is in $\mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^m)$. An analogous argument shows that $\left((N^{T_{m+1}})^{d,\widehat{P}^{m+1}}\right)^{T_m}$ is in $\mathcal{M}_{\mathrm{loc}}(\widehat{P}^m)$. By the uniqueness of (3.6) with respect to $\widehat{P}^m$, (3.11) will be proved once we show that $R := \left((N^{T_{m+1}})^{d,\widehat{P}^{m+1}}\right)^{T_m}$ is strongly orthogonal to every $Y \in \mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^m)$. But since $\widehat{P}^{m+1}$ and $\widehat{P}^m$ are equivalent, Y is a continuous $\widehat{P}^{m+1}$-semimartingale and can therefore be written as

$$Y = U^{m+1} + B^{m+1}$$

with $U^{m+1} \in \mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^{m+1})$ and B^{m+1} continuous and of finite variation. Thus we obtain

$$\langle Y, R\rangle^{\widehat{P}^m} = [Y, R] = \left[U^{m+1}, (N^{T_{m+1}})^{d,\widehat{P}^{m+1}}\right]^{T_m} = \left(\left\langle U^{m+1}, (N^{T_{m+1}})^{d,\widehat{P}^{m+1}}\right\rangle^{\widehat{P}^{m+1}}\right)^{T_m} = 0$$

from (3.3) and the fact that U^{m+1} is continuous and $(N^{T_{m+1}})^{d,\widehat{P}^{m+1}}$ purely discontinuous with respect to $\widehat{P}^{m+1}$. This proves (3.11). Now consider (3.7). By (3.11),

$$\begin{aligned}
(N^{T_m})^{c,\widehat{P}^m} &= \left((N^{T_{m+1}})^{c,\widehat{P}^{m+1}}\right)^{T_m} \\
&= \left(\int \xi^{m+1} dX^{T_{m+1}}\right)^{T_m} + (L^{m+1})^{T_m} \\
&= \int \xi^{m+1} I_{[\![0,T_m]\!]}\, dX^{T_m} + (L^{m+1})^{T_m},
\end{aligned}$$

where the second equality uses (3.7) with $m+1$ instead of m. By the uniqueness of (3.7) with respect to $\widehat{P}^m$, it is therefore enough to show that $(L^{m+1})^{T_m}$ is strongly $\widehat{P}^m$-orthogonal to $(X^{T_m})^i$ for each i, since this implies both that $L^m = \left(L^{m+1}\right)^{T_m}$ and that $\xi^m = \xi^{m+1} I_{[\![0,T_m]\!]}$. But $L^{m+1} \in \mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^{m+1})$, so $(L^{m+1})^{T_m} \in \mathcal{M}^c_{\mathrm{loc}}(\widehat{P}^m)$ and therefore

$$\begin{aligned}
\left\langle (L^{m+1})^{T_m}, (X^{T_m})^i \right\rangle^{\widehat{P}^m} &= \left\langle (L^{m+1})^{T_m}, (X^{T_m})^i \right\rangle^{\mathrm{qv}} \\
&= \left(\left\langle L^{m+1}, X^i \right\rangle^{\mathrm{qv}}\right)^{T_m} \\
&= \left(\left\langle L^{m+1}, (X^{T_{m+1}})^i \right\rangle^{\mathrm{qv}}\right)^{T_m} \\
&= \left(\left\langle L^{m+1}, (X^{T_{m+1}})^i \right\rangle^{\widehat{P}^{m+1}}\right)^{T_m} \\
&= 0
\end{aligned}$$

by (3.3) and the strong $\widehat{P}^{m+1}$-orthogonality of L^{m+1} to $(X^{T_{m+1}})^i$ for each i. Thus ξ^H and L^H are indeed well-defined by (3.9) and (3.10), respectively.

3) Since each ξ^m is predictable and X^{T_m}-integrable, ξ^H is also predictable and X-integrable by Théorème 4 of Chou/Meyer/Stricker [4]. If we set

$$H_0 := E[N_0], \tag{3.12}$$

then (3.6), (3.7), (3.9) and (3.10) show that

$$N = H_0 + \int \xi^H dX + L^H = \widehat{V}$$

by (3.2), so (3.1) holds by the definition of N, and $\widehat{Z}\widehat{V} = \widehat{Z}N$ is a P-martingale. Since

$$[L^H, M^i]^{T_m} = [L^H, X^i]^{T_m} = \left(\langle L^H, X^i \rangle^{\mathrm{qv}}\right)^{T_m} = \langle L^m + (N^{T_m})^d, (X^{T_m})^i \rangle^{\mathrm{qv}} = 0 \tag{3.13}$$

for all m, i by the continuity of A, (3.3), (3.10) and (3.8), it only remains to show that L^H is a local P-martingale. To that end, it is enough to show that

$$(L^H)^{T_m} = L^m + (N^{T_m})^d + N_0 - E[N_0]$$

is a local P-martingale for each m, and since $\widehat{P}^m \approx P$ with density process $\widehat{Z}^{T_m}$, this is equivalent to showing that $L^m + (N^{T_m})^d$ is strongly $\widehat{P}^m$-orthogonal to $\frac{1}{\widehat{Z}^{T_m}}$ for each m. But (2.11) implies that

$$\frac{1}{\widehat{Z}^{T_m}} = \mathcal{E}\left(\int \widehat{\lambda}\, dX\right)^{T_m} = \mathcal{E}\left(\int \widehat{\lambda}\, dX^{T_m}\right),$$

hence the required strong orthogonality follows from (3.8), and this completes the proof.

q.e.d.

Remarks. 1) As mentioned above, Theorem 9 was already obtained by Ansel/Stricker [1] for the case $d = 1$. Their proof is considerably shorter since for $d = 1$, ξ^H can be defined directly by setting

$$\xi^H = \frac{d\langle X, N \rangle^{\mathrm{qv}}}{d\langle X \rangle^{\mathrm{qv}}}.$$

The properties of ξ^H and L^H are then derived by showing that on $[\![0, T_m]\!]$, ξ^H coincides with the integrand ξ^m in the Kunita-Watanabe decomposition (3.7). For $d > 1$, no such explicit formula for ξ^H is available and thus ξ^H and L^H must be pasted together as in the preceding proof.

2) If $E[\widehat{Z}_T] = 1$ so that $\widehat{Z}$ is not only a local martingale, but a true martingale under P, the proof of Theorem 9 also simplifies considerably. In fact, we can then argue directly with the minimal equivalent local martingale measure $\widehat{P}$ instead of using $\widehat{P}^m$. Thus ξ^H and L^H can immediately be constructed globally, and part 2) of the above proof can be dispensed with. In addition, the constant H_0 is then given by $H_0 = \widehat{E}[H]$, due to (3.12) and (3.5).

In some situations, it is desirable to have more integrability for ξ^H and L^H in the decomposition (3.1) of H; see for instance Föllmer/Schweizer [10] or Schweizer [16] for applications to hedging problems in financial mathematics. The next result shows how this can be deduced from assumptions on $\widehat{K}$ and H.

Corollary 10. *Suppose that X is a continuous $\mathbb{F}$-adapted process admitting a strict martingale density. If the mean-variance tradeoff process $\widehat{K}$ of X is bounded and if $H \in \mathcal{L}^p(P, \mathcal{F}_T)$ for some $p > 1$, then H admits a generalized Föllmer-Schweizer decomposition with $\xi^H \in L^r(M)$ and $L^H \in \mathcal{M}^r(P)$ for every $r < p$.*

Proof. Since $\widehat{K}$ is bounded, (2.12) holds; thus $\widehat{Z}$ is a martingale and the minimal equivalent local martingale measure $\widehat{P}$ exists. Now fix $p > 1$ and $H \in \mathcal{L}^p(P, \mathcal{F}_T)$. Then (2.12) implies by Hölder's inequality that $H\widehat{Z}_T$ is in $\mathcal{L}^1(P)$, so H admits a decomposition (3.1) by Theorem 9, and it only remains to prove the integrability assertions concerning ξ^H and L^H. But (3.2), (3.3), continuity of A and (3.13) imply that

$$\int_0^T (\xi_s^H)^{\mathrm{tr}} \sigma_s \xi_s^H \, dB_s = \left\langle \int \xi^H \, dM \right\rangle_T^P = \left[\int \xi^H dM \right]_T = \left[\int \xi^H dX \right]_T \leq [\widehat{V}]_T$$

and

$$[L^H]_T \leq [\widehat{V}]_T,$$

and since L^H is a local P-martingale we have

$$E\left[\left(\sup_{0 \leq t \leq T} |L_t^H| \right)^r \right] \leq \text{ const. } E\left[[L^H]_T^{\frac{r}{2}} \right]$$

for $r > 1$ by the Burkholder-Davis-Gundy inequality. Hence $\xi^H \in L^r(M)$ and $L^H \in \mathcal{M}^r(P)$ will both follow for every $r < p$ once we have proved that

$$[\widehat{V}]_T \in \mathcal{L}^{\frac{r}{2}}(P) \qquad \text{for every } r < p. \tag{3.14}$$

But $\widehat{V}\widehat{Z}$ is a P-martingale by Theorem 9, so $\widehat{V}$ is a $\widehat{P}$-martingale and thus

$$\widehat{E}\left[\left([\widehat{V}]_T \right)^s \right] \leq \text{ const. } \widehat{E}\left[\left(\sup_{0 \leq t \leq T} |\widehat{V}_t| \right)^{2s} \right] \leq \text{ const. } \widehat{E}\left[|\widehat{V}_T|^{2s} \right]$$

for $2s > 1$ by the Burkholder-Davis-Gundy inequality and Doob's inequality. Since $p > 1$ and $\widehat{V}_T = H \in \mathcal{L}^p(P)$ by (3.1), (2.12) implies that $\widehat{V}_T \in \mathcal{L}^{2s}(\widehat{P})$ for $2s < p$, hence $[\widehat{V}]_T \in \mathcal{L}^s(\widehat{P})$ for every $s < \frac{p}{2}$, so (3.14) follows by (2.14), and this completes the proof.

q.e.d.

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