Option Hedging for Semimartingales

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Abstract: We consider a general stochastic model of frictionless continuous trading. The price process is a semimartingale and the model is incomplete. Our objective is to hedge contingent claims by using trading strategies with a small riskiness. To this end, we introduce a notion of local R-minimality and show its equivalence to a new kind of stochastic optimality equation. This equation is solved by a Girsanov transformation to a minimal equivalent martingale measure. We prove existence and uniqueness of the solution, and we provide several examples. Our approach contains previous treatments of option trading as special cases.

Key words: option hedging
semimartingales
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optimality equation
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Black/Scholes model
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0. Introduction

The formula of Black and Scholes [1] for the valuation of options has led to the development of a general hedging method for contingent claims in a complete financial market by Harrison and Pliska [8]. In such a market, any claim can be replicated by a self-financing dynamic portfolio strategy which only makes use of the existing assets; in this sense, the claim is redundant. In an *incomplete* market, however, there exist non-redundant claims which carry an intrinsic risk, and any portfolio strategy generating such a claim will involve a random process of cumulative costs. In order to compare these strategies, a measure R of riskiness in terms of a conditional mean square error was introduced in Föllmer and Sondermann [6]. Although somewhat ad hoc from an economic point of view, this formulation permits one to apply martingale theory in a natural way. In particular, R-minimizing trading strategies turn out to be *mean-self-financing*, i.e., their cost process is a martingale. In the case where the stock price process X is a martingale under the basic probability measure P, existence and uniqueness of an R-minimizing strategy were proved in Föllmer and Sondermann [6], using the Kunita-Watanabe projection technique.

In this paper, we consider a general incomplete model where the price process X is assumed to be a *semimartingale* under P. This assumption is quite natural because it is implied by the existence of an equivalent martingale measure for X, i.e., a probability measure P^* equivalent to P such that X is a martingale under P^*. In turn, the existence of P^* corresponds to assuming the absence of arbitrage opportunities. Our purpose is to analyze the riskiness of non-redundant contingent claims and to determine optimal hedging strategies in this context. To this end, we have to modify the approach taken in Föllmer and Sondermann [6] since the Kunita-Watanabe projection technique does not apply directly to the case of a semimartingale. In section 2, the idea of keeping conditional variances as small as possible is now formalized in a *local* manner, leading to the notion of a locally R-minimizing strategy. We show that these strategies are mean-self-financing, and that they can be characterized by a stochastic *optimality equation*. This involves new results in Schweizer [13] on the differentiation of semimartingales and their connection to the orthogonality of martingales. In section 3 we solve the optimality equation. Existence and uniqueness of the solution are established under the assumption that P admits a *minimal* equivalent martingale measure $\widetilde{P}$. This measure has the property that it only turns X into a martingale and otherwise does not disturb the structure of the model. The optimal strategy for P is identified with the unique strategy which is R-minimizing for $\widetilde{P}$ in the sense of Föllmer and

Sondermann [6]. It should be pointed out that the minimal martingale measure plays here only the role of a tool; the option writer's subjective assessment of the market structure is given by the measure P, and so our formulation of the optimality criterion also uses P. But the identification of the optimal strategy in terms of $\widetilde{P}$ has an important consequence. It shows the invariance of an R-minimizing strategy within a certain class of equivalent semimartingale models. Thus a key feature of the complete market situation treated by Harrison and Pliska [8] is preserved in our incomplete model. The preceding results are illustrated in section 4 by two examples. The first one is an incomplete version of the Black/Scholes model with two sources of uncertainty, but only one stock. The second involves point processes and is related to dynamic reinsurance policies for stop-loss contracts; this approach was initiated by Sondermann [14].

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1. The basic model

This section has two purposes. First of all, we describe our model for option trading and introduce the required notation and terminology. After that, we review some previous results in the literature in order to motivate the subsequent development.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right-continuity and completeness. $T \in I\!R$ denotes a fixed and finite time horizon; furthermore, we assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$. Let $X = (X_t)_{0\leq t\leq T}$ be a semimartingale with a decomposition

$$X = X_0 + M + A \;, \tag{1.1}$$

such that

(X1) $M = (M_t)_{0\leq t\leq T}$ is a square-integrable martingale with $M_0 = 0$

and $A = (A_t)_{0\leq t\leq T}$ is a predictable process of finite variation $|A|$ with $A_0 = 0$. Additional assumptions on X will be introduced later on when the need arises. By (X1), M has a variance process $\langle M\rangle$ with respect to P, and we denote by P_M the measure $P\times\langle M\rangle$ on the product space $\overline{\Omega} := \Omega\times[0, T]$ with the σ-algebra $\mathcal{P}$ of predictable sets.

Definition. A *trading strategy* φ is a pair of processes $\xi = (\xi_t)_{0 \le t \le T}$, $\eta = (\eta_t)_{0 \le t \le T}$ satisfying the following conditions:

(1.2) ξ is predictable.

(1.3) The process $\int_0^t \xi_u \, dX_u \quad (0 \le t \le T)$ is a semimartingale of class $\mathcal{S}^2$.

(1.4) η is adapted.

(1.5) The process $V(\varphi)$ defined by $V_t(\varphi) := \xi_t \cdot X_t + \eta_t$ $(0 \le t \le T)$ is right-continuous and satisfies $V_t(\varphi) \in \mathcal{L}^2(P)$, $0 \le t \le T$.

The integrability condition (1.3) is equivalent to

$$E\left[\int_0^T \xi_u^2 \, d\langle M \rangle_u + \left(\int_0^T |\xi_u| \, d|A|_u\right)^2\right] < \infty$$

which means that

$$\xi \in \mathcal{L}^2(P_M) \quad \text{and} \quad \int_0^T |\xi_u| \, d|A|_u \in \mathcal{L}^2(P) \, .$$

In accordance with the usual terminology, the process $V(\varphi)$ will be called the *value process* of φ and the right-continuous square-integrable process $C(\varphi)$ defined by

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \xi_u \, dX_u \qquad , \qquad 0 \le t \le T \tag{1.6}$$

the (cumulative) *cost process* of φ. See Harrison and Pliska [8] and Föllmer and Sondermann [6] for a detailed motivation.

Interpretation. The process X is a model for the price evolution of a risky asset (called *stock*). We tacitly assume that there also exists a riskless asset (called *bond*) whose value is 1 at all times. Actually, any strictly positive continuous process of finite variation will do for this purpose; the normalization to 1 simply means that we work directly with discounted prices and helps to avoid more complicated notations. A trading strategy is interpreted as a *dynamic portfolio* of stock and bond: at time t, we hold ξ_t shares of stock and η_t unit bonds, and clearly $V_t(\varphi)$ is the value of this portfolio. (1.6) expresses the fact that the cumulative costs up to time t are equal to the current value of the portfolio reduced by the accumulated trading gains. The predictability condition (1.2) on ξ means that we have to determine the amount of shares before the next infinitesimal stock price movement is actually known. On the other hand, η is allowed to be adapted; this relaxation of the measurability condition on η was introduced and motivated by Föllmer and Sondermann [6]. In a complete financial market, the distinction does not matter because there the relevant trading strategies turn out to be predictable in both components. But in an incomplete situation, it will give us some extra freedom in adjusting the portfolio value to a desired level, and this will be essential.

A *contingent claim* H is intended to model the payoff at time T of some financial instrument. The simplest example is given by a European call option with fixed strike price $K \in I\!R$ where

$$H = (X_T - K)^+ .$$

This claim has the special form $H = h(X_T)$ for some function h. More generally, H could depend on the whole evolution of X up to time T. One example of such a path-dependent option would be a call on the average value of the stock, i.e.,

$$H = \left(X_T - \frac{1}{T} \cdot \int_0^T X_u \, du \right)^+ .$$

Depending on the structure of the filtration $(\mathcal{F}_t)$, even some external events could play a role. Note, however, that payoffs will be made at the terminal time T only. Assuming that we have sold such a claim, this means that we shall have to pay the amount H at time T. However, the exact size of this obligation is in general still uncertain at any time $t < T$, and it makes sense that we should like to reduce the inherent dangers of this uncertainty. To achieve this end, we have to use the available means: buying and selling stocks and bonds. A natural approach is therefore to look for a trading strategy which generates the required payoff H and at the same time minimizes some measure of riskiness. This idea will now be made precise.

In mathematical terms, a contingent claim is a random variable $H \in \mathcal{L}^2(P)$. We shall concentrate on strategies which are H*-admissible* in the sense that

$$V_T(\varphi) = H \qquad P - a.s. ; \tag{1.7}$$

φ is then said to *generate* H. Note that an H-admissible strategy always exists: we can simply choose $\xi \equiv 0$ and $\eta = 0$ except for $\eta_T = H$. This corresponds to "doing nothing until one has to pay up". As a measure of riskiness, we introduce for each strategy the *conditional mean square error process*

$$R_t(\varphi) := E\left[\left(C_T(\varphi) - C_t(\varphi)\right)^2 \middle| \mathcal{F}_t\right] \quad , \quad 0 \le t \le T, \tag{1.8}$$

defined as a right-continuous version. Following Harrison and Pliska [8], a strategy φ is called *self-financing* if its cost process $C(\varphi)$ is constant P-a.s. It is called *mean-self-financing* if $C(\varphi)$ is a martingale; this notion was introduced by Föllmer and Sondermann [6].

Remarks. 1) The criterion (1.8) is essentially a mean-variance criterion; in fact, $R_t(\varphi)$ is simply the conditional variance of the total cost $C_T(\varphi)$, given the information up to time t, if the strategy φ is mean-self-financing. Lemma 1.2 below will show that we can restrict ourselves to this class of strategies.

2) The main reason for our use of P in the definition of $R_t(\varphi)$ is that P is intended to model the subjective beliefs of the option writer. It has been suggested by the referee to use a risk-neutral probability (i.e., a martingale measure) $\widehat{P}$ instead of P in (1.8). In that case, the subsequent discussion would simply reduce to the martingale case treated by Föllmer and Sondermann [6]. As it turns out from Theorem 3.2 below, the optimal strategy can in fact be described in terms of a certain minimal martingale measure $\widetilde{P}$, and this proves the robustness of the criterion (1.8) and of the corresponding optimal strategy under certain equivalent changes of measure. But this result is only meaningful if our analysis is done in terms of P. If we had used a martingale measure in the definition of $R_t(\varphi)$, robustness would only hold by definition.

Definition. A contingent claim H is called *attainable* if it is of the form

$$H = H_0 + \int_0^T \xi^*_u \, dX_u \qquad P-a.s. \tag{1.9}$$

with a constant H_0 and a predictable process ξ^* satisfying (1.3).

Proposition 1.1. *Let H be a contingent claim. The following statements are equivalent:*

1) *There exists a self-financing H-admissible trading strategy φ.*

2) *There exists an H-admissible trading strategy φ with $R_0(\varphi) = 0$.*

3) *There exists an H-admissible trading strategy φ with*

$$R_t(\varphi) = 0 \quad P - a.s. \quad , \quad 0 \le t \le T.$$

4) *H is attainable.*

Proof. Since 1), 2) and 3) are clearly equivalent, it is enough to show the equivalence of 1) and 4). But 1) yields

$$H = V_T(\varphi) = C_T(\varphi) + \int_0^T \xi_u \, dX_u = C_0(\varphi) + \int_0^T \xi_u \, dX_u \qquad P - a.s. \, ,$$

and 4) allows us to define $\varphi = (\xi^*, \eta)$ by

$$V_t(\varphi) = H_0 + \int_0^t \xi_u^* \, dX_u \qquad , \qquad 0 \le t \le T.$$

q.e.d.

Remark. The crucial equivalence between 1) and 4) was proved by Harrison and Pliska [8]; the explicit use of the process $R(\varphi)$ in 2) and 3) appears in Föllmer and Sondermann [6]. An attainable claim H has some very special features. First of all, it is riskless in the following sense: If we start with the non-random initial amount $C_0(\varphi) = H_0$ and then use the above self-financing strategy, we can exactly duplicate the cash-flow induced by H. In our idealized model of frictionless continuous trading, H and φ are therefore equivalent. This implies that the price of H is uniquely determined and must be H_0 in order to exclude the possibility of arbitrage. An easy way to compute both the price H_0 and the generating strategy ξ^* is provided by the use of an equivalent martingale measure P^* for X. (1.9) then implies

$$H_0 = E^*[H]$$

and

$$\xi_t^* = \frac{d\langle V^*, X\rangle_t^{P^*}}{d\langle X\rangle_t^{P^*}} \qquad , \qquad 0 \le t \le T$$

where V^* denotes a right-continuous version of the process

$$V_t^* := E^*[H|\mathcal{F}_t] \qquad , \qquad 0 \le t \le T.$$

Thus, the optimal strategy can be expressed in terms of P^* alone. In particular, it does not depend on the subjective beliefs described by the original measure $P \approx P^*$.

The preceding discussion of attainable claims is of course well-known. The arbitrage argument yielding H_0 as the fair option price in this model is exactly the one made already by Black and Scholes [1], with the small difference that they used stock and option to form a portfolio earning the riskless rate of return. They worked with a specific model where X follows a geometric Brownian motion. This was generalized by Harrison and Pliska [8], [9] who treated the case of a so-called *complete* market, i.e., a situation where every contingent claim is attainable. Harrison and Pliska [9] also proved that a model is complete if and only if it admits a unique equivalent martingale measure P^* for X. However, all these contributions were limited to attainable claims. But as Hakansson [7] pointed out, any attainable claim is essentially redundant because it can be duplicated by using the already existing assets, namely stock and bond. Starting from this observation, Föllmer and Sondermann [6] introduced the process $R(\varphi)$ and formulated the following optimization problem.

Definition. Let $\varphi = (\xi, \eta)$ be a trading strategy and $t \in [0, T)$. An *admissible continuation of φ from t on* is a trading strategy $\widetilde{\varphi} = (\widetilde{\xi}, \widetilde{\eta})$ satisfying

$$\begin{aligned} \widetilde{\xi}_s &= \xi_s \qquad \text{for} \quad s \le t \\ \widetilde{\eta}_s &= \eta_s \qquad \text{for} \quad s < t \end{aligned} \tag{1.10}$$

and

$$V_T(\widetilde{\varphi}) = V_T(\varphi) \qquad P - a.s. \tag{1.11}$$

An *admissible variation of φ from t on* is a trading strategy $\Delta = (\delta, \varepsilon)$ such that $\varphi + \Delta$ is an admissible continuation of φ from t on.

Definition. A trading strategy φ is called *R-minimizing* if for any $t \in [0, T)$ and for any admissible continuation $\widetilde{\varphi}$ of φ from t on we have

$$R_t(\widetilde{\varphi}) \ge R_t(\varphi) \qquad P - a.s.$$

or equivalently if

$$R_t(\varphi + \Delta) - R_t(\varphi) \ge 0 \qquad P - a.s. \tag{1.12}$$

for every admissible variation Δ of φ from t on.

Problem: Given a contingent claim H, find an H-admissible R-minimizing strategy.

Remark. R-minimization should be viewed as a sequential regression procedure in the following sense: at any time t, one "lets the past be the past" and concentrates instead on those strategies which differ from the reference strategy only on the remaining time interval $(t, T]$. Note that this is an extension of the Black/Scholes approach where $R(\varphi)$ can be reduced to 0; cf. Proposition 1.1. In an *incomplete* model, this is in general not possible; it is therefore important to analyze the riskiness of a given contingent claim in more detail. Before recalling the central result on R-minimizing strategies, we give a technical lemma on the improvement of trading strategies which will be useful later on. It says that for any H-admissible strategy we can find another H-admissible strategy which is mean-self-financing and which has smaller conditional mean square error. Thus, it suggests that any "good" strategy ought to be mean-self-financing. Note that this excludes for example a strategy which is self-financing on $[0, T)$ and makes up the balance at the end.

Lemma 1.2. *Let $\varphi = (\xi, \eta)$ be a trading strategy and $t \in [0, T]$. Then there exists a trading strategy $\widehat{\varphi}$ satisfying*

a) $V_T(\widehat{\varphi}) = V_T(\varphi) \quad P - a.s.$

b) $C_s(\widehat{\varphi}) = E\big[\, C_T(\widehat{\varphi}) \,\big|\, \mathcal{F}_s \,\big] \quad P - a.s. \qquad \text{for } t \le s \le T.$

c) $R_s(\widehat{\varphi}) \le R_s(\varphi) \quad P - a.s. \qquad \text{for } t \le s \le T.$

If we choose $t := 0$, then $\widehat{\varphi}$ is mean-self-financing.

Proof. Set $\widehat{\xi} := \xi$ and

$$\widehat{\eta}_s := \begin{cases} \eta_s & \text{for } s < t \\ E\left[V_T(\varphi) - \displaystyle\int_0^T \xi_u \, dX_u \,\middle|\, \mathcal{F}_s \right] + \displaystyle\int_0^s \xi_u \, dX_u - \xi_s \cdot X_s & \text{for } s \ge t \;, \end{cases}$$

choosing right-continuous versions. Then $V(\widehat{\varphi})$ is given by

$$V_s(\widehat{\varphi}) = \begin{cases} V_s(\varphi) & \text{for } s < t \\ E\left[V_T(\varphi) - \displaystyle\int_0^T \xi_u \, dX_u \,\middle|\, \mathcal{F}_s \right] + \displaystyle\int_0^s \xi_u \, dX_u & \text{for } s \ge t \end{cases}$$

and therefore right-continuous, since both parts are. Furthermore, since

$$C_T(\widehat{\varphi}) = V_T(\widehat{\varphi}) - \int_0^T \widehat{\xi}_u \, dX_u = V_T(\varphi) - \int_0^T \xi_u \, dX_u = C_T(\varphi) \;,$$

we have by the above that

$$C_s(\widehat{\varphi}) = E\big[\, C_T(\widehat{\varphi}) \,\big|\, \mathcal{F}_s \,\big] \qquad \text{for } s \ge t$$

and therefore

$$\begin{aligned} R_s(\widehat{\varphi}) &= E\Big[\big(C_T(\widehat{\varphi}) - C_s(\widehat{\varphi})\big)^2 \,\Big|\, \mathcal{F}_s \Big] \\ &= E\Big[\big(C_T(\varphi) - C_s(\varphi) + C_s(\varphi) - C_s(\widehat{\varphi})\big)^2 \,\Big|\, \mathcal{F}_s \Big] \\ &= R_s(\varphi) + \big(C_s(\varphi) - C_s(\widehat{\varphi})\big)^2 + 2 \cdot \big(C_s(\widehat{\varphi}) - C_s(\varphi)\big) \cdot \big(C_s(\varphi) - C_s(\widehat{\varphi})\big) \\ &= R_s(\varphi) - \big(C_s(\varphi) - C_s(\widehat{\varphi})\big)^2 \\ &\le R_s(\varphi) \quad P-a.s. \qquad \text{for } s \ge t. \end{aligned}$$

q.e.d.

In the martingale case, the problem of R-minimization was completely solved by Föllmer and Sondermann [6]. In order to state their result, we need to recall the *Kunita-Watanabe decomposition*: If X is a square-integrable martingale, then every $H \in \mathcal{L}^2(P)$ can be written as

$$H = E[H] + \int_0^T \xi_u^H \, dX_u + L_T^H \qquad P-a.s. \,, \tag{1.13}$$

where $\xi^H \in \mathcal{L}^2(P_X)$ and $L^H = (L_t^H)_{0 \le t \le T}$ is a square-integrable martingale orthogonal to X with $L_0^H = 0$ P-a.s.

Proposition 1.3. *Assume that X is a square-integrable martingale. For every contingent claim H, there exists a unique H-admissible R-minimizing trading strategy φ^H. It is mean-self-financing, and its ξ-component is given by the integrand ξ^H in (1.13).*

Proof. Let us first remark that φ^H is determined by this description since any H-admissible mean-self-financing trading strategy can be characterized by its ξ-component. Now fix $t \in [0,T)$ and consider an admissible continuation $\varphi = (\xi, \eta)$ of φ^H from t on. For the strategy $\widehat{\varphi}$ constructed from φ as in Lemma 1.2, we obtain by (1.13)

$$C_T(\widehat{\varphi}) - C_t(\widehat{\varphi}) = V_T(\widehat{\varphi}) - \int_0^T \xi_u \, dX_u - E\big[\, C_T(\widehat{\varphi}) \,\big|\, \mathcal{F}_t \,\big]$$

$$
\begin{aligned}
&= H - \int_0^T \xi_u \, dX_u - E\left[H - \int_0^T \xi_u \, dX_u \,\middle|\, \mathcal{F}_t \right] \\
&= \int_t^T \left(\xi_u^H - \xi_u\right) \, dX_u + \left(L_T^H - L_t^H\right) \qquad P - a.s.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
C_T(\varphi^H) - C_t(\varphi^H) &= H - \int_0^T \xi_u^H \, dX_u - E\left[H - \int_0^T \xi_u^H \, dX_u \,\middle|\, \mathcal{F}_t \right] \\
&= L_T^H - L_t^H \qquad P - a.s.
\end{aligned}
$$

and therefore

$$
\begin{aligned}
R_t(\varphi) &\geq R_t(\widehat{\varphi}) \\
&= E\left[\int_t^T \left(\xi_u^H - \xi_u\right)^2 \, d\langle X\rangle_u \,\middle|\, \mathcal{F}_t \right] + E\left[\left(L_T^H - L_t^H\right)^2 \,\middle|\, \mathcal{F}_t \right] \\
&\geq R_t\left(\varphi^H\right) \qquad P - a.s.
\end{aligned}
$$

which shows that φ^H is R-minimizing. To prove uniqueness, we first note that any R-minimizing trading strategy φ^* must be mean-self-financing; this follows from Lemma 1.2. But then the same argument as above yields

$$
R_0(\varphi^*) = E\left[\int_0^T \left(\xi_u^H - \xi_u^*\right)^2 \, d\langle X\rangle_u \right] + E\left[\left(L_T^H\right)^2 \right] > R_0\left(\varphi^H\right)
$$

unless $\xi^* = \xi^H$ P_X-a.e.

q.e.d.

Remark. The idea of using Lemma 1.2 in the proof is taken from Schweizer [12] and exploited again in the next section. In Proposition 1.3, X is assumed to be a martingale. Unfortunately, the following example shows that the general case of a semimartingale is less pleasant: Consider a discrete-time model with three trading dates 0,1,2, and assume that the price increments $X_k - X_{k-1}$ can take three distinct values in each period. This will prevent the model from being complete. If there exists an R-minimizing trading strategy φ^* for a given claim H, it must minimize both $R_0(\varphi)$ and $R_1(\varphi)$ over all admissible continuations of

φ^* from 0 and 1 on, respectively. But if P is not a martingale measure for X, it is easy to find a claim H and an H-admissible mean-self-financing trading strategy $\overline{\varphi}$ such that either $R_0(\varphi^*) > R_0(\overline{\varphi})$ or $R_1(\varphi^*) > R_1(\overline{\varphi})$. This reflects the fact that it is impossible to minimize simultaneously $R_0(\varphi)$ and $R_1(\varphi)$ by the same strategy. Hence, there cannot exist any R-minimizing strategy in such a situation. For explicit computations in this example, see Schweizer [12].

Technically speaking, the above concept of R-minimization fails in the general case because we cannot control the influence of the term $\int \xi \, dA$ on the process $R(\varphi)$. More precisely, there is no analogue to the Kunita-Watanabe projection theorem allowing us to decompose a claim H into a stochastic integral $\int \xi \, dX$ (with respect to X) and an orthogonal component. From an intuitive point of view, the class of permissible variations of a trading strategy is too large. We must use a weaker approach by restricting our attention to variations which are small enough in some sense. This is quite straightforward in a discrete-time model; see Schweizer [12], and Föllmer and Schweizer [4] for an expository account. The rather delicate situation in continuous time will be treated in the next section.

2. Local R-minimization and the optimality equation

In this section, we introduce the concept of a locally R-minimizing trading strategy. Being an infinitesimal concept, it will involve limit considerations, and under suitable assumptions on the price process, the required limits actually exist. This will enable us to prove that a trading strategy is locally R-minimizing if and only if it is mean-self-financing and satisfies a certain equation. We shall call this the optimality equation.

Definition. A trading strategy $\Delta = (\delta, \varepsilon)$ is called a *small perturbation* if it satisfies the following conditions:

(2.1) $$\delta \text{ is bounded.}$$

(2.2) $$\int_0^T |\delta_u| \, d|A|_u \text{ is bounded.}$$

(2.3) $$\delta_T = \varepsilon_T = 0\,.$$

Remark. Due to (1.1), the price process X can be thought of as having two components: the "unpredictable" martingale term M and the "drift" or "trend" A. $\int_0^T \delta_u \, dA_u$ therefore represents the systematic part of the trading gains from Δ, and condition (2.2) says that Δ is meant to be small in the sense of limited systematic gains. (2.3) has two consequences: $V_T(\Delta) = 0$ P-a.s., so that $\varphi + \Delta$ is an H-admissible trading strategy for every H-admissible φ, and the restriction of Δ to any subinterval of $[0,T]$ is again a small perturbation.

As mentioned in the last section, our idea is to introduce the notion of a local variation of a trading strategy. To this end, we consider partitions $\tau = (t_i)_{0 \le i \le N}$ of the interval $[0,T]$. Such partitions will always satisfy

$$0 = t_0 < t_1 < \ldots < t_N = T \ ,$$

and their mesh will be defined by $|\tau| := \max\limits_{1 \le i \le N} (t_i - t_{i-1})$. A sequence $(\tau_n)_{n \in I\!N}$ of partitions will be called *increasing* if $\tau_n \subseteq \tau_{n+1}$ for all n; it will be called *0-convergent* if it satisfies

$$\lim_{n \to \infty} |\tau_n| = 0 \ .$$

If Δ is a small perturbation and $(s,t]$ a subinterval of $[0,T]$, we define the small perturbation

$$\Delta\big|_{(s,t]} := \Big(\delta\big|_{(s,t]}, \varepsilon\big|_{[s,t)} \Big)$$

by setting

$$\begin{aligned} \delta\big|_{(s,t]}(\omega,u) &:= \delta_u(\omega) \cdot I_{(s,t]}(u) \\ \varepsilon\big|_{[s,t)}(\omega,u) &:= \varepsilon_u(\omega) \cdot I_{[s,t)}(u) \ . \end{aligned}$$

The asymmetry corresponds to the fact that δ is predictable and ε merely adapted.

Definition. Let φ be a trading strategy, Δ a small perturbation and τ a partition of $[0,T]$. Then we can define the *R-quotient*

$$(2.4) \qquad r^\tau[\varphi,\Delta](\omega,t) := \sum_{t_i \in \tau} \frac{R_{t_i}\big(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\big) - R_{t_i}(\varphi)}{E\big[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\big|\, \mathcal{F}_{t_i} \big]}(\omega) \cdot I_{(t_i,t_{i+1}]}(t) \ .$$

The strategy φ is called *locally R-minimizing* if

$$(2.5) \qquad \liminf_{n \to \infty} r^{\tau_n}[\varphi,\Delta] \ge 0 \qquad P_M - a.e.$$

for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0,T]$.

Remark. Obviously, $r^\tau[\varphi, \Delta]$ is a stochastic process which is well-defined P_M-a.e. on $\overline{\Omega}$. It can be interpreted as a measure for the total change of riskiness if φ is locally perturbed by Δ along the partition τ. The denominator in (2.4) describes the appropriate time scale for these measurements. Note that (2.5) is the infinitesimal analogue of the condition (1.12).

For our next result, we need an additional assumption on X.

(X2) For P-almost all ω, the measure on $[0,T]$ induced by $\langle M\rangle_.(\omega)$ has the whole interval $[0,T]$ as its support.

Equivalently, we could postulate that $\langle M\rangle_.(\omega)$ is strictly increasing P-a.s. This nondegeneracy condition prevents the martingale M from being locally constant. For example, both a diffusion process with a strictly positive diffusion coefficient and a point process with a strictly positive intensity satisfy (X2).

Lemma 2.1. *Assume that X satisfies (X1) and (X2). If a trading strategy φ is locally R-minimizing, it is mean-self-financing.*

Proof. Construct $\widehat{\varphi}$ from φ as in Lemma 1.2 with $t = 0$. $\Delta := \widehat{\varphi} - \varphi$ is then a small perturbation. Let τ_n be the n-th dyadic partition of $[0,T]$, and denote by $d' := (d + 2^{-n} \cdot T) \wedge T$ the successor in τ_n of $d \in \tau_n$. Since

$$V_d\left(\varphi + \Delta\big|_{(d,d']}\right) = V_d(\varphi) + \widehat{\eta}_d - \eta_d = V_d(\widehat{\varphi}) \;,$$

it follows from (2.3) that

$$C_T\left(\varphi + \Delta\big|_{(d,d']}\right) - C_d\left(\varphi + \Delta\big|_{(d,d']}\right) = C_T(\widehat{\varphi}) - C_d(\widehat{\varphi})$$

for any $n \in I\!N$ and $d \in \tau_n$. The proof of Lemma 1.2 now yields

$$\begin{aligned} R_d\left(\varphi + \Delta\big|_{(d,d']}\right) - R_d(\varphi) &= R_d(\widehat{\varphi}) - R_d(\varphi) \\ &= -\Big(C_d(\varphi) - E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_d \,\big]\Big)^2 \end{aligned}$$

and therefore

$$r^{\tau_n}[\varphi, \Delta] = -\sum_{d \in \tau_n} \frac{\Big(C_d(\varphi) - E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_d \,\big]\Big)^2}{E\big[\, \langle M\rangle_{d'} - \langle M\rangle_d \,\big|\, \mathcal{F}_d \,\big]} \cdot I_{(d,d']} \;. \tag{2.6}$$

Now assume that for some dyadic rational d_0, there is a set B of positive probability such that

$$C_{d_0}(\varphi)(\omega) \neq E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_{d_0} \,\big](\omega)$$

for all $\omega \in B$. Since both $C_{\cdot}(\varphi)$ and $E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_{\cdot} \,\big]$ have been chosen to be right-continuous, there exist for any $\omega \in B$ constants $\gamma(\omega) > 0$ and $\beta(\omega) > 0$ such that

$$\Big| C_d(\varphi) - E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_d \,\big] \Big|(\omega) \geq \gamma(\omega) > 0$$

for every dyadic rational $d \in \big[d_0, d_0 + \beta(\omega)\big]$. But then (2.6) implies for all $\omega \in B$ that

$$\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta](\omega, t) < 0$$

for any t in the open interval $\big(d_0, d_0 + \beta(\omega)\big)$, in contradiction to (X2) and to the assumption that

$$\liminf_{n \to \infty} r^{\tau_n}[\varphi, \Delta](\omega, t) \geq 0$$

holds for $\langle M \rangle_{\cdot}(\omega)$-almost all t outside of a set of probability 0. Hence we conclude that

$$C_d(\varphi) = E\big[\, C_T(\varphi) \,\big|\, \mathcal{F}_d \,\big] \qquad \text{for every dyadic rational } d$$

holds P-a.s., and the assertion follows from right-continuity.

q.e.d.

The next step furnishes us with the key result of this section. It is technically somewhat involved, but essentially it tells us that we can find a locally R-minimizing trading strategy by varying only the ξ-component. This turns out to be very important since it enables us to use again martingale techniques. Let H be a fixed contingent claim and $\varphi = (\xi, \eta)$ an H-admissible mean-self-financing trading strategy. Since $C(\varphi)$ is a martingale with terminal value

$$C_T(\varphi) = H - \int_0^T \xi_u \, dX_u \qquad P - a.s.\,, \tag{2.7}$$

φ is uniquely determined by ξ, and we write $C(\xi) := C(\varphi)$ and $R(\xi) := R(\varphi)$. Now take a small perturbation $\Delta = (\delta, \varepsilon)$ and a partition τ of $[0, T]$. For $t_i \in \tau$, we are going to compare the H-admissible (but not necessarily mean-self-financing) trading strategy $\varphi + \Delta\big|_{(t_i, t_{i+1}]}$ with the H-admissible mean-self-financing trading strategy associated to $\xi + \delta\big|_{(t_i, t_{i+1}]}$. These strategies have the same ξ-component, but may differ in the η-component. Because of (2.3), we obtain

$$\begin{aligned} C_T\left(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\right) &= H - \int_0^T \xi_u \, dX_u - \int_{t_i}^{t_{i+1}} \delta_u \, dX_u \\ &= C_T(\varphi) - \int_{t_i}^{t_{i+1}} \delta_u \, dX_u \\ &= C_T\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) . \end{aligned}$$

Furthermore, we have

$$C_{t_i}\left(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\right) = C_{t_i}(\varphi) + \varepsilon_{t_i}$$

and

$$\begin{aligned} C_{t_i}\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) &= E\left[C_T\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) \,\middle|\, \mathcal{F}_{t_i} \right] \\ &= C_{t_i}(\varphi) - E\left[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \end{aligned}$$

since φ is mean-self-financing. This implies

$$\begin{aligned} & C_T\left(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\right) - C_{t_i}\left(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\right) \\ &= C_T\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) - C_{t_i}\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) - \left(\varepsilon_{t_i} + E\left[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \right) \end{aligned}$$

and therefore by the martingale property of $C\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right)$

$$R_{t_i}\left(\varphi + \Delta\big|_{(t_i,t_{i+1}]}\right) = R_{t_i}\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right) + \left(\varepsilon_{t_i} + E\left[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2 .$$

Summing up, we conclude that

$$\begin{aligned} r^\tau[\varphi, \Delta] &= \sum_{t_i \in \tau} \frac{R_{t_i}(\xi + \delta\big|_{(t_i,t_{i+1}]}) - R_{t_i}(\xi)}{E\left[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i} \right]} \cdot I_{(t_i,t_{i+1}]} \\ &\quad + \sum_{t_i \in \tau} \frac{\left(\varepsilon_{t_i} + E\left[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2}{E\left[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\middle|\, \mathcal{F}_{t_i} \right]} \cdot I_{(t_i,t_{i+1}]} . \end{aligned} \tag{2.8}$$

In accordance with previous notations, we denote the first term on the right-hand side of (2.8) by $r^{\tau}[\xi, \delta]$.

Now it is time to introduce our final assumptions on X. They will enable us to show that the last term in (2.8) is asymptotically negligible.

(X3) A is continuous.

(X4) A is absolutely continuous with respect to $\langle M \rangle$ with a density α satisfying $E_M\left[|\alpha| \cdot \log^+ |\alpha|\right] < \infty$.

(X5) X is continuous at T P-a.s.

Condition (X5) means that X does not have a fixed time of discontinuity at T. Because of (X3), it implies that M does not jump at T so that $\langle M \rangle$ does not have any mass in T.

Lemma 2.2. *Assume that X satisfies (X1) – (X5). Let H be a contingent claim and $\varphi = (\xi, \eta)$ an H-admissible trading strategy. Then the following statements are equivalent:*

1) *φ is locally R-minimizing.*

2) *φ is mean-self-financing, and*

$$\liminf_{n \to \infty} r^{\tau_n}[\xi, \delta] \geq 0 \qquad P_M - a.e. \tag{2.9}$$

for every bounded predictable process δ satisfying (2.2) and (2.3), and for every increasing 0-convergent sequence (τ_n) of partitions of $[0, T]$.

Proof. Due to Lemma 2.1, we may assume φ to be mean-self-financing. But then (2.8) immediately shows that 1) follows from 2). For the converse, we first note that we may choose all ε_{t_i} to be 0 in (2.8). The estimate

$$\left(E\left[\int_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2 \leq \|\delta\|_\infty^2 \cdot E\Big[\left(|A|_{t_{i+1}} - |A|_{t_i} \right)^2 \,\Big|\, \mathcal{F}_{t_i} \Big]$$

then yields

$$\sum_{t_i \in \tau_n} \frac{\left(E\left[\int\limits_{t_i}^{t_{i+1}} \delta_u \, dA_u \,\middle|\, \mathcal{F}_{t_i} \right] \right)^2}{E\big[\, \langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\big|\, \mathcal{F}_{t_i} \,\big]} \cdot I_{(t_i, t_{i+1}]}$$

$$\leq \|\delta\|_\infty^2 \cdot \sum_{t_i \in \tau_n} \frac{E\big[\, \big(|A|_{t_{i+1}} - |A|_{t_i} \big)^2 \,\big|\, \mathcal{F}_{t_i} \,\big]}{E\big[\, \langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\big|\, \mathcal{F}_{t_i} \,\big]} \cdot I_{(t_i, t_{i+1}]} \, .$$

If $\langle M \rangle_t = t$ and A is absolutely continuous with respect to Lebesgue measure with a bounded density, it is easy to see that this last expression converges to 0 P_M-a.e. In the general case, the required convergence follows from (X1) – (X5) by Proposition 3.1 of Schweizer [13]. Therefore, (2.8) shows that 2) follows from 1).

q.e.d.

By its definition, local R-minimality is a variational concept involving two variables ξ and η. Lemma 2.2 splits this into two separate and simpler problems; it tells us to vary only ξ and to determine η from the side condition that φ is mean-self-financing. Put differently, this amounts to studying the following question: If we consider the martingale $C(\xi)$ and the locally perturbed process

$$C_t \left(\xi + \delta\big|_{(t_i, t_{i+1}]} \right) = E\left[C_T(\xi) - \int\limits_{t_i}^{t_{i+1}} \delta_u \, dX_u \,\middle|\, \mathcal{F}_t \right] \qquad , \qquad 0 \leq t \leq T \, ,$$

how do their R-quotients compare? This problem is resolved in Schweizer [13] for a general martingale Y instead of $C(\xi)$. It is shown there that R-minimality under such local perturbations is equivalent to orthogonality of Y and M. Hence, we now obtain a martingale-theoretic characterization of locally R-minimizing trading strategies.

Proposition 2.3. *Assume that X satisfies (X1) – (X5). Let H be a contingent claim and φ an H-admissible trading strategy. Then the following statements are equivalent:*

1) *φ is locally R-minimizing.*

2) *φ is mean-self-financing, and the martingale $C(\varphi)$ is orthogonal to M.*

Proof. Due to Lemma 2.2, this follows directly from Theorem 3.2 of Schweizer [13].

q.e.d.

Having established Proposition 2.3, it is now straightforward to derive the *optimality equation* for a locally R-minimizing trading strategy. All we have to do is to find the Kunita-Watanabe decomposition of $C_T(\varphi)$ with respect to P and M. From the decompositions

$$H = E[H] + \int_0^T \mu_u^{H;P}\, dM_u + L_T^{H;P} \qquad P-a.s. \tag{2.10}$$

and (using (1.3))

$$\int_0^T \xi_u\, dA_u = E\left[\int_0^T \xi_u\, dA_u\right] + \int_0^T \mu_u^{\xi,A;P}\, dM_u + L_T^{\xi,A;P} \qquad P-a.s.\ , \tag{2.11}$$

we conclude that it is given by

$$C_T(\varphi) = C_0(\varphi) + \int_0^T \left(\mu_u^{H;P} - \xi_u - \mu_u^{\xi,A;P}\right) dM_u + L_T^{H;P} - L_T^{\xi,A;P} \qquad P-a.s. \tag{2.12}$$

due to (2.7).

Theorem 2.4. *Assume that X satisfies (X1) – (X5). Let H be a contingent claim and $\varphi = (\xi, \eta)$ an H-admissible trading strategy. Then φ is locally R-minimizing if and only if φ is mean-self-financing and ξ satisfies the optimality equation*

$$\mu^{H;P} - \xi - \mu^{\xi,A;P} = 0 \qquad P_M - a.e. \tag{2.13}$$

Proof. This follows immediately from Proposition 2.3 and the decomposition (2.12).

q.e.d.

The importance of Theorem 2.4 lies in the fact that it reduces the variational problem of finding a locally R-minimizing trading strategy to the solving of a stochastic optimality equation. Of course, equation (2.13) still remains to be solved. We shall give an existence and uniqueness result in the next section.

Remark. Since the concept of an R-minimizing strategy has a direct and intuitive interpretation, it is natural to ask in which sense a locally R-minimizing strategy is optimal. To answer this question, we shall focus on mean-self-financing strategies which is quite reasonable in view of Lemmas 1.2 and 2.2.

Using Proposition 3.1 of Schweizer [13], we first note that

$$\lim_{n\to\infty} r^{\tau_n}[\xi,\delta] = \delta^2 - 2\cdot\delta\cdot\left(\mu^{H;P} - \xi - \mu^{\xi,A;P}\right) \qquad P_M - a.e. \tag{2.14}$$

holds for all ξ satisfying (1.3) and for all bounded predictable δ satisfying (2.2). If φ is locally R-minimizing, this implies by Theorem 2.4 that

$$\lim_{n\to\infty} r^{\tau_n}[\xi,\delta] = \delta^2 \qquad P_M - a.e.$$

Therefore, we conclude that for P_M-almost all (ω,t) the inequality

$$R_{t_i}\left(\xi + \delta\big|_{(t_i,t_{i+1}]}\right)(\omega) \geq R_{t_i}(\xi)(\omega)$$

holds for all $n \geq n_0(\omega,t)$, $t_i \in \tau_n$ and $t \in (t_i,t_{i+1}]$. This means that any bounded perturbation of ξ on a small enough time interval leads to an increase of R.

This formulation is still not quite satisfactory since it does not allow us to compare ξ directly with another trading strategy $\widetilde{\xi}$. But if we assume that both α and $\langle M\rangle_T$ are bounded, (2.14) holds even for all predictable processes δ satisfying (1.3), and choosing $\delta := \widetilde{\xi} - \xi$ then implies that for P_M-almost all (ω,t),

$$R_{t_i}\left(\xi + (\widetilde{\xi} - \xi)\big|_{(t_i,t_{i+1}]}\right)(\omega) \geq R_{t_i}(\xi)(\omega) \tag{2.15}$$

for all $n \geq n_0(\omega,t)$, $t_i \in \tau_n$ and $t \in (t_i,t_{i+1}]$. Hence, we can say that any modification of ξ by another mean-self-financing trading strategy on a small interval will increase R, and this is exactly what the term "locally R-minimizing" suggests.

3. Solving the optimality equation

In this section, we show how the optimality equation (2.13) can be solved. This yields a locally R-minimizing trading strategy by Theorem 2.4. The basic idea for solving (2.13) is both simple and intuitive; however, it requires quite a lot of technical machinery in the general case. To provide a better insight, we therefore concentrate here on a situation with additional explicit structure.

Definition. We say that M *and* N *form a* P*-basis of* $\mathcal{L}^2(P)$ if the following conditions are satisfied:

(3.1) Both M and N are square-integrable martingales under P.

(3.2) $M - M_0$ and $N - N_0$ are P-orthogonal (as martingales).

(3.3) Every $H \in \mathcal{L}^2(P)$ has a unique representation

$$H = E[H] + \int_0^T \mu_u^{H;P}\, dM_u + \int_0^T \nu_u^{H;P}\, dN_u \qquad P-a.s.$$

for two predictable processes $\mu^{H;P} \in \mathcal{L}^2(P_M)$ and $\nu^{H;P} \in \mathcal{L}^2(P_N)$.

Condition (3.3) is equivalent to assuming that the stable subspace generated by M and N coincides with the whole space of square-integrable martingales under P.

From now on we add the following assumptions to our initial model:

(P1) There exists a process $N = (N_t)_{0 \le t \le T}$ such that M and N form a P-basis of $\mathcal{L}^2(P)$.

(P2) There exists a probability measure $\widetilde{P}$ equivalent to P such that X and N form a $\widetilde{P}$-basis of $\mathcal{L}^2(\widetilde{P})$.

Remarks. **1)** (3.1) alone would entail that $\widetilde{P}$ is an equivalent martingale measure for X, i.e., a probability measure $\widetilde{P} \approx P$ such that X is a martingale under $\widetilde{P}$. Assuming the existence of such a measure is quite familiar in this context since it corresponds to a no-arbitrage condition. Here, however, we require a *minimal* martingale measure: Apart from turning X into a martingale, it should leave intact the remaining structure of the model; in particular, orthogonality relations should be preserved. See Föllmer and Schweizer [5] for a generalization of this concept.

2) (3.3) and Theorem 11.2 of Jacod [10] clearly show that $\widetilde{P}$ is not extremal in the set $\mathcal{M}(X)$ of martingale measures for X since X does not span $\mathcal{L}^2(\widetilde{P})$. But due to Corollary 11.4 of Jacod [10] and the remark following it, assumption (P2) implies that $\widetilde{P}$ is extremal in the set $\mathcal{M}(X,N)$ of probability measures turning *both* X and N into (local) martingales.

Lemma 3.1. *Assume that X satisfies (X1) and that (P1) and (P2) hold. If in addition*

$$\widetilde{Z}_T := \frac{d\widetilde{P}}{dP} \in \mathcal{L}^2(P)\ , \tag{3.4}$$

then A is absolutely continuous with respect to $\langle M \rangle^P$ with a density α, and a *right-continuous version of the process* $\widetilde{Z}_t := E\big[\,\widetilde{Z}_T \,\big|\, \mathcal{F}_t\,\big]$ $(0 \le t \le T)$ *is given by*

$$\text{(3.5)} \qquad \widetilde{Z}_t = \mathcal{E}\left(-\int \alpha\, dM\right)_t$$

$$:= \exp\left(-\int_0^t \alpha_u\, dM_u - \frac{1}{2}\cdot\int_0^t |\alpha_u|^2\, d\langle M^c\rangle_u^P\right)\cdot \prod_{0\le u\le t}(1-\alpha_u\cdot\Delta M_u)\cdot e^{\alpha_u\cdot\Delta M_u}$$

P-a.s. *for all* $t \in [0,T]$*. Here* M^c *denotes the continuous martingale part of* M*, and* $\Delta M_u := M_u - M_{u-}$ *is the jump of* M *in* u*.*

Proof. (3.4) and (3.3) yield the Kunita-Watanabe decomposition

$$\widetilde{Z}_T = 1 + \int_0^T \mu_u\, dM_u + \int_0^T \nu_u\, dN_u \qquad P-a.s.$$

Since N is a $\widetilde{P}$-martingale by (P2), it must be P-orthogonal to $\widetilde{Z}$. This implies that $\widetilde{Z}\cdot\int \nu\, dN$ is a P-martingale, hence

$$0 = E\left[\widetilde{Z}_T\cdot\int_0^T \nu_u\, dN_u\right] = E\left[\int_0^T \nu_u^2\, d\langle N\rangle_u^P\right]$$

by the orthogonality of M and $N - N_0$ under P. Therefore $\nu = 0$ P_N-a.e., and $\widetilde{Z}$ is given by

$$\text{(3.6)} \qquad \widetilde{Z}_t = 1 + \int_0^t \mu_u\, dM_u \qquad , \qquad 0 \le t \le T.$$

Furthermore, the process $\langle M, \widetilde{Z}\rangle^P$ exists and is given by

$$\langle M, \widetilde{Z}\rangle_t^P = \int_0^t \mu_u\, d\langle M\rangle_u^P \qquad , \qquad 0 \le t \le T.$$

By Theorem 13.14 of Elliott [3], this implies that M is a special $\widetilde{P}$-semimartingale with the canonical decomposition

$$M_t = \left(M_t - \int_0^t \frac{1}{\widetilde{Z}_{u-}}\, d\langle M, \widetilde{Z}\rangle_u^P\right) + \int_0^t \frac{1}{\widetilde{Z}_{u-}}\, d\langle M, \widetilde{Z}\rangle_u^P$$

$$= \left(M_t - \int_0^t \frac{\mu_u}{\widetilde{Z}_{u-}}\, d\langle M\rangle_u^P\right) + \int_0^t \frac{\mu_u}{\widetilde{Z}_{u-}}\, d\langle M\rangle_u^P \qquad , \qquad 0 \le t \le T.$$

But M can also be written as

$$M_t = X_t - X_0 - A_t \qquad , \qquad 0 \le t \le T$$

under $\widetilde{P}$; since A is predictable, uniqueness of the canonical decomposition implies that

$$A_t = -\int_0^t \frac{\mu_u}{\widetilde{Z}_{u-}}\, d\langle M\rangle_u^P = \int_0^t \alpha_u\, d\langle M\rangle_u^P \qquad , \qquad 0 \le t \le T$$

with

$$\alpha := -\frac{\mu}{\widetilde{Z}_-} .$$

Inserting this into (3.6), we conclude that $\widetilde{Z}$ satisfies the equation

$$\widetilde{Z}_t = 1 - \int_0^t \widetilde{Z}_{u-} \cdot \alpha_u\, dM_u \qquad , \qquad 0 \le t \le T$$

whose unique solution is given by (3.5); cf. Elliott [3], Theorem 13.5.

q.e.d.

Remarks. 1) Lemma 3.1 has several implications. First of all, it tells us that $\widetilde{P}$, if it exists, is essentially unique. Secondly, (3.5) shows the effect of switching from P to $\widetilde{P}$: this change of measure is achieved by a Girsanov transformation which removes the drift A from X. Furthermore, (3.5) can be used as a starting point for constructing $\widetilde{P}$. We simply define $\widetilde{Z}$ by (3.5), and the question to decide is then whether this yields an equivalent probability measure $\widetilde{P}$ or not. General integrability conditions for this are given by Jacod [10] and Novikov [11]. Finally, (3.5) shows another minimality property of $\widetilde{P}$: only the information about X is required for its construction.

2) The existence of any equivalent martingale measure P^* for X already implies the absolute continuity of A with respect to $\langle M\rangle^P$; this can be seen from the proof of Lemma 3.1. Thus, the assumption (X4) reduces to an integrability condition.

Let us now consider a contingent claim $H \in \mathcal{L}^2(\widetilde{P})$. Due to (P2) and Proposition 1.3, there exists a unique trading strategy $\varphi^{H;\widetilde{P}} = (\xi^{H;\widetilde{P}}, \eta^{H;\widetilde{P}})$ which is R-minimizing with respect to $\widetilde{P}$. The process $\xi^{H;\widetilde{P}}$ is given by the Kunita-Watanabe decomposition (with respect to $\widetilde{P}$)

$$(3.7) \qquad H = \widetilde{E}[H] + \int_0^T \xi_u^{H;\widetilde{P}}\, dX_u + \int_0^T \nu_u^{H;\widetilde{P}}\, dN_u \qquad \widetilde{P} - a.s. ;$$

$\eta^{H;\widetilde{P}}$ is then determined by the condition

$$V_t(\varphi^{H;\widetilde{P}}) = \widetilde{E}\left[\, H \,\middle|\, \mathcal{F}_t \,\right] \qquad \widetilde{P} - a.s. \qquad , \qquad 0 \le t \le T. \tag{3.8}$$

$\varphi^{H;\widetilde{P}}$ might seem to be a candidate for a locally R-minimizing strategy under P, and the next result tells us that this is indeed the case.

Theorem 3.2. *Assume that X satisfies (X1) and that (P1) and (P2) hold. Let $H \in \mathcal{L}^2(P)$ be a contingent claim and assume that $H \in \mathcal{L}^2(\widetilde{P})$, $\nu^{H;\widetilde{P}} \in \mathcal{L}^2(P_N)$ and that $\xi^{H;\widetilde{P}}$ satisfies (1.3). Then the following assertions hold:*

1) *$\xi^{H;\widetilde{P}}$ is a solution of the optimality equation (2.13).*

2) *If $\nu^{H;P} \in \mathcal{L}^2(\widetilde{P}_N)$ and if ξ is a solution of (2.13) which satisfies the conditions (1.3), $\xi \in \mathcal{L}^2(\widetilde{P}_X)$ and $\nu^{\xi,A;P} \in \mathcal{L}^2(\widetilde{P}_N)$, then $\xi = \xi^{H;\widetilde{P}}$ $\widetilde{P}_X$-a.e.*

3) *If X also satisfies (X2) – (X5), then $\varphi^{H;\widetilde{P}}$ is locally R-minimizing with respect to P.*

Proof. 1) Let us first note that we need not qualify a.s. because P and $\widetilde{P}$ are equivalent. Since $\xi^{H;\widetilde{P}}$ satisfies (1.3), we obtain from (P1)

$$\int_0^T \xi_u^{H;\widetilde{P}}\, dA_u = E\left[\int_0^T \xi_u^{H;\widetilde{P}}\, dA_u\right] + \int_0^T \mu_u^{\xi^{H;\widetilde{P}},A;P}\, dM_u + \int_0^T \nu_u^{\xi^{H;\widetilde{P}},A;P}\, dN_u\ .$$

Again using (1.3), we can therefore rewrite (3.7) as

$$\begin{aligned} H &= \widetilde{E}[H] + \int_0^T \xi_u^{H;\widetilde{P}}\, dM_u + \int_0^T \xi_u^{H;\widetilde{P}}\, dA_u + \int_0^T \nu_u^{H;\widetilde{P}}\, dN_u \\ &= E[H] + \int_0^T \left(\xi_u^{H;\widetilde{P}} + \mu_u^{\xi^{H;\widetilde{P}},A;P}\right) dM_u + \int_0^T \left(\nu_u^{H;\widetilde{P}} + \nu_u^{\xi^{H;\widetilde{P}},A;P}\right) dN_u\ ; \end{aligned}$$

note that (P2) implies

$$E[H] = \widetilde{E}[H] + E\left[\int_0^T \xi_u^{H;\widetilde{P}}\, dA_u\right]\ .$$

From the uniqueness of the Kunita-Watanabe decomposition (2.10), we now conclude that

$$\mu^{H;P} = \xi^{H;\widetilde{P}} + \mu^{\xi^{H;\widetilde{P}},A;P} \qquad P_M - a.e.$$

so that (2.13) is indeed satisfied. ∎

2) Setting $\delta := \xi^{H;\widetilde{P}} - \xi$, we obviously have

$$\mu^{\delta,A;P} = \mu^{\xi^{H;\widetilde{P}},A;P} - \mu^{\xi,A;P}$$

and therefore

$$\delta + \mu^{\delta,A;P} = 0 \qquad P_M - a.e. \ ,$$

since both $\xi^{H;\widetilde{P}}$ and ξ are solutions of (2.13). From the Kunita-Watanabe decomposition (with respect to P)

$$\int_0^T \delta_u \, dX_u = E\left[\int_0^T \delta_u \, dX_u\right] + \int_0^T \left(\delta_u + \mu_u^{\delta,A;P}\right) dM_u + \int_0^T \nu_u^{\delta,A;P} \, dN_u \ ,$$

we conclude that

$$\int_0^T \delta_u \, dX_u = E\left[\int_0^T \delta_u \, dX_u\right] + \int_0^T \nu_u^{\delta,A;P} \, dN_u \qquad \widetilde{P} - a.s.$$

But since $X - X_0$ and $N - N_0$ are $\widetilde{P}$-orthogonal by (P2), we must have

$$\delta = 0 \qquad \widetilde{P}_X - a.e.$$

so that 2) holds. ∎

3) Because of 1) and Theorem 2.4, it is sufficient to show that $\varphi^{H;\widetilde{P}}$ is mean-self-financing with respect to P. But (3.7) yields

$$C_T(\varphi^{H;\widetilde{P}}) = H - \int_0^T \xi_u^{H;\widetilde{P}} \, dX_u = \widetilde{E}[H] + \int_0^T \nu_u^{H;\widetilde{P}} \, dN_u \ ,$$

and (3.8) implies by (3.7)

$$\begin{aligned} C_t(\varphi^{H;\widetilde{P}}) &= \widetilde{E}\left[\, H \,\middle|\, \mathcal{F}_t \,\right] - \int_0^t \xi_u^{H;\widetilde{P}} \, dX_u \\ &= \widetilde{E}[H] + \int_0^t \nu_u^{H;\widetilde{P}} \, dN_u \qquad , \qquad 0 \le t \le T \ , \end{aligned}$$

since both X and N are martingales under $\widetilde{P}$. Hence, the assertion follows from $\nu^{H;\widetilde{P}} \in \mathcal{L}^2(P_N)$ and the fact that N is also a martingale under P. ∎

q.e.d.

Theorem 3.2 has several aspects. First of all, it gives an existence and uniqueness result for the solution of the optimality equation (2.13). Furthermore, it also provides us with a procedure for finding a locally R-minimizing trading strategy. In a first step, we have to look for the minimal equivalent martingale measure $\widetilde{P}$. Then we can take the strategy $\varphi^{H;\widetilde{P}}$ which is (strongly) R-minimizing with respect to $\widetilde{P}$ and whose existence and uniqueness is guaranteed by Proposition 1.3. Due to (3.7) and (3.8), this optimal strategy can be described in terms of $\widetilde{P}$ alone, in analogy to the complete case where ξ^* and V^* were determined by P^*. The uniqueness of P^* in the complete case now corresponds to the uniqueness of the minimal martingale measure $\widetilde{P}$, and the optimal value process

$$\widetilde{V}_t := \widetilde{E}\left[\, H \,\middle|\, \mathcal{F}_t \,\right] \qquad , \qquad 0 \le t \le T \tag{3.9}$$

of (3.8) can therefore be viewed as a plausible candidate for the valuation of the option H.

If we combine Proposition 1.3 and Theorem 3.2, we obtain an interesting *stability result*. In a complete model, every contingent claim H can be reproduced with $R \equiv 0$, and the generating self-financing strategy is independent of the initial measure P in the sense that any equivalent measure will yield the same optimal strategy. If the model is incomplete, both these aspects become more subtle. In a martingale model, Proposition 1.3 shows that we can at least still generate H with an R-minimizing strategy. Furthermore, Theorem 3.2 tells us that this strategy is *robust*: it will again be optimal for a whole class of semimartingale models P, namely all those which admit $\widetilde{P}$ as their minimal equivalent martingale measure. In this sense, two key properties of complete models are at least partially preserved.

4. Special cases and examples

This section is devoted to several special cases and examples of the preceding results. We begin by showing that local R-minimization can be viewed as an extension of R-minimization and then give two explicit examples where we use the methods of section 3.

4.1. Let us first consider an attainable claim H with a representation (1.9).

Combining this with (2.11), we obtain the Kunita-Watanabe decomposition

$$H = H_0 + E\left[\int_0^T \xi^*_u \, dA_u\right] + \int_0^T \left(\xi^*_u + \mu_u^{\xi^*,A;P}\right) dM_u + L_T^{\xi^*,A;P} \qquad P-a.s. \tag{4.1}$$

The resulting optimality equation is

$$\xi^* + \mu^{\xi^*,A;P} - \xi - \mu^{\xi,A;P} = 0 \qquad P_M - a.e. \tag{4.2}$$

with the obvious solution $\xi = \xi^*$. Of course, this is not surprising: The self-financing trading strategy φ in Proposition 1.1 has $R \equiv 0$ and is therefore a fortiori locally R-minimizing.

4.2. Next we examine the case where X is not a general semimartingale, but a martingale under P. This means that $M \equiv X - X_0$ and $A \equiv 0$. The optimality equation (2.13) simplifies to

$$\xi^H - \xi = 0 \qquad P_X - a.e.$$

so that the unique locally R-minimizing trading strategy coincides with the R-minimizing strategy φ^H given by Proposition 1.3. Hence, local R-minimization generalizes R-minimization.

4.3. Example. Let (W^1, W^2) be a 2-dimensional Brownian motion, $(\mathcal{F}_t)$ its natural filtration and $\beta = (\beta_t)_{0 \le t \le T}$ a bounded adapted process. Defining X and N by

$$dX_t = X_t \, dW^1_t + X_t \cdot \beta_t \, dt$$
$$dN_t = N_t \, dW^2_t$$

yields

$$M_t = \int_0^t X_u \, dW^1_u \quad , \qquad 0 \le t \le T$$

$$\langle M \rangle_t = \int_0^t X_u^2 \, du \quad , \qquad 0 \le t \le T$$

$$A_t = \int_0^t \beta_u \cdot X_u \, du \quad , \qquad 0 \le t \le T$$

$$\alpha_t = \frac{\beta_t}{X_t} \quad , \qquad 0 \le t \le T.$$

Furthermore, it is clear that W^1 and W^2 form a P-basis of $\mathcal{L}^2(P)$. A suitable Girsanov transformation will remove the drift β and yield the unique equivalent measure $\widetilde{P}$ with respect to which

$$\widetilde{W}_t^1 := W_t^1 + \int_0^t \beta_u \, du \qquad , \qquad 0 \le t \le T$$

and W^2 form a 2-dimensional Brownian motion and therefore a $\widetilde{P}$-basis of $\mathcal{L}^2(\widetilde{P})$. Note that X satisfies

$$dX_t = X_t \, d\widetilde{W}_t^1$$

with respect to $\widetilde{P}$ so that stochastic integrals with respect to $\widetilde{W}^1$ can be rewritten as stochastic integrals with respect to X. Hence, for every contingent claim H satisfying certain integrability conditions, there exists by Theorem 3.2 a unique locally R-minimizing trading strategy. Its ξ-component can be computed quite explicitly: If we set

$$\widetilde{V}_t := \widetilde{E}\left[H \,\middle|\, \mathcal{F}_t \right] \qquad , \qquad 0 \le t \le T,$$

it is given as

$$\xi_t^{H;\widetilde{P}} = \frac{d\langle \widetilde{V}, X \rangle_t}{d\langle X \rangle_t} = \frac{1}{X_t^2} \cdot \frac{d}{dt} \langle \widetilde{V}, X \rangle_t \qquad , \qquad 0 \le t \le T. \tag{4.3}$$

Note that (4.3) can be evaluated path by path since both $\widetilde{V}$ and X can be taken as continuous.

If the claim H is of the special form

$$H = h(X_T, N_T)$$

for some function h, we can give even more explicit formulas. Let us denote by $g(x, y, t)$ the solution of the partial differential equation

$$g_t + \frac{1}{2} \cdot \left(x^2 \cdot g_{xx} + y^2 \cdot g_{yy} \right) = 0 \tag{4.4}$$

with the boundary condition

$$g(x, y, T) = h(x, y) \qquad \text{for all } x, y \in I\!R. \tag{4.5}$$

Then it is well-known that

$$\widetilde{V}_t = g(X_t, N_t, t) \qquad , \qquad 0 \le t \le T,$$

and Itô's formula implies

$$\xi_t^{H;\widetilde{P}} = g_x(X_t, N_t, t) \qquad , \qquad 0 \le t \le T. \tag{4.6}$$

Thus, we have solved the optimality equation (2.13) by solving a partial differential equation which is independent of β — although (2.13) does depend on β because of the term $\mu^{\xi,A;P}$. This is explained by Theorem 3.2 which allows us to work again in a martingale model. Of course, there are many equivalent martingale measures P^* for X in this example; removing the drift β in the first coordinate and adding any drift γ in the second is enough. Only the minimal martingale measure $\widetilde{P}$, however, will give such a simple solution. Finally, note that the η-component of the optimal strategy $\varphi^{H;\widetilde{P}}$ is also independent of β due to (3.8). This "stability" of the model $\widetilde{P}$ corresponds exactly to the fact that in the Black/Scholes model, both price and hedging strategy do not depend on the drift parameter.

Remark. In this example, one could also think of translating the optimality equation (2.13) directly into a partial differential equation. Assuming that β is of the form

$$\beta_t = b(X_t, N_t, t) \ ,$$

that the optimal strategy ξ can be written as

$$\xi_t = f(X_t, N_t, t)$$

and that H is of the form

$$H = h(X_T, N_T)$$

for suitably regular functions b, f and h, one would then need explicit expressions for the integrands $\mu^{H;P}$ and $\mu^{\xi,A;P}$ in the representations (2.10) and (2.11), respectively. Such expressions are provided by the Haussmann filtering formula (cf. Davis [2]); however, the required computations turn out to be rather involved. It is fortunate that they are not really necessary for our purposes: thanks to Theorem 3.2, we can work with the martingale measure $\widetilde{P}$, and this easily yields (4.4) and (4.5).

4.4. Example. Let (S^1, S^2) be a 2-variate point process with P-intensities $\lambda^i = (\lambda_t^i)_{0 \le t \le T}$ $(i = 1, 2)$, and take $(\mathcal{F}_t)$ to be the natural filtration of (S^1, S^2). Let $\dot{p}^1 = (\dot{p}_t^1)_{0 \le t \le T}$ be a positive adapted process and define

$$X_t := X_0 + S_t^1 - \int_0^t \dot{p}_u^1 \, du \qquad , \qquad 0 \le t \le T. \tag{4.7}$$

Setting

$$\begin{aligned} M_t^i &:= S_t^i - \int_0^t \lambda_u^i \, du \qquad , \qquad 0 \le t \le T \qquad (i = 1, 2) \\ M &:= M^1 \\ N &:= M^2 \, , \end{aligned}$$

this yields

$$\begin{aligned} X &= X_0 + M + A \\ \langle M \rangle_t^P &= \int_0^t \lambda_u^1 \, du \qquad , \qquad 0 \le t \le T \\ A_t &= \int_0^t (\lambda_u^1 - \dot{p}_u^1) \, du = \int_0^t \left(1 - \frac{\dot{p}_u^1}{\lambda_u^1} \right) d\langle M \rangle_u^P \qquad , \qquad 0 \le t \le T. \end{aligned}$$

Under suitable integrability and boundedness conditions (see Schweizer [12] for more specific details) on the processes λ^1, λ^2 and $\dot{p}^1$, M and N will form a P-basis of $\mathcal{L}^2(P)$. Now we use a Girsanov transformation to construct the unique equivalent measure $\widetilde{P}$ such that S^1 has the $\widetilde{P}$-intensity $\dot{p}^1$. It can then be shown that X and N form a $\widetilde{P}$-basis of $\mathcal{L}^2(\widetilde{P})$ so that again every suitable contingent claim H admits a unique locally R-minimizing trading strategy. Its ξ-component can be computed as

$$\xi_t^{H;\widetilde{P}} = \frac{1}{\dot{p}_t^1} \cdot \frac{d}{dt} \langle \widetilde{V}, X \rangle_t^{\widetilde{P}} \qquad , \qquad 0 \le t \le T \tag{4.8}$$

with

$$\widetilde{V}_t := \widetilde{E} \left[H \, \middle| \, \mathcal{F}_t \right] \qquad , \qquad 0 \le t \le T$$

as before. In the special case where $\dot{p}^1$ is deterministic and $H = (S_T^1 - K)^+$, (4.8) can be written explicitly as

$$\xi_t^{H;\widetilde{P}} = \sum_{j \ge \max(0, K - S_{t-}^1)} e^{-\left(p^1(T) - p^1(t)\right)} \cdot \frac{\left(p^1(T) - p^1(t)\right)^j}{j!}$$

with $p^1(t) := \int_0^t \dot{p}_u^1 \, du$ $(0 \le t \le T)$.

Remark. If one thinks of S^1 as a cumulative claim process and of p^1 as the corresponding cumulative premium process, then the special contingent claim H in our example describes a stop-loss contract. The observation that this is the exact analogue of a call option was made by Sondermann [14]. He used the methods of option pricing in order to analyze stop-loss contracts in a complete model with a single process. Example 4.4 shows how this approach can be extended to an incomplete situation.

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