Risk-Minimality and Orthogonality of Martingales

Martin Schweizer
Universität Bonn
Institut für Angewandte Mathematik
Wegelerstraße 6
D – 5300 Bonn 1

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Abstract: We characterize the orthogonality of martingales as a property of risk-minimality under certain perturbations by stochastic integrals. The integrator can be either a martingale or a semimartingale; in the latter case, the finite variation part must be continuous. This characterization is based on semimartingale differentiation techniques.

Key words: orthogonality of martingales
risk-minimality
semimartingales
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0. Introduction

Two square-integrable martingales Y and M are called *orthogonal* if their product is again a martingale. For a fixed M, an equivalent condition is that the projection of Y on the stable subspace generated by M is 0. This means that the integrand in the Kunita-Watanabe decomposition of Y with respect to M must vanish. In this paper, we characterize orthogonality by a variational approach. We show that Y is orthogonal to M if and only if the conditional quadratic risk

$$R_t(Y) := E\big[\,(Y_T - Y_t)^2 \,\big|\, \mathcal{F}_t\,\big]$$

is always increased by a perturbation of Y along M. Such a perturbation consists of adding to Y the stochastic integral (with respect to M) of a bounded predictable process. This result is proved in section 1.

Now consider a *semimartingale*

$$X = X_0 + M + A$$

and suppose that every perturbation of Y along X leads to an increase of risk. Can we then still conclude that Y is orthogonal to M? The ultimate answer will be a qualified yes, and the key to the argument is provided by a technical differentiation result in section 2. On a finite time interval $[0,T]$ with a partition τ, we consider a process C of finite variation and an increasing process B. For $p > 0$, we define the quotient

$$Q_p[C,B,\tau] := \sum_{t_i \in \tau} \frac{\big|C_{t_i} - C_{t_{i-1}}\big|^p}{B_{t_i} - B_{t_{i-1}}} \cdot I_{(t_{i-1},t_i]}$$

as well as a conditional version $\widetilde{Q}_p[C,B,\tau]$. We then provide sufficient conditions for their convergence to 0 as $|\tau| \to 0$. This result is applied in section 3 to solve the orthogonality problem. We first introduce a risk quotient $r^\tau[Y,\delta]$ to measure the change of risk under a local perturbation of Y by δ. Under some continuity and integrability assumptions on A, we show that $r^{\tau_n}[Y,\delta]$ converges along all suitable sequences (τ_n), and we identify the limit. The main result is then that Y and M are orthogonal if and only if

$$\liminf_{n\to\infty}\, r^{\tau_n}[Y,\delta] \geq 0$$

for all small perturbations δ. This equivalence has an immediate application in the mathematical theory of option trading. The latter property corresponds there to the variational concept of an infinitesimal increase of risk; the orthogonality statement, on the other hand, can be translated into a stochastic optimality equation. See Schweizer [2], [3] for a detailed discussion of these aspects.

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1. Orthogonality of square-integrable martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions of right-continuity and completeness; $T \in \mathbf{R}$ denotes a fixed and finite time horizon. Let $M = (M_t)_{0 \le t \le T}$ be a square-integrable martingale with $M_0 = 0$. A square-integrable martingale $Y = (Y_t)_{0 \le t \le T}$ is called *orthogonal to* M if $M \cdot Y$ is a martingale. In the sequel, we shall give other equivalent formulations of this property.

Let us introduce the product space $\overline{\Omega} := \Omega \times [0, T]$ with the product σ-algebra $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}([0, T])$ and the σ-algebra $\mathcal{P}$ of predictable sets. The variance process $\langle M \rangle$ associated with M induces a finite measure $P_M := P \times \langle M \rangle$ on $\left(\overline{\Omega}, \overline{\mathcal{F}}\right)$. Note that P_M is already determined by its restriction to $\left(\overline{\Omega}, \mathcal{P}\right)$ and gives measure 0 to the sets $A_0 \times \{0\}$ with $A_0 \in \mathcal{F}_0$. Now consider the *Kunita-Watanabe decomposition* of Y_T with respect to M and P:

$$Y_T = Y_0 + \int_0^T \mu_u^Y \, dM_u + L_T^Y \qquad P-a.s. \, , \tag{1.1}$$

where $Y_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$, $\mu^Y \in \mathcal{L}^2\left(\overline{\Omega}, \mathcal{P}, P_M\right)$ and $L^Y = (L_t^Y)_{0 \le t \le T}$ is a square-integrable martingale with $L_0^Y = 0$ which is orthogonal to M. It is obvious from (1.1) that Y is orthogonal to M if and only if

$$\mu^Y = 0 \qquad P_M - a.e. \tag{1.2}$$

In order to give a third formulation of orthogonality, we introduce the processes

$$R_t(Y) := E\Big[(Y_T - Y_t)^2 \,\Big|\, \mathcal{F}_t \Big] = E\big[\langle Y \rangle_T - \langle Y \rangle_t \,\big|\, \mathcal{F}_t \big] \qquad , \qquad 0 \le t \le T \tag{1.3}$$

(this is the potential associated to $\langle Y \rangle$) and

$$Y_t^\delta := E\left[Y_T - \int_0^T \delta_u \, dM_u \,\middle|\, \mathcal{F}_t \right] = Y_t - \int_0^t \delta_u \, dM_u \qquad , \qquad 0 \le t \le T \tag{1.4}$$

for any bounded predictable process $\delta = (\delta_t)_{0 \le t \le T}$. Then we obtain the following perturbational characterization:

Proposition 1.1. *Y is orthogonal to M if and only if*

$$R_t(Y^\delta) - R_t(Y) \ge 0 \quad P-a.s. \quad , \quad 0 \le t \le T \tag{1.5}$$

for every bounded predictable process δ.

Proof. Since

$$\langle Y^\delta \rangle_t = \langle Y \rangle_t + \int_0^t \left(\delta_u^2 - 2\cdot\delta_u\cdot\mu_u^Y \right) d\langle M \rangle_u \quad , \quad 0 \le t \le T$$

by (1.1), we obtain for a fixed δ and $t \le s \le T$

$$R_t\left(Y^{\delta\cdot I_{(t,s]}}\right) - R_t(Y) = E\left[\int_t^s \left(\delta_u^2 - 2\cdot\delta_u\cdot\mu_u^Y \right) d\langle M \rangle_u \middle| \mathcal{F}_t \right].$$

Therefore, (1.5) is equivalent to

$$E_M\Big[\left(\delta^2 - 2\cdot\delta\cdot\mu^Y \right) \cdot I_D \Big] \ge 0 \tag{1.6}$$

for all bounded predictable δ and all sets D of the form $D = A_t \times (t,s]$ ($A_t \in \mathcal{F}_t$, $0 \le t \le s \le T$) or $D = A_0 \times \{0\}$ ($A_0 \in \mathcal{F}_0$). But since the class of these sets generates $\mathcal{P}$ and $\mathcal{P}$ determines P_M, (1.6) is equivalent to

$$\delta^2 - 2\cdot\delta\cdot\mu^Y \ge 0 \qquad P_M - a.e.$$

for every bounded predictable δ. Choosing $\delta := \varepsilon\cdot\text{sign}\ \mu^Y$ and letting ε tend to 0 now yields (1.2).

q.e.d.

Remark. $R(Y)$ can be interpreted as the *risk* entailed by Y; for example, this is appropriate if Y represents a cost process. (1.5) then expresses the idea that any perturbation of Y along M will increase risk, and Proposition 1.1 relates orthogonality of Y and M to a condition of risk-minimality. See Schweizer [3] for details on an application of this aspect.

Let us now consider a semimartingale

$$X = X_0 + M + A \ ,$$

and let us examine perturbations of Y along X instead of M. If the contributions from the quadratic increments of A are not too big, we may hope to find a similar connection between orthogonality and minimization of risk under such perturbations. In the following sections, we shall give precise results in this direction.

2. A convergence lemma

In this section, we prove a preliminary result which will help us solve the above problem. First we need to introduce some notation. If $\tau = (t_i)_{0 \le i \le N}$ is a partition of $[0,T]$, i.e.,

$$0 = t_0 < t_1 < \ldots < t_N = T \ ,$$

we denote by $|\tau| := \max\limits_{1 \le i \le N} (t_i - t_{i-1})$ the mesh of τ. Such a partition gives rise to the σ-algebras

$$\mathcal{B}^\tau := \sigma\Big(\big\{ D_0 \times \{0\}, D_i \times (t_{i-1}, t_i] \,\big|\, D_0 \in \mathcal{F}_0 \, , t_i \in \tau \, , D_i \in \mathcal{F}_{t_i} \big\} \Big)$$

and

$$\mathcal{P}^\tau := \sigma\Big(\big\{ D_0 \times \{0\}, D_{i-1} \times (t_{i-1}, t_i] \,\big|\, D_0 \in \mathcal{F}_0 \, , t_i \in \tau \, , D_{i-1} \in \mathcal{F}_{t_{i-1}} \big\} \Big)$$

on $\overline{\Omega}$. From now on, we shall work with an arbitrary but fixed sequence $(\tau_n)_{n \in \mathbf{N}}$ of partitions which is increasing (i.e., $\tau_n \subseteq \tau_{n+1}$ for all n) and satisfies $\lim\limits_{n \to \infty} |\tau_n| = 0$. Note that these properties together imply

$$\mathcal{P} = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{P}^{\tau_n} \right) \ . \tag{2.1}$$

Now let $C = (C_t)_{0 \le t \le T}$ be an adapted process with $C_0 = 0$. For $p > 0$, the p-variation of C on $[0,T]$ is defined by

$$W_p(C,T) := \sup_\tau \sum_{i=1}^{N(\tau)} \left| C_{t_i} - C_{t_{i-1}} \right|^p \ ,$$

with the supremum taken over all partitions τ of $[0,T]$. If $B = (B_t)_{0 \le t \le T}$ is an increasing adapted process with $B_0 = 0$ and $E[B_T] < \infty$, we denote by P_B the finite measure $P{\times}B$ on $\left(\overline{\Omega}, \overline{\mathcal{F}}\right)$ and by E_B expectation with respect to P_B. Finally, we define the processes

$$Q_p[C,B,\tau](\omega,t) := \sum_{t_i \in \tau} \frac{\left|C_{t_i} - C_{t_{i-1}}\right|^p}{B_{t_i} - B_{t_{i-1}}}(\omega) \cdot I_{(t_{i-1},t_i]}(t)$$

and

$$\widetilde{Q}_p[C,B,\tau](\omega,t) := \sum_{t_i \in \tau} \frac{E\Big[\,\left|C_{t_i} - C_{t_{i-1}}\right|^p \,\Big|\, \mathcal{F}_{t_{i-1}}\Big]}{E\big[\,B_{t_i} - B_{t_{i-1}} \,\big|\, \mathcal{F}_{t_{i-1}}\,\big]}(\omega) \cdot I_{(t_{i-1},t_i]}(t) \;;$$

both are nonnegative and well-defined P_B-a.e. The following result then gives sufficient conditions for the convergence to 0 of $Q_p[C,B,\tau_n]$ and $\widetilde{Q}_p[C,B,\tau_n]$:

Lemma 2.1. *Let $1 \le r < p$ and assume that C is continuous and has integrable r-variation. Then*

$$\lim_{n\to\infty} Q_p[C,B,\tau_n] = 0 \qquad P_B - a.e.$$

If in addition

$$\sup_n Q_p[C,B,\tau_n] \in \mathcal{L}^1(P_B) \tag{2.2}$$

and

(2.3) *C is constant over any interval on which B is constant ,*

then

$$\lim_{n\to\infty} \widetilde{Q}_p[C,B,\tau_n] = 0 \qquad P_B - a.e.$$

Proof. We have

$$Q_p[C,B,\tau_n] = Q_r[C,B,\tau_n] \cdot \sum_{t_i \in \tau_n} \left|C_{t_i} - C_{t_{i-1}}\right|^{p-r} \cdot I_{(t_{i-1},t_i]} \;,$$

and the second term on the right-hand side converges to 0 by the continuity of C. Hence, it is enough to show that $\sup_n Q_r[C,B,\tau_n] < \infty$ P_B-a.e. But

$$Q_r[C,B,\tau_n] \le Q_1\big[W_r(C,.),B,\tau_n\big] = \frac{d\big(P{\times}W_r(C,.)\big)}{dP_B}\Bigg|_{\mathcal{B}^{\tau_n}} \;,$$

and the last term is a nonnegative $(P_B, \mathcal{B}^{\tau_n})$-supermartingale, hence bounded in n P_B-a.e. The second assertion now follows immediately from Hunt's Lemma (cf. Dellacherie/Meyer [1], V.45) and the fact that

$$\widetilde{Q}_p[C, B, \tau_n] \le E_B\left[Q_p[C, B, \tau_n] \,\middle|\, \mathcal{P}^{\tau_n} \right]$$

if (2.3) holds.

q.e.d.

3. Orthogonality in a semimartingale setting

In this section, we apply the preceding result to derive a new characterization of orthogonality. We shall assume that $X = (X_t)_{0 \le t \le T}$ is a semimartingale with a decomposition

$$X = X_0 + M + A \ , \tag{3.1}$$

where $M = (M_t)_{0 \le t \le T}$ is a square-integrable martingale with $M_0 = 0$ and $A = (A_t)_{0 \le t \le T}$ is a continuous process of finite variation $|A| := W_1(A, .)$ with $A_0 = 0$. A bounded predictable process $\delta = (\delta_t)_{0 \le t \le T}$ will be called a *small perturbation* if the process $\int |\delta| \, d|A|$ is bounded. If δ is a small perturbation, the process

$$\int_0^t \delta_u \, dX_u \qquad (0 \le t \le T)$$

is well-defined as a stochastic integral and square-integrable. For a square-integrable martingale $Y = (Y_t)_{0 \le t \le T}$ and a partition τ of $[0, T]$, we define the processes

$$Y_t(\delta, \tau, i) := E\left[Y_T - \int_{t_{i-1}}^{t_i} \delta_u \, dX_u \,\middle|\, \mathcal{F}_t \right] \qquad , \qquad 0 \le t \le T \quad , \quad 1 \le i \le N$$

(choosing right-continuous versions) and

$$r^\tau[Y, \delta](\omega, t) := \sum_{t_i \in \tau} \frac{R_{t_i}\big(Y(\delta, \tau, i+1)\big) - R_{t_i}(Y)}{E\big[\, \langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \,\big|\, \mathcal{F}_{t_i} \,\big]}(\omega) \cdot I_{(t_i, t_{i+1}]}(t) \ .$$

Our objective now is to study the behaviour of $r^{\tau_n}[Y, \delta]$ along (τ_n).

Remark. $Y(\delta,\tau,i)$ can be viewed as a local perturbation of Y along X by $\delta\big|_{(t_{i-1},t_i]}$, and this corresponds exactly to the notion introduced in (1.4). If we again interpret $R(Y)$ as the risk of Y, then $r^\tau[Y,\delta]$ is a measure for the total change of risk under a local perturbation of Y along X by δ. The denominator in $r^\tau[Y,\delta]$ gives the "time scale" which should be used for these measurements.

Proposition 3.1. *Assume that A is absolutely continuous with respect to $\langle M\rangle$ with a density α satisfying*

$$E_M\big[|\alpha|\cdot\log^+|\alpha|\big] < \infty\,. \tag{3.2}$$

Then

$$\lim_{n\to\infty} r^{\tau_n}[Y,\delta] = \delta^2 - 2\cdot\delta\cdot\mu^Y \qquad P_M - a.e. \tag{3.3}$$

for every small perturbation δ.

Proof. 1) Inserting the definitions yields

$$\begin{aligned}
&Y_T(\delta,\tau_n,i+1) - Y_{t_i}(\delta,\tau_n,i+1)\\
&= Y_T - Y_{t_i} - \int_{t_i}^{t_{i+1}} \delta_u\,dM_u - \left(\int_{t_i}^{t_{i+1}} \delta_u\,dA_u - E\left[\int_{t_i}^{t_{i+1}} \delta_u\,dA_u \,\middle|\, \mathcal{F}_{t_i}\right]\right)
\end{aligned}$$

and therefore by (1.1)

$$\begin{aligned}
&R_{t_i}\big(Y(\delta,\tau_n,i+1)\big) - R_{t_i}(Y)\\
&= E\left[\int_{t_i}^{t_{i+1}} \big(\delta_u^2 - 2\cdot\delta_u\cdot\mu_u^Y\big)\,d\langle M\rangle_u \,\middle|\, \mathcal{F}_{t_i}\right] + \mathrm{Var}\left[\int_{t_i}^{t_{i+1}} \delta_u\,dA_u \,\middle|\, \mathcal{F}_{t_i}\right]\\
&\quad + 2\cdot\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_u\,dM_u - \big(Y_{t_{i+1}} - Y_{t_i}\big)\,,\ \int_{t_i}^{t_{i+1}} \delta_u\,dA_u \,\middle|\, \mathcal{F}_{t_i}\right).
\end{aligned}$$

This allows us to write $r^{\tau_n}[Y,\delta]$ as

$$\begin{aligned}
r^{\tau_n}[Y,\delta] &= E_M\left[\delta^2 - 2\cdot\delta\cdot\mu^Y \,\middle|\, \mathcal{P}^{\tau_n}\right]\\
&\quad + \sum_{t_i\in\tau_n} \frac{\mathrm{Var}\left[\int_{t_i}^{t_{i+1}} \delta_u\,dA_u \,\middle|\, \mathcal{F}_{t_i}\right]}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}\,\big|\,\mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]}\\
&\quad + 2\cdot\sum_{t_i\in\tau_n} \frac{\mathrm{Cov}\left(\int_{t_i}^{t_{i+1}} \delta_u\,dM_u - \big(Y_{t_{i+1}} - Y_{t_i}\big)\,,\ \int_{t_i}^{t_{i+1}} \delta_u\,dA_u \,\middle|\, \mathcal{F}_{t_i}\right)}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}\,\big|\,\mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]}\,.
\end{aligned}$$

By martingale convergence, the first term on the right-hand side tends to $\delta^2 - 2\cdot\delta\cdot\mu^Y$ P_M-a.e., due to (2.1). The second term is dominated by

$$\sum_{t_i\in\tau_n} \frac{E\Big[\Big(\int_{t_i}^{t_{i+1}} \delta_u\, dA_u\Big)^2 \,\Big|\, \mathcal{F}_{t_i}\Big]}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\big|\, \mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]} = \widetilde{Q}_2\left[\int \delta\, dA, \langle M\rangle, \tau_n\right] .$$

For the third term, we use the Cauchy-Schwarz inequality for sums to get

$$\left|\sum_{t_i\in\tau_n} \frac{\mathrm{Cov}\Big(\int_{t_i}^{t_{i+1}} \delta_u\, dM_u - \big(Y_{t_{i+1}} - Y_{t_i}\big)\;,\; \int_{t_i}^{t_{i+1}} \delta_u\, dA_u \,\Big|\, \mathcal{F}_{t_i}\Big)}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\big|\, \mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]}\right|$$

$$\leq \left(\sum_{t_i\in\tau_n} \frac{\mathrm{Var}\Big[\int_{t_i}^{t_{i+1}} \delta_u\, dA_u \,\Big|\, \mathcal{F}_{t_i}\Big]}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\big|\, \mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]}\right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\sum_{t_i\in\tau_n} \frac{E\Big[\int_{t_i}^{t_{i+1}} \delta_u^2\, d\langle M\rangle_u + \big(\langle Y\rangle_{t_{i+1}} - \langle Y\rangle_{t_i}\big) \,\Big|\, \mathcal{F}_{t_i}\Big]}{E\big[\,\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i} \,\big|\, \mathcal{F}_{t_i}\,\big]}\cdot I_{(t_i,t_{i+1}]}\right)^{\frac{1}{2}}$$

$$\leq \left(\widetilde{Q}_2\left[\int \delta\, dA, \langle M\rangle, \tau_n\right]\right)^{\frac{1}{2}} \cdot \left(\widetilde{Q}_1\left[\int \delta^2\, d\langle M\rangle + \langle Y\rangle, \langle M\rangle, \tau_n\right]\right)^{\frac{1}{2}} .$$

But

$$\widetilde{Q}_1\left[\int \delta^2\, d\langle M\rangle + \langle Y\rangle, \langle M\rangle, \tau_n\right] = E_M\left[\,\delta^2 \,\Big|\, \mathcal{P}^{\tau_n}\right] + \frac{dP_Y}{dP_M}\bigg|_{\mathcal{P}^{\tau_n}}$$

is a nonnegative $(P_M, \mathcal{P}^{\tau_n})$-supermartingale and therefore bounded in n P_M-a.e. Hence, it only remains to show that

$$\lim_{n\to\infty} \widetilde{Q}_2\left[\int \delta\, dA, \langle M\rangle, \tau_n\right] = 0 \qquad P_M - a.e. \tag{3.4}$$

2) The process $\int \delta \, dA$ is continuous and has bounded variation. Furthermore,

$$Q_2\left[\int \delta \, dA, \langle M \rangle, \tau_n\right] = Q_1\left[\int \delta \, dA, \langle M \rangle, \tau_n\right] \cdot \sum_{t_i \in \tau_n} \left| \int\limits_{t_{i-1}}^{t_i} \delta_u \, dA_u \right| \cdot I_{(t_{i-1}, t_i]}$$

$$\leq \|\delta\|_\infty \cdot Q_1\left[|A|, \langle M \rangle, \tau_n\right] \cdot \int\limits_0^T |\delta_u| \, d|A|_u$$

$$= \|\delta\|_\infty \cdot E_M\left[\, |\alpha| \,\middle|\, \mathcal{B}^{\tau_n} \right] \cdot \int\limits_0^T |\delta_u| \, d|A|_u$$

implies by (3.2) and Doob's inequality that

$$\sup_n Q_2\left[\int \delta \, dA, \langle M \rangle, \tau_n\right] \in \mathcal{L}^1(P_M) \, . \tag{3.5}$$

This yields (3.4) by Lemma 2.1.

q.e.d.

We can now use Proposition 3.1 to give the announced characterization of those square-integrable martingales Y which are orthogonal to M:

Theorem 3.2. *Under the assumptions of Proposition 3.1, the following statements are equivalent:*

1) $\liminf_{n \to \infty} r^{\tau_n}[Y, \delta] \geq 0 \quad P_M$-a.e. *for every small perturbation* δ.

2) $\mu^Y = 0 \quad P_M$-a.e.

3) *Y is orthogonal to M.*

Proof. Proposition 3.1 shows that the limit in 1) exists P_M-a.e. and equals $\delta^2 - 2 \cdot \delta \cdot \mu^Y$. To prove that 1) implies 2), we choose $\delta := \varepsilon \cdot \operatorname{sign} \mu^Y \cdot I_{\{|A| \leq k\}}$ and then let $\varepsilon \to 0$ and $k \to \infty$.

q.e.d.

Remarks. 1) As mentioned above, the original inspiration for this work comes from an application to the theory of option trading. In this context, Y represents the cost process of a trading strategy so that $R(Y)$ can indeed be interpreted as risk. The relevant trading strategies can be parametrized by a certain class of predictable processes ξ, and the ultimate goal is to determine an optimal ξ^* in this class. Statement 1) of Theorem 3.2 is then an optimality criterion expressing a notion of *risk-minimality* under local perturbations of a trading strategy. The equivalent statement 2) translates into a complicated *stochastic optimality equation* which ξ^* must satisfy. Hence, Theorem 3.2 reduces the variational problem of finding an optimal strategy to the task of solving this optimality equation. For a more detailed account of these aspects, we refer to Schweizer [2], [3].

2) In the theory of option trading, the process X represents the price fluctuations of a stock, and a standard assumption which excludes arbitrage opportunities is the existence of an equivalent martingale measure P^* for X. A closer look at the Girsanov transformation from P to P^* then reveals that A must be absolutely continuous with respect to $\langle M \rangle^P$, at least if the density process corresponding to the change of measure is locally square-integrable. The hypotheses of Proposition 3.1 are therefore quite natural within such a framework.

3) We have assumed the perturbations δ to be bounded. However, some applications make it desirable to admit predictable processes δ such that $\int \delta \, dX$ is a semimartingale of class $\mathcal{S}^2$. If for example both α and $\langle M \rangle_T$ are bounded, then a slight modification of the proof shows that the assertions of Proposition 3.1 still hold true for these more general δ.

4) If A has square-integrable variation, continuity of A is equivalent to the assumption that A has *2-energy 0* in the sense that

$$\lim_{n \to \infty} E\left[\sum_{t_i \in \tau_n} \left(A_{t_i} - A_{t_{i-1}} \right)^2 \right] = 0 \, .$$

This is a more precise formulation of the intuitive condition that the quadratic increments of A should be asymptotically negligible.

References

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