On Feedback Effects from Hedging Derivatives[*]

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Abstract: This paper proposes a new explanation for the smile and skewness effects in implied volatilities. Starting from a microeconomic equilibrium approach, we develop a diffusion model for stock prices explicitly incorporating the technical demand induced by hedging strategies. This leads to a stochastic volatility endogenously determined by agents' trading behaviour. By using numerical methods for stochastic differential equations, we quantitatively substantiate the idea that option price distortions can be induced by feedback effects from hedging strategies.

Key words: option pricing, Black-Scholes formula, implied volatility, smile, skewness, stochastic volatility, feedback effects

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0. Introduction

"I sometimes wonder why people still use the Black-Scholes formula, since it is based on such simple assumptions — unrealistically simple assumptions" (Black (1990)). This well-known formula expresses the (discounted) price $u_{BS}(t, S_t; K, T, \sigma)$ of a European call option as a function of five parameters. The current time t and the current (discounted) stock price S_t are observable, and so are the option's characteristics, the (discounted) strike price K and the expiration date T. The volatility σ is not observable, but one can numerically invert the Black-Scholes formula to define the so-called *implied volatility* corresponding to a given option price $u(t, S_t; K, T)$:

$$u(t, S_t; K, T) = u_{BS}\Big(t, S_t; K, T, \sigma_{\text{impl}}(t, S_t; K, T)\Big). \tag{0.1}$$

If market prices were computed by the Black-Scholes formula, σ_{impl} would be a constant function. But many empirical studies show that this is not the case in real markets; see for instance Rubinstein (1985), Fung and Hsieh (1991), the paper by Clewlow and Xu (1993) which contains a comprehensive list of references, or Dumas et al. (1995). For fixed t, S_t, T, the graph of the mapping $K \mapsto \sigma_{\text{impl}}(t, S_t; K, T)$ is typically U-shaped: implied volatilities *"smile"*. Moreover, this smile can be more or less lopsided; this is called *"skewness"*. It is also generally acknowledged that the smile effect decreases with increasing time to maturity; see for instance Rubinstein (1985), Clewlow and Xu (1993) or Derman and Kani (1994a,b). To our knowledge, the nonlinearity was first explicitly pointed out by Shastri and Wethyavivorn (1987); a colourful account and intuitive description is given by Hicks (1992).

Apart from a few papers on stock price models with jumps and apart from informal suggestions along the lines of transaction costs, liquidity problems or fat tails in the stock price distributions, the existing literature has predominantly focussed on directly modelling volatility as a stochastic process in order to explain such option price distortions; see for instance Hobson and Rogers (1994), Renault and Touzi (1996) or Taylor and Xu (1993). But the description of the volatility, though always plausible, is typically ad hoc. Our approach in the present paper concentrates instead on the criticism that "The formula assumes that you can't affect either stock or option prices by placing orders" (Black (1990)). We offer a new explanation based on the idea that a substantial use of hedging strategies *will* affect the dynamics of the underlying stock. Since our resulting model can formally still be viewed as a stochastic volatility model, we start by briefly discussing a number of previous papers before explaining our alternative approach.

0.1. Stochastic volatility models without an additional noise term

A first group of models in the literature describes stock prices by a stochastic differential equation of the general form

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t + \mu_t \, dt. \tag{0.2}$$

The important point here is that the stochastic processes (σ_t) and (μ_t) are assumed to be adapted to the filtration $\mathbb{F}^W$ generated by W. Examples include the Constant Elasticity of Variance (CEV) model of Cox and Ross (1976) investigated by Beckers (1980), the Displaced Diffusion model of Rubinstein (1983) or the recently proposed model of Hobson and Rogers (1994). The last paper also contains further references and a more detailed discussion of the various families of models. One big advantage of the specification (0.2) with $(\mu_t), (\sigma_t)$ $\mathbb{F}^W$-adapted is the fact that under fairly weak assumptions, one obtains a *complete market* and therefore unique option prices determined by arbitrage considerations alone. For a suitable choice of (0.2), Hobson and Rogers (1994) show that these option prices can indeed exhibit smile and skewness effects. From our point of view, however, this argument is still rather unsatisfactory. Even if volatility evolves in a very plausible way (including dependence on the current stock price level as well as the past values of stock prices and volatilities), it still has to be modelled explicitly in this approach. There is no truly endogenous derivation of the basic model, and so we feel that it does not really provide a deeper explanation for the emergence of smile and skewness.

Other papers formally belonging to this group of models are those by Derman and Kani (1994a,b) and Dupire (1994); see also Rubinstein (1994). They consider the stochastic differential equation

$$\frac{dS_t}{S_t} = \sigma(S_t)\, dW_t, \tag{0.3}$$

but their approach is completely opposite to the one above. Assuming that the prices of call options are given for all strikes, they identify a unique function $\sigma(s)$ such that the arbitrage-free call option prices corresponding to (0.3) coincide with the given prices; this is very similar in spirit to the construction of a term structure model of interest rates. The knowledge of the *volatility function* $\sigma(s)$ can then be used to price and hedge derivative securities other than call options in a way that is consistent with the given smile pattern. Although this looks very attractive, we think that it suffers from two drawbacks. From a theoretical point of view, it does not explain smile or skewness at all; they are simply given facts. More seriously, though, the method does not really appear to be feasible in practice, because a sufficiently exact observation of call prices for sufficiently many strikes seems impossible in most real options markets.

0.2. Stochastic volatility models with extra noise

A second group of models uses an additional stochastic process for the volatility; this typically entails the introduction of at least one extra source of randomness, usually a second (possibly correlated) Brownian motion. To quote a few papers in this area, we mention Hull and White (1987, 1988), Stein and Stein (1991), Dupire (1993) and Heston (1993); a survey is given by Clewlow and Xu (1992). The use of implied volatilities as forecasts for future volatilities in such models is discussed by Stein (1989), Scott (1992) and Heynenet al. (1994), among others. Quite recently, there have also been attempts to explain smile and skewness as consequences of stochastic volatility. Renault and Touzi (1996) consider a generalized version of the model proposed by Hull and White (1988) and show by theoretical arguments that qualitative smile effects will appear under reasonable assumptions. For the case of the Hull and White (1988) model, similar (but only approximate) results are obtained by Taylor and Xu (1993), Ball and Roma (1994) or Heynen (1994); see also Paxson (1994) and Duan (1995) for related work.

Despite these advances, we think that all the preceding stochastic volatility models have two serious disadvantages. They are unsatisfactory from a theoretical point of view, because volatility is *exogenously* given by some stochastic process chosen more or less ad hoc. The same criticism unfortunately also applies to the rich literature on ARCH models and their various relatives. From a practical point of view, the existing results are unsatisfactory since they can explain only a relatively small percentage of the observed smile effect. Clewlow and Xu (1992) mention that "empirically observed smile effects are typically characterized by implied volatilities around 10% higher for away-from-the-moneyness of around 5%", while "the Hull and White (1988) model gives implied volatilities of only 1% higher for this level of away-from-the-moneyness". Taylor and Xu (1993) find that the magnitudes of empirical smiles are approximately twice the magnitude of their theoretical smiles. They suggest that this can be ascribed to market imperfections or violations of assumptions (e.g., continuity) about the price process. Unfortunately, such an argument brings us back to the level of an informal discussion.

In contrast to most previous papers, we do not model volatility explicitly. Since we want to understand how hedging strategies and the underlying asset influence each other, we start from a microeconomic equilibrium approach as in Föllmer (1991) and Föllmer and Schweizer (1993). This allows us to develop a diffusion model for stock prices which explicitly incorporates the technical demand induced by hedgers. In particular, we obtain a stochastic volatility which is *endogenously* determined by the trading behaviour of the agents in our economy. This part of the paper is closely related to Frey and Stremme (1995) who use a similar approach to study the influence of portfolio insurance on the asset's volatility, a question which is also discussed in Grossman (1988) and Brennan and Schwartz (1989). But in contrast to these authors, we then go on to examine the issue of *implied* volatilities. Numerical results from our model exhibit clear smiles and skews whose magnitudes agree very well with empirical observations. Thus we are able to quantitatively substantiate the idea that option price distortions can be induced by *feedback effects from hedging strategies*.

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1. The basic model

This section presents a method for constructing asset price models in which one can study how hedging strategies for derivatives interact with the evolution of the underlying financial instruments. The general formulation is worked out in detail for one particular example which yields our new explanation for smile and skewness. Basically, the idea is to obtain the asset dynamics implicitly from the equilibrium condition of market clearing. This approach is due to Föllmer (1991); it was subsequently taken up and refined in Föllmer and Schweizer (1993), Föllmer (1994), Platen and Schweizer (1994), Frey and Stremme (1995) and Platen and Rebolledo (1996). For different approaches to similar questions, see also Brennan and Schwartz (1989), De Long et al. (1990), Grossman (1988), Jarrow (1994) and the references in the papers mentioned above.

To keep the exposition as simple as possible, we consider a stylized economy in continuous time t with just one risky asset. We call it stock, denote its time t price by S_t and write $L_t = \log S_t$. There is also a riskless asset with zero interest rate so that S is actually the discounted stock price. We work with L rather than S to simplify expressions. Agents in our economy are price-takers. Given a proposed log stock price ℓ at time t, let $D(t,\ell,U_t)$ denote their cumulative demand for stock up to time t, summed over all agents. The term U_t here summarizes all other parameters except ℓ that may influence the demand for stock. Typically, U will be a stochastic process that represents some noise; its main characteristic is that it is either exogenous or at least not modelled in much more detail.

Stock prices in this economy are determined by the *equilibrium* condition of *market clearing*,

$$D(t,L_t,U_t) = \text{const.} \qquad , \qquad t \geq 0. \tag{1.1}$$

If we assume that L and U are continuous semimartingales and that D is sufficiently regular, we can use the implicit function theorem and Itô's formula to solve (1.1) for L_t and to obtain the stochastic differential equation

$$dL_t = -\frac{1}{D_L}\left(D_U\,dU_t + D_t\,dt + \frac{1}{2}\left(D_{LL}\left(\frac{D_U}{D_L}\right)^2 - 2D_{LU}\frac{D_U}{D_L} + D_{UU}\right)d\langle U\rangle_t\right). \tag{1.2}$$

In (1.2), subscripts of D denote partial derivatives, and the argument (t,L_t,U_t) has been suppressed throughout. Details and regularity assumptions for this derivation can be found in the appendix. Applying Itô's formula to $S = e^L$ immediately yields the evolution of S.

In order to study feedback effects from derivatives on an underlying asset, we now have to specify the demand D in more detail. Our subsequent approach can be viewed as a *partial equilibrium* analysis in the following sense. In some large economy, derivatives written on our stock may be valued, traded and hedged, thus inducing among other things a demand for stock. But we are only interested in the evolution of our particular stock, and so we restrict our attention to a sub-economy where only this single stock is traded. From that perspective, options are not traded assets (in the small economy), and the demand for stock induced by them can be viewed as exogenous (for the small economy). In particular, the total demand for stock will only depend on the evolution of the stock price itself.

To see in an example how this works, let us consider the demand function

$$D(t,L_t,U_t) = U_t + \gamma(L_t - L_0) + \xi(t,L_t) \tag{1.3}$$

for some constant $\gamma \neq 0$, where

$$U_t = vW_t + mt \tag{1.4}$$

is a Brownian motion with drift $m \in I\!R$ and variance $v^2 > 0$. Each of the three components of D has a natural interpretation. First of all, U_t represents a *random* error term whose origin is not specified in more detail. Typical examples are noise, mis-specifications in the model, a demand for liquidity purposes etc. The second component is the cumulative demand of *arbitrage-based* agents or speculators; the parameter γ describes the way they react to changes in (logarithmic) stock prices. Finally, the term $\xi(t,L_t)$ can be viewed as the *hedging* or technical part of the demand in the sense that it results from strategies used to hedge options written on our stock. With this in mind, we shall presently describe ξ in more detail.

For the specification (1.3) of D with U as in (1.4), the model in (1.2) takes the form

$$dL_t = -\frac{v}{\gamma + \xi'(t,L_t)}\,dW_t - \left(\frac{m + \dot{\xi}(t,L_t)}{\gamma + \xi'(t,L_t)} + \frac{1}{2}\frac{v^2\xi''(t,L_t)}{\big(\gamma + \xi'(t,L_t)\big)^3}\right)dt, \tag{1.5}$$

where $\dot{}$ and $'$ denote partial derivatives with respect to t and ℓ, respectively. In terms of S, (1.5) can be written as

$$\frac{dS_t}{S_t} = \sigma(t, S_t)\, dW_t + \mu(t, S_t)\, dt \tag{1.6}$$

with

$$\sigma(t, s) = -\frac{v}{\gamma + \xi'(t, \log s)} \tag{1.7}$$

and

$$\begin{aligned} \mu(t, s) &= -\left(\frac{m + \dot{\xi}(t, \log s)}{\gamma + \xi'(t, \log s)} - \frac{1}{2} \frac{v^2}{\Big(\gamma + \xi'(t, \log s)\Big)^2} + \frac{1}{2} \frac{v^2 \xi''(t, \log s)}{\Big(\gamma + \xi'(t, \log s)\Big)^3} \right) \\ &= \frac{m + \dot{\xi}(t, \log s)}{v} \sigma(t, s) + \frac{1}{2} \sigma^2(t, s) + \frac{\xi''(t, \log s)}{2v} \sigma^3(t, s). \end{aligned} \tag{1.8}$$

Let us now turn to the description of the hedging demand ξ. Our option hedgers are assumed to be *program traders* working with a fictitious model $S^{(0)}$. For simplicity, we take $S^{(0)}$ to be geometric Brownian motion,

$$\frac{dS_t^{(0)}}{S_t^{(0)}} = \sigma_0\, dB_t + \mu_0\, dt \tag{1.9}$$

for some Brownian motion B. By assumption, then, our hedgers act according to the well-known Black-Scholes formula with an a priori volatility σ_0. This is of course a simplification, and we shall comment below on its significance and the problems involved in relaxing it.

Consider first an individual European call option on a stock S with (contract) size V, strike price K and maturity T. The payoff at time T of this instrument is given by

$$h_{\text{call}}(S_T) = V \max(S_T - K, 0) = V(S_T - K)^+ \tag{1.10}$$

so that the size V simply represents the number of shares that can be bought at time T for the price K. For a hedger working with the model (1.9), the hedging strategy for the individual call in (1.10) is thus to hold at time t

$$\tilde{\xi}_{\text{call}}(t, \ell; V, K, T) = V \Phi\left(\frac{\ell - \log K + \frac{1}{2}\sigma_0^2 (T - t)}{\sigma_0 \sqrt{T - t}} \right) \tag{1.11}$$

shares of the underlying stock, where $\ell = \log S_t$ is the logarithm of the asset price at time t. Since the left-hand side of (1.11) depends on t and T only via the *time to maturity* $\tau := T - t$, we write

$$\xi_{\text{call}}(\ell; V, K, \tau) = \tilde{\xi}_{\text{call}}(t, \ell; V, K, \tau + t). \tag{1.12}$$

For a put option with the same parameters, the hedging strategy would be

$$\xi_{\text{put}}(\ell; V, K, \tau) = \xi_{\text{call}}(\ell; V, K, \tau) - V$$

by put-call parity. Since we shall only need the derivatives of ξ with respect to ℓ, we can treat calls and puts as equivalent for our purposes, and therefore we only work with call options from now on.

The function ξ required in (1.3) is now obtained by summing the individual hedging demands in (1.12) over all outstanding options. We shall assume that options are traded in a liquid market with many agents or option contracts constantly entering or leaving the scene. Intuitively, this means that we consider a stationary market in which we see at each date the same distribution of outstanding options to be hedged. For concreteness, we shall work with the specification

$$\xi(\ell) = \sum_{i=1}^{M}\sum_{j=1}^{N} \xi_{\text{call}}(\ell; V_{ij}, K_{ij}, \tau_i) = \sum_{i=1}^{M}\sum_{j=1}^{N} V_{ij}\Phi\left(\frac{\ell - \log K_{ij} + \frac{1}{2}\sigma_0^2\tau_i}{\sigma_0\sqrt{\tau_i}}\right) \tag{1.13}$$

which means that we work with MN option series: At each point in time, there are M types of outstanding (call) options with "typical" times to maturity τ_i, each subdivided into N sub-types with sizes V_{ij} and strikes K_{ij} corresponding to τ_i. This kind of distribution is a fairly good approximation to reality in an active options market. More generally, we could describe our stationary market by specifying the total option exposure as a measure ν over the space of triples (V, K, τ) and then defining ξ as the integral of ξ with respect to ν. This formulation is due to Frey and Stremme (1995) and was also used in Platen and Schweizer (1994); the special case (1.13) corresponds to ν being a finite sum of Dirac measures. Note that the sizes V_{ij} will typically be nonnegative; this reflects the fact that far more option writers than buyers will hedge their positions.

The specification of ξ in (1.13) together with (1.3) and (1.4) induces via the equilibrium condition (1.1) a new stock price model which we denote by $S^{(1)}$. According to (1.6) – (1.8), $S^{(1)}$ is given by the stochastic differential equation

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \sigma(S_t^{(1)})\,dW_t + \mu(S_t^{(1)})\,dt, \tag{1.14}$$

where W is some Brownian motion under P,

$$\sigma(s) = -\frac{v}{\gamma + \xi'(\log s)}. \tag{1.15}$$

and

$$\begin{aligned} \mu(s) &= -\left(\frac{m}{\gamma + \xi'(\log s)} - \frac{1}{2}\frac{v^2}{\big(\gamma + \xi'(\log s)\big)^2} + \frac{1}{2}\frac{v^2\xi''(\log s)}{\big(\gamma + \xi'(\log s)\big)^3}\right) \\ &= \frac{m}{v}\sigma(s) + \frac{1}{2}\sigma^2(s) + \frac{\xi''(\log s)}{2v}\sigma^3(s). \end{aligned} \tag{1.16}$$

If the filtration is generated by $S^{(1)}$ or (equivalently) by W, then $S^{(1)}$ admits a unique equivalent martingale measure. It removes the drift μ by a Girsanov transformation and this leads directly to the description

$$\frac{dS_t}{S_t} = \sigma(S_t)\,d\widehat{W}_t \tag{1.17}$$

for some Brownian motion $\widehat{W}$ with respect to some measure $\widehat{P}$. Uniqueness implies that $S^{(1)}$ is *complete*, and so the familiar martingale approach suggests to define the value at time t of a European call option with size V, strike price K and maturity T as

$$u(t,s;K,T) := V\widehat{E}\left[(S_T-K)^+\middle|S_t=s\right] = V\widehat{E}\left[\left(S_T^{t,s}-K\right)^+\right]; \tag{1.18}$$

see for instance Harrison and Pliska (1981). In (1.18), $S^{t,s}$ denotes as usual the solution of (1.17) starting from stock price s at time t. For a general filtration, we typically lose completeness, but we could then for instance refer to the argument in Hofmann et al. (1992) to motivate the use of the minimal equivalent martingale measure $\widehat{P}$. In both cases, we can use the formula (1.18) as a proxy for option values.

At this point, a number of comments seems appropriate. We first want to emphasize that the above approach to the construction of a stock price model is of course not limited to the study of smile and skewness effects. It should rather be viewed as a flexible framework in which one obtains an equilibrium price process from microeconomic specifications. These are embodied in the choice of the demand D, and our present model is one particular example. For ease of exposition, we have started directly with a continuous-time economy and (via the argument in the appendix) an infinitesimal formulation for the market clearing condition. As in Föllmer and Schweizer (1993) or Frey and Stremme (1995), one could also begin with a discrete-time equilibrium model and derive (1.2) via a suitable diffusion limit. For the model (1.14) – (1.16), this limiting approach is discussed in more detail in Platen and Schweizer (1994).

From a conceptual point of view, the preceding approach derives a model $S^{(k+1)}$ from a model $S^{(k)}$, where k has an intuitive significance as the *level of sophistication* of the agents using hedging strategies. More precisely, one starts from a model $S^{(k)}$, uses this to compute option values $u^{(k)}$ and hedging strategies $\xi^{(k)}$, and then employs an equilibrium argument to derive a new model $S^{(k+1)}$. This new model takes into account the hedging induced by the old one, and thus includes the feedback from already existing options. Ideally, hedgers should therefore base their strategies not on $S^{(0)}$, but rather on a fixed point $S^{(\infty)}$ of this mechanism. Such a model $S^{(\infty)}$ would a priori take into account the feedback effects of hedging strategies on the underlying asset. However, this ideal solution is at present not feasible for several reasons. For one thing, it is an open problem if the sequence $(S^{(k)})_{k\in\mathbb{N}}$ converges in some sense or if there exists a fixed point $S^{(\infty)}$. The passage from $S^{(0)}$ to $S^{(1)}$ is still possible because the constant volatility function $\sigma^{(0)}(s) \equiv \sigma_0$ in (1.9) permits a closed-form expression for the strategy $\xi^{(0)}$ and therefore also for the next volatility function $\sigma^{(1)}$ given by (1.15). At higher levels of iteration, this tractability is no longer present. Passing from level k to $k+1$ requires the computation of the second derivative of the function $u^{(k)}$ to obtain $\sigma^{(k+1)}$ via (1.15), and the currently available numerical methods are not sufficient to guarantee the accuracy and stability required for a numerical iteration of such computations. Our approach thus raises challenging theoretical and numerical problems well beyond the scope of the present paper, and we plan to address these issues in the future.

The preceding discussion also illuminates the difference between our approach and the work by Frey and Stremme (1995). They basically compare the two models resulting from (1.6) – (1.8) when ξ is first omitted (their "reference model") and then included (their "alternative economy that includes portfolio insurance"). In contrast to that, we compare the fictitious initial model $S^{(0)}$ (which is in general *not* the same as their reference model) to the actually resulting model $S^{(1)}$. This implies in particular that in our framework, one cannot draw conclusions by simply omitting one component of D in (1.3) and leaving all others unchanged.

The next comment concerns the sign of the parameter γ in (1.3) which describes the way that arbitrageurs or speculators react to price changes in the underlying asset. The case $\gamma < 0$ corresponds to *impatient* agents who want to take their profits immediately; they thus react with selling orders to a rise in stock prices and vice versa. On the other hand, $\gamma > 0$ describes *patient* speculators who buy the stock after an increase in its value since they hope for a further rise and thus a larger profit later on. We shall assume for the rest of the paper that

$$\gamma > 0, \tag{1.19}$$

and this deserves some more discussion.

In Frey and Stremme (1995), the authors study similar models; the assumptions made there correspond to a negative γ. Moreover, they argue (against $\gamma > 0$) that "working with an increasing demand function for the reference traders yields ... an unstable equilibrium". While this is true, it is not an argument against *our* positive γ. As pointed out above, we do *not* introduce our program traders into a reference economy as in Frey and Stremme (1995). Quite in contrast, we want to model the interaction between hedgers and speculators and its effect on stock prices. Since our hedgers are Black-Scholes program traders, their demand is procyclic: they buy with rising prices. Once a speculator realizes this fact, he may be quite happy to follow along and bet on an increasing stock price. Of course, not every single arbitrageur will behave like this. But in a situation with interaction between speculators and procyclic hedgers, it seems reasonable that a majority of the speculators will be patient rather than impatient, and this is exactly condition (1.19). For a discussion on the relaxation of this assumption, we also refer to the end of the next section.

2. Numerical results

In this section, we present the numerical results obtained from the computation of the option values in (1.16). Let us begin with some more technical remarks. Since we assume that $\gamma > 0$ and that the sizes V_{ij} in (1.13) are all nonnegative, the volatility function σ in (1.15) is sufficiently regular so that the stochastic differential equation (1.17) has a unique strong solution for S. For the actual computation of (1.18), we can then apply stochastic numerical methods as explained for example in Kloeden and Platen (1992). For our subsequent results, we used a weak Euler predictor-corrector approximation method of weak order 1.0. Together with a combination of several variance reduction techniques, this led to a rather reliable and efficient evaluation for the parameters given below.

To completely specify the model (1.13) – (1.16), we now fix the parameter values $v = 0.2$, $\gamma = 0.5$, $t = 0$ and $S_0 = 100$. The hedging demand function ξ is given by (1.13) with $M = 1$ and $N = 3$. By (1.13) and (1.15), the volatility function can then be written as

$$\sigma(s) = -\frac{v}{\gamma + \sum_{j=1}^{3} V_j \frac{1}{\sqrt{2\pi\sigma_0^2\tau}} \exp\left(-\frac{\left(\log s - \log K_j + \frac{1}{2}\sigma_0^2\tau\right)^2}{2\sigma_0^2\tau}\right)}. \tag{2.1}$$

According to information obtained from experienced traders, the average time to maturity of the most intensively hedged options can usually be found between 0.05 and 0.32 years. As representative time to maturity, we therefore choose $\tau = 0.16$. The traders also said that they usually only hedge options whose strikes are at most 10% in or out of the money. For that reason, we work with the strikes $K_1 = 90$, $K_2 = 100$ and $K_3 = 110$. The sizes V_j will be chosen below in two different ways to obtain a smile with a more or less pronounced skewness effect. Their actual values are not important for our numerical results; what matters is how big they are relative to each other and relative to the patience parameter γ. Finally, the volatility σ_0 of the Black-Scholes model (1.9) in the background is taken to be $\sigma_0 = 0.2$.

2.1. The smile effect

Let us first consider the case where the sizes V_j lie symmetrically around the current stock price S_0, i.e., $V_1 = V_3 = 0.01$. The size $V_2 = 0.09$ is chosen considerably larger to model a situation where most of the options held in the market are at the money.

Figure 1 Volatility function $\sigma(s)$ for the sizes (0.01, 0.09, 0.01).

For this case, Figure 1 shows the (negative of the) volatility function $\sigma(s)$ in (2.1). We see that σ varies between a value of approximately 0.2 for the current stock price $s = S_0 = 100$ and goes as high as 0.4 for values of s far away from S_0. Furthermore, we observe that $\sigma(s)$ is almost symmetric around the current stock price $S_0 = 100$.

Next we compare the option values obtained from (1.18) to the Black-Scholes prices computed with the a priori volatility σ_0, i.e., to $u_{BS}(0, S_0; K, T, \sigma_0)$. Since we are at date $t = 0$, the parameter T may be interpreted either as expiration date or as time to maturity. Figure 2 displays the difference $u(0, S_0; K, T) - u_{BS}(0, S_0; K, T, \sigma_0)$ for strike prices K between 80 and 120 and for expiration dates up to 1 year away from now. We note that in the considered range of parameters, the difference of our value to the Black-Scholes price is always nonnegative and goes up to the fairly large amount of 2 monetary units. To give an idea of the relative importance of these deviations, we remark that typical figures for Black-Scholes prices range here from 0 to around 21 monetary units.

Figure 2 Option value minus Black-Scholes price.

The most interesting result is given by Figure 3. It shows the implied volatilities for the option values $u(0, S_0; K, T)$ from (1.18), i.e., the surface defined by the mapping $(K, T) \mapsto \sigma_{\text{impl}}(0, S_0; K, T)$. Looking at Figure 3, we observe a strong *smile effect:* strikes at the money have a distinctly lower implied volatility than those in or out of the money. Furthermore, the difference also decreases with increasing time to maturity.

Figure 3 Implied volatilities for the sizes (0.01, 0.09, 0.01).

These numerical results coincide with the empirical findings of Rubinstein (1985), Clewlow and Xu (1993) and Taylor and Xu (1993). Extensive simulation studies with different volatility functions of the above type always showed a pronounced smile effect. Thus we conclude that at least with the definition (1.18) of an option value, our new stock price model has the potential to produce *implied volatility smiles* as a *consequence of dynamical hedging.*

2.2. The skewness effect

Let us now turn to the case where the distribution of the option sizes in the market is not symmetric around S_0. Intuitively, this could represent either a bear market or a bull market, depending on the direction of the asymmetry. To be specific, we shall now take $V_1 = 0.001$, $V_2 = 0.05$ and $V_3 = 0.09$. All other parameters are left unchanged. The (negative of the) corresponding volatility function $\sigma(s)$ is shown in Figure 4; it has a negative slope at the current stock price S_0.

Figure 4 Volatility function for the sizes (0.001, 0.05, 0.09).

Figure 5 shows the implied volatilities for this specification. We clearly see a strong *skewness effect:* implied volatilities tend to rise more for decreasing than for increasing strike prices, and the implied volatility curve is not centered around S_0, but rather around a strike larger than S_0. Another skew is obtained in Figure 6 by choosing now $V_1 = 0.09$, $V_2 = 0.05$ and $V_3 = 0.001$. Plotting the negative of the volatility function for this case would reveal that its slope at S_0 is positive which causes the reversal of the skew compared to Figure 5.

Rubinstein (1985) applied nonparametric tests on various alternative option pricing models, using as data all reported trades and quotes on the 30 most active option classes on the Chicago Board Options Exchange from 1976 through 1978. He found in the period until about October 1977 that implied volatilities tended to rise with *decreasing* strike price for strikes in the neighbourhood of the current stock price. A reversal of this bias was observed in the second period, where implied volatilities tended to rise with *increasing* strike price. As shown in Figures 5 and 6, our model has the ability to explain this kind of bias reversals in implied volatilities as being caused by changes in the distribution of options in the market. As a rough rule, one could say that if options out of the money are expected to be more intensively hedged than those in the money, dynamical hedging should usually induce a rise of implied volatilities for decreasing strike, and vice versa.

Figure 5 Implied volatility for the sizes (0.001, 0.05, 0.09).

Figure 6 Implied volatility for the sizes (0.09, 0.05, 0.001).

To conclude this subsection, we exhibit the cross-section of Figure 3 along the plane $K = S_0$.

Figure 7 Implied volatility at the money for the sizes (0.01,0.09, 0.01).

In other words, Figure 7 shows the implied volatility for an option at the money as a function of the time to maturity. We observe that this curve is increasing; this is also the case in both our skewness examples. Prices of options at the money are usually easy to observe in the market. To fix the parameter γ, one could therefore try to fit the curve of at-the-money implied volatilities computed from our model to the curve corresponding to market prices.

Remark. All the preceding results are based on the assumption (1.19) of a positive patience parameter γ, and so one can ask what happens if γ is negative. For strongly negative γ and sufficiently small sizes V_{ij}, the numerator in (1.15) will become negative instead of positive. This implies that the negative of the volatility function will decrease away from the current stock price, thus leading to a picture which is just the reverse of Figure 1. Numerical computations then show that the smile in Figure 3 also changes sign and becomes a "frown". For intermediate values of γ and V_{ij}, it is less clear what happens since the numerator in (1.15) may become 0.

Further numerical evidence suggests that the implied volatility function σ_{impl} has a shape quite similar to the volatility function σ, but with a considerably smaller amplitude. This can even be observed if σ is allowed to depend on current time t in addition to the current stock price s. For instance, we could consider a time-dependent patience function $\gamma(t)$ or an option exposure distribution $\nu(t)$ changing over time to represent a non-stationary market as in Frey and Stremme (1995) or Platen and Schweizer (1994). Depending on the specific market under consideration, one could therefore also explain more general patterns of volatility deformations as results of feedback effects from hedging strategies. This is a topic for future research in both theoretical and empirical directions.

Appendix

This section provides the details for the derivation of the stochastic differential equation (1.2) for L from the market clearing condition (1.1). We assume that L and U are continuous semimartingales and that D is in C^2 with

$$(A1) \qquad D_L(t,\ell,u) \neq 0 \qquad \text{for all } (t,\ell,u).$$

As in section 1, subscripts of D denote partial derivatives. Condition (A1) is for instance satisfied for the function D in (1.3) if $\xi' = \frac{\partial \xi}{\partial \ell}$ is nonnegative and γ is positive as in (1.19). For $\gamma < 0$, one could guarantee (A1) by suitable boundedness conditions on ξ'; this is the approach taken in Frey and Stremme (1995).

A derivation of (1.2) can now be based on the implicit function theorem. Thanks to the assumption (A1), the equation

$$(A2) \qquad D(t,\ell,u) = \text{const.}$$

can be solved for ℓ to give

$$(A3) \qquad \ell = f(t,u)$$

for a function f defined by

$$(A4) \qquad D(t,f(t,u),u) = \text{const.}$$

Differentiating the identity (A4) and using again (A1) yields the partial derivatives of f as

$$(\text{A5}) \qquad \begin{aligned} f_t &= -\frac{D_t}{D_L}, \\ f_U &= -\frac{D_U}{D_L}, \\ f_{UU} &= -\frac{1}{D_L}\left(D_{UU} - 2D_{LU}\frac{D_U}{D_L} + D_{LL}\left(\frac{D_U}{D_L}\right)^2\right), \end{aligned}$$

with the right-hand sides of (A5) evaluated at $(t,f(t,u),u)$. Since L_t is defined by the equilibrium condition

$$(1.1) \qquad D(t,L_t,U_t) = \text{const.} \quad , \quad t \geq 0,$$

equations (A2) and (A3) yield

$$L_t = f(t,U_t) \quad , \quad t \geq 0.$$

Applying Itô's formula and using (A5) then leads to the stochastic differential equation (1.2).

References

C. A. Ball and A. Roma (1994), "Stochastic Volatility Option Pricing", *Journal of Financial and Quantitative Analysis 29*, 589–607

S. Beckers (1980), "The Constant Elasticity of Variance Model and its Implications for Options Pricing", *Journal of Finance XXXV*, 661–673

F. Black (1990), "Living up to the Model", *RISK*; reprinted in "From Black-Scholes to Black Holes. New Frontiers in Options", Risk Magazine Ltd, London (1992), 17–21

M. J. Brennan and E. Schwartz (1989), "Portfolio Insurance and Financial Market Equilibrium", *Journal of Business 62*, 455–476

L. Clewlow and X. Xu (1992), "A Review of Option Pricing with Stochastic Volatility", *FORC preprint 92/35, University of Warwick*

L. Clewlow and X. Xu (1993), "The Dynamics of Stochastic Volatility", *preprint, University of Warwick*

J. C. Cox and S. A. Ross (1976), "The Valuation of Options for Alternative Stochastic Processes", *Journal of Financial Economics 3*, 145–166

J. B. De Long, A. Shleifer, L. H. Summers and R. J. Waldmann (1990), "Positive Feedback Investment Strategies and Destabilizing Rational Speculation", *Journal of Finance XLV*, 379–395

E. Derman and I. Kani (1994a), "The Volatility Smile and Its Implied Tree", *Goldman Sachs Quantitative Strategies Research Notes*

E. Derman and I. Kani (1994b), "Riding on the Smile", *RISK 7*, 32–39

J.-C. Duan (1995), "The GARCH Option Pricing Model", *Mathematical Finance 5*, 13–32

B. Dumas, J. Fleming and R. E. Whaley (1995), "Implied Volatility Smiles: Empirical Tests", *preprint, Fuqua School of Business, Duke University*

B. Dupire (1993), "Model Art", *RISK 6*, 118–124

B. Dupire (1994), "Pricing with a Smile", *RISK 7*, 18–20

H. Föllmer (1991), "Probabilistic Aspects of Options", *SFB 303 discussion paper No. B–202, University of Bonn*

H. Föllmer (1994), "Stock Price Fluctuations as a Diffusion in a Random Environment", *Philosophical Transactions of the Royal Society, London, Series A, 347*, 471–483

H. Föllmer and M. Schweizer (1993), "A Microeconomic Approach to Diffusion Models for Stock Prices", *Mathematical Finance 3*, 1–23

R. Frey and A. Stremme (1995), "Market Volatility and Feedback Effects from Dynamic Hedging", *SFB 303 discussion paper No. B–310, University of Bonn, to appear in Mathematical Finance*

W. K. H. Fung and D. A. Hsieh (1991), "Empirical Analysis of Implied Volatility: Stocks, Bonds and Currencies", *Proceedings of the Fourth Options Conference, FORC, University of*

Warwick

S. J. Grossman (1988), “An Analysis of the Implications for Stock and Futures Price Volatility of Program Trading and Dynamic Hedging Strategies”, *Journal of Business 61*, 275–289

J. M. Harrison and S. R. Pliska (1981), “Martingales and Stochastic Integrals in the Theory of Continuous Trading”, *Stochastic Processes and their Applications 11*, 215–260

S. L. Heston (1993), “A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”, *Review of Financial Studies 6*, 327–343

R. Heynen (1994), “An Empirical Investigation of Observed Smile Patterns”, *preprint, Erasmus University Rotterdam*

R. Heynen, A. Kemna and T. Vorst (1994), “Analysis of the Term Structure of Implied Volatilities”, *Journal of Financial and Quantitative Analysis 29*, 31–56

A. Hicks (1992), “Volatility Smiles, but Beware the Teeth!”, *Futures and Options World: Foreign Exchange Special*, 11–12

D. G. Hobson and L. C. G. Rogers (1994), “Complete Models with Stochastic Volatility”, *preprint, University of Bath*

N. Hofmann, E. Platen and M. Schweizer (1992), “Option Pricing under Incompleteness and Stochastic Volatility”, *Mathematical Finance 2*, 153–187

J. Hull and A. White (1987), “The Pricing of Options on Assets with Stochastic Volatilities”, *Journal of Finance XLII*, 281–300

J. Hull and A. White (1988), “An Analysis of the Bias in Option Pricing Caused by a Stochastic Volatility”, *Advances in Futures and Options Research 3*, 29–61

R. A. Jarrow (1994), “Derivative Security Markets, Market Manipulation, and Option Pricing Theory”, *Journal of Financial and Quantitative Analysis 29*, 241–261

P. E. Kloeden and E. Platen (1992), “Numerical Solution of Stochastic Differential Equations”, *Springer*

D. A. Paxson (1994), “ ‘Wipe the Smile off’ Implied Volatilities”, *preprint, Manchester Business School*

E. Platen and R. Rebolledo (1996), “Principles for Modelling Financial Markets”, *Journal of Applied Probability 33*, 601–613

E. Platen and M. Schweizer (1994), “On Smile and Skewness”, *SFB 303 discussion paper No. B–302, University of Bonn*

E. Renault and N. Touzi (1996), “Option Hedging and Implicit Volatilities in a Stochastic Volatility Model”, *Mathematical Finance 6*, 279–302

M. Rubinstein (1983), “Displaced Diffusion Option Pricing”, *Journal of Finance XXXVIII*, 213–217

M. Rubinstein (1985), “Nonparametric Tests of Alternative Option Pricing Models Using all Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978”, *Journal of Finance XL*, 455–480

M. Rubinstein (1994), “Implied Binomial Trees”, *Journal of Finance XLIX*, 771–818

L. O. Scott (1992), “Stock Market Volatility and the Pricing of Index Options: An Analysis of Implied Volatilities and the Volatility Risk Premium in a Model with Stochastic Interest Rates and Volatility”, *preprint, University of Georgia*

K. Shastri and K. Wethyavivorn (1987), “The Valuation of Currency Options for Alternate Stochastic Processes”, *Journal of Financial Research X*, 283–293

E. M. Stein and J. C. Stein (1991), “Stock Price Distributions with Stochastic Volatility: An Analytical Approach”, *Review of Financial Studies 4*, 727–752

J. C. Stein (1989), “Overreactions in the Options Market”, *Journal of Finance XLIV*, 1011–1023

S. J. Taylor and X. Xu (1993), “The Magnitude of Implied Volatility Smiles: Theory and Empirical Evidence for Exchange Rates”, *Review of Futures Markets 13*, 355-380