RATIONAL PERIOD FUNCTIONS AND CYCLE INTEGRALS

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1. Introduction

In this paper, which is an outgrowth of [3], we will give some applications of the theory of weakly holomorphic modular forms to rational period functions for the modular group and their modular integrals. First we give a simple basis when $k > 2$ is an even integer for the space W_{k-2} of period polynomials, which can be defined as the space of all polynomials that satisfy

$$\psi(z) = \psi(1+z) + (z+1)^{k-2}\psi(\tfrac{z}{z+1}). \tag{1}$$

The existence of such a basis is well-known, and our aim here is to illustrate the effectiveness of using weakly holomorphic forms in providing one. Our main goal is to construct modular integrals for certain rational solutions ψ to (1) for any $k \in \mathbb{Z}$ made out of indefinite binary quadratic forms. A modular integral for ψ is a periodic function F holomorphic on the upper half-plane $\mathcal{H}$ and meromorphic at $i\infty$ that satisfies

$$\psi(z) = F(z) - z^{k-2}F(-1/z). \tag{2}$$

Knopp [8], [9] introduced rational period functions and proved the first results about them. In particular, he showed using results from [7] that they have modular integrals, but his construction arises from a meromorphic Poincaré series formed out of cocycles and is very difficult to compute (see also [6]). The modular integrals we construct are canonical in the sense that their order at $i\infty$ is maximal among all modular integrals with period function ψ. Their Fourier coefficients are given by cycle integrals of certain weakly holomorphic forms and are quite explicit. When $k > 2$ this construction gives, in combination with the basis for W_{k-2}, some well-known evaluations of the zeta functions attached to indefinite binary forms. A special case when $k = 0$ of this construction was given in [3]. Among the many additional references for the theory of rational period functions are [10], [2] and [13].

2. Statement of results

For any $k \in 2\mathbb{Z}$ let $M_k^!$ be the space of weakly holomorphic modular forms of weight k for the modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. Each $f \in M_k^!$ is a holomorphic function f on $\mathcal{H}$ of the form

$$f(z) = \sum_{n \geq n_0} a(n)e(nz),$$

where $e(z) = e^{2\pi i z}$, that satisfies $f = f|_k g$ for all $g \in \Gamma$, where we define as usual

$$f|_k g(z) = (cz+d)^{-k} f(\tfrac{az+b}{cz+d})$$

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for any $g = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathbb{R})$. Thus a weakly holomorphic modular form is a meromorphic modular form with no pole except possibly at the cusp $i\infty$. Set $\mathrm{ord}_\infty f = n_0$ if $a(n_0) \neq 0$ and let

$$\ell = \ell_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor & \text{otherwise.} \end{cases} \tag{3}$$

Note that $\ell_{2-k} + \ell_k = -1$. The space $M_k^!$ has a canonical basis in the following sense.

Theorem 1. *For each $m \geq -\ell$ there is a unique $f_{k,m} \in M_k^!$ of the form*

$$f_{k,m}(z) = e(-mz) + \sum_{n>\ell} a_k(m,n)\, e(nz) \tag{4}$$

where $a_k(m,n) \in \mathbb{Z}$ satisfy $a_k(m,n) = -a_{2-k}(n,m)$. The set $\{f_{k,m}\}_{m\geq -\ell}$ is a basis for $M_k^!$.

When $\ell = 0$ this result was given in [1]. For $k = 0$ the first three basis elements are

$$f_{0,0}(z) = 1 \quad f_{0,1}(z) = j(z) - 744 \quad f_{0,2}(z) = j(z)^2 - 1488j(z) + 159768,$$

where j is the usual modular function (19). For general k a proof of Theorem 1 was given in [4], and for convenience we will sketch the proof below in section 3. Let $S_k \subset M_k$ be the subspace of cusp forms, which are those forms that vanish at ∞. It is easily seen that $\{f_{k,m}\}_{-\ell\leq m<0}$ is a basis for S_k, hence $\dim S_k = \ell$, provided that $k > 2$.

Suppose that $k > 2$. For each non-zero m with $m \geq -\ell$ we may define the Eichler integral of the basis element $f_{k,m}$ by

$$F_{k,m}(z) = (-m)^{1-k} e(-mz) + \sum_{n>\ell} a_k(m,n) n^{1-k} e(nz). \tag{5}$$

Bol's identity states that for any $g \in \mathrm{PSL}(2,\mathbb{R})$

$$D^{k-1}(F|_{2-k}g) = (D^{k-1}F)|_k g \quad \text{where} \quad D = \tfrac{1}{2\pi i}\tfrac{d}{dz}.$$

This (or a direct calculation) shows that

$$\psi_{k,m}(z) \stackrel{\text{(def)}}{=} F_{k,m}(z) - z^{k-2} F_{k,m}(-\tfrac{1}{z}) \in W_{k-2}. \tag{6}$$

A proof of the following result is given in Section 3.

Theorem 2. *For $k > 2$ the set $\{1 - z^{k-2}\} \cup \{\psi_{k,m}(z)\}_{0<|m|\leq\ell}$ gives a basis for the space W_{k-2}. In particular,* $\dim W_{k-2} = 2\ell + 1$.

Apparently it was Poincaré who first had the idea that the classical theory of abelian integrals and their periods on a compact Riemann surface can be extended to integrals of higher order, but now having polynomial periods.[1] Poincaré worked in the context of quotients of $\mathcal{H}$ by Fuchsian groups and gave details for the modular group, but indicated that the results hold more generally. Roughly speaking, he gave a correspondence between essentially distinct sets of polynomial periods and meromorphic modular forms of the second kind modulo exact forms. In particular, as vector spaces he showed that they have the same finite dimension, which is explicitly computable by means of the Riemann-Roch theorem. Our proof of Theorem 2 is similar except that we employ forms with no poles except in the cusp at $i\infty$.

[1]See [14, p. 99-108 or Œuvres V. p. 213-223]. This volume of Crelle's Journal from 1905, which was dedicated to the centenary of Dirichlet's birth, contains papers by Dedekind, Frobenius, Hensel, Hilbert, Hurwitz, Klein, Minkowski, Picard, Poincaré, and Weber, among others.

Poincaré's (general) result was independently rediscovered by Eichler [5] about 50 years later. In this paper Eichler also interpreted the sets of polynomials as cohomology classes and applied the theory to obtain the trace formula for Hecke operators, thereby opening up its arithmetic side. Because of the ensuing connections with L-functions, it is now more usual to identify the even and odd parts of a period polynomial with those of two different cusp forms, the resulting isomorphism being the Eichler-Shimura Theorem.

Turning now to our main result, for $d > 0$ a non-square discriminant let $\mathcal{Q}_d$ be the set of all binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2 = [a, b, c]$ with integral coefficients and $d = b^2 - 4ac$. The modular group acts naturally on $\mathcal{Q}_d$ splitting it into finitely many classes $\Gamma\backslash\mathcal{Q}_d$. The group $\Gamma_Q = \{g \in \Gamma; \gamma Q = Q\}$ of automorphs of Q is infinite cyclic with a distinguished generator g_Q given below in (25). For $f \in M_k^!$ and $Q \in \mathcal{Q}_d$ define the cycle integral

$$r_Q(f) = \int_{C_Q} f(z)Q(z,1)^{k/2-1}dz, \tag{7}$$

where the integral is over any smooth curve C_Q from τ to $g_Q\tau$ for some fixed $\tau \in \mathcal{H}$. This integral is a class invariant of Q that does not depend on the choice of $\tau \in \mathcal{H}$. For fixed $Q \in \mathcal{Q}_d$ define the generating function

$$F(z, Q) = F_k(z, Q) = \sum_{m\geq -\ell} r_Q(f_{k,m})e(mz). \tag{8}$$

Our main result is that *for any weight* $k \in 2\mathbb{Z}$, $F(z, Q)$ defines a modular integral with a rational period function.

Theorem 3. *For any $k \in 2\mathbb{Z}$ and form $Q \in \mathcal{Q}_d$ the function $F(z, Q)$ is holomorphic on $\mathcal{H}$ and satisfies*

$$F(z, Q) - z^{k-2}F(-1/z, Q) = \psi_Q(z), \tag{9}$$

where

$$\psi_Q(z) = \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \operatorname{sgn}(c)(az^2 + bz + c)^{k/2-1}.$$

Here $(Q) \in \Gamma\backslash\mathcal{Q}_d$ denotes the class containing Q.

The proof of this result, which is given in Section 3, is elementary and uses mainly contour deformation techniques. If $k \geq 2$ then $\psi_Q(z) \in W_{k-2}$ so by Theorem 2 we know that ψ_Q is a linear combination of the $\psi_{k,m}$ with $m \neq 0$ and $1 - z^{k-2}$. In fact we have the following explicit expansion

$$\psi_Q(z) = r_Q(f_{k,0})(1 - z^{k-2}) + \sum_{0<|m|\leq\ell} m^{k-1}r_Q(f_{k,m})\,\psi_{k,-m}(z). \tag{10}$$

This has some well-known consequences in case $k \in \{4, 6, 8, 10, 14\}$. Here we have that $\ell = 0$ and $f_{k,0} = E_k$, the Eisenstein series defined for any $k \in 2\mathbb{Z}$ by

$$E_k(z) = 1 + \tfrac{2}{\zeta(1-k)} \sum_{n\geq 1} \sigma_{k-1}(n)e(nz), \tag{11}$$

where as usual σ is the sum of divisors function. Thus $r_Q(E_k)(1 - z^{k-2}) = \psi_Q(z)$ giving the remarkable fact that for such k

$$r_Q(E_k) = \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} \operatorname{sgn}(c)c^{k/2-1}, \tag{12}$$

which is an integer. After Siegel [15, p.12, Hilfssatz 1] we get analogues of Euler's zeta evaluations in case $k \in \{4, 6, 8, 10, 14\}$:

$$\sum_{(m,n)\in\mathbb{Z}^2/\Gamma_Q}{}' Q(m,n)^{-k/2} = c_k\, d^{\frac{1}{2}(1-k)}\, r_Q(E_k) \qquad c_k = -\tfrac{(2\pi)^k B_k}{k\Gamma(k/2)^2}, \tag{13}$$

the prime indicating that $(0,0)$ be omitted from the sum. Here B_k is the Bernoulli number. Thus for example

$$\sum_{(m,n)\in\mathbb{Z}^2/\Gamma_Q}{}' Q(m,n)^{-2} = \frac{2\pi^4 d^{-3/2}}{15} \sum_{\substack{[a,b,c]\in(Q)\\ ac<0}} |c|.$$

This evaluation is equivalent to one of Kohnen-Zagier [11, Cor p.223], who make use of period polynomials of hyperbolic Poincaré series of the type

$$\sum_{[a,b,c]\in(Q)} (az^2+bz+c)^{-k/2} \tag{14}$$

in their proof and express their result in terms of reduced forms. Summing (13) over classes and applying the functional equation gives Siegel's well-known formulas for special values of the Dedekind zeta function of a real quadratic field K of discriminant d when $k \in \{4, 8\}$

$$\zeta_K(1-\tfrac{k}{2}) = \frac{1}{15k} \sum_{b\equiv d(2)} \sigma_{\frac{k}{2}-1}\big(\tfrac{d-b^2}{4}\big). \tag{15}$$

When $k \le 0$ our construction of the modular integral $F(z,Q)$ should be compared with that of Parson [12], who noticed that for some k a partial version of the Poincaré series in (14) is a modular integral. In fact we have when $k < 0$ that

$$F(z,Q) = \sum_{\substack{[a,b,c]\in(Q)\\ a>0}} (az^2+bz+c)^{k/2-1} + f(z) \tag{16}$$

for some $f \in S_{2-k}$. It appears to be difficult to make Hecke's convergence trick work here when $k = 0$.

3. Weakly holomorphic forms and period polynomials

In this section we will prove Theorems 1 and 2. Any nonzero $f \in M_k^!$ satisfies the Riemann-Roch formula

$$\frac{k}{12} = \mathrm{ord}_\infty f + \tfrac{1}{2}\,\mathrm{ord}_i f + \tfrac{1}{3}\,\mathrm{ord}_\rho f + \sum_{z\in\mathcal{F}\setminus\{i,\rho\}} \mathrm{ord}_z f, \tag{17}$$

where $\rho = e(1/6) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $\mathcal{F}$ is the usual fundamental domain for Γ given by

$$\mathcal{F} = \{z \in \mathcal{H}; -\tfrac{1}{2} \le \mathrm{Re}\, z \le 0, |z| \ge 1\} \cup \{z \in \mathcal{H}; 0 < \mathrm{Re}\, z < \tfrac{1}{2}, |z| > 1\}.$$

One obtains this easily by integrating the Γ-invariant differential $(\frac{f'(z)}{f(z)} - \frac{k}{4\pi y})dz$ around the boundary of $\mathcal{F}$, with the usual detours. In particular, for a nonzero $f \in M_k^!$ we have

$$\mathrm{ord}_\infty f \le \ell. \tag{18}$$

For each integer $m \ge -\ell$ we have $f_{k,m} = \Delta^\ell E_{k-12\ell} P(j)$, where P is a monic polynomial of degree $\ell + m$ with integer coefficients,

$$\Delta = \tfrac{1}{1728}(E_4^3 - E_6^2) \in S_{12} \quad \text{and} \quad j = E_4^3/\Delta \in M_0^!. \tag{19}$$

Since $E_{k-12\ell}, \Delta, j$ and P all have integer coefficients it follows that $a_k(m,n)$ is integral. It follows from (18) that the $f_{k,m}$ are unique and that they form a basis for $M_k^!$. Set

$$f_k = \Delta^\ell E_{k-12\ell}.$$

We have the following expansion (see [4])

$$\frac{f_k(\tau)f_{2-k}(z)}{j(z)-j(\tau)} = \sum_{m\geq -\ell} f_{k,m}(\tau)e(mz), \tag{20}$$

which converges uniformly on compacta in z for fixed $\tau \in \mathcal{H}$ if $\operatorname{Im} z > \operatorname{Im}\tau$. This yields the duality relation $a_k(m,n) = -a_{2-k}(n,m)$ and completes the proof of Theorem 1.

For fixed z with $E_{14}(z) \neq 0$ the expression on the left hand side of (20) has a simple pole at $\tau = z$. Since

$$f_k(z)f_{2-k}(z) = E_{14}(z)/\Delta(z) = -Dj(z),$$

in this case we have

$$\operatorname{Res}_{\tau=z}\frac{f_k(\tau)f_{2-k}(z)}{j(z)-j(\tau)} = \lim_{\tau\to z}\frac{(\tau-z)f_k(z)f_{2-k}(z)}{j(z)-j(\tau)} = \frac{1}{2\pi i}. \tag{21}$$

We now prove Theorem 2. The proof we give follows that of [14] rather closely, except that Poincaré made use of forms with poles in $\mathcal{H}$, which he constructed via his famous series. Note that $1 - z^{k-2} \in W_{k-2}$ is associated to the constant 1 through (2). The set $\{1 - z^{k-2}\} \cup \{\psi_{k,m}\}_{0<|m|\leq\ell}$ must be linearly independent since otherwise one could construct a nonzero form $F \in M_{2-k}^!$ having $\operatorname{ord}_\infty F > -1-\ell = \ell_{2-k}$ from a linear combination of a constant and the $F_{k,m}$ from (5) with $0 < |m| \leq \ell$, contradicting (18). Theorem 2 will follow if we can show that $\{1 - z^{k-2}\} \cup \{\psi_{k,m}\}_{0<|m|\leq\ell}$ spans W_{k-2}, and for this it is enough to show that

$$\dim W_{k-2} \leq 2\ell + 1. \tag{22}$$

As is well known, Γ is generated by the elliptic transformations $S = \pm\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ and $U = \pm\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)$ with the defining relations $S^2 = U^3 = \pm 1$. Note that $T = US = \pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Let P_{k-2} be the space of all complex polynomials of degree at most $k-2$. For $\psi(z) \in P_{k-2}$ define the action of $A \in \operatorname{PSL}(2,\mathbb{C})$ in the usual way as

$$\psi|A = (cz+d)^{k-2}\psi\left(\frac{az+b}{cz+d}\right)$$

and extend the definition by linearity to the group ring. Observe that

$$W_{k-2} = \{\psi \in P_{k-2}; \psi|(1+S) = \psi|(1+U+U^2) = 0\}.$$

See [**?**, Proposition p. 249] for the short proof. Let

$$W_{k-2}^S = \{\psi \in P_{k-2}; \psi|(1+S) = 0\} \quad \text{and} \quad W_{k-2}^U = \{\psi \in P_{k-2}; \psi|(1+U+U^2) = 0\}.$$

It is not hard to check that the linear map from $W_{k-2} \oplus P_{k-2}$ to $W_{k-2}^S \oplus W_{k-2}^U$ defined by

$$(\psi,\phi) \mapsto \big(\psi + \phi|(1-S), \psi + \phi|(1-U)\big)$$

is an injection. Hence

$$\dim W_{k-2} \leq \dim W_{k-2}^S + \dim W_{k-2}^U + 1 - k. \tag{23}$$

Note that S and U are conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to $\tilde{S}=\pm\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$ and $\tilde{U}=\pm\left(\begin{smallmatrix} \rho & 0\\ 0 & \bar\rho\end{smallmatrix}\right)$, respectively, where $\rho=e(1/6)=\frac{1}{2}+\frac{i\sqrt{3}}{2}$. Now a brief calculation shows that

$$\left(\sum_{r=0}^{k-2} a_r z^r\right)|(1+\tilde{S}) = 2\sum_{r\equiv k/2+1\ (2)} a_r z^r.$$

and

$$\left(\sum_{r=0}^{k-2} a_r z^r\right)|(1+\tilde{U}+\tilde{U}^2) = 3\sum_{r\equiv k/2-1\ (3)} a_r z^r.$$

Hence

$$\dim W^S_{k-2}=\#\{0\le n\le k-2|\ n\equiv \tfrac{k}{2}\ (2)\}\quad\text{and}\quad \dim W^U_{k-2}=\#\{0\le n\le k-2|\ n\equiv \tfrac{k}{2},\tfrac{k}{2}{+}1\ (3)\},$$

and a simple counting argument gives that

$$\dim W^S_{k-2}=2\lceil\tfrac{k-2}{4}\rceil\quad\text{and}\quad \dim W^U_{k-2}=2\lceil\tfrac{k-2}{3}\rceil$$

from which we deduce (22), hence Theorem 2, in view of (23).

4. Cycle integrals

In this section we will prove Theorem 3. First we make some preparatory observations about quadratic forms and cycle integrals. Recall that $\mathcal{Q}_d$ is the set of integral binary quadratic forms $Q(x,y)=ax^2+bxy+cy^2=[a,b,c]$ of discriminant $d=b^2-4ac$. Let $Q\mapsto gQ$ be the usual action of Γ that is compatible with linear fractional action on the roots of $Q(\tau,1)=0$. Explicitly, and with an obvious abuse of notation,

$$(g^{-1}Q)(\tau,1)=Q(g\tau,1)(c\tau+d)^2\quad\text{if}\quad g=\pm\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma. \tag{24}$$

As is well known, the resulting set of classes $\Gamma\backslash\mathcal{Q}_d$ is finite and those classes consisting of primitive forms form an abelian group under composition. Let $\Gamma_Q=\{g\in\Gamma; gQ=Q\}$ be the group of automorphs of Q. If $d>0$ then Γ_Q is infinite cyclic with a distinguished generator denoted by g_Q, which for primitive Q is given by

$$g_Q=\pm\begin{pmatrix} \frac{t+bu}{2} & cu\\ -au & \frac{t-bu}{2}\end{pmatrix} \tag{25}$$

where t,u are the smallest positive integral solutions of $t^2-du^2=4$. If $\delta=\gcd(a,b,c)$ then $g_Q=g_{Q/\delta}$. For $Q\in\mathcal{Q}_d$ with $d>0$ not a square let S_Q be the oriented semi-circle defined by

$$a|\tau|^2+(\operatorname{Re}\tau)b+c=0,$$

directed counterclockwise if $a>0$ and clockwise if $a<0$. Clearly

$$S_{gQ}=gS_Q, \tag{26}$$

for any $g\in\Gamma$. Given $z\in S_Q$ let $C_Q=C_Q(\tau_0)$ be the directed arc on S_Q from τ_0 to $g_Q\tau_0$. It can easily be checked that C_Q has the same orientation as S_Q. Let[2]

$$d\tau_Q=Q(\tau,1)^{k/2-1}\,d\tau.$$

For any continuous function f on $\mathcal{H}$ that satisfies

$$f|_k g=f|g=f \tag{27}$$

[2]Here we omit the factor $\sqrt{d}$ that was given in the corresponding definition in [3].

for all $g \in \Gamma$, the integral $\int_{C_Q} f(\tau)d\tau_Q$ is both independent of $\tau_0 \in S_Q$ and is a class invariant. This is an immediate consequence of the following lemma that expresses this cycle integral as a sum of integrals over arcs in a fixed fundamental domain for Γ.

Lemma 1. *Let $Q \in \mathcal{Q}_d$ be a form with $d > 0$ not a square and $\mathcal{F}' = g\mathcal{F}$ be the image of $\mathcal{F}$ under any fixed $g \in \Gamma$. Suppose that f satisfies (27) and is continuous on S_Q. Then for any $\tau_0 \in S_Q$ we have*

$$\int_{C_Q(\tau_0)} f(\tau)d\tau_Q = \sum_{q\in(Q)} \int_{S_q\cap\mathcal{F}'} f(\tau)d\tau_q, \tag{28}$$

where (Q) denote the class of Q.

Proof. Let $\tilde{f}(\tau) = f(\tau)$ if $\tau \in \mathcal{F}'$ and $\tilde{f}(\tau) = 0$ otherwise, so $f(\tau) = \sum_{g\in\Gamma} \tilde{f}|g(\tau)$ with only a discrete set of exceptions, Thus

$$\begin{aligned}\int_{C_Q} f(\tau)d\tau_Q &= \int_{C_Q} \sum_{g\in\Gamma} \tilde{f}|g(\tau)d\tau_Q = \sum_{g\in\Gamma/\Gamma_Q} \sum_{\sigma\in\Gamma_Q} \int_{C_Q} \tilde{f}|g|\sigma(\tau)d\tau_Q \\ &= \sum_{g\in\Gamma/\Gamma_Q} \int_{S_Q} \tilde{f}|g(\tau)d\tau_Q = \sum_{g\in\Gamma/\Gamma_Q} \int_{S_{gQ}} \tilde{f}(\tau)d\tau_{gQ},\end{aligned}$$

where in the last step we have changed variable $\tau \mapsto g^{-1}\tau$ and have used (26) and (24). This immediately yields (28). $\square$

We now prove Theorem 3. First we prove that

$$F(z, Q) = \sum_{m\geq-\ell} r_Q(f_{k,m})e(mz)$$

has an analytic continuation to $\mathcal{H}$. Now from (20)

$$\sum_{m\geq-\ell} f_{k,m}(\tau)e(mz) = \frac{f_k(\tau)f_{2-k}(z)}{j(z) - j(\tau)}$$

converges uniformly on compacta in z for fixed $\tau \in \mathcal{H}$ if $\mathrm{Im}(z) > \mathrm{Im}(\tau)$. Thus for $\mathrm{Im}(z)$ sufficiently large we have that

$$F(z, Q) = \int_{C_Q} \frac{f_k(\tau)f_{2-k}(z)}{j(z) - j(\tau)}d\tau_Q \tag{29}$$

where we may take for C_Q any smooth curve connecting an arbitrary point $\tau_0 \in \mathcal{H}$ and $g_Q\tau_0$. Let $z_0 \in \mathcal{H}$ be an arbitrary point. The images of C_Q under Γ separate $\mathcal{H}$ into infinitely many connected components each with a piecewise smooth boundary. Clearly we may choose C_Q so that z_0 is in the interior of one of these components. The integral in (29) defines in each component an analytic function. To get the analytic continuation it is enough to show that we can continue across finitely many adjacent components and thus reach a neighborhood of z_0. By the monodromy theorem we will get a unique continuation since $\mathcal{H}$ is simply connected.

Let Ω_1 be a component in which (29) holds and let Ω_2 be an adjacent component. Let $F_2(\tau, Q)$ be the analytic function in Ω_2 defined by the integral in (29). Choose any simple point z_1 on the edge between Ω_1 and Ω_2. Make a small semi-circular deformation of C_Q so that the new integral continues $F(z, Q)$ to a neighborhood A of z_1. Now assuming that A is chosen to not contain any elliptic point of Γ, for each z in $A \cap \Omega_2$ the value of the analytic continuation of $F(z, Q)$ is given by

$$F(z, Q) = F_2(z, Q) \pm Q(z, 1)^{k/2-1}, \tag{30}$$

coming from the pole of the integrand at $\tau = z$. Here the sign is the same for all such z and depends only on the orientation of the edge. Since $Q(z, 1)^{k/2-1}$ is holomorphic in $\mathcal{H}$, we see that (30) gives the continuation of $F(z, Q)$ to all of Ω_2. Continue this process starting with $F_2(\tau, Q) \pm Q(z, 1)^{k/2-1}$ and repeat until we have continued F to the component Ω_n containing z_0. It follows that $F(z, Q)$ is holomorphic in $\mathcal{H}$ and that its Fourier expansion converges in $\mathcal{H}$.

In order to prove the functional equation of Theorem 3 let P be the closure of $\mathcal{F} \cup \mathcal{F}'$, where $\mathcal{F}' = -\mathcal{F}^{-1}$ is the image of the standard fundamental domain $\mathcal{F}$ under inversion $z \mapsto -1/z$. Take z with $\operatorname{Im} z$ sufficiently large. By Lemma 1 we have

$$F(z, Q) = \tfrac{1}{2} \sum_q \int_{A_q} \frac{f_k(\tau) f_{2-k}(z)}{j(z) - j(\tau)} d\tau_q$$

where the sum runs over all $q \in (Q)$ for which S_q intersects the interior of P, giving the arc $A_q = S_q \cap P$. It is easily seen that $S_{[a,b,c]}$ intersects P if and only if $ac < 0$. Now we deform each arc A_q in the sum to a curve B_q that is still within P and has the same endpoints as A_q, but leaves z and $-1/z$ in the same connected component. The images of these curves under inversion will also do this. By evaluating each resulting residue at z using (21) we get the formula

$$F(z, Q) = \tfrac{1}{2} \sum_{\substack{[a,b,c] \in (Q) \\ ac<0}} \int_{B_{[a,b,c]}} \frac{f_k(\tau) f_{2-k}(z)}{j(z) - j(\tau)} (a\tau^2 + b\tau + c)^{\frac{k}{2}-1} d\tau - \tfrac{1}{2} \sum_{\substack{[a,b,c] \in (Q) \\ ac<0}} \operatorname{sgn}(a)(az^2 + bz + c)^{\frac{k}{2}-1},$$

which is also valid at $-1/z$. A simple calculation now shows that (9) holds in a neighborhood of z, hence for all $z \in \mathcal{H}$. Thus Theorem 3 follows.

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