On the behaviour of eigenvalues of Hecke operators

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January 6, 2010

Abstract

In this paper we use the Selberg trace formula for Hecke operators in order to obtain information on the distribution of the eigenvalues of Hecke operators in the situation of the hyperbolic 3-space. We prove that the eigenvalues are equidistributed with respect to a measure that tends to the Sato-Tate measure.

1 Introduction

Let E be an elliptic curve E over $\mathbb{Q}$ without complex multiplication and let $N(p)$ denote the number of points of E modulo p, p a prime. The Sato-Tate conjecture says that, as $p \to \infty$, the angles $\theta(p) = \arccos\left(\frac{1+p-N(p)}{2\sqrt{p}}\right)$ are equidistributed with respect to the measure $d\mu_\theta = \frac{2}{\pi}\sin^2\theta\, d\theta$. Similarly, for a Hecke eigenform $f(z) = \sum \rho_f(n)e^{2\pi i n z}$ of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$, the generalized Sato-Tate conjecture predicts that the angles $\theta_f(p)$ in $2\cos\theta_f(p) = p^{(1-k)/2}\rho_f(p)$ are uniformly distributed with respect to $d\mu_\theta$ (cf. [Ser68]). This is a deep conjecture which is intimately related to the analytic properties of symmetric power L-functions ([Mu]). Recently in a major breakthrough R. Taylor [T], building on important work by Clozel, Harris, Shepard-Barron and Taylor([CHT], [HS-BT]), proved the Sato-Tate conjecture for an elliptic curve E over any totally real field with multiplicative reduction at some prime.

For a general holomorphic cusp form or a Maaß form f one can also change the viewpoint and try to understand the much easier problem of distribution of the Hecke eigenvalues $\rho_f(p)$ of non-CM f for fixed p and varying f. This situation has been studied for holomorphic cusp forms in [CDF], [Ser97] and for Maaß forms in [Sar]. Denote the set of cusp forms of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$ that are also eigenfunctions of the Hecke operators by $\mathcal{S}_k$. Then Conrey, Duke and Farmer [CDF] proved that for a fixed prime p, $\{\rho(f,p) := p^{(1-k)/2}\rho_f(p) : \; f \in \mathcal{S}_k\}$ is equidistributed with respect to

$$d\mu_p(x) := \begin{cases} \frac{1}{2\pi}\left(1+\frac{1}{p}\right)\frac{\sqrt{4-x^2}}{\left(1+\frac{1}{p}\right)^2-\frac{1}{p}x^2}\,dx & \text{if } |x|<2, \\ 0 & \text{otherwise} \end{cases}$$

as $k \to \infty$ and that as $p \to \infty$, p prime, and $k \to \infty$ with $k > e^p$, $\{\rho(f,p) : \; f \in \mathcal{S}_k\}$ becomes equidistributed with respect to the Sato-Tate measure. We note that μ_p is essentially the Plancherel measure of $GL_2(\mathbb{Q}_p)$. Apart from these results a discussion for congruence subgroups can also be found in [Ser97]. See also [Go] and [MS]. In both of these papers the authors obtained error terms for the Serre and Conrey-Duke-Farmer theorem. The error bound in [MS] is effective. As already mentioned the analogue situation for Maaß forms is treated in [Sar]. If Δ is the Laplace operator corresponding to the hyperbolic metric on the upper half-plane $\mathbb{H}^2$ and the eigenfunctions $(u_j)_{j\geq 0}$ of the discrete spectrum of $-\Delta$ on $L^2(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}^2)$ are chosen such that $T_n u_j = \rho_j(n) u_j$ with T_n being the Hecke operator, then Sarnak [Sar] showed that if $x_j \in \prod_p [-(p^{-1/2}+p^{1/2}),\, p^{-1/2}+p^{1/2}]$ is defined by $x_j := (\rho_j(2), \rho_j(3), \dots)$, then $(x_j)_j$ is equidistributed with respect to $\mu = \prod_p \mu_p$.

In this paper we turn to the case of $\mathrm{SL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field K of class number one. We consider the hyperbolic 3-space $\mathbb{H}^3$ equipped with the hyperbolic metric and regard it as a subset of Hamilton's quaternions. The Laplace-Beltrami operator corresponding to this metric is given by

$$\Delta := r^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) - r\frac{\partial}{\partial r}.$$

A careful treatment of its spectral theory can be found in [EGM]. The groups $\mathrm{SL}_2(\mathcal{O}) := \left\{M \in \mathrm{Mat}\,(2\times 2,\, \mathcal{O}) : \, \det M = 1\right\}$ and $\Gamma := \mathrm{PSL}_2(\mathcal{O}) := \mathrm{SL}_2(\mathcal{O})/\{\pm I\}$ act on $\mathbb{H}^3$ and they can be thought of as the analogue of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z})$, respectively. It is well-known that the spectrum of $-\Delta$ on $L^2(\Gamma \setminus \mathbb{H}^3)$ consists of a discrete part and an absolutely continuous part (cf. e.g. [EGM]). We denote the eigenfunctions belonging to the discrete spectrum by $(e_m)_{m\geq 0}$ and the corresponding eigenvalues by $(\lambda_m)_{m\geq 0}$ where the eigenvalues are counted with multiplicity and ordered according to their size. Moreover, we choose the eigenfunctions $(e_m)_{m\geq 0}$ so that they are also eigenfunctions of the Hecke operators $T_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{O}$ a prime. For the definition and basic properties of the Hecke operators see section 2. Let $\rho_m(\mathfrak{p})$ be the eigenvalue of the Hecke operator $T_{\mathfrak{p}}$ that belongs to e_m, i.e. $T_{\mathfrak{p}} e_m = \rho_m(\mathfrak{p}) e_m$. Then our main theorem shows that there exists a measure $d\mu_{\mathfrak{p}}$ with respect to which $(\rho_m(\mathfrak{p}))_{m\geq 1}$ is equidistributed. More precisely, we prove

Main Theorem. For a prime $\mathfrak{p} \in \mathcal{O}$ let $\rho_m(\mathfrak{p})$ be the eigenvalue of the Hecke operator $T_\mathfrak{p}$ with corresponding eigenfunction e_m. Then $(\rho_m(\mathfrak{p}))_{m\geq 1}$ is equidistributed with respect to the measure

$$d\mu_\mathfrak{p}(x) := \begin{cases} \frac{1}{2\pi}\left(1+\frac{1}{N(\mathfrak{p})}\right)\frac{\sqrt{4-x^2}}{\left(1+\frac{1}{N(\mathfrak{p})}\right)^2-\frac{x^2}{N(\mathfrak{p})}}\,dx & \text{if } |x|<2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this measure tends to the Sato-Tate measure as $N(\mathfrak{p}) \to \infty$.

Another natural question in this context is the dependence of the distribution of Hecke eigenvalues on the level structure. The second author has already made some progress in this direction.

2 The Hecke Operators

To introduce the Hecke operators we follow the approach of Heitkamp [Hei]. First, we recall the action of $\mathrm{GL}_2(\mathbb{C}) := \{M \in \mathrm{Mat}(2\times 2,\mathbb{C}) : \det M \neq 0\}$ on $\mathbb{H}^3$: If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ and $P \in \mathbb{H}$, we set $q := \sqrt{\det M}$ and

$$MP := q^{-1}(aP+b)(cP+d)^{-1}q,$$

with the inverse being taken in the skew field of quaternions. Moreover, for $v \in \mathcal{O}\setminus\{0\}$ let $\mathcal{M}_v := \{M \in \mathrm{Mat}\,(2\times 2,\mathcal{O}) : \det M = v\}$ and $\mathcal{V}_v$ be a system of representatives for the right cosets of $\mathcal{M}_v$ modulo $\mathrm{SL}_2(\mathcal{O})$. Then the Hecke operator T_v is defined by:

$$(T_v f)(P) := \frac{1}{\sqrt{N(v)}} \sum_{M\in\mathcal{V}_v} f(MP),$$

f being a Γ-invariant function. Note that the factor $1/\sqrt{N(v)}$ does not appear in the definition of the Hecke operators in [Hei]. We prefer to work with this factor because it simplifies the recurrence relation satisfied by the Hecke operators. The theory of Hecke operators as it is needed for the discussion in this paper is developed in [Hei]. Let $\mathcal{O}^*$ be the set of units of $\mathcal{O}$. Then it follows from [Hei], p. 83 that for any prime $\mathfrak{p} \in \mathcal{O}$ and $n \in \mathbb{N}$ the set

$$\mathcal{V}_{\mathfrak{p}^n} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : d \in \mathcal{O}/\mathcal{O}^*,\, ad = \mathfrak{p}^n,\, b \in \mathcal{O}/\langle d\rangle \right\}$$

is a set of representatives of $\mathcal{M}_{\mathfrak{p}^n}$ mod $\mathrm{SL}_2(\mathcal{O})$. Thus

$$(T_{\mathfrak{p}^n} f)(P) = \frac{1}{N(\mathfrak{p})^{n/2}} \sum_{\substack{d\in\mathcal{O}/\mathcal{O}^*,\, ad=\mathfrak{p}^n, \\ b\in\mathcal{O}/\langle d\rangle}} f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} P\right).$$

Similarly to the two-dimensional case (cf. [Gun], p. 60) we find the following recurrence relation for the Hecke operators.

2.1 Lemma. *For $n \in \mathbb{N}$ and a prime $\mathfrak{p} \in \mathcal{O} \setminus \{0\}$ we obtain*

$$T_{\mathfrak{p}} T_{\mathfrak{p}^n} = T_{\mathfrak{p}^{n+1}} + T_{\mathfrak{p}^{n-1}}.$$

Proof. The proof of this identity is very similar to the proof of the relation of the Hecke operators for $\mathrm{SL}_2(\mathbb{Z})$. Let f be a Γ-invariant function and note that

$$\begin{aligned} \begin{pmatrix} a_2\mathfrak{p} & b_2\mathfrak{p} \\ 0 & d_2 \end{pmatrix} (z,r) &= \left(\frac{a_2\mathfrak{p} z + b_2\mathfrak{p}}{d_2}, \frac{|\mathfrak{p}^n|}{|d_2|}\, r \right) \\ &= \left(\frac{a_2 z + b_2}{d_2/\mathfrak{p}}, \frac{|\mathfrak{p}^{n-1}|}{|d_2/\mathfrak{p}|}\, r \right) \end{aligned}$$

for $P = (z,r) \in \mathbb{H}^3$. Then we get

$$\begin{aligned} & N(\mathfrak{p})^{\frac{n+1}{2}} (T_{\mathfrak{p}} T_{\mathfrak{p}^n} f)(P) \\ = & \sum_{\substack{d_1 \in \mathcal{O}/\mathcal{O}^*,\, a_1 d_1 = \mathfrak{p}, \\ b_1 \in \mathcal{O}/\langle d_1 \rangle}} \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} P \right) \\ = & \sum_{\substack{d_1 \in \mathcal{O}/\mathcal{O}^*,\, a_1 d_1 = \mathfrak{p}, \\ b_1 \in \mathcal{O}/\langle d_1 \rangle}} \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{pmatrix} P \right) \\ = & \sum_{b_1 \in \mathcal{O}/\langle \mathfrak{p} \rangle} \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_2 & b_2 + b_1 d_2 \\ 0 & d_2 \mathfrak{p} \end{pmatrix} P \right) \\ & + \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_2\mathfrak{p} & b_2\mathfrak{p} \\ 0 & d_2 \end{pmatrix} P \right) \\ = & \sum_{b_1 \in \mathcal{O}/\langle \mathfrak{p} \rangle} \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_2 & b_2 + b_1 d_2 \\ 0 & d_2 \mathfrak{p} \end{pmatrix} P \right) \\ & + f\left(\begin{pmatrix} \mathfrak{p}^{n+1} & 0 \\ 0 & 1 \end{pmatrix} P \right) + \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ \mathfrak{p} \mid d_2,\, b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_2\mathfrak{p} & b_2\mathfrak{p} \\ 0 & d_2 \end{pmatrix} P \right) \\ = & N(\mathfrak{p})^{\frac{n+1}{2}} (T_{\mathfrak{p}^{n+1}} f)(P) + \sum_{\substack{d_2 \in \mathcal{O}/\mathcal{O}^*,\, a_2 d_2 = \mathfrak{p}^n, \\ \mathfrak{p} \mid d_2,\, b_2 \in \mathcal{O}/\langle d_2 \rangle}} f\left(\begin{pmatrix} a_2\mathfrak{p} & b_2\mathfrak{p} \\ 0 & d_2 \end{pmatrix} P \right) \\ = & N(\mathfrak{p})^{\frac{n+1}{2}} (T_{\mathfrak{p}^{n+1}} f)(P) + N\mathfrak{p} \sum_{\substack{d' \in \mathcal{O}/\mathcal{O}^*,\, a' d' = \mathfrak{p}^{n-1}, \\ b' \in \mathcal{O}/\langle d' \rangle}} f\left(\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} P \right) \\ = & N(\mathfrak{p})^{\frac{n+1}{2}} (T_{\mathfrak{p}^{n+1}} f)(P) + N(\mathfrak{p}) N(\mathfrak{p})^{\frac{n-1}{2}} (T_{\mathfrak{p}^{n-1}} f)(P). \end{aligned}$$

This proves the lemma. □

2.2 Remark. This lemma can be compared to [Hei], Theorem 6. 5, p. 40.

With the help of Lemma 2.1 we infer the following lemma for the Hecke operators.

2.3 Lemma. *For any $n \in \mathbb{N}$ and a prime $\mathfrak{p} \in \mathcal{O}$ the following identity holds:*

$$(T_\mathfrak{p})^{2n} = \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) T_{\mathfrak{p}^{2j}}.$$

Moreover, for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and a prime $\mathfrak{p} \in \mathcal{O}$

$$(T_\mathfrak{p})^{2n+1} = \sum_{j=0}^{n} \left(\binom{2n+1}{n-j} - \binom{2n+1}{n-j-1} \right) T_{\mathfrak{p}^{2j+1}}.$$

Proof. If we consider the identity of Lemma 2.1 then induction yields

$$(T_\mathfrak{p})^{2n} = \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) T_{\mathfrak{p}^{2j}}.$$

The second formula follows by applying the Hecke operator to this identity.

2.4 Remark. A similar relation can be found in [Hei], Corollary 6. 6, p. 43.

□

3 The Selberg trace formula

In order to obtain information on the statistical properties of the eigenvalues of the Hecke operators we use the Selberg trace formula. Our notation in this section is the same as in [EGM] and in [Rau] where the trace of a Hecke operator on a fixed eigenspace of the Laplace operator is determined. The appropriate Selberg transform for our problem is the same as the one used for proving Weyl's law. In the situation of the hyperbolic 3-space this function can e. g. be found in [EGM], p. 307.

3.1 Definition. For $\epsilon > 0$ define the function h_ϵ by

$$h_\epsilon(1 + t^2) = e^{-\epsilon(1+t^2)}.$$

Furthermore, the Fourier transform g_ϵ of h_ϵ can be computed and we have:

3.2 Lemma. *For $\epsilon > 0$ the following identity holds:*

$$g_\epsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\epsilon\left(1+t^2\right) e^{-itx} dt = \frac{e^{-\epsilon - x^2/(4\epsilon)}}{\sqrt{4\pi\epsilon}}.$$

Proof. See [EGM], p. 307. □

By the inversion formulas of [EGM], Lemma 5. 5, p. 121 we can also determine the corresponding point-pair invariant to h_ϵ.

3.3 Lemma. *For $\epsilon > 0$ and $t > 1$ we have*

$$k_\epsilon(t) = \frac{\log(t+\sqrt{t^2-1})}{(4\pi\epsilon)^{3/2} e^\epsilon \sqrt{t^2-1}} \left(\frac{1}{t+\sqrt{t^2-1}}\right)^{\frac{1}{4\pi}\log(t+\sqrt{t^2-1})}.$$

and $k_\epsilon(1) = \frac{1}{(4\pi\epsilon)^{3/2}e^\epsilon}$. The corresponding point-pair invariant is given by $K_\epsilon(P,Q) := k_\epsilon \circ \delta(P,Q)$.

Then the function h_ϵ is the Selberg transform of the point-pair invariant K_ϵ.

As in [Rau], p. 114 we introduce the following subset of $\mathrm{PSL}_2(\mathbb{C})$.

3.4 Definition. For every prime $\mathfrak{p} \in \mathcal{O} \setminus \{0\}$ and $l \in \mathbb{N}_0$ let

$$\Gamma^*_{\mathfrak{p}^l} := \frac{1}{\sqrt{\mathfrak{p}^l}} \bigcup_{\substack{d \in \mathcal{O}/\mathcal{O}^*,\, ad = \mathfrak{p}^l, \\ b \in \mathcal{O}/\langle d \rangle}} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \Gamma.$$

Furthermore, as in [Rau], p. 114 we define the following invariant functions that depend on the chosen point-pair invariant and the Hecke operators.

3.5 Definition. For $P,\, Q \in \mathbb{H}$, $\mathfrak{p} \in \mathcal{O} \setminus \{0\}$, $l \in \mathbb{N}_0$ and $\epsilon > 0$ let

1. $K_\Gamma(P,\, Q) := \sum_{\gamma \in \Gamma} K_\epsilon(P,\, \gamma Q),$
2. $H_\Gamma(P,\, Q) := \frac{[\Gamma_\infty : \Gamma'_\infty]}{4\pi\, |\mathcal{O}|} \int_{-\infty}^{\infty} h_\epsilon(1+t^2) E(P,\, it) \overline{E(Q,\, it)}\, dt,$
3. $K_{\Gamma^*_{\mathfrak{p}^l}}(P,\, Q) := \sum_{\gamma \in \Gamma^*_{\mathfrak{p}^l}} K_\epsilon(P,\, \gamma Q),$
4. $H_{\Gamma^*_{\mathfrak{p}^l}}(P,\, Q) := \frac{[\Gamma_\infty : \Gamma'_\infty]}{4\pi\, |\mathcal{O}|} \sum_{M \in \mathcal{V}_{\mathfrak{p}^l}} \int_{-\infty}^{\infty} h_\epsilon(1+t^2) E(MP,\, it) \overline{E(Q,\, it)}\, dt.$

Here $E(P,\, s)$ denotes the Eisenstein series where we use the definition of [EGM], chapter 6, p. 266, $[\Gamma_\infty : \Gamma'_\infty]$ is the index of Γ'_∞ in Γ_∞ and M runs through a set $\mathcal{V}_{\mathfrak{p}^l}$ of representatives of $\mathcal{M}_{\mathfrak{p}^l}$ modulo $\mathrm{SL}_2(\mathcal{O})$. Moreover, $|\mathcal{O}|$ is the Euclidean measure of a fundamental domain of the lattice $\mathcal{O}$.

The following function that appears in the Fourier expansion of the Eisenstein series will also be important in our discussion.

3.6 Definition. Let ζ_K denote the zeta function of K and d_K be the discriminant of K. For $s \in \mathbb{C}$ we set:

$$\phi(s) := \frac{2\pi}{s\sqrt{|d_K|}} \frac{\zeta_K(s)}{\zeta_K(1+s)}.$$

3.7 Theorem. *Let $\mathcal{F}$ be a fundamental domain of Γ and $\mathfrak{p} \in \mathcal{O} \setminus \{0\}$ a prime. Then for $\epsilon > 0$ and $n \in \mathbb{N}$ the equation*

$$\begin{aligned} \sum_{m \geq 0} \rho_m^{2n}(\mathfrak{p}) h_\epsilon(\lambda_m) &= \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) N(\mathfrak{p})^{-j} \\ &\quad \times \int_{\mathcal{F}} \left(K_{\Gamma^*_{\mathfrak{p}^{2j}}}(P,P) - H_{\Gamma^*_{\mathfrak{p}^{2j}}}(P,P) \right) dv(P) \end{aligned}$$

holds and for $n \in \mathbb{N}_0$ we have

$$\begin{aligned} \sum_{m \geq 0} \rho_m^{2n+1}(\mathfrak{p}) h_\epsilon(\lambda_m) &= \sum_{j=0}^{n} \left(\binom{2n+1}{n-j} - \binom{2n+1}{n-j-1} \right) N(\mathfrak{p})^{-j-\frac{1}{2}} \\ &\quad \times \int_{\mathcal{F}} \left(K_{\Gamma^*_{\mathfrak{p}^{2j+1}}}(P,P) - H_{\Gamma^*_{\mathfrak{p}^{2j+1}}}(P,P) \right) dv(P). \end{aligned}$$

Proof. If $(e_m)_{m \geq 0}$ is a complete orthonormal set of eigenfunctions of the discrete spectrum of $-\Delta$ in $L^2(\Gamma \setminus \mathbb{H}^3)$ with corresponding eigenvalues $(\lambda_m)_{m \geq 0}$ and if every eigenfunction e_m of $-\Delta$ is also an eigenfunction of every Hecke operator $T_{\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{O} \setminus \{0\}$, with corresponding eigenvalue $\rho_m(\mathfrak{p})$, then we infer from the decomposition

$$K_\Gamma(P, Q) - H_\Gamma(P, Q) = \sum_{m \geq 0} h_\epsilon(\lambda_m) e_m(P) \overline{e_m(Q)}$$

(cf. [EGM], Proposition 4.1, pp. 278-279) and from Lemma 2.3

$$\begin{aligned} \sum_{m \geq 0} h_\epsilon(\lambda_m) \rho_m^{2n}(\mathfrak{p}) e_m(P) \overline{e_m(Q)} &= (T_{\mathfrak{p}})^{2n} \sum_{m \geq 0} h_\epsilon(\lambda_m) e_m(P) \overline{e_m(Q)} \\ &= \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) \\ &\quad \times N(\mathfrak{p})^{-j} \left(K_{\Gamma^*_{\mathfrak{p}^{2j}}}(P,Q) - H_{\Gamma^*_{\mathfrak{p}^{2j}}}(P,Q) \right). \end{aligned}$$

Thus the first formula of the theorem follows if we set $P = Q$ and integrate over the fundamental domain $\mathcal{F}$. In order to obtain the second formula we just have to replace the identity for $(T_{\mathfrak{p}})^{2n}$ by the formula for $(T_{\mathfrak{p}})^{2n+1}$. □

For each fixed $l \in \{0, \dots, n\}$ the integrals appearing in Theorem 3.7 can be computed using the approach of the Selberg trace formula, i. e. by dividing the sets $\Gamma^*_{\mathfrak{p}^{2j}}$ and $\Gamma^*_{\mathfrak{p}^{2j+1}}$ into Γ-conjugacy classes. For this let $l \in \mathbb{N}$ and let $\mathcal{F}(\mathcal{C}(M))$ denote a fundamental domain of $\mathcal{C}(M)$, $M \in \Gamma^*_{\mathfrak{p}^l}$, with $\mathcal{C}(M)$ being the centralizer of M in Γ. Furthermore, if $M \in \Gamma^*_{\mathfrak{p}^l}$ has no cusps of Γ as fixed points, we define:

$$c_{\epsilon,l}(M) := \int_{\mathcal{F}(\mathcal{C}(M))} K_\epsilon(P,\, MP)\, dv(P).$$

It can be seen that the leading term in the asymptotics comes from the contribution of the identity element. Since the contribution to the trace of the identity, the elliptic elements and the loxodromic elements that do not stabilize cusps of Γ are very similar to the ones in [Rau] we give the results as a series of propositions below.

3.8 Proposition. *If $\Gamma^*_{\mathfrak{p}^l}$ contains the identity I, i. e. if $l \equiv 0 \mod 2$, we get:*

$$c_{\epsilon,l}(I) = \frac{|d_K|^{3/2}\zeta_K(2)}{(4\pi^2)^2} \int_{-\infty}^{\infty} h_\epsilon(1+t^2)\, t^2\, dt = \frac{|d_K|^{3/2}\zeta_K(2)}{32\, \pi^{7/2}\, \epsilon^{3/2}}\, e^{-\epsilon}.$$

Proof. See [EGM], p. 307 and [Rau], Theorem 3. 2, p. 118. □

3.9 Proposition. *Let T be a loxodromic element of $\Gamma^*_{\mathfrak{p}^l}$ whose centralizer $\mathcal{C}(T) = \langle T_0 \rangle \times \mathcal{E}(T)$ contains at least one element of infinite order. Then we obtain:*

$$c_{\epsilon,l}(T) = \frac{\log NT_0}{|\mathcal{E}(T)||a(T) - a(T)^{-1}|^2}\, g_\epsilon(\log NT)$$

where $|\mathcal{E}(T)|$ is the order of the set $\mathcal{E}(T)$ of all elements having finite order contained in $\mathcal{C}(T)$, and T_0 is a primitive loxodromic element generating the infinite cyclic group $\langle T_0 \rangle$. Here we call a loxodromic element $T_0 \in \Gamma$ primitive if NT_0 is minimal among all norms of loxodromic elements from Γ having the same fixed points as T.

Proof. This proposition follows by generalizing the arguments of [EGM], pp. 191–192. See also [Rau2] or [Rau], Theorem 3. 3, p. 119. □

3.10 Proposition. *If R is an elliptic element of $\Gamma^*_{\mathfrak{p}^l}$ having no cusps of Γ as fixed points, then its contribution to the trace is given by:*

$$c_{\epsilon,l}(R) = \frac{\log NT_0}{4|\mathcal{E}(R)|\sin^2(\theta)}\, g_\epsilon(0).$$

Here $|\mathcal{E}(R)|$ is the order of the maximal finite subgroup of $\mathcal{C}(R)$, T_0 is a primitive loxodromic element from $\mathcal{C}(R)$ and θ is defined by $\operatorname{tr} R = 2\cos\theta$.

Proof. This proposition can be proved by generalizing the arguments of [EGM], pp. 193–198. See also [Rau2] or [Rau], Theorem 3. 5, p. 119. □

We next introduce the numbers $c_\epsilon(\infty, l)$, $l \in \mathbb{N}_0$, coming from the continuous spectrum.

3.11 Definition. Let $\mathcal{R}$ denote a set of representatives of $\mathcal{O}/\{\pm 1\}$ and for $l \in \mathbb{N}_0$ and a prime $\mathfrak{p} \in \mathcal{O}$ set $d(\mathfrak{p}^l) := \sum_{\substack{ad=\mathfrak{p}^l,\\ d\in\mathcal{O}}} 1 = l+1$. Then for $\epsilon > 0$ we define

$$\begin{aligned} c_\epsilon(\infty, l) \;&:=\; \frac{|\mathfrak{p}^l|}{[\Gamma_\infty : \Gamma'_\infty]} \Bigg(\delta_2(l) f_\epsilon(l) + \frac{(l+1)h_\epsilon(1)}{8} \\ &\quad + \frac{1}{8\pi} \int_{-\infty}^{\infty} h_\epsilon(1+t^2) \sum_{\substack{ad=\mathfrak{p}^l,\\ d\in\mathcal{O}}} \left(\frac{|a|}{|d|}\right)^{it} \frac{\phi'}{\phi}(it)\, dt \\ &\quad + \frac{1}{2} \sum_{\substack{ad=\mathfrak{p}^l,\\ a\neq d\in\mathcal{R}}} \int_{|\log(\frac{|a|}{|d|})|}^{\infty} g_\epsilon(x) \frac{\sinh x\, dx}{\cosh x + \frac{|a-d|^2}{2|\mathfrak{p}^l|} - \frac{|a|^2+|d|^2}{2|\mathfrak{p}^l|}} \\ &\quad + \frac{1}{2} \sum_{\substack{ad=\mathfrak{p}^l,\\ a\neq d\in\mathcal{R},\\ b\in\mathcal{O}/<a-d>}} \frac{\log\left(\left|\frac{a-d}{(b,a-d)}\right|^2\right)}{|a-d|^2} g_\epsilon\left(\log\left(\frac{|a|}{|d|}\right)\right)\Bigg) \end{aligned}$$

with

$$f_\epsilon(l) := g_\epsilon(0)\left(\frac{\log|\mathfrak{p}^l|}{2} + \frac{\kappa_{\mathcal{O}}}{2} - \gamma\right) + \frac{h_\epsilon(1)}{4} - \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\epsilon(1+t^2) \frac{\Gamma'}{\Gamma}(1+it)\, dt$$

and $\delta_2(l) := 1$ if l is even and $\delta_2(l) := 0$ otherwise.

With the help of this definition, Theorem 3.7, Proposition 3.8, Proposition 3.9, Proposition 3.10 and the arguments of [Rau] we finally get:

3.12 Theorem. *Let $\mathfrak{p} \in \mathcal{O}$ be a prime. For $\epsilon > 0$ and $n \in \mathbb{N}$ we obtain*

$$\begin{aligned} \sum_{m\geq 0} \rho_m^{2n}(\mathfrak{p}) e^{-\epsilon\lambda_m} \;&=\; \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) N(\mathfrak{p})^{-j} \\ &\quad \times \left(c_{\epsilon,2j}(I) + {\sum_{\{R\}}}' c_{\epsilon,2j}(R) + {\sum_{\{T\}}}' c_{\epsilon,2j}(T) + c_\epsilon(\infty, 2j) \right) \end{aligned}$$

and for $n \in \mathbb{N}_0$

$$\begin{aligned}\sum_{m\geq 0} \rho_m^{2n+1}(\mathfrak{p}) e^{-\epsilon\lambda_m} &= \sum_{j=0}^{n} \left(\binom{2n+1}{n-j} - \binom{2n+1}{n-j-1} \right) N(\mathfrak{p})^{-j-\frac{1}{2}} \\ &\quad \times \left(\sum_{\{R\}}{}' c_{\epsilon,2j+1}(R) + \sum_{\{T\}}{}' c_\epsilon(T) + c_\epsilon(\infty, 2j+1) \right).\end{aligned}$$

Here the primes indicate that the sums on the right-hand side are extended over the Γ-conjugacy classes of those elliptic or loxodromic elements in $\Gamma^*_{\mathfrak{p}^{2j}}$ resp. $\Gamma^*_{\mathfrak{p}^{2j+1}}$ not having cusps of Γ as fixed points.

In order to infer our main theorem we have to determine the behaviour of the two expressions appearing in Theorem 3.12 as $\epsilon \to 0$.

3.13 Lemma. *For $n \in \mathbb{N}$ and any $\epsilon' > 0$ we get, as $\epsilon \to 0$,*

$$\sum_{m\geq 0} \rho_m^{2n}(\mathfrak{p}) e^{-\epsilon\lambda_m} = \frac{\mathrm{vol}\,(\Gamma)}{8\pi^{3/2}} \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) N(\mathfrak{p})^{-j} \epsilon^{-3/2} + O\left(\epsilon^{-1/2-\epsilon'}\right)$$

and for $n \in \mathbb{N}_0$ we have, as $\epsilon \to 0$,

$$\sum_{m\geq 0} \rho_m^{2n+1}(\mathfrak{p}) e^{-\epsilon\lambda_m} = O\left(\epsilon^{-1/2-\epsilon'}\right).$$

Proof. First of all note that for each $l \in \{1, \ldots, 2n\}$ there are only finitely many Γ-conjugacy classes of elliptic elements $R \in \Gamma^*_{\mathfrak{p}^l}$ that do not stabilize cusps of Γ. Hence due to the form of $c_{\epsilon,l}(R)$ their contribution to the trace is $O\left(\epsilon^{-1/2}\right)$. For the contribution of the loxodromic elements that do not stabilize cusps of Γ we infer from Lemma 3.9

$$\begin{aligned}\sum_{\{T\}} c_{\epsilon,l}(T) &= \sum_{\{T\}} \frac{\log NT_0}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} \frac{e^{-\epsilon-(\log NT)^2/(4\epsilon)}}{\sqrt{4\pi\epsilon}} \\ &\ll \frac{1}{e^\epsilon \sqrt{4\pi\epsilon}} \sum_{\{T\}} \frac{\log NT_0\, e^{-(\log NT)^2/(4\epsilon_0)}}{|\mathcal{E}(T)||a(T)-a(T)^{-1}|^2} \\ &= O\left(\frac{e^{-\epsilon}}{\sqrt{\epsilon}}\right)\end{aligned}$$

as $\epsilon \to 0$ since the last series on the right-hand converges as a result of [Rau], Lemma 3. 1 if ϵ_0 is sufficiently small (see also [Rau], Lemma 3. 25). The simple terms of $c_\epsilon(\infty, l)$ that contain h_ϵ or g_ϵ also yield $O\left(\epsilon^{-1/2}\right)$ so that we only have to discuss the integrals appearing in $c_\epsilon(\infty, l)$. According to [EGM], p. 307 we have

$$\int_{-\infty}^{\infty} h_\epsilon(1+t^2)\frac{\Gamma'}{\Gamma}(1+it)dt = O\left(\epsilon^{-1/2-\epsilon'}\right)$$

for any $\epsilon' > 0$. In order to treat the contribution of

$$\int_{-\infty}^{\infty} h_\epsilon(1+t^2) \sum_{\substack{ad=\mathfrak{p}^l,\\ d\in\mathcal{O}}} \left(\frac{|a|}{|d|}\right)^{it} \frac{\phi'}{\phi}(it)\, dt$$

we recall that

$$\begin{aligned}\frac{\phi'}{\phi}(it) &= 2\log\left(\frac{2\pi}{\sqrt{|d_K|}}\right) - \frac{\Gamma'}{\Gamma}(1+it) - \frac{\Gamma'}{\Gamma}(1-it) - \left(\frac{\zeta_K'}{\zeta_K}(1+it) + \frac{1}{it}\right)\\ &\quad - \left(\frac{\zeta_K'}{\zeta_K}(1-it) - \frac{1}{it}\right).\end{aligned}$$

Then $\frac{\zeta_K'}{\zeta_K}(1-it) = O\left(\log|t|\right)$ (cf. [Pr], Theorem 7. 1, pp. 131–132) and the form of $h_\epsilon(1+t^2)$ imply

$$\int_{-\infty}^{\infty} h_\epsilon(1+t^2) \sum_{\substack{ad=\mathfrak{p}^l,\\ d\in\mathcal{O}}} \left(\frac{|a|}{|d|}\right)^{it} \frac{\phi'}{\phi}(it)\, dt = O\left(\epsilon^{-1/2-\epsilon'}\right)$$

for any $\epsilon' > 0$. The statement of the lemma finally follows if we also consider Lemma 3.8 and Theorem 3.12. □

By means of a Tauberian theorem we then deduce

3.14 Theorem. *Let $\mathfrak{p} \in \mathcal{O}$ be a prime. For $n \in \mathbb{N}$ we obtain, as $N \to \infty$,*

$$\begin{aligned}\sum_{\lambda_m \le N} \rho_m^{2n}(\mathfrak{p}) &= \frac{\mathrm{vol}\,(\Gamma)}{6\pi^2} \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1}\right) N(\mathfrak{p})^{-j} N^{3/2}\\ &\quad + O\left(\frac{N^{3/2}}{\log N}\right)\end{aligned}$$

and for $n \in \mathbb{N}_0$

$$\sum_{\lambda_m \le N} \rho_m^{2n+1}(\mathfrak{p}) = O\left(\frac{1}{\log N}\right).$$

Proof. The two results of the theorem follow from Lemma 3.13 and from the Theorem of Karamata-Freud that can be found in the following form in [Te], p. 231:

$A(t)$ be a non-decreasing function such that the integral

$$F(\sigma) := \int_0^\infty e^{-\sigma t} dA(t)$$

converges for all $\sigma > 0$. Suppose that two real numbers $c \geq 0, \omega \geq 0$ and an increasing function $\psi(t)$ exist such that

$$\psi(t) \to \infty,\ t^{-\omega}\psi(t) \text{ decreases} \qquad (t \to \infty)$$

and

$$F(\sigma) = \frac{c + O\left(\psi\left(\frac{1}{\sigma}\right)^{-1}\right)}{\sigma^\omega} \qquad (\sigma \to 0+).$$

Then

$$A(x) = \left(c + O\left(\frac{1}{\log \psi(x)}\right)\right) \frac{x^\omega}{\Gamma(\omega + 1)} \qquad (x \to \infty).$$

See also [EGM], p. 308. □

We recall Weyl's law:

3.15 Theorem. *Let* $\mathcal{N}(\Gamma, N) := |\{m \geq 0 :\ \lambda_m \leq N\}|$. *Then*

$$\mathcal{N}(\Gamma, N) \sim \frac{\mathrm{vol}(\Gamma \setminus \mathbb{H}^3)}{6\pi^2} N^{3/2}.$$

Proof. See [EGM], Theorem 9.2, p. 405. □

This yields

3.16 Theorem. *For* $k \in \mathbb{N}_0$ *and a prime* $\mathfrak{p}$ *of* $\mathcal{O}$ *we have, as* $N \to \infty$,

$$\lim_{N \to \infty} \frac{1}{\mathcal{N}(\Gamma, N)} \sum_{\lambda_m \leq N} \rho_m^k(\mathfrak{p}) = \begin{cases} \sum_{j=0}^n \left(\binom{2n}{n-j} - \binom{2n}{n-j-1}\right) N(\mathfrak{p})^{-j}, & \text{if } k = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

4 Equidistribution of the eigenvalues

Having obtained the analogue of Weyl's law for powers of eigenvalues of Hecke operators in the last section we determine the limit distribution in this section. First of all, we recall the definition of equidistribution:

4.1 Definition. A sequence (x_n) is equidistributed on a space X with respect to the measure μ or simply μ-equidistributed if for every $f \in C_c(X)$ the following identity holds:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(x_n) = \int_X f(x)d\mu(x).$$

We want to show that we have equidistribution with respect to the measure

$$d\mu_{\mathfrak{p}}(x) := \begin{cases} \frac{1}{2\pi}\left(1+\frac{1}{N(\mathfrak{p})}\right)\frac{\sqrt{4-x^2}}{\left(1+\frac{1}{N(\mathfrak{p})}\right)^2-\frac{x^2}{N(\mathfrak{p})}}\,dx & \text{if } |x|\leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

To this end we first note that an easy computation along the lines of [CDF], p. 408 gives

4.2 Lemma. *For $k \in \mathbb{N}$ and a prime $\mathfrak{p} \in \mathcal{O}$ we have*

$$\int_{-\infty}^{\infty} x^k d\mu_{\mathfrak{p}}(x) = \begin{cases} \sum_{j=0}^{n}\left(\binom{2n}{n-j}-\binom{2n}{n-j-1}\right)N(\mathfrak{p})^{-j}, & \textit{if } k=2n, \\ 0, & \textit{otherwise.} \end{cases}$$

Note that the polynomial functions are dense in the space $C_c([-(\sqrt{N(\mathfrak{p}^l)}+1/\sqrt{N(\mathfrak{p})^l}), \sqrt{N(\mathfrak{p}^l)}+1/\sqrt{N(\mathfrak{p})^l}]$. Now using the method of moments (cf. e.g. [Fe], pp. 225–227 and [Fe], p. 251), Theorem 3.16, Lemma 4.2 and the definition of equidistribution proves the main theorem stated in the introduction.

5 The error term

As is in the case of Weyl's law, the error term in Theorem 3.14 can be improved. Recently Lapid and Müller [LM], in a beautiful paper, proved a general Weyl's law with an error term for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Their method is more general than the method involving Selberg's zeta function and can also be applied in our situation.

In this section we use Hörmander's method as in [M] to improve the error term in Theorem 3.14 and prove

5.1 Theorem. *For $n \in \mathbb{N}$ we obtain, as $N \to \infty$,*

$$\sum_{\lambda_m \le N} \rho_m^{2n}(\mathfrak{p}) = \frac{\mathrm{vol}\,(\Gamma)}{6\pi^2} \sum_{j=0}^{n} \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) N(\mathfrak{p})^{-j} N^{3/2} + O\left(N\right).$$

The rest of this section is devoted to the proof of this theorem where we follow closely the arguments of [M]. Let $g \in C_c^\infty(\mathbb{R})$ be an even function and $h(z) := \int_{-\infty}^{\infty} g(u)e^{-iuz}du$. For $t \in \mathbb{R}$ we set $h_t(z) = h(t-z) + h(t+z)$. Then $h_t(z) = h_t(-z)$ and $g_t(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_t(r)e^{iru}dr = e^{-itu}g(u) + e^{itu}g(-u)$. Furthermore, we symmetrize the spectrum $\lambda_m = 1 + r_m^2$, $m \ge 0$, of $-\Delta$ by $r_{-m} := -r_m$, $m \in \mathbb{N}$, so that $\lambda_{-m} = \lambda_m$ and we set $\rho_{-m}(\mathfrak{p}) = \rho_m(\mathfrak{p})$, $m \in \mathbb{N}$. Then the trace formula for Hecke operators will read as in Theorem (3.7) with g_t, h_t instead of the functions g_ϵ, h_ϵ that we had before. Namely,

$$\sum_{m \in \mathbb{Z}} \rho_m^{2n}(\mathfrak{p}) h(t - r_m) = \tag{1}$$
$$\sum_{j=0}^{n} \mathcal{M}(j,n,\mathfrak{p}) \left(c_{t,2j}(I) + \sum_{\{R\}}{}' c_{t,2j}(R) + \sum_{\{T\}}{}' c_{t,2j}(T) + c(t,\infty,2j) \right)$$

with

$$\mathcal{M}(j,n,\mathfrak{p}) := \left(\binom{2n}{n-j} - \binom{2n}{n-j-1} \right) N(\mathfrak{p})^{-j}.$$

Here the $c_{t,2j}$'s and $c(t,\infty,2j)$ are given as in Proposition 3.8, 3.9, 3.10 and Definition 3.11. The primes on the right-hand side of (1) indicate that the sums on the right-hand side are extended over the Γ-conjugacy classes of those elliptic or loxodromic elements in $\Gamma^*_{\mathfrak{p}^{2j}}$ not having cusps of Γ as fixed points. Now choose $\epsilon > 0$ so small that $\log NT > \epsilon$ for all loxodromic $T \in \Gamma^*_{\mathfrak{p}^{2j}}$ that do not fix cusps of Γ and let the function g be supported on $(-\epsilon, \epsilon)$. Then formula (1) simplifies to

$$\sum_{m \in \mathbb{Z}} \rho_m^{2n}(\mathfrak{p}) h(t - r_m) = \sum_{j=0}^{n} \mathcal{M}(j,n,\mathfrak{p}) \Bigg(\frac{\mathrm{vol}(\Gamma)}{2\pi^2} \int_{-\infty}^{\infty} h(t-r) r^2 dr + 4\pi \sum_{\{R\}}{}' \frac{\log NT_0}{4|\mathcal{E}(R)| \sin^2(\theta)} + c(t,\infty,2j) \Bigg). \tag{2}$$

As before $|\mathcal{E}(R)|$ denotes the order of the maximal finite subgroup of $\mathcal{C}(R)$, T_0 is a primitive loxodromic element from $\mathcal{C}(R)$ and θ is defined by $\mathrm{tr}\, R = 2\cos\theta$.

Moreover, we can assume that $h \in \mathcal{S}(\mathbb{R})$ was chosen in such a way that $h \geq 0$, $h > 0$ on $[-a, a]$, $a \in \mathbb{R}$, $\hat{h}(0) = 1$ and that $\operatorname{supp}\hat{h}$ is contained in $(-\epsilon, \epsilon)$ (see proof of Lemma 2.3 in [DG] or [M], p. 140). Here $\hat{h}(x) = \int_{\mathbb{R}} h(r)e^{-irx}dr$ is the Fourier transform of h.
In order to derive the asymptotics stated in the theorem we first determine the behaviour of the right-hand side of (2) as $|t| \to \infty$.

5.2 Lemma. *For h as above we have*

$$\sum_{m \in \mathbb{Z}} \rho_m^{2n}(\mathfrak{p}) h(t - r_m) = O\left(|t|^2\right).$$

Proof. First of all consider the contribution coming from the identity. Here we get

$$\int_{-\infty}^{\infty} h(t-r) r^2 dr = O\left(t^2\right)$$

as $|t| \to \infty$. Since there are only finitely many Γ-conjugacy classes of non-cuspidal elliptic elements of $\Gamma^*_{\mathfrak{p}^{2j}}$ we easily see that

$$4\pi \sum_{\{R\}}{}' \frac{\log NT_0}{4|\mathcal{E}(R)| \sin^2(\theta)} = O(1)$$

as $|t| \to \infty$.

In order to treat $c(t, \infty, 2j)$ recall that it has the following form. If $\mathcal{R}$ denotes a set of representatives of $\mathcal{O}/\{\pm 1\}$ then

$$\begin{aligned} c(t,\infty,2j) \;=\; & \frac{|\mathfrak{p}^{2j}|}{[\Gamma_\infty : \Gamma'_\infty]} \Bigg(f_t(2j) + \frac{(2j+1)h(t)}{4} \\ & + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t-r) \sum_{\substack{ad=\mathfrak{p}^{2j},\\ d \in \mathcal{O}}} \left(\frac{|a|}{|d|}\right)^{ir} \frac{\phi'}{\phi}(ir)\, dr \\ & + \frac{1}{2} \sum_{\substack{ad=\mathfrak{p}^{2j},\\ a \neq d \in \mathcal{R}}} \int_{\left|\log\left(\frac{|a|}{|d|}\right)\right|}^{\infty} g_t(x) \frac{\sinh x\, dx}{\cosh x + \frac{|a-d|^2}{2|\mathfrak{p}^{2j}|} - \frac{|a|^2+|d|^2}{2|\mathfrak{p}^{2j}|}} \\ & + \frac{1}{2} \sum_{\substack{ad=\mathfrak{p}^{2j},\\ a \neq d \in \mathcal{R},\\ b \in \mathcal{O}/<a-d>}} \frac{\log\left(\left|\frac{a-d}{(b,a-d)}\right|^2\right)}{|a-d|^2} g_t\left(\log\left(\frac{|a|}{|d|}\right)\right)\Bigg) \end{aligned}$$

with

$$f_t(2j) = 2\pi\left(\frac{\log|\mathfrak{p}^{2j}|}{2} + \frac{\kappa_{\mathcal{O}}}{2} - \gamma\right) + \frac{h(t)}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t-r) \left(\frac{\Gamma'}{\Gamma}(1+ir) + \frac{\Gamma'}{\Gamma}(1-ir)\right) dr.$$

As in [M], p. 19 we have as $|t| \to \infty$

$$\int_{-\infty}^{\infty} h(t-r)\,\frac{\Gamma'}{\Gamma}(1 \pm ir)\,dr = O(\log|t|) \tag{3}$$

and

$$\int_{-\infty}^{\infty} h(t-r) \sum_{\substack{ad=\mathfrak{p}^{2j},\\ d\in\mathcal{O}}} \left(\frac{|a|}{|d|}\right)^{ir} \frac{\phi'}{\phi}(ir)\,dr = O(\log|t|) \tag{4}$$

since

$$\phi(s) = \frac{2\pi}{s\sqrt{|d_K|}}\,\frac{\zeta_K(s)}{\zeta_K(1+s)} = \frac{2\pi}{s\sqrt{|d_K|}}\,\frac{\zeta(s)L(s,\chi_K)}{\zeta(1+s)L(1+s,\chi_K)}.$$

Here d_K is the discriminant of K and χ_K denotes the character of K. Using the functional equation of the zeta function and the L-function it follows that in order to estimate the growth of $\phi'/\phi(ir)$ we have to determine the behaviour of terms of the form

$$\frac{\Gamma'}{\Gamma}(1 \pm ir),\ \frac{\zeta'}{\zeta}(1 \pm ir) \text{ and } \frac{L'}{L}(1 \pm ir, \chi_K).$$

This can be done with Stirling's formula (see e. g. [Brü], p. 55) and [Pr], Theorem 7. 1, pp. 131–132 and then (4) follows. The remaining terms of $c(t,\infty,2j)$ contribute only $O(1)$ as $|t| \to \infty$. Thus we deduce

$$\sum_{m\in\mathbb{Z}} \rho_m^{2n}(\mathfrak{p})h(t-r_m) = O\left(|t|^2\right)$$

as $|t| \to \infty$. □

This estimate is valid if, in particular, the considered Hecke operator is the identity and then by the same arguments as in the proof of [M], Lemma 2. 2, p. 139 we obtain that for each $a > 0$ there exists a $C > 0$ such that

$$\sharp\{m : |r_m - \mu| \le a\} \le C(1 + |\mu|^2) \tag{5}$$

for all $\mu \in \mathbb{R}$. With the help of estimate (5), we obtain the following analogue of [M], Lemma 2. 3, p. 140.

5.3 Lemma. *For every h as above there exists a constant $C = C(\mathfrak{p}, n)$ depending on $\mathfrak{p}$ and n such that for $N \ge 1$ the following inequalities hold:*

$$\sum_{|r_m|\le N} \left| \int_{\mathbb{R}\setminus[-N,N]} \rho_m^{2n}(\mathfrak{p})h(t-r_m)\,dt \right| \ \le\ CN^2, \tag{6}$$

$$\sum_{|r_m|> N} \left| \int_{-N}^{N} \rho_m^{2n}(\mathfrak{p})h(t-r_m)\,dt \right| \ \le\ CN^2. \tag{7}$$

Proof. With some minor modifications the proof of the lemma follows the proof of [M], Lemma 2. 3, p. 140. For the convenience of the reader we give the details. Using the trivial bound on the Hecke eigenvalues $\rho_m(\mathfrak{p}) \leq 2N(\mathfrak{p})$, (5) and the fact that $h(r) \leq C(1+|r|)^{-5}$ for some $C > 0$ we get

$$\begin{aligned}
\sum_{|r_m|\leq N} \left| \int_N^\infty \rho_m^{2n}(\mathfrak{p}) h(t-r_m)\,dt \right| &\leq C \sum_{|r_m|\leq N} \int_N^\infty |h(t-r_m)|\;dt \\
&\leq C \sum_{|r_m|\leq N} \int_{N-r_m}^\infty \frac{1}{(1+|t|^2)^5} dt \\
&= C \sum_{k=-\lfloor N\rfloor}^{\lfloor N\rfloor-1} \sum_{k\leq r_m\leq k+1} \frac{1}{(1+N-r_m)^4} \\
&\leq C \sum_{k=-\lfloor N\rfloor}^{\lfloor N\rfloor-1} \frac{\sharp\{m : |r_m-k|\leq 1\}}{(N-k)^4} \\
&\leq C \sum_{k=-\lfloor N\rfloor}^{\lfloor N\rfloor-1} \frac{1+k^2}{(N-k)^4} \leq O\left(N^2\right).
\end{aligned}$$

The inequality (7) follows from a similar argument. ☐

Now we collect the above lemmata to prove Theorem 5.1.

Proof. We start with the following elementary identity

$$\begin{aligned}
\int_{-N}^N \sum_{m=-\infty}^\infty \rho_m^{2n}(\mathfrak{p}) h(t-r_m)\,dt &= \sum_{|r_m|\leq N} \int_{-\infty}^\infty \rho_m^{2n}(\mathfrak{p}) h(t-r_m)\,dt \\
&\quad - \sum_{|r_m|\leq N} \rho_m^{2n}(\mathfrak{p}) \int_{\mathbb{R}\setminus[-N,N]} h(t-r_m)\,dt \\
&\quad + \sum_{|r_m|>N} \rho_m^{2n}(\mathfrak{p}) \int_{-N}^N h(t-r_m)\,dt \qquad (8)
\end{aligned}$$

and note that this identity together with $\hat{h}(0) = 1$ implies

$$\begin{aligned}
\sum_{|r_m|\leq N} \rho_m^{2n}(\mathfrak{p}) &= \int_{-N}^N \sum_{m\in\mathbb{Z}} \rho_m^{2n}(\mathfrak{p}) h(t-r_m)\,dt \\
&\quad + \sum_{|r_m|\leq N} \rho_m^{2n}(\mathfrak{p}) \int_{\mathbb{R}\setminus[-N,N]} h(t-r_m)\,dt \\
&\quad - \sum_{|r_m|>N} \rho_m^{2n}(\mathfrak{p}) \int_{-N}^N h(t-r_m)\,dt. \qquad (9)
\end{aligned}$$

To get the asymptotics of the theorem we integrate (2) from $-N$ to N with respect to the variable t:

$$\int_{-N}^{N} \sum_{m\in\mathbb{Z}} \rho_m^{2n}(\mathfrak{p}) h(t-r_m)\,dt =$$

$$\sum_{j=0}^{n} \mathcal{M}(j,n,\mathfrak{p}) \Bigg(\frac{\mathrm{vol}(\Gamma)}{2\pi^2} \int_{-N}^{N} \int_{-\infty}^{\infty} h(t-r) r^2 dr\,dt \quad (10)$$

$$+ 4\pi {\sum_{\{R\}}}' \frac{\log NT_0}{4|\mathcal{E}(R)| \sin^2(\theta)} \int_{-N}^{N} dt$$

$$+ \int_{-N}^{N} c(t,\infty,2j)\,dt \Bigg).$$

By the estimate for the non-cuspidal elliptic elements, (3) and (4) the contribution of the last two terms of (10) to the asymptotics is $O(N \log N)$. Furthermore, using (6), (7), (9) and (10), yield

$$2 \sum_{r_m \le N} \rho_m^{2n}(\mathfrak{p}) \quad = \quad \frac{\mathrm{vol}(\Gamma)}{2\pi^2} \sum_{j=0}^{n} \mathcal{M}(j,n,\mathfrak{p}) \int_{-N}^{N} \int_{-\infty}^{\infty} h(t-r) r^2 dr\,dt$$

$$+ O\left(N^2\right). \quad (11)$$

For the term involving the identity we obtain

$$\int_{-N}^{N} \int_{-\infty}^{\infty} h(t-r) r^2 dr\,dt = \int_{-N}^{N} r^2 dr + O\left(N^2\right) = \frac{2}{3} N^3 + O\left(N^2\right)$$

as $N \to \infty$. For justifying this identity we note that there exists a $C > 0$ such that $|h(r)| < C(1+|r|)^{-5}$ and imitate the arguments of [M], p. 140 with $p(r) = r^2$. Inserting this result into (11) we finally get

$$\sum_{r_m \le N} \rho_m^{2n}(\mathfrak{p}) = \frac{\mathrm{vol}(\Gamma)}{6\pi^2} \sum_{j=0}^{n} \mathcal{M}(j,n,\mathfrak{p}) N^3 + O\left(N^2\right)$$

as $N \to \infty$. Since $\lambda_m = 1 + r_m^2$ this proves the theorem. □

Acknowledgements

Part of this work was done while the second author was a guest at the Institute for Mathematical Research at the ETH in Zürich and she gratefully acknowledges its hospitality. During the preparation of the final version of the paper the second author visited the Max Planck Institute for Mathematics in Bonn and she thanks the institute for its hospitality. Furthermore, the authors want to thank Y. Petridis for his comments on an earlier version of this paper, B. Winn for useful discussions and the referee for his/her careful reading of the paper which resulted in a correction of an error.

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