MODULAR COCYCLES AND LINKING NUMBERS

W. DUKE, Ö. IMAMOḠLU, AND Á . TÓTH

Abstract. It is known that the 3-manifold $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to the complement of the trefoil knot in S^3. E. Ghys showed that the linking number of this trefoil knot with a modular knot is given by the Rademacher symbol, which is a homogenization of the classical Dedekind symbol. The Dedekind symbol arose historically in the transformation formula of the logarithm of Dedekind's eta function under $\mathrm{SL}(2,\mathbb{Z})$. In this paper we give a generalization of the Dedekind symbol associated to a fixed modular knot. This symbol also arises in the transformation formula of a certain modular function. It can be computed in terms of a special value of a certain Dirichlet series and satisfies a reciprocity law. The homogenization of this symbol, which generalizes the Rademacher symbol, gives the linking number between two distinct symmetric links formed from modular knots.

1. Introduction

Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. The homogeneous space $\Gamma\backslash G$ is diffeomorphic to the 3-manifold $\mathcal{M}$ the complement of a trefoil knot in the 3-sphere S^3. In [31] Milnor gives a proof (that he attributes to Quillen) of this remarkable fact. The diagonal geodesic flow on $\Gamma\backslash G$ has arithmetically interesting periodic orbits. Suppose that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ is a primitive hyperbolic element with an eigenvalue $\epsilon > 1$. Fix a $g \in G$ so that $g^{-1}\gamma g = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{smallmatrix}\right)$. Then

$$\Gamma g \mapsto \Gamma g \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

where $t \in [0, \log\epsilon]$ gives a primitive oriented closed orbit in $\Gamma\backslash G$ which depends only on the conjugacy class of γ. The image of this orbit in M is a modular knot. Ghys [20] gave the beautiful result that the linking number of this knot with the trefoil (with some orientation) is given by the Rademacher symbol

$$\Psi(\gamma) = \Phi(\gamma) - 3\,\mathrm{sign}(c(a+d)). \tag{1.1}$$

Here $\Phi(\gamma)$ is the Dedekind symbol defined for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by

$$\Phi(\gamma) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12\,\mathrm{sign}\, c \cdot s(a,c) & \text{if } c \neq 0, \end{cases} \tag{1.2}$$

where $s(a,c)$ is the Dedekind sum defined for $\gcd(a,c) = 1$, $c \neq 0$ by

$$s(a,c) = \sum_{n=1}^{|c|-1} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right). \tag{1.3}$$

As usual, $((x)) = 0$ if $x \in \mathbb{Z}$ and otherwise $((x)) = x - \lfloor x \rfloor - 1/2$.

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The Rademacher symbol defined for all $\gamma \in \Gamma$ by (1.1) is a conjugacy class invariant [35] and, for γ hyperbolic, it is the homogenization of the Dedekind symbol $\Phi(\gamma)$ [5] [9]. More precisely,

$$\Psi(\gamma) = \lim_{n\to\infty} \frac{\Phi(\gamma^n)}{n} \tag{1.4}$$

In addition to its role here, the Dedekind sum $s(a,c)$ occurs in surprisingly diverse contexts (see e.g. [4], [35], [25]). Among its many properties we note here only the famous reciprocity formula for $a, c > 0$

$$s(a,c) - s(-c,a) = \tfrac{1}{12}\left(\tfrac{a^2+c^2+1}{ac}\right) - \tfrac{1}{4}. \tag{1.5}$$

The Dedekind symbol arose in Dedekind's [11] evaluation of the transformation law for the logarithm of

$$\Delta(z) = q \prod_{m\geq 1}(1-q^m)^{24}$$

where as usual $q = e(z) = e^{2\pi i z}$ for $z \in \mathbb{H}$. Thus for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$ we have

$$\log \Delta(\gamma z) - \log \Delta(z) = 6\log(-(cz+d)^2) + 2\pi i \Phi(\gamma), \tag{1.6}$$

where $\Phi(\gamma)$ is given by the formula (1.2) and where we choose $\arg(-(cz+d)^2) \in (-\pi,\pi)$. Sarnak [36] applied the modular forms connection to study the distribution of modular knots with a given linking number by means of the trace formula. See also [32]. At the end of his paper Ghys mentions the problem of interpreting the linking number between two modular knots.

In this paper we approach this question from the modular point of view by giving an appropriate generalization of the Dedekind symbol. Perhaps surprisingly this also leads to a linking number; in this case that of two symmetric links. To outline our method we first give an equivalent but slightly different approach to the above results about the Dedekind symbol: it arises as a limiting value of the weight 0 cocycle whose derivative is $\frac{12c}{cz+d}$. This limiting value is an integer and its homogenization is also an integer that gives the linking number with the trefoil.

To put this into perspective, let $\mathcal{P}$ be the space of holomorphic functions f on $\mathbb{H}$ such that $f(z) \ll y^\alpha + y^{-\alpha}$ for some α depending on f. For any integer $k \in 2\mathbb{Z}$, $\gamma \in \Gamma$ acts on $\mathcal{P}$ by the usual slash action defined via $f|_k\gamma = (cz+d)^{-k} f(\gamma z)$. A 1-cocycle of weight k for Γ with coefficients in $\mathcal{P}$ is a map $\Gamma \to \mathcal{P}$ given by $\gamma \mapsto r(\gamma, z)$ with

$$r(\sigma\gamma, z) = r(\sigma, z)|_k\gamma + r(\gamma, z)$$

for all $\gamma, \sigma \in \Gamma$. Now given a 1-cocycle $r(\gamma, z)$ of weight 2 for Γ there will be a unique 1-cocycle $R(\gamma, z)$ of weight 0 for Γ such that

$$\frac{d}{dz} R(\gamma, z) = r(\gamma, z), \tag{1.7}$$

the uniqueness following from the fact that $H^1(\Gamma, \mathbb{C}) = \{0\}$. We call $R(\gamma, z)$ the primitive of $r(\gamma, z)$.

The weight 2 cocycle relevant to the Dedekind sum is given for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ by

$$r(\gamma, z) = \frac{12c}{cz+d}$$

which, up to a constant, appears in the transformation formula of the weight 2 Eisenstein series $E_2(z)$ (a multiple of Δ'/Δ). It follows from (1.6) that the primitive for this cocycle is

$$R(\gamma, z) = 6\log(-(cz+d)^2) + 2\pi i \Phi(\gamma),$$

provided $c \neq 0$, from which we have the limit formula for $\Phi(\gamma)$ in (1.2):

$$\Phi(\gamma) = \tfrac{1}{2\pi} \lim_{y \to \infty} \operatorname{Im} R(\gamma, iy). \tag{1.8}$$

As an attempt to generalize the linking number formula of Ghys to two closed orbits, we will associate to any conjugacy class $\mathcal{C}$ of hyperbolic $\sigma \in \Gamma$ with $\operatorname{tr}\sigma > 2$ the weight two 1-cocycle defined for $c \neq 0$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by

$$r_{\mathcal{C}}(\gamma, z) := \varepsilon_{\mathcal{C}} \sum \frac{1}{z - w} - \frac{1}{z - w'}, \tag{1.9}$$

where the sum is over the fixed points w', w of $\sigma \in \mathcal{C}$, satisfying $w' < -d/c < w$ and

$$\varepsilon_{\mathcal{C}} = \begin{cases} 1 & \text{if } \sigma \not\sim \sigma^{-1} \\ 2 & \text{if } \sigma \sim \sigma^{-1} \end{cases}. \tag{1.10}$$

If $c = 0$ we let $r_{\mathcal{C}}(\gamma, z) = 0$. We then have

Theorem 1. *Let $r_{\mathcal{C}}(\gamma, z)$ be defined as in (1.9). Then $r_{\mathcal{C}}(\gamma, z)$ is a weight 2 cocycle for Γ.*

Let $R_{\mathcal{C}}(\gamma, z)$ be the unique primitive of $r_{\mathcal{C}}(\gamma, z)$. Next we define the Dedekind symbol for $\mathcal{C}$ and any $\gamma \in \Gamma$ by

$$\Phi_{\mathcal{C}}(\gamma) = \tfrac{2}{\pi} \lim_{y \to \infty} \operatorname{Im} R_{\mathcal{C}}(\gamma, iy) \tag{1.11}$$

Then we have

Theorem 2. *$\Phi_{\mathcal{C}}(\gamma)$ exists and is an integer.*

The homogenization of $\Phi_{\mathcal{C}}$ possesses a linking number interpretation. Even though our point of view is two dimensional this is not unexpected as this symbol is closely related to a Green function. (See (3.2) and the paragraph that follows.) In order to define the linking number of two cycles in a manifold we must assume that they are each homologous to 0 and that they don't intersect. For two orbits as above one can either fill in the trefoil appropriately to get S^3, as is done in [21], or restrict attention to orbits that are null-homologous as in [12]. It is not immediately clear how modular forms may enter in the first approach. We follow the second course and use a theorem that goes back to Birkhoff that shows that the link determined by a primitive hyperbolic element and its inverse is null-homologous and the linking number of two such links is given by the number of unsigned intersections (with appropriate multiplicities) of their porojection on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Based on this and properties of our new symbol we show that for two such distinct symmetric links, denoted also by $\mathcal{C}_\gamma$, and $\mathcal{C}_\sigma$, their linking number $Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma)$ is given by the homogenization of $\Phi_{\mathcal{C}_\sigma}$. More precisely;

Theorem 3. *Let $\mathcal{C}_\sigma$ and $\mathcal{C}_\gamma$ denote also the links associated to two different primitive conjugacy classes and let*

$$\Psi_{\mathcal{C}_\sigma}(\gamma) = \lim_{n \to \infty} \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n}.$$

Then

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = \Psi_{\mathcal{C}_\sigma}(\gamma)$$

Of course it is desirable to have a simple closed form expression for $\Phi_{\mathcal{C}}(\gamma)$ like that for $\Phi(\gamma)$ in (1.2). While it seems unlikely that such a simple sum can be given in general, we are able to express $\Phi_{\mathcal{C}}(\gamma)$ in terms of a special value of a certain Dirichlet series that has some properties analogous to the Dedekind sum $s(a, c)$ from (1.3), including the reciprocity formula (1.5). That something like this might be possible is indicated by the fact that for the Dirichlet series

$$L(s,a/c)=\sum_{n\geq 1}\sigma(n)e(\tfrac{a}{c}n)n^{-s},$$

where $\sigma(n)$ is the usual divisor sum, we have the limit formula (proven below in Corollary 2.2)

$$s(a,c)=\tfrac{1}{2\pi i}\lim_{s\to 1}\left[L(s,\tfrac{a}{c})+\tfrac{1}{2s-2}\right], \tag{1.12}$$

assuming $c>0$.

The Dirichlet series associated to the cocycles of Theorem 1 are given explicitly as follows. For each $m\geq 0$ let j_m be the unique modular function holomorphic on $\mathbb{H}$ whose Fourier expansion begins

$$j_m(z)=q^{-m}+\mathrm{O}(q)$$

and define for $\alpha\in\mathbb{Q}$ the Dirichlet series

$$L_{\mathcal{C}}(s,\alpha)=\sum_{n\geq 1}a_{\mathcal{C}}(n)e(n\alpha)n^{-s}, \tag{1.13}$$

where the coefficient $a_{\mathcal{C}}(n)$ is given by the cycle integral

$$a_{\mathcal{C}}(m)=\sqrt{D'}\int_{z_0}^{\sigma z_0} j_m(z)\tfrac{dz}{Q_\sigma(z)}. \tag{1.14}$$

Here $\sigma=\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)\in\mathcal{C}$ is primitive and we set $Q_\sigma(z)=c'z^2+(d'-a')z-b'$ and $D'=(a'+d')^2-4$. The path of integration can be taken as any path from z_0 to σz_0. Note that the integral is independent of the choice $\sigma\in\mathcal{C}$ and z_0. In particular, if λ is the eigenvalue >1 of σ^2 then

$$a_{\mathcal{C}}(0)=\log\lambda,$$

assuming that $\operatorname{tr}\sigma>2$.

Our next theorem gives the connection of this Dirichlet series to $\Phi_{\mathcal{C}}(\gamma)$.

Theorem 4. *Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ with $c\neq 0$ and and $L_{\mathcal{C}}(s,a/c)$ be the Dirichlet series as in (1.13). Then $L_{\mathcal{C}}(s,a/c)$ converges for* $\mathrm{Re}(s)>9/4$, *has a meromorphic continuation to $s>0$ and is holomorphic at $s=1$. Moreover*

$$\Phi_{\mathcal{C}}(\gamma)=-\tfrac{1}{\pi^2}\operatorname{Re}L_{\mathcal{C}}(1,a/c). \tag{1.15}$$

It is interesting that $\Phi_{\mathcal{C}}(\gamma)$ depends only on $a/c \bmod 1$. Furthermore, we have the following reciprocity formula, which will be proved in Theorem 4.3:

For $z_i\in\mathbb{C}\cup\{\infty\}$, let $[z_1,z_2,z_3,z_4]=\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}$ denote the cross ratio. We assume that $(a,c)=1$ and $ac\neq 0$. Then

$$\frac{1}{i\pi}\left[L_{\mathcal{C}}(1,a/c)-L_{\mathcal{C}}(1,-c/a)\right]=-2\left(\tfrac{a^2+c^2+1}{ac}\right)\log\lambda+\varepsilon_{\mathcal{C}}\sum_{w'<0<w}\log[\tfrac{a}{c},w,w',-\tfrac{c}{a}] \tag{1.16}$$

Here we interpret the imaginary part of the logarithm of a negative real number to be π. Note that (1.16) is in some sense analogous to (1.5) and allows for a fast calculation of $L_{\mathcal{C}}(1,a/c)$ and hence also of $\Phi_{\mathcal{C}}(\gamma)$.

The rest of the paper is organized as follows. First in section 2 we define the Dirichlet series associated to a general modular integral, prove its analytic properties and express the weight 0 cocycle in terms of it. In section 3 we prove Theorem 1 that the function $r_{\mathcal{C}}(\gamma,z)$ defined by (1.9) is a weight 2 parabolic cocycle for Γ and introduce the modular integral $F_{\mathcal{C}}(z)$ associated to the rational period function $r_{\mathcal{C}}(\gamma,z)$. In the next section we give a formula for the unique weight zero cocycle $R_{\mathcal{C}}(\gamma, z)$ in terms of special values of the Dirichlet series $L_{\mathcal{C}}(s, a/c)$ associated to $F_{\mathcal{C}}(z)$. In this section we also give two applications of the cocycle relation for $R_{\mathcal{C}}(\gamma, z)$. The first one gives the reciprocity formula for $L_{\mathcal{C}}(1, a/c)$ where as the second one provides a geometric interpretation for $L_{\mathcal{C}}(1, a/c)+L_{\mathcal{C}}(1, -d/c)$. In section 5 we turn our attention to the analog of the Dedekind symbol, $\Phi_{\mathcal{C}}(\gamma)$ and establish that $\Phi_{\mathcal{C}}(\gamma)$ is an intersection number, hence an integer. In the last section we review some properties of $\Gamma\backslash SL_2(\mathbb{R})$ that are used in the paper. To make the paper self-contained we also give here an elementary demonstration of the important result of Birkhoff that identifies linking numbers of modular knots with intersection numbers of closed geodesics. We finish section 6 by proving that homogenization of $\Phi_{\mathcal{C}}$ gives the linking number of two symmetric links formed from modular knots. Finally for the convenience of the reader, in Appendix A we give Ghys' argument for the identification of the Rademacher symbol with the linking number.

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2. Dirichlet series associated to weight 2 cocycles

Recall that a (strongly) parabolic cocycle of weight k for Γ with coefficients in $\mathcal{P}$ is a map $\Gamma \to \mathcal{P}$ given by $\gamma \mapsto r(\gamma, z)$ with

$$r(\sigma\gamma, z) = r(\sigma, z)|_k\gamma + r(\gamma, z)$$

for all $\gamma, \sigma \in \Gamma$ which also satisfies $r(T, z) \equiv 0$.

It follows from a more general result of Knopp [29] that given a parabolic cocycle $r(\gamma, z)$ for Γ there is $F(z) = \sum_{n\geq 0} a_n e^{2\pi i n z}$ with $a_n \ll n^C$ for some $C > 0$ such that $\forall\gamma \in \Gamma$,

$$F|_k\gamma(z) = F(z) + r(\gamma, z).$$

The function $F(z)$ is called the modular integral associated to $r(\gamma, z)$. We now restrict ourselves to the case of $k = 2$ and let $r(\gamma, z)$ be a cocycle of weight 2. We associate to $r(\gamma, z)$ and its modular integral a Dirichlet series

$$L(F, s, a/c) = \sum_{n\geq 1} a_n e(\frac{an}{c})n^{-s}.$$

In this section we will first prove a general theorem giving the relation of the special value of $L(F, 1, a/c)$ to the unique weight 0 cocycle $R(\gamma, z)$ which satisfies $R'(\gamma, z) = r(\gamma, z)$.

This is based on the fact the function $G(z) = a_0 z + \sum_{n>0} \frac{a_n}{2\pi i n} e^{2\pi i n z}$ is a primitive of $F(z)$ and satisfies $\frac{d}{dz}(G(\gamma z) - G(z)) = r(\gamma, z)$. This gives a relation between $R(\gamma, z)$ and $\int_z^{\gamma z}(F(w) - a_0)dw$, which in turn expresses $\lim_{y\to\infty} R(\gamma, iy)$ in terms of the "period-integral" $\int_{a/c}^{i\infty}(F(w) - a_0)dw$. If F were a weight 2 cusp form, it is well known that this period integral is expressible in terms of the central value of a twisted Dirichlet series of F. The next theorem shows the case of modular integrals is similar. More precisely we have the following theorem.

Theorem 2.1. *Let $r(\gamma, z) \in \mathcal{P}$ be a cocycle of weight 2 and $F(z) = \sum_{n\geq 0} a_n q^n$ be its modular integral. Assume that $a_n \ll n^\alpha$ for some $\alpha > 0$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, let*

$$\Lambda(s, \tfrac{a}{c}) = \Lambda(F, s, \tfrac{a}{c}) = \left(\tfrac{2\pi}{c}\right)^{-s} \Gamma(s) \sum_{n\geq 1} a_n e(\frac{an}{c}) n^{-s} \tag{2.1}$$

and

$$H(s, \tfrac{a}{c}) = \Lambda(s, \tfrac{a}{c}) + \int_1^\infty r(\gamma, -d/c + it/c) t^{1-s} dt + \frac{a_0}{s} - \frac{a_0}{2-s}. \tag{2.2}$$

Then $H(s, \frac{a}{c})$ is entire and satisfies the functional equation $H(s, \frac{a}{c}) = H(2-s, \frac{-d}{c})$. Moreover if

$$R(\gamma, z) = \tfrac{-i}{c} H(1, \tfrac{a}{c}) + \int_{-\frac{d}{c}+\frac{i}{c}}^{z} r(\gamma, w) dw + a_0 \left(\tfrac{a+d}{c}\right) \tag{2.3}$$

Then $R(\gamma, z)$ is the weight zero cocycle such that $R'(\gamma, z) = r(\gamma, z)$.

Proof. Let $z_t = \frac{-d}{c} + \frac{i}{ct}$ so that $\gamma z_t = \frac{a}{c} + \frac{it}{c}$ and $cz_t + d = i/t$. Then

$$\begin{aligned}
\Lambda(s, a/c) &= \int_0^\infty (F(\gamma z_t) - a_0) t^{s-1} dt \\
&= \int_0^1 (F(\gamma z_t) - a_0) t^{s-1} dt + \int_1^\infty (F(\gamma z_t) - a_0) t^{s-1} dt \\
&= -\frac{a_0}{s} - \int_1^\infty F(\gamma z_{1/t}) (it)^{-2} t^{1-s} dt + \int_1^\infty (F(\gamma z_t) - a_0) t^{s-1} dt \\
&= -\frac{a_0}{s} - \int_1^\infty [F(z_{1/t}) + r(\gamma, z_{1/t})] t^{1-s} dt + \int_1^\infty (F(\gamma z_t) - a_0) t^{s-1} dt \\
&= -\frac{a_0}{s} + \frac{a_0}{2-s} - \int_1^\infty r(\gamma, z_{1/t}) t^{1-s} dt \\
&\quad - \int_1^\infty (F(z_{1/t}) - a_0) t^{1-s} dt + \int_1^\infty (F(\gamma z_t) - a_0) t^{s-1} dt
\end{aligned}$$

Hence

$$\begin{aligned}
H(s, \tfrac{a}{c}) &= \Lambda(s, \tfrac{a}{c}) + \int_1^\infty r(\gamma, -d/c + it/c) t^{1-s} dt + \frac{a_0}{s} - \frac{a_0}{2-s} \\
&= -\int_1^\infty (F(z_{1/t}) - a_0) t^{1-s} dt + \int_1^\infty (F(\gamma z_t) - a_0) t^{s-1} dt \qquad (2.4)
\end{aligned}$$

Both integrals in (2.4) converge for all $s \in \mathbb{C}$ due to the exponential decay of the integrands proving the analytic continuation of $H(s, \frac{a}{c})$ to the whole complex plane. The functional equation $H(s, a/c) = H(2-s, -d/c)$ also follows easily from (2.4) since $z_{1/t} = \frac{-d}{c} + \frac{it}{c}$ and $\gamma z_t = \frac{a}{c} + \frac{it}{c}$.

We next take the limit $s \to 1$ to get

$$\begin{aligned}
H(1, \tfrac{a}{c}) &= -\frac{c}{i} \int_{z_1}^{i\infty} (F(z) - a_0) dz + \frac{c}{i} \int_{\gamma z_1}^{i\infty} (F(z) - a_0) dz \\
&= -\frac{c}{i} \left(G(\gamma z_1) - G(z_1) - a_0 \left(\frac{a+d}{c} \right) \right)
\end{aligned}$$

where $G(z) = a_0 z + \sum_{n\geq 1} \frac{a_n}{2\pi i n} q^n$. Since $G'(z) = F(z)$,

$$G(\gamma z) - G(z) = \int_{z_1}^{z} r(\gamma, w)dw + \Phi(\gamma)$$

with $\Phi(\gamma) = (G(\gamma z_1) - G(z_1))$.

Hence

$$R(\gamma, z) = \int_{z_1}^{z} r(\gamma, w)dw + (G(\gamma z_1) - G(z_1)) = G(\gamma z) - G(z)$$

is a cocycle being the boundary of a function G. This finishes the proof of the theorem since clearly $R'(\gamma, z) = r(\gamma, z)$. □

As an immediate corollary of Theorem 2.1 we prove the limit formula (1.12) for the classical Dedekind sums defined as in (1.3).

Corollary 2.2. *Let $s(a, c)$ be the Dedekind sum and*

$$L(s, a/c) = \sum_{n\geq 1} \sigma(n) e(\tfrac{a}{c} n) n^{-s},$$

Then

$$s(a, c) = \tfrac{1}{2\pi i} \lim_{s\to 1} \left[L(s, \tfrac{a}{c}) + \tfrac{1}{2s-2} \right].$$

Proof. We apply Theorem 2.1 in the case of Eisenstein series $F(z) = E_2(z) = 1 - 24 \sum \sigma(n) q^n$ and its cocycle $r(\gamma, z) = \frac{6}{\pi i} \frac{c}{cz+d}$, so that $L(F, s, a/c) = -24 L(s, a/c)$. For simplicity assume $c > 0$. As a primitive of $r(\gamma, z)$ we choose $\frac{6}{\pi i} \log\left(\frac{cz+d}{i}\right)$. Using (2.2) and (2.3) we have

$$R(\gamma, z) = \lim_{s\to 1} \left[-\frac{24}{2\pi i} L(s, a/c) - \frac{6}{\pi i} \frac{1}{s-1} \right] + \frac{6}{\pi i} \int_{-d/c+i/c}^{z} \frac{c}{cw+d} dw + \left(\frac{a+d}{c} \right) \tag{2.5}$$

$$= \frac{12}{2\pi i} \log \frac{cz+d}{i} + \Phi(\gamma) \tag{2.6}$$

where

$$\Phi(\gamma) = \lim_{s\to 1} \left[-\frac{12}{\pi i} L(s, a/c) - \frac{6}{\pi i} \frac{1}{s-1} \right] + \left(\frac{a+d}{c} \right)$$

The limit formula (1.12) now follows from Dedekind's formula (1.2) for $\Phi(\gamma)$. □

3. Weight 2 rational cocycles for the modular group

In this section we restrict ourselves to cocycles of weight 2 which are rational functions. The simplest example is $r(\gamma, z) = 12c/(cz + d)$ whose poles are in $\mathbb{Q}$. In the case $r(\gamma, z)$ is a rational cocycle whose poles are not rational it is known that $r(S, z)$ can be written as a finite linear combination of functions of the form

$$\sqrt{D} \sum_{AC<0} \frac{\operatorname{sign} A}{Az^2 + Bz + C} \tag{3.1}$$

where $Q(X, Y) = AX^2 + BXY + CY^2$ runs over quadratic forms in the class $\mathcal{C}$ (see [3, 10, 33]). Rational period functions were introduced by Knopp in the 1970s [27, 28] who showed using results from [26] that they have modular integrals. His construction arises from a meromorphic Poincaré series formed out of cocycles and is very difficult to compute (see also [16]). On the other hand in [14] and [15] certain explicit modular integrals were constructed whose Fourier coefficients are given by cycle integrals of weakly holomorphic forms. These functions are parametrized by classes of indefinite quadratic forms $\mathcal{C}$ and are given by the Fourier expansion

$$F_{\mathcal{C}}(z) = \sum_{m\geq 0} a_{\mathcal{C}}(m)e(mz). \tag{3.2}$$

with

$$a_{\mathcal{C}}(m) = \sqrt{D}\int_{z_0}^{\sigma z_0} j_m(z)\tfrac{dz}{Q(z)}. \tag{3.3}$$

Here j_m is the unique modular function whose Fourier expansion has the form $q^{-m}+O(q)$, Q is any quadratic form in the class $\mathcal{C}$, $\sigma = \sigma_Q$ is a distinguished generator of the group of automorphs of Q. The value of the integral is independent of the path and the point $z_0 \in \mathbb{H}$. In [14] it is shown that the function $F_{\mathcal{C}}$ arises from the cycle integral of the Green function $\frac{j'(z)}{j(z)-j(w)}$. The cycle integral of this Green function is modular but with jump singularities along the geodesic. $F_{\mathcal{C}}$ is then the analytic continuation from the connected component of the cusp. It is holomorphic, but no longer invariant.

The association $Q \mapsto \sigma_Q$ sets up a bijection between elements of the class $\mathcal{C}$ of the quadratic form Q and the conjugacy class of σ_Q, which by abuse of notation will also be denoted by $\mathcal{C}$. Since it is more convenient for us to express our results in terms of the hyperbolic conjugacy class, we briefly recall this correspondence. If $Q(X,Y) = AX^2 + BXY + CY^2$ has discriminant $D = B^2 - 4AC$, and t, u are the smallest positive solutions of Pell's equation $t^2 - Du^2 = 4$ then

$$\sigma_Q = \begin{pmatrix} \frac{t+Bu}{2} & Cu \\ -Au & \frac{t-Bu}{2} \end{pmatrix}.$$

Conversely if $\sigma = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathcal{C}$ is a primitive hyperbolic element and we set $Q_\sigma(z) = (c'X^2 + (d'-a')XY - b'Y^2)$, and $Q = \frac{-1}{u}Q_\sigma$ with $u = \gcd(c', d'-a', b')$ then $\sigma_Q = \sigma$. It follows that with $D' = (a'+d')^2 - 4$ we also have

$$a_{\mathcal{C}}(m) = \sqrt{D'}\int_{z_0}^{\sigma z_0} j_m(z)\tfrac{dz}{Q_\sigma(z)}$$

as in (1.14).

As in [14] one can show that $a_{\mathcal{C}}(m) \ll m^{5/4+\epsilon}$ for any $\epsilon > 0$ and $F_{\mathcal{C}}$ satisfies the transformation property

$$z^{-2}F_{\mathcal{C}}(-1/z) - F_{\mathcal{C}}(z) = \varepsilon_{\mathcal{C}} \sum_{w'_Q<0<w_Q} \frac{1}{z-w} - \frac{1}{z-w'}. \tag{3.4}$$

Note that the rational function on the right hand side above is the same as in (3.1).

Here for $Q \in \mathcal{C}$, $w'_Q < w_Q$ are the two roots of $Q(t,1) = 0$. If $\sigma = \sigma_Q$ then these are also the fixed points $w'_\sigma < w_\sigma$ of σ, and $\varepsilon_{\mathcal{C}}$ is defined as in (1.10).

If $\mathcal{C}$ denotes the class of Q or the class of the hyperbolic element σ_Q we let

$$\mathcal{W}_{\mathcal{C}} = \big\{(w'_Q, w_Q) : Q \in \mathcal{C}\big\} = \{(w'_\sigma, w_\sigma) : \sigma \in \mathcal{C}\} \tag{3.5}$$

the ordered pairs of roots of $Q \in \mathcal{C}$ or equivalently the fixed points of σ.

For a fixed $\gamma \in SL_2(\mathbb{Z})$, we let as in (1.9),

$$r_{\mathcal{C}}(\gamma, z) := \varepsilon_{\mathcal{C}} \sum \frac{1}{z-w} - \frac{1}{z-w'}$$

where the sum is over $(w', w) \in \mathcal{W}_{\mathcal{C}}$, satisfying $w' < -d/c < w$ if $c \neq 0$ and $r_{\mathcal{C}}(\gamma, z) \equiv 0$ otherwise.

Remark 3.1. *Although the set $\mathcal{W}_{\mathcal{C}}$ is infinite, the sum defining $r_{\mathcal{C}}(\gamma, z)$ is finite. To see this note that in the case that $-d/c$ is an integer the number of terms is the same as the number of quadratic forms $[A, B, C]$ for which $AC < 0$. Otherwise consider a form $[A, B, C]$ satisfying $\frac{-B-\sqrt{D}}{2A} < \frac{-d}{c} < \frac{-B+\sqrt{D}}{2A}$, then the form $[A, cB, c^2C]$ has discriminant c^2D and its roots are separated by $-d$, and integer.*

For later use we give another description of $r_{\mathcal{C}}(\gamma, z)$. For $\sigma \in \mathcal{C}$ a fixed hyperbolic element, let w_σ, w'_σ be its two fixed points, $\Gamma_\sigma = \{g \in \Gamma : g^{-1}\sigma g = \sigma\}$, and S_σ be the semicircle whose endpoints are w_σ and w'_σ. Let $\partial\mathbb{H} = \mathbb{R} \cup i\infty$ and $\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$.

For $z_1, z_2 \in \overline{\mathbb{H}}$ we denote the geodesic segment joining z_1 and z_2 by ℓ_{z_1,z_2}. Let

$$I_{\mathcal{C}}(z_1, z_2) = \{\alpha \in \Gamma/\Gamma_\sigma : \alpha S_\sigma \text{ intersects } \ell_{z_1,z_2}\}. \tag{3.6}$$

and let $|I_{\mathcal{C}}(z_1, z_2)|$ denote the cardinality of $I_{\mathcal{C}}(z_1, z_2)$.

Note that if we define the net of σ, $\mathcal{N}_\sigma$ as the preimage of the closed geodesic associated to σ in $\mathbb{H}$ so that

$$\mathcal{N}_\sigma := \bigcup_{g\in\Gamma} gS_\sigma = \bigcup_{g\in\Gamma} S_{g^{-1}\sigma g}, \tag{3.7}$$

then $|I_{\mathcal{C}}(\alpha, \beta)|$ counts the number of intersections of the geodesic segment $\ell_{\alpha,\beta}$ with the semicircles in $\mathcal{N}_\sigma$, the net of σ. Moreover $\mathcal{W}_{\mathcal{C}}$ is simply the set of end points of the geodesics in the net $\mathcal{N}_\sigma$.

With the above notation we also have

$$r_{\mathcal{C}}(\gamma, z) = \sum_{\alpha\in I_{\mathcal{C}}(-d/c, i\infty)} \operatorname{sign}(\alpha w_\sigma - \alpha w'_\sigma)\left(\frac{1}{z-\alpha w_\sigma} - \frac{1}{z-\alpha w'_\sigma}\right) \tag{3.8}$$

Theorem 3.2. *For any $\gamma, \sigma \in \Gamma$, with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$*

$$r_{\mathcal{C}}(\sigma\gamma, z) = r_{\mathcal{C}}(\sigma, \gamma z)(cz+d)^{-2} + r_{\mathcal{C}}(\gamma, z) \tag{3.9}$$

Proof. To ease the notation the dependence on $\mathcal{C}$, which is fixed, is suppressed. As usual let T and S denote the two generators of Γ corresponding to the translation $z \to z+1$ and the inversion $z \to -1/z$ respectively. First note that $r(T\gamma, z) = r(\gamma, z)$. Hence if we prove

$$r(S\gamma, z) = r(S, \gamma z)(cz+d)^{-2} + r(\gamma, z) \tag{3.10}$$

the proposition follows by induction on the word length expressing σ in terms of the generators S and T. Recall that for $z, w \in \mathbb{C}$ and $\gamma \in \Gamma$

$$\frac{w-w'}{(\gamma z - w)(\gamma z - w')}(cz+d)^{-2} = \frac{\gamma^{-1}w - \gamma^{-1}w'}{(z-\gamma^{-1}w)(z-\gamma^{-1}w')} \tag{3.11}$$

Since $S\gamma = \left(\begin{smallmatrix} -c & -d \\ a & b \end{smallmatrix}\right)$ to prove (3.10), using (3.11) we have to prove that

$$\sum_{w'<-b/a<w}\left(\frac{1}{z-w} - \frac{1}{z-w'}\right) - \sum_{w'<-d/c<w}\left(\frac{1}{z-w} - \frac{1}{z-w'}\right) = \sum_{w'<0<w}\left(\frac{1}{z-\gamma^{-1}w} - \frac{1}{z-\gamma^{-1}w'}\right) \tag{3.12}$$

all sums over pairs $(w', w) \in \mathcal{W}$.

Assume first that $ac > 0$ so that $-d/c < -b/a$. On the left hand side of (3.12) we have

$$\sum_{-d/c<w'<-b/a<w}\left(\frac{1}{z-w} - \frac{1}{z-w'}\right) - \sum_{w'<-d/c<w<-b/a}\left(\frac{1}{z-w} - \frac{1}{z-w'}\right) \tag{3.13}$$

On the other hand we can write for the right hand side of (3.12)

$$\sum_{w'<0<w}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right)=$$

$$\sum_{w'<0<w<a/c}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right)+\sum_{w'<0<a/c<w}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right) \tag{3.14}$$

Now note that

$$\gamma^{-1}z=-\frac{d}{c}-\frac{1}{c^2(z-a/c)}$$

and the function $x\to -\frac{d}{c}-\frac{1}{c^2(x-a/c)}$ is monotonic for $x\in(-\infty,a/c)$ and also for $x\in(a/c,\infty)$.

It follows that for $w'<0<w<a/c$,

$$-d/c<\gamma^{-1}w'<-b/a<\gamma^{-1}w \tag{3.15}$$

and similarly that for $w'<0<a/c<w$

$$\gamma^{-1}w<-d/c<\gamma^{-1}w'<-b/a. \tag{3.16}$$

Using (3.15) and (3.16) in (3.14) we get that

$$\begin{aligned}&\sum_{w'<0<w}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right) \\ &=\sum_{w'<0<w<a/c}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right)+\sum_{w'<0<a/c<w}\left(\frac{1}{z-\gamma^{-1}w}-\frac{1}{z-\gamma^{-1}w'}\right) \\ &\qquad=\sum_{-d/c<w'<-b/a<w}\left(\frac{1}{z-w}-\frac{1}{z-w'}\right)-\sum_{w'<-d/c<w<-b/a}\left(\frac{1}{z-w}-\frac{1}{z-w'}\right)\end{aligned} \tag{3.17}$$

This proves (3.12) when $ac>0$. The case $ac<0$ follows in the same manner. The case $ac=0$ can be checked easily since $c=0$ corresponds to $\gamma=T^m$ whereas $a=0$ rises from $\gamma=ST^m$.

This proves Theorem 3.2 and hence also Theorem 1 from the introduction. □

Extending our earlier work we show that

Theorem 3.3. *For any hyperbolic conjugacy class $\mathcal{C}$ the function $F_{\mathcal{C}}(z)$ is holomorphic on $\mathbb{H}$ and satisfies*

$$(cz+d)^{-2}F_{\mathcal{C}}(\gamma z)=F_{\mathcal{C}}(z)+r_{\mathcal{C}}(\gamma,z) \tag{3.18}$$

Proof. The claim is trivial for T and has been established for the generator $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ in [15]. It is possible to give a proof of the general case along the lines of the proof of (3.4) given in [15]. However the algebraic proof above already established that the rational function $r_{\mathcal{C}}(\gamma,z)$ defined in (1.9) is a weight 2 cocycle. Since it agrees with the cocycle associated to $F_{\mathcal{C}}(z)$ for the generators $\gamma=S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ the difference is a 1-cocycle that vanishes on both S and T and so must vanish identically. □

4. The Dirichlet Series associated to $F_{\mathcal{C}}(z)$

Guided by the example of the Eisenstein series $E_2(z)$ and its primitive $\log\Delta(z)$, it is natural to study a primitive of a general modular integral, and the associated weight zero cocycle that appears in its transformation property.

We look at this problem in the case of the function $F_{\mathcal{C}}(z)$ and determine the unique weight 0 primitive $R_{\mathcal{C}}(\gamma, z)$ of the cocycles $r_{\mathcal{C}}(\gamma, z)$ in terms of the special values of the Dirichlet series $L(F_{\mathcal{C}}, s, a/c)$.

The next theorem and its corollary, which are based on Theorem 2.1, proves Theorem 4 from the introduction.

Theorem 4.1. *Let $F_{\mathcal{C}}(z)$ be the modular integral in (3.2) and $L_{\mathcal{C}}(s, a/c) := L(F_{\mathcal{C}}, s, a/c)$ be its associated Dirichlet series. Then $L_{\mathcal{C}}(s, a/c)$ converges for $\mathrm{Re}(s) > 9/4$, has a meromorphic continuation to $s > 0$ and is holomorphic at $s = 1$. Moreover if $R_{\mathcal{C}}(\gamma, z)$ is the unique weight 0 cocycle such that $R'_{\mathcal{C}}(\gamma, z) = r_{\mathcal{C}}(\gamma, z)$ then*

$$R_{\mathcal{C}}(\gamma, z) = \varepsilon_{\mathcal{C}} \sum_{w < \frac{-d}{c} < w'} \log(z - w) - \log(z - w') + \tfrac{1}{2\pi i} L_{\mathcal{C}}(1, a/c) + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right) \tag{4.1}$$

Proof. The convergence of $L_{\mathcal{C}}(s, a/c)$ for $\mathrm{Re}(s) > 9/4$ follows from the bound $a_{\mathcal{C}}(m) \ll m^{5/4+\epsilon}$ which was proved in Proposition 6 of [14].

To prove (4.1), in Theorem 2.1 we let $r(\gamma, z) = r_{\mathcal{C}}(\gamma, z) = \varepsilon_{\mathcal{C}} \sum_{w' < -d/c < w} \frac{1}{z-w} - \frac{1}{z-w'}$. As a primitive of $r_{\mathcal{C}}(\gamma, z)$ we choose

$$\varepsilon_{\mathcal{C}} \sum_{w' < -d/c < w} \log(z - w) - \log(z - w').$$

Once again using (2.2) and (2.3) we have

$$\begin{aligned} R_{\mathcal{C}}(\gamma, z) =& \frac{-i}{c} \lim_{s \to 1} \left[\left(\frac{2\pi}{c}\right)^{-s} \Gamma(s) L_{\mathcal{C}}(s, a/c) + \int_1^\infty r_{\mathcal{C}}(\gamma, -d/c + it/c) t^{1-s} dt \right] \\ &+ \int_{z_1}^{z} r_{\mathcal{C}}(\gamma, w) dw + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right) \end{aligned} \tag{4.2}$$

where $z_1 = -d/c + i/c$.

Contrary to the case of E_2, the Dirichlet series $L_{\mathcal{C}}(s, a/c)$ has no pole at $s = 1$. This is due to the fact that at $s = 1$ the first integral in (4.2) has the finite value

$$\varepsilon_{\mathcal{C}} \sum_{w' < -d/c < w} \log(z_1 - w) - \log(z_1 - w').$$

To finish the proof of Theorem 4.1 we combine the two integrals in (4.2) to get

$$\begin{aligned} R_{\mathcal{C}}(\gamma, z) =& \frac{1}{2\pi i} L_{\mathcal{C}}(1, a/c) + \int_\infty^z r_{\mathcal{C}}(\gamma, z) dw + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right) \\ =& \frac{1}{2\pi i} L_{\mathcal{C}}(1, a/c) + \varepsilon_{\mathcal{C}} \sum_{w' < -d/c < w} \log(z - w) - \log(z - w') + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right). \end{aligned} \tag{4.3}$$

□

Since $a_{\mathcal{C}}(0) = \log\lambda$ is real, the following corollary easily follows from (4.1)

Corollary 4.2. *Let* $\Phi_{\mathcal{C}}(\gamma) = \frac{2}{\pi}\lim_{y\to\infty} \operatorname{Im} R_{\mathcal{C}}(\gamma, iy)$. *Then*

$$\Phi_{\mathcal{C}}(\gamma) = -\tfrac{1}{\pi^2} \operatorname{Re} L_{\mathcal{C}}(1, a/c).$$

In the rest of the section we will give two applications of Theorem 4.1 and the cocycle relation

$$R_{\mathcal{C}}(\sigma\gamma, z) = R_{\mathcal{C}}(\sigma, \gamma z) + R_{\mathcal{C}}(\gamma, z).$$

The first one is an analog of the Dedekind's reciprocity formula (1.5) for the Dirichlet series $L_{\mathcal{C}}(1, a/c)$. More precisely we have

Theorem 4.3. *For* $z_i \in \mathbb{C} \cup \{\infty\}$, *let* $[z_1, z_2, z_3, z_4] = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}$ *denote the cross ratio. We assume that* $(a, c) = 1$ *and* $ac \neq 0$. *Then*

$$\frac{1}{i\pi}\left[L_{\mathcal{C}}(1, a/c) - L_{\mathcal{C}}(1, -c/a)\right] = -2\left(\tfrac{a^2+c^2+1}{ac}\right)\log\lambda + \varepsilon_{\mathcal{C}} \sum_{w'<0<w} \log[\tfrac{a}{c}, w, w', -\tfrac{c}{a}] \tag{4.4}$$

Here we interpret the imaginary part of the logarithm of a negative real number to be π.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. From (4.3) we have

$$R_{\mathcal{C}}(\gamma, z) = \frac{1}{2\pi i} L_{\mathcal{C}}(1, a/c) + \int_{i\infty}^{z} r_{\mathcal{C}}(\gamma, w)dw + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right)$$

Since $R_{\mathcal{C}}(\gamma, z)$ is a cocycle it satisfies

$$R_{\mathcal{C}}(S\gamma, z) = R_{\mathcal{C}}(S, \gamma z) + R_{\mathcal{C}}(\gamma, z). \tag{4.5}$$

Hence

$$\begin{aligned} R_{\mathcal{C}}(S\gamma, z) =& \frac{1}{2\pi i} L_{\mathcal{C}}(1, -c/a) + \int_{i\infty}^{z} r_{\mathcal{C}}(S\gamma, w)dw + a_{\mathcal{C}}(0)\left(\frac{b-c}{a}\right) \\ =& \frac{1}{2\pi i} L_{\mathcal{C}}(1, 0) + \int_{i\infty}^{\gamma z} r_{\mathcal{C}}(S, w)dw \\ &+ \frac{1}{2\pi i} L_{\mathcal{C}}(1, a/c) + \int_{i\infty}^{z} r_{\mathcal{C}}(\gamma, w)dw + a_{\mathcal{C}}(0)\left(\frac{a+d}{c}\right) \end{aligned} \tag{4.6}$$

We let $z \to i\infty$ to get

$$\frac{1}{2\pi i}\left[L_{\mathcal{C}}(1, -c/a) - L_{\mathcal{C}}(1, a/c)\right] = a_{\mathcal{C}}(0)\left(\frac{a^2+c^2+1}{ac}\right) \tag{4.7}$$

$$+ \frac{1}{2\pi i} L_{\mathcal{C}}(1, 0) + \int_{i\infty}^{a/c} r_{\mathcal{C}}(S, w)dw \tag{4.8}$$

Hence

$$\begin{aligned} &\frac{1}{2\pi i}\left[L_{\mathcal{C}}(1, -c/a) - L_{\mathcal{C}}(1, a/c)\right] \\ &\quad = a_{\mathcal{C}}(0)\left(\frac{a^2+c^2+1}{ac}\right) + \frac{1}{2\pi i} L_{\mathcal{C}}(1, 0) \\ &\qquad + \varepsilon_{\mathcal{C}} \sum_{w'<0<w} \log(\frac{a}{c} - w) - \log(\frac{a}{c} - w') \end{aligned} \tag{4.9}$$

Now replacing the roles of $-c$ with a and a with c gives

$$\begin{aligned}(4.10)\quad \frac{1}{2\pi i}\left[L_{\mathcal{C}}(1,a/c)-L_{\mathcal{C}}(1,-a/c)\right] \\ = -a_{\mathcal{C}}(0)\left(\frac{a^2+c^2+1}{ac}\right)+\frac{1}{2\pi i}L_{\mathcal{C}}(1,0) \\ +\varepsilon_{\mathcal{C}}\sum_{w'<0<w}\log(\frac{-c}{a}-w)-\log(\frac{-c}{a}-w')\end{aligned}$$

Finally noting that $a_{\mathcal{C}}(0)=\log\lambda$ and taking the difference of the last two equations prove (4.4). ☐

As a second application we have the following geometric interpretation of the special value of the Dirichlet series $L_{\mathcal{C}}(s,a/c)$.

Theorem 4.4. *Let $L_{\mathcal{C}}(s,a/c)$ be the Dirichlet series associated to $F_{\mathcal{C}}(z)$. Then*

$$(4.11)\quad \frac{1}{2\pi i}\left[L_{\mathcal{C}}(1,a/c)+L_{\mathcal{C}}(1,-d/c)\right]$$

$$(4.12)\quad = -\varepsilon_{\mathcal{C}}\sum_{w'<\frac{-d}{c}<w}\log(\tfrac{-d}{c}-w)-\log(\tfrac{-d}{c}-w')$$

$$(4.13)\quad = -\,\varepsilon_{\mathcal{C}}\left(2\log\left|\prod_{w'<\frac{-d}{c}<w}\tan\left(\frac{\theta_w}{2}\right)\right|+i\pi\sum_{w'<\frac{-d}{c}<w}1\right)$$

where the sum and the product runs over elements $(w',w)\in\mathcal{W}_{\mathcal{C}}$ that are separated by $\frac{-d}{c}$. θ_w is the angle of intersection of the vertical line $(-d/c,-d/c+i\infty)$ with the semicircle with end points w' and w. Here θ_w is the angle containing the line segment connecting this intersection to w'.

Proof. Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. Using the cocycle relation $0=R_{\mathcal{C}}(\gamma,\gamma^{-1}z)+R_{\mathcal{C}}(\gamma^{-1},z)$, the formula (4.1) and taking the limit as $z\to i\infty$ leads to the first equality (4.12). Since $-d/c,w,w'$ all lie on the real axis, the argument of each logarithm term in the sum in (4.12) is π. Here we interpret the imaginary part of the logarithm of a negative real number to be π. This proves that the imaginary part of (4.12) is indeed given by $\pi\sum_{w'<\frac{-d}{c}<w}1$.

The fact that the real part (4.12) is given as in (4.13) follows easily using elementary geometry. (See also [6] p.116.)

☐

5. Intersection numbers

In this section we restrict ourselves to the imaginary part of $R_{\mathcal{C}}(\gamma,z)$. Recall from (4.2) that

$$\Phi_{\mathcal{C}}(\gamma)=\tfrac{2}{\pi}\lim_{y\to\infty}\operatorname{Im}R_{\mathcal{C}}(\gamma,iy)=-\tfrac{1}{\pi^2}\operatorname{Re}L_{\mathcal{C}}(1,a/c).$$

Our first goal is to prove that $\Phi_{\mathcal{C}}(\gamma)$ is an intersection number, hence an integer.

We start by noting that Theorem 4.4 gives

$$\Phi_{\mathcal{C}}(\gamma)+\Phi_{\mathcal{C}}(\gamma^{-1})=2\sum_{w'<\frac{-d}{c}<w}1$$

and hence as a simple corollary we have

Proposition 5.1. *Let $\mathcal{C}$ be the conjugacy class of a hyperbolic element σ, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ another hyperbolic element in Γ and $I_{\mathcal{C}}(\gamma^{-1}(i\infty), i\infty) = I_{\mathcal{C}}(-d/c, i\infty)$ be as defined in (3.6). Then*

$$\Phi_{\mathcal{C}}(\gamma) + \Phi_{\mathcal{C}}(\gamma^{-1}) = -2|I_{\mathcal{C}}(-d/c, i\infty)|$$

The next result shows that $\Phi_{\mathcal{C}}(\gamma)$ is already an integer.

Theorem 5.2. *Let $\gamma \in \Gamma$ be a hyperbolic element. Then $\Phi_{\mathcal{C}}(\gamma) = -|I_{\mathcal{C}}(-d/c, i\infty)|$ and hence $\Phi_{\mathcal{C}}(\gamma) \in \mathbb{Z}$.*

Proof. For $\gamma_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$, two not necessarily hyperbolic elements of Γ, let

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = \Phi_{\mathcal{C}}(\gamma_1\gamma_2) - \Phi_{\mathcal{C}}(\gamma_1) - \Phi_{\mathcal{C}}(\gamma_2).$$

Note that $I_{\mathcal{C}}(-d_1/c_1, i\infty) = I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty)$. We will show that

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = |I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty)| + |I_{\mathcal{C}}(\gamma_2^{-1} i\infty, i\infty)| - |I_{\mathcal{C}}((\gamma_1\gamma_2)^{-1} i\infty, i\infty)|. \tag{5.1}$$

This will prove the theorem since this then $\gamma \mapsto \Phi_{\mathcal{C}}(\gamma) + |I_{\mathcal{C}}(\gamma^{-1} i\infty, i\infty)|$ is a homomorphism of Γ into $\mathbb{C}$ and so is identically 0.

First note that if either γ_1 or γ_2 is T^n for some $n \in \mathbb{Z}$ then $\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = 0$ and the identity holds trivially. So we assume that γ_1, γ_2 are not parabolic.

To prove (5.1) note that from definition (1.11) of $\Phi_C(\gamma)$ and the cocycle property we have

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = \frac{2\varepsilon_{\mathcal{C}}}{\pi} \lim_{y\to\infty} \mathrm{Im}(R_{\mathcal{C}}(\gamma_1, \gamma_2 iy) - R_{\mathcal{C}}(\gamma_1, iy))$$

which by (4.3) equals

$$\frac{2\varepsilon_{\mathcal{C}}}{\pi} \lim_{y\to\infty} \left[\sum \arg\left(\frac{\gamma_2 iy - w}{\gamma_2 iy - w'}\right) - \sum \arg\left(\frac{iy - w}{iy - w'}\right)\right]$$

the sums are over $(w', w) \in \mathcal{W}_{\mathcal{C}}$, $w' < -d_1/c_1 < w$. The second sum in the limit clearly goes to zero. Since $\gamma_2 iy \to a_2/c_2$ when $y \to \infty$

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = 2\varepsilon_{\mathcal{C}} n(\gamma_1^{-1}, \gamma_2) \tag{5.2}$$

where $n(\gamma_1^{-1}, \gamma_2)$ is the number of $(w', w) \in \mathcal{W}_{\mathcal{C}}$, for which $w' < -d_1/c_1, a_2/c_2 < w$. By the definition (3.6) we have

$$\varepsilon_{\mathcal{C}} n(\gamma_1^{-1}, \gamma_2) = |I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty) \cap I_{\mathcal{C}}(\gamma_2 i\infty, i\infty)|$$

Any geodesic that does not go through the vertices of an ideal hyperbolic triangle intersects exactly two sides of the triangle if it intersects the triangle at all. Applying this fact to the ideal hyperbolic triangle with vertices $i\infty, a_2/c_2 = \gamma_2 i\infty$ and $-d_1/c_1 = \gamma_1^{-1} i\infty$ shows that the sets

$$I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty) \cap I_{\mathcal{C}}(\gamma_2 i\infty, i\infty),$$
$$I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty) \cap I_{\mathcal{C}}(\gamma_2 i\infty, \gamma_1^{-1} i\infty) \text{ and}$$
$$I_{\mathcal{C}}(\gamma_2 i\infty, \gamma_1^{-1} i\infty) \cap I_{\mathcal{C}}(\gamma_2 i\infty, i\infty)$$

are mutually disjoint. A standard inclusion exclusion argument then gives

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = |I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty)| + |I_{\mathcal{C}}(\gamma_2 i\infty, i\infty)| - |I_{\mathcal{C}}(\gamma_1^{-1} i\infty, \gamma_2 i\infty)|$$

Finally we use that $|I_{\mathcal{C}}(z_2, z_1)| = |I_{\mathcal{C}}(z_1, z_2)| = |I_{\mathcal{C}}(\gamma z_1, \gamma z_2)|$ for all $\gamma \in \Gamma$ to establish that

$$\delta_{\mathcal{C}}(\gamma_1, \gamma_2) = |I_{\mathcal{C}}(\gamma_1^{-1} i\infty, i\infty)| + |I_{\mathcal{C}}(\gamma_2^{-1} i\infty, i\infty)| - |I_{\mathcal{C}}(\gamma_2^{-1}\gamma_1^{-1} i\infty, i\infty)|. \tag{5.3}$$

□

Note that formula (5.2) for the co-boundary $\delta_{\mathcal{C}}$ of $\Phi_{\mathcal{C}}$ allows one to calculate $\Phi_{\mathcal{C}}(\gamma)$ successively by writing γ in terms of some set of generators of the group Γ. We give an alternative approach for establishing that $\Phi_{\mathcal{C}}$ takes integer values. This method does not identify $\Phi_{\mathcal{C}}$ geometrically but also gives a fast algorithm to compute it.

Note that since $L_{\mathcal{C}}(1, a/c)$ depends only on $a/c \bmod 1$, so does $\Phi_{\mathcal{C}}(\gamma)$ and hence for $c \neq 0$ we can write $\Phi_{\mathcal{C}}(a/c) = \Phi_{\mathcal{C}}(\gamma)$. The following is a simple corollary of Theorem 4.4 and Corollary 4.2.

Lemma 5.3. *Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then*

$$\Phi_{\mathcal{C}}(0) = \Phi_{\mathcal{C}}(S) = -\varepsilon_{\mathcal{C}} \sum_{w'<0<w} 1$$

The following theorem is an analogue of Dedekind's reciprocity formula. It allows for, via Euclid's algorithm, a quick computation of $\Phi_{\mathcal{C}}(\gamma)$.

Theorem 5.4. *Let $\mathcal{C}$ be a hyperbolic conjugacy class and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For $ac \neq 0$ we have*

$$\Phi_{\mathcal{C}}(a/c) = \Phi_{\mathcal{C}}(-c/a) + \frac{\epsilon_{\mathcal{C}}}{2} \sum_{w'<0<w} (1 - \operatorname{sign}[\tfrac{a}{c}, w, w', -\tfrac{c}{a}]) \tag{5.4}$$

Proof. The formula follows from Theorem 4.3 and Corollary 4.2. Note that our definition of the argument gives $\operatorname{Im} \log x = 0$ for a positive real number x, and $\operatorname{Im} \log x = \pi$ for a negative real number x. $\square$

Remark 5.5. *Note that*

$$\frac{1}{2} \sum_{w'<0<w} (1 - \operatorname{sign}[\tfrac{a}{c}, w, w', -\tfrac{c}{a}])$$

counts those $w' < 0 < w$ for which exactly one of $\{\frac{a}{c}, -\frac{c}{a}\}$ is in the open interval $(-w', w)$. Therefore once all the conjugates of $\sigma \in \mathcal{C}$ whose fixed points are separated by 0 are listed (an easy task, see Remark 3.1) the right hand side in the above theorem is an easily computable elementary function of $\frac{a}{c}$. This in turn allows a fast calculation of $\Phi_{\mathcal{C}}(a/c)$ in view of $\Phi_{\mathcal{C}}(\frac{a}{c}) = \Phi_{\mathcal{C}}(\frac{a+nc}{c})$ for any $n \in \mathbb{Z}$. Since $\Phi_{\mathcal{C}}(0)$ is an integer, it also establishes that $\Phi_{\mathcal{C}}(\frac{a}{c})$ is an integer.

We finish this section by collecting some results about the hyperbolic geometry that will be needed to prove Theorem 3 from the introduction, In particular it will be important for us to compare $|I_{\mathcal{C}_\sigma}(\gamma^{-1}z_0, z_0)|$ and $|I_{\mathcal{C}_\sigma}(\gamma^{-1}i\infty, i\infty)|$. We start with a simple lemma about hyperbolic quadrangles. Recall that for $z_1, z_2 \in \overline{\mathbb{H}}$ the geodesic segment connecting z_1 and z_2 is denoted by ℓ_{z_1,z_2}.

Lemma 5.6. *Let $z_1, z_2 \in \mathbb{H}$ and $x_1, x_2 \in \partial\mathbb{H}$. If ℓ is a geodesic that intersects neither the geodesic half line ℓ_{z_1,x_1} nor the geodesic half line ℓ_{z_2,x_2} then ℓ intersects either both ℓ_{x_1,x_2} and ℓ_{z_1,z_2} or it intersects neither of them.*

Proof. By applying a hyperbolic isometry if necessary we may assume that $\ell = \ell_{0,i\infty}$. The geodesic arc from z_1 to x_1 does not intersect $\ell = (0, i\infty)$, so x_1 and $\operatorname{Re}(z_1)$ have the same sign. Similarly the geodesic arc from z_2 to x_2 does not intersect $(0, i\infty)$, so x_2 and $\operatorname{Re}(z_2)$ have the same sign. Finally the arc from z_1 to z_2 intersects $(0, i\infty)$ if and only if their real parts have opposite signs. This proves that ℓ either intersects both the arc from z_1 to z_2 and the geodesic from x_1 to x_2 or that intersects neither of them.

$\square$

Proposition 5.7. *Let σ, γ be hyperbolic elements, and fix a point $z_0 \in S_\gamma$. Then*

$$\big|\, |I_{\mathcal{C}_\sigma}(\gamma^{-1}z_0, z_0)| - |I_{\mathcal{C}_\sigma}(\gamma^{-1}i\infty, i\infty)| \,\big| \le 2|I_{\mathcal{C}_\sigma}(z_0, i\infty)|. \tag{5.5}$$

Note that we do not assume γ to be primitive.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Consider the geodesic circular arc L_1 connecting $\gamma^{-1}z_0$ to $\gamma^{-1}i\infty = -d/c$ and the half-line L_2 connecting z_0 to $i\infty$. Assume that αS_σ intersects neither L_1 nor L_2. Then it follows from Lemma 5.6 that either αS_σ intersects both the arc from z_0 to $\gamma^{-1}z_0$ and the line from $-d/c$ to $i\infty$ or αS_σ intersects neither of them.

Hence we have shown that the symmetric difference of the sets $I_{\mathcal{C}}(-d/c, i\infty)$ and $I_{\mathcal{C}}(z_0, \gamma^{-1}z_0)$ is a subset of $I_{\mathcal{C}}(z_0, i\infty) \cup I_{\mathcal{C}}(-d/c, \gamma^{-1}z_0)$;

$$I_{\mathcal{C}}(-d/c, i\infty) \triangle I_{\mathcal{C}}(z_0, \gamma^{-1}z_0) \subset I_{\mathcal{C}}(z_0, i\infty) \cup I_{\mathcal{C}}(-d/c, \gamma^{-1}z_0,)$$

Since

$$\big|\, |I_{\mathcal{C}}(z_0, \gamma^{-1}z_0)| - |I_{\mathcal{C}}(-d/c, i\infty)| \,\big| \le |I_{\mathcal{C}}(-d/c, i\infty) \triangle I_{\mathcal{C}}(z_0, \gamma^{-1}z_0)|$$

and $I_{\mathcal{C}}(z_0, i\infty)$ and $I_{\mathcal{C}}(-d/c, \gamma^{-1}z_0)$ have the same cardinality $|I_{\mathcal{C}_\sigma}(z_0, i\infty)|$ this proves the proposition.

□

6. Linking numbers in $\Gamma \backslash SL_2(\mathbb{R})$

In this section we prove Theorem 3. This is based on results of the previous section and a theorem of Birkhoff [7].

If γ is a primitive hyperbolic element such that $\operatorname{tr}\gamma > 2$ there is an associated closed periodic orbit of the geodesic flow whose linking number with the trefoil is given by the Rademacher symbol (see [4], [5],[20]).

$$\Psi(\gamma) = \Phi(\gamma) - 3\operatorname{sign} c(a+d) = \lim_{n\to\infty} \frac{\Phi(\gamma^n)}{n}.$$

For the convenience of the reader we sketch Ghys' argument for the identification of the Rademacher symbol with the linking number with the trefoil in the Appendix.

Our goal in this section is to provide the background for a similar interpretation for the homogenization of $\Phi_{\mathcal{C}}(\gamma)$ of Theorem 3,

$$\Psi_{\mathcal{C}}(\gamma) := \lim_{n\to\infty} \frac{\Phi_{\mathcal{C}}(\gamma^n)}{n}$$

as a linking number.

As alluded above this is based on Theorem 6.3, originally due to Birkhoff, (cf. [7]) which relates this linking number to the geometry of the net $\mathcal{N}_\sigma$ of a primitive hyperbolic element $\sigma \in \mathcal{C}$. Birkhoff's theorem [7, Section 27] which proves the existence of a certain surface bounding symmetric curves which is a surface of section of the geodesic flow, is more general than what is needed for us. This theorem was popularized by Fried [17] who named them Birkhoff sections. The theorem holds in even more generality as shown in [1, 2, 22, 24]. As is clear from this rich history there are a number of proofs of this theorem especially for compact hyperbolic surfaces (see e.g. [8] and the references therein, also [12] and esp. section 3 of [13]). For the convenience of the reader we also give one which is self contained and very elementary; it is based on a simple computation of the sign of the triple product of three vectors in the Lie-algebra $\mathfrak{sl}_2(\mathbb{R})$, (Proposition 6.2). The relation to the invariant $\Psi_{\mathcal{C}}(\gamma)$ follows from a careful book-keeping of potential multiplicities (Lemmas 6.5, 6.6, and 6.7).

To make this explicit note that if $\gamma \in \Gamma$ has $\operatorname{tr}\gamma > 2$ and fixed points $w' < w$ then both γ and γ^{-1} are diagonalized by $M = \frac{1}{\sqrt{w-w'}}\left(\begin{smallmatrix} w & w' \\ 1 & 1 \end{smallmatrix}\right)$. By replacing γ with γ^{-1} if necessary we may assume that

$$\gamma M = M \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}$$

where $\varepsilon > 1$. When $a + d > 2$ this is equivalent to $\operatorname{sign} c > 0$. Both

$$\tilde{\gamma}_+(t) = M\phi(t) \text{ and } \tilde{\gamma}_-(t) = MS\phi(t)$$

are periodic orbits of the geodesic flow $g \mapsto g\phi(t)$ on $\Gamma\backslash SL_2(\mathbb{R})$. Here $\phi(t) = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$.

We now move on to interpret linking numbers combinatorially as intersection numbers. Let $[\tilde{\gamma}_+]$ and $[\tilde{\gamma}_-]$ be the homology class of the curves $t \mapsto M\phi(t)$, $t \in [0, \log\varepsilon]$ and $t \mapsto MS\phi(t)$, $t \in [0, \log\varepsilon]$, respectively. Note that $\tilde{\gamma}_+(t)i$, $t \in [0 \log\varepsilon]$ maps into a geodesic arc in $\mathbb{H}$ connecting Mi to γMi on the semicircle with endpoints w and w'. On the quotient space $\Gamma\backslash\mathbb{H}$ this is a closed geodesic, and $\tilde{\gamma}_-(t)i$ simply travels this closed geodesic backwards. The natural Seifert surface bounding $[\tilde{\gamma}_+]$ and $[\tilde{\gamma}_-]$ is just formed by the collection of unit tangent vectors rotating counterclockwise continuously through 180 degrees from the one orientation of the circle to the other. This is the geometric content of the following

Lemma 6.1. *$[\tilde{\gamma}_+] + [\tilde{\gamma}_-]$ is null-homologous in $\Gamma\backslash SL_2(\mathbb{R})$.*

Proof. In fact we even have that $M\phi(t)$ and $MS\phi(-t)$ are homotopic via

$$\begin{aligned} h: \quad [0, \log\varepsilon] \times [0, \pi/2] &\to G \\ (t, \theta) &\mapsto M\phi(t)k(\theta) \end{aligned}$$

where as usual

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

□

Note the image of h is an immersed sub-manifold X_γ in the quotient space $\Gamma\backslash SL_2(\mathbb{R})$. This follows readily from the fact that $\phi(t_1)k(\theta_1) = \phi(t_2)k(\theta_2)$, for $\theta_i \in [0, \pi/2]$ implies $t_1 = t_2$, $\theta_1 = \theta_2$ and so the image of h when viewed in $SL_2(\mathbb{R})$ is an embedded submanifold.

Now assume that $\mathcal{C}_\sigma$ and $\mathcal{C}_\gamma$ are two (different) primitive conjugacy classes. The above construction of the null-homologous chains associated to σ, γ have a well-defined linking number [19], [30] which we denote by $Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma)$. (This is well defined as the chains themselves depend only on the conjugacy class.) A geometric interpretation of this linking number between the trivial homology class $[\tilde{\gamma}_+] + [\tilde{\gamma}_-]$ and $[\tilde{\sigma}_+] + [\tilde{\sigma}_-]$ is given as the number of signed intersections of X_γ (the surface defined above by the homotopy map h) and $\tilde{\sigma}_+(s)$ and $\tilde{\sigma}_-(s)$, $s \in [0, \log\lambda]$, the closed orbits associated to σ. The geodesic flow has the interesting property that all intersections of X_γ and $\tilde{\sigma}_+$ have the same sign.

We fix the sign by fixing an orientation as follows. We think of $SL_2(\mathbb{R})$ as a subspace of the space of real 2×2 matrices. The tangent space at the identity is the set of 2×2 real matrices with trace 0 where we fix the basis (see [23] pg 27)

$$\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we say the orientation of three tangent vectors tangent to $SL_2(\mathbb{R})$ at g is positive, i.e. three matrices v_1, v_2, v_3 are positively oriented if $g^{-1}v_1, g^{-1}v_2, g^{-1}v_3$, are positively oriented at the identity. We then have the following proposition.

Proposition 6.2. *Let* $N = \frac{1}{\sqrt{w_\sigma - w'_\sigma}} \left(\begin{smallmatrix} w_\sigma & w'_\sigma \\ 1 & 1 \end{smallmatrix}\right)$, *where* w_σ, w'_σ *are the two fixed points of* σ. *Assume that the trajectory* $N\phi(s)$ *is disjoint from* $[\tilde{\gamma}_+] + [\tilde{\gamma}_-]$ *and intersects* $X_{\tilde{\gamma}}$ *at a point* g. *Then the sign of the intersection is negative.*

Proof. Let

$$g = M\phi(t)k(\theta) = N\phi(s).$$

To compute the sign of the intersection we have to compute the determinant of the coefficient matrix of the tangent vectors

$$g^{-1}M\phi'(t)k(\theta), \quad g^{-1}M\phi(t)\kappa'(\theta) \text{ and } g^{-1}N\phi'(s).$$

Since $\phi'(t) = \phi(t)\mathbf{h}$ and $\kappa'(\theta) = \kappa(\theta)(\mathbf{y} - \mathbf{x})$ we have

$$g^{-1}M\phi'(t)k(\theta) = k(-\theta)\mathbf{h}k(\theta),$$
$$g^{-1}M\phi(t)k'(\theta) = (\mathbf{y} - \mathbf{x}),$$

and

$$g^{-1}N\phi'(s) = \mathbf{h}.$$

Since $k(-\theta)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)k(\theta) = \left(\begin{smallmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{smallmatrix}\right) = -\sin 2\theta\mathbf{x} - \sin 2\theta\mathbf{y} + \cos 2\theta\mathbf{h}$ the value of the determinant we need to compute is $-2\sin 2\theta$, always negative since $\theta \in (0, \pi/2)$. □

An immediate consequence of Proposition 6.2 is the following theorem.

Theorem 6.3. *Let* $M = \frac{1}{\sqrt{w_\gamma - w'_\gamma}} \begin{pmatrix} w_\gamma & w'_\gamma \\ 1 & 1 \end{pmatrix}$, $N = \frac{1}{\sqrt{w_\sigma - w'_\sigma}} \left(\begin{smallmatrix} w_\sigma & w'_\sigma \\ 1 & 1 \end{smallmatrix}\right)$, *with* $\{w_\gamma, w'_\gamma\}$ *and* $\{w_\sigma, w'_\sigma\}$, *the fixed points of* γ *and* σ *respectively so that*

$$\gamma M = M \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}, \quad \sigma N = N \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

and let

$$A = \{(s, t, \theta) \in [0, \log\lambda) \times [0, \log\varepsilon) \times [0, \pi/2) : \exists\alpha \in \Gamma, M\phi(t)k(\theta) = \alpha N\phi(s)\} \tag{6.1}$$

and

$$B = \{(s, t, \theta) \in [0, \log\lambda) \times [0, \log\varepsilon) \times [0, \pi/2) : \exists\alpha \in \Gamma, M\phi(t)k(\theta) = \alpha NS\phi(s)\}. \tag{6.2}$$

For the linking number we have

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = -|A| - |B|.$$

Proof. By definition each point in the set A corresponds to an intersection of the surface X_γ with the curve $[\tilde{\sigma}^+]$ and similarly points in B correspond to intersections of X_γ with the curve $[\tilde{\sigma}^-]$. Hence for the linking number, using Proposition 6.2, we have

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = -|A| - |B|$$

which proves Theorem 6.3. □

Note that it is natural to interpret (see for example [12]) the elements of A as values $\{(s, t) \in [0, \log\lambda) \times [0, \log\varepsilon) : M\phi(t)i = N\phi(s)i \in \Gamma\backslash\mathbb{H}\}$, i.e. the number of intersections of the closed geodesics in $\Gamma\backslash\mathbb{H}$ associated to γ, σ, and similarly for B, since each time the underlying path of σ in H crosses the underlying curve of γ, precisely one of its two lifts will intersect the Seifert surface. The proper interpretation of this geometric idea requires care due to both multiplicities arising from self-intersections and the presence of elliptic elements in $\Gamma = SL_2(\mathbb{Z})$. To avoid these complications we go directly to $|I_{\mathcal{C}}(z_0, \gamma z_0)|$ which counts the group elements in $I_{\mathcal{C}}(z_0, \gamma z_0)$. In this notation Birkhoff's theorem takes the following form:

Theorem 6.4 (Birkhoff)**.** *If we let $z_0 = Mi \in S_\gamma$ then*

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = -|I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)|$$

The theorem will follow from a series lemmas relating $|A| + |B|$ to $|I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)|$.

Lemma 6.5. *For A, B as in* (6.1), (6.2) *we have $A \cap B = \emptyset$.*

Proof. Recall that each point in A, (resp in B) corresponds to an intersection of X_γ with the curve $\tilde{\sigma}_+$ (resp $\tilde{\sigma}_-$).

Assume that $(s, t, \theta) \in A \cap B$ with $M\phi(t)k(\theta) = \alpha N\phi(s)$ and $M\phi(t)k(\theta) = \beta NS\phi(s)$ for some $\alpha, \beta \in \Gamma$. It follows that $\beta^{-1}\alpha = NSN^{-1}$. Recall that $N = \frac{1}{\sqrt{w_\sigma - w'_\sigma}}\left(\begin{smallmatrix} w_\sigma & w'_\sigma \\ 1 & 1 \end{smallmatrix}\right)$, where w_σ, w'_σ are the two fixed points of σ. Now a simple matrix multiplication shows that the matrix NSN^{-1} cannot have integer entries, contradicting $\beta^{-1}\alpha \in \mathrm{SL}(2, \mathbb{Z})$. Hence $A \cap B = \emptyset$. □

Lemma 6.6. *There is a bijection between B in* (6.2) *and*

$$B' = \{(s, t, \theta) \in [0, \log\lambda) \times [0, \log\varepsilon) \times [\pi/2, \pi) : \exists\alpha \in \Gamma, M\phi(t)k(\theta) = \alpha N\phi(s)\}$$

given by for $s \neq 0$

$$(s, t, \theta) \mapsto (\log\lambda - s, t, \theta + \pi/2).$$

and for $s = 0$ by

$$(0, t, \theta) \mapsto (0, t, \theta + \pi/2)$$

Proof. Assume $(s, t, \theta) \in B$. The case $s = 0$ is trivial and otherwise $\exists\alpha \in \Gamma$ such that

$$M\phi(t)k(\theta) = \alpha NS\phi(s).$$

Since $\sigma N = N\phi(\log\lambda)$

$$M\phi(t)k(\theta) = \alpha\sigma^{-1}N\phi(\log\lambda - s)S.$$

This gives the claim since $S^{-1} = -k(\pi/2)$. □

Lemma 6.7. *There is a bijection between the set $A \cup B'$ and $I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)$ and hence $|A \cup B'| = |I_{\mathcal{C}}(z_0, \gamma z_0)|$.*

Proof. We define a map

$$f : A \cup B' \to \Gamma/\Gamma_\sigma \tag{6.3}$$

$$(s, t, \theta) \mapsto \alpha\Gamma_\sigma. \tag{6.4}$$

Here α is the unique element in Γ given by

$$M\phi(t)k(\theta)\phi(-s)N^{-1} = \alpha. \tag{6.5}$$

To see that f is injective let $f(s, t, \theta) = f(s', t', \theta')$ with $M\phi(t)k(\theta)\phi(-s)N^{-1} = \alpha$ and $M\phi(t')k(\theta')\phi(-s')N^{-1} = \beta$. Then $\alpha\sigma^k = \beta$ for some $k \in \mathbb{Z}$. Hence

$$\phi(t)k(\theta)\phi(-s)N^{-1}\sigma^k N = \phi(t')k(\theta')\phi(-s').$$

Since $N^{-1}\sigma^k N = \phi(k\log\lambda)$ we have

$$\phi(t - t')k(\theta)\phi(k\log\lambda - s + s') = k(\theta').$$

Now a simple matrix multiplication shows that this equality holds only if $(s,t,\theta) = (s',t',\theta')$, proving the injectivity of f.

To show that $f(s,t,\theta) \in I_{\mathcal{C}}(z_0, \gamma z_0)$, let $(s,t,\theta), \alpha$ be such that

$$M\phi(t)k(\theta) = \alpha N\phi(s).$$

Now $M\phi(t)i$ is in $\mathcal{A}_\gamma$, the geodesic arc connecting $z_0 = Mi$ and γz_0 where as $N\phi(s)i$ is in S_σ and hence $\alpha\Gamma_\sigma \in I_{\mathcal{C}}(z_0, \gamma z_0)$.

Finally to see that this map is onto $I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)$, let α be such that $\alpha\Gamma_\sigma \in I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)$ so that there is $\tau \in S_\sigma$ for which $\alpha\tau \in A_\gamma$, and so $\alpha\tau = M\phi(t)i$ for some $t \in [0, \log\varepsilon)$, and also $\tau = \sigma^k N\phi(s)i$ for some $s \in [0, \log\lambda)$. Since the stabilizer of i in $SL_2(R)$ is $SO(2)$, there exists $\theta \in [0, 2\pi)$, such that

$$M\phi(t)k(\theta) = \alpha\sigma^k N\phi(s).$$

Replacing α by $-\alpha$ if necessary we may assume that $\theta \in [0,\pi)$ proving surjectivity.

□

Proof of the Theorem 6.4.

By Birkhoff's theorem for the linking number we have

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = -|A| - |B|.$$

By Lemma 6.5 , $A \cap B = \emptyset$ and we have $Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = -|A \cup B|$. Finally by Lemma 6.6 and Lemma 6.7, $|A \cup B| = |I_{\mathcal{C}_\sigma}(z_0, \gamma z_0)|$.

This finishes the proof of the Theorem 6.4. □

We are now ready to prove

Theorem 6.8. *Let $\mathcal{C}_\sigma$ and $\mathcal{C}_\gamma$ be different primitive conjugacy classes. Then*

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = \Psi_{\mathcal{C}_\sigma}(\gamma)$$

Proof. By Theorem 6.4 we have

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_{\gamma^n}) = -|I_{\mathcal{C}}(z_0, \gamma^n z_0)|$$

and by Theorem 5.2

$$\Phi_{\mathcal{C}}(\gamma^n) = -|I_{\mathcal{C}}(\gamma^{-n} i\infty, i\infty)|$$

Clearly $I_{\mathcal{C}}(z_0, \gamma^{-n} z_0) = I_{\mathcal{C}}(z_0, \gamma^n z_0)$ and hence

$$|nLk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) - \Phi_{\mathcal{C}_\sigma}(\gamma^n)| = |\,|I_{\mathcal{C}}(z_0, \gamma^{-n} z_0)| - |I_{\mathcal{C}}(\gamma^{-n} i\infty, i\infty)|\,|$$

Now using Proposition 5.7 we have

$$|Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) - \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n}| \leq \frac{2|I_{\mathcal{C}}(z_0, i\infty)|}{n}$$

Since $|I_{\mathcal{C}}(z_0, i\infty)|$ is independent of n this proves

$$Lk(\mathcal{C}_\sigma, \mathcal{C}_\gamma) = \lim_{n\to\infty} \frac{\Phi_{\mathcal{C}_\sigma}(\gamma^n)}{n} = \Psi_{\mathcal{C}_\sigma}(\gamma).$$

□

Appendix A. Ghys' theorem

We sketch Ghys' argument for the identification of the Rademacher symbol with the linking number. Let
$$\Delta(z) = q\prod_{m\geq 1}(1-q^m)^{24}$$
and define $\tilde{\Delta} : SL_2(\mathbb{R}) \to \mathbb{C}$ by
$$\tilde{\Delta}(g) = \Delta(gi)j_{12}(g,i)$$
where for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(R)$
$$j_{12}(g,z) = (cz+d)^{-12}.$$
Similar lifts $\tilde{E}_4, \tilde{E}_6$ of the classical Eisenstein series E_4 and E_6 give an embedding of $\Gamma\backslash SL_2(\mathbb{R})$ into $\mathbb{C}^2$. The 3-manifold $\{(\tilde{E}_4(g), \tilde{E}_6(g) : g \in SL_2(\mathbb{R})\}$ is disjoint from the hypersurface $\mathcal{V} = \{(z,w) : z^3 = w^2\}$ and is easily seen to be homeomorphic to the complement of $\mathcal{V} \cap S^3$, the trefoil knot, in S^3. Let $\gamma \in SL_2(\mathbb{R})$ be hyperbolic, with $\operatorname{tr}\gamma > 2$. We are looking for the linking number of the closed periodic orbit $\tilde{\gamma}_+$ with the trefoil (after the above identification). Since $\tilde{E}_4{}^3 - \tilde{E}_6{}^2 = \tilde{\Delta}$, a general topological argument shows that this linking number is the same as the winding number of $\tilde{\Delta}(\tilde{\gamma}_+(t))$ around 0. This in turn can be computed as follows
$$2\pi i \operatorname{ind}(\tilde{\Delta}(\tilde{\gamma}_+(t)),0) = \int_{\tilde{\gamma}_+}\frac{d\tilde{\Delta}}{\tilde{\Delta}} = \int_{\tilde{\gamma}_+}\frac{d\Delta}{\Delta} + \int_{\tilde{\gamma}_+}\frac{dj_{12}}{j_{12}}.$$

The first integral can be evaluated from the transformation formula of $\log\Delta$ from $\tilde{\gamma}_+(0)i = Mi = z_0$ to $\tilde{\gamma}_+(\log\varepsilon)i = \gamma z_0$
$$\log\Delta(\gamma z_0) - \log\Delta(z_0) = 12\log\left(\frac{cz_0+d}{i\operatorname{sign} c}\right) + 2\pi i\Phi(\gamma)$$
with $\Phi(\gamma)$ as in (1.2). (See [35] equation (60) on page 49.)

Similarly the value of the second integral is $12\log(cz_0+d)$ and the linking number of the closed orbit of a hyperbolic γ is given by
$$\frac{6}{\pi i}\left(\log\left(\frac{cz_0+d}{i\operatorname{sign} c}\right) - \log(cz_0+d)\right) + \Phi(\gamma)$$
Finally for $\operatorname{Im} z_0 > 0$
$$\frac{6}{\pi i}\left(\log\left(\frac{cz_0+d}{i\operatorname{sign} c}\right) - \log(cz_0+d)\right) = -3\operatorname{sign} c$$
leading to Ghys' theorem.

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UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
E-mail address: wdduke@ucla.edu

ETH, Mathematics Dept., CH-8092, Zürich, Switzerland
E-mail address: ozlem@math.ethz.ch

Eotvos Lorand University, Department of Analysis, Budapest, Hungary
E-mail address: toth@cs.elte.hu