KRONECKER'S FIRST LIMIT FORMULA, REVISITED

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To Don Zagier, with admiration

Abstract. We give some new applications of Kronecker's first limit formula to real quadratic fields. In particular, we give a surprising geometrical relationship between the CM points associated to two imaginary quadratic fields with discriminants d and d' and certain winding number functions coming from the closed geodesics associated to the real quadratic field of discriminant $d'd$.

1. Introduction

The Kronecker limit formulas have been studied intensively and have inspired several generations of mathematicians since Kronecker's paper [30] of 1863. Weil devoted a book [52] to their historical development. Siegel's lucid treatment [44] makes their significance in number theory, especially in the study of L-functions, apparent. They directly influenced works by Weber [50], Lerch [34], Landau [32], Hecke [21], Herglotz [25], Chowla and Selberg [7], Ramachandra [37], Siegel [45], Stark [47], Ray and Singer [38], Zagier [54] and Shintani [42], to list chronologically a dozen prominent ones published at least forty years ago. The last forty years has mostly seen a (substantial) development of various generalizations of the classical limit formulas and the application of these generalizations in number theory, geometry and physics.

In this paper we confine our attention almost exclusively to results closely connected to Kronecker's classical *first* limit formula (KLF for short). After stating KLF, we will illustrate its application in some very special (and pretty) cases. Then we will sketch its proof and recall its relation to certain L functions for quadratic fields. This build-up is to motivate and place into context some new results we will present in the real quadratic case. In a sense this paper is a companion piece to [15], where other new results on geometric invariants associated to real quadratic fields can be found.

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2. The first limit formula

Kronecker's first limit formula is a two dimensional version of a familiar one for the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at $s = 1$:

$$\lim_{s\to 1^+}\left(\zeta(s) - \tfrac{1}{s-1}\right) = \gamma_0, \tag{1}$$

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where $\gamma_0 = 0.577216\ldots$ is Euler's constant. Let

$$Q(x,y) = ax^2 + bxy + cy^2$$

be a real binary quadratic form with discriminant $D = b^2 - 4ac$. In case Q is positive definite, that is $a > 0$ and $D < 0$, Kronecker obtained a limit formula analogous to (1) for the zeta function

$$\zeta_Q(s) = \sum_{m,n\in\mathbb{Z}}{}' Q(m,n)^{-s}. \tag{2}$$

Here the prime indicates that $(0,0)$ is omitted from the sum. Since Q is positive definite, the sum defining $\zeta_Q(s)$ converges absolutely for $\mathrm{Re}\, s > 1$, yet it blows up at $s = 1$, as does the Riemann zeta function. Kronecker's limit formula has the same shape as (1), except that on the right hand side we have a term that depends on Q in a non-trivial way. This dependence is necessarily modular in that $\zeta_Q(s) = \zeta_{\gamma Q}(s)$ for $\gamma \in \Gamma = \mathrm{PSL}(2,\mathbb{Z})$ with γQ arising from Q by a unimodular change of variables. When the form Q is represented by

$$z_Q = \frac{-b+\sqrt{D}}{2a} \in \mathcal{H},$$

$\mathcal{H}$ the upper half-plane, it can be checked that $z_{\gamma^{-1}Q} = \gamma z_Q$ where $\gamma \in \Gamma$ acts as usual on $z \in \mathcal{H}$ as a linear fractional map. The modular function that appears is

$$H(z) = -\log(y|\eta(z)|^4) \tag{3}$$

(H *for height*) where, for $q = e(z) = \exp(2\pi i z)$,

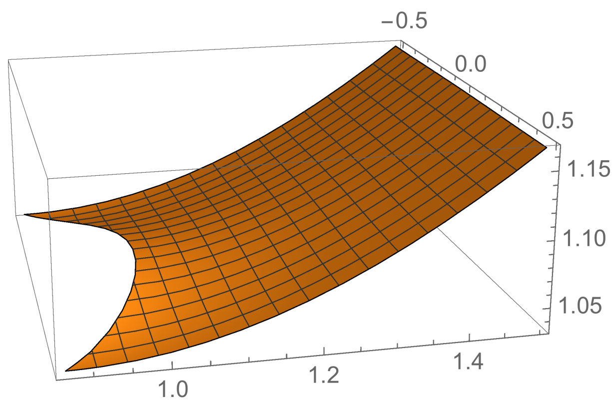
$$\eta(z) = q^{1/24} \prod_{m\geq 1}(1-q^m)$$

is the Dedekind eta function. This H is a counterpart to the usual modular j-function

$$j(z) = q^{-1} + 744 + 196844q + \cdots .$$

Although it is non-holomorphic, H is bi-harmonic with respect to the hyperbolic Laplacian.

FIGURE 1. The function $H(z)$



Theorem 1 (KLF)**.** *For $a > 0$ and $D < 0$ we have*

$$\lim_{s\to 1^+}\left(|D|^{\frac{s}{2}}\zeta_Q(s) - \tfrac{2\pi}{s-1}\right) = 2\pi\big(2\gamma_0 - \log 4 + H(z_Q)\big), \tag{4}$$

where γ_0 is Euler's constant.

Some examples

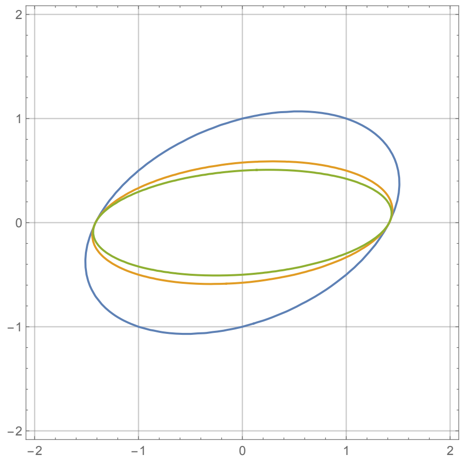
Before describing a proof, here are some attractive classical applications of KLF that do not require a lot of preparation to appreciate. All of these examples involve an interplay between $\zeta_Q(s)$ and Dirichlet L-functions.

Consider for a prime $q \equiv 3 \pmod 4$ the positive definite *integral* quadratic form

$$Q(x,y) = x^2 - xy + \tfrac{q+1}{4}y^2$$

whose discriminant is $D = -q$. For a positive integer n plot the ellipse in $\mathbb{R}^2$ determined by the equation $Q(x,y) = n$. It is a lovely fact that for some q we have a simple exact formula

Figure 2. Ellipses $x^2 - xy + \frac{q+1}{4}y^2 = n$ for $q = 7, 23, 31$ and $n = 2$



as a function of n for the number $r_q(n)$ of lattice points in $\mathbb{Z}^2$ that lie on the ellipse:

$$r_q(n) = r_q(1) \sum_{d|n} \left(\tfrac{d}{q}\right), \tag{5}$$

where $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol. Since the nineteenth century it has been known that formula (5) holds for all n when

$$q \in \{3, 7, 11, 19, 43, 67, 163\}.$$

Unfortunately, (5) does not work for all q. It obviously does for $n = 1$. For $n = 2$ the right hand side of formula (5) gives the value 4 when $q \equiv 7 \pmod 8$ and Figure 2 illustrates that this works when $q = 7$ since there are 4 lattice points on the outermost ellipse, which is given by $x^2 - xy + 2y^2 = 2$. However, it fails for any larger $q \equiv 7 \pmod 8$ as the resulting narrowing ellipses (illustrated for $q = 23$ and 31) cannot possibly contain any lattice points. It is instructive to try to extend this method to show that for any fixed $q \equiv 3 \pmod 8$ other than $q = 3, 11, 19, 43, 67, 163$, formula (5) must fail for some integer $n > 0$.

For fixed $q \in \{3, 7, 11, 19, 43, 67, 163\}$ we can translate (5) into the form

$$\zeta_Q(s) = r_q(1)\zeta(s)L_{-q}(s), \tag{6}$$

where for any fundamental discriminant D the Dirichlet L-function is defined by

$$L_D(s) = \sum_{n \geq 1} \chi_D(n) n^{-s},$$

with $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ the Kronecker symbol. Here we have applied quadratic reciprocity. Of course $r_q(1) = 2$ unless $q = 3$, when $r_3(1) = 6$. To avoid writing $r_q(1)$ we now leave out the case $q = 3$ and also, for the rest of this section, write $L(s)$ for $L_D(s)$, as long as the value of D is understood.

For these forms Q, KLF together with (1) and (6) implies both the evaluation $L(1) = \frac{\pi}{\sqrt{q}}$ and the deeper fact that

$$\frac{\sqrt{q}}{\pi}L'(1) = \gamma_0 - \tfrac{1}{2}\log 4q + H\left(\tfrac{1+\sqrt{-q}}{2}\right). \tag{7}$$

One way to go further with (7) is to utilize the Euler product

$$L(s) = \prod_{p \text{ prime}} (1 - \left(\tfrac{p}{q}\right)p^{-s})^{-s}.$$

Write $\frac{\sqrt{q}}{\pi}L'(1) = D\log L(s)|_{s=1}$ in (7) and observe that it implies

$$\sum_{p \text{ prime}} \left(\tfrac{p}{q}\right)\tfrac{\log p}{p} = \log q - H\left(\tfrac{1+\sqrt{-q}}{2}\right) + O(1) = -\tfrac{\pi}{6}\sqrt{q} + \log q + O(1). \tag{8}$$

Roughly speaking, since the negative term $-\frac{\pi}{6}\sqrt{q}$ on the RHS of (8) dominates, this asymptotic formula indicates that for (6) to hold for large q, there must be many quadratic non-residues modulo q among the first few primes. For instance, the first 12 primes are non-residues modulo 163. This is one way to become convinced that formula (5) cannot remain true for large q, since the resulting imbalance in the distribution of quadratic residues should contradict the generalized Riemann hypothesis.

A different kind of application of KLF to (7) makes use of the identity

$$\frac{\sqrt{q}}{\pi}L'(1) = \gamma_0 + \log 2\pi - \sum_{n=1}^{q-1} \left(\tfrac{n}{q}\right)\log\Gamma\left(\tfrac{n}{q}\right), \tag{9}$$

which was derived in 1883 by Berger [2] and independently by Lerch [34] in 1897 using Kummer's formula

$$\log\left(\tfrac{1}{\sqrt{2\pi}}\Gamma(x)\right) = \left(\tfrac{1}{2} - x\right)(\gamma_0 + \log 2\pi) + \tfrac{1}{2}\log(2\sin \pi x) + \tfrac{1}{\pi}\sum_{n\geq 1} n^{-1}\log n\ \sin(2\pi n x). \tag{10}$$

Taken together, (9) and (7) give for $q = 7, 11, 19, 43, 67, 163$ some remarkable evaluations:

$$\left|\eta\left(\tfrac{1+\sqrt{-q}}{2}\right)\right|^4 = \tfrac{1}{2\pi q}\prod_{n=1}^{q-1} \Gamma\left(\tfrac{n}{q}\right)^{\left(\frac{n}{q}\right)}, \tag{11}$$

also due to Lerch [34]. Chowla and Selberg [7, 8] later independently derived (11) and applied it to evaluate elliptic integrals having singular moduli, a connection already glimpsed by Landau in 1902 [32, p.313]. An equivalent formulation can be given in terms of certain hypergeometric series:

$$F(a, b, c; x) = 1 + \sum_{n\geq 1} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},$$

where $(a)_n = a(a + 1)\cdots(a + n - 1)$. Their result can be put into an elegant form using the Schläfli modular function

$$f(\tau) = e\left(-\tfrac{1}{48}\right)\frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}. \tag{12}$$

For $q = 7, 11, 19, 43, 67, 163$ and $f = f(\sqrt{-q})$ it can be shown that (11) yields

$$F(\tfrac{1}{4}, \tfrac{1}{4}, 1; 2^6 f^{-24}) = (2q\pi)^{-\frac{1}{2}} f^2 \prod_{n=1}^{q-1} \Gamma\big(\tfrac{n}{q}\big)^{\frac{1}{2}(\frac{n}{q})}. \tag{13}$$

Weber (see Tabelle VI. of [51]) computed explicitly all these values of f. Thus $f(\sqrt{-7}) = \sqrt{2}$ and the others are algebraic integers of degree 3 over $\mathbb{Q}$ that may be expressed in the (shockingly) simple form

$$3f(\sqrt{-q}) = \alpha_q + (\beta_q + 3\sqrt{3q})^{\frac{1}{3}} + (\beta_q - 3\sqrt{3q})^{\frac{1}{3}},$$

where $\alpha_q = 2$ except that $\alpha_{19} = 0$ and $\beta_{11} = 17, \beta_{19} = 27, \beta_{43} = 35, \beta_{67} = 53, \beta_{163} = 135$.

In fact, diophantine properties of f are crucial in "resurrections" of Heegner's proof ([24]) that 163 is the largest q for which (5) holds. See Stark's ICM paper [46] for such a proof and also [48] for his recent description of some of the drama surrounding this result. See also Birch's papers [3], [4] and that of Deuring [12].

Weber's table of values of f also gives that for $q = 5, 13, 37$

$$\tfrac{1}{2} f(\sqrt{-q})^8 = t + u\sqrt{q}, \tag{14}$$

where $(t, u) = (3, 1), (11, 3), (146, 24)$, respectively. It turns out that these (t, u) solve the Pell equation

$$t^2 - qu^2 = 4$$

with t minimal among all solutions with $t, u > 0$. These give samples of Kronecker's "solution" of the Pell equation using modular functions. The connection with KLF comes from the fact that for

$$Q(x, y) = x^2 + qy^2 \quad \text{and} \quad Q'(x, y) = 2x^2 - 2xy + \tfrac{q+1}{2} y^2$$

when $q = 5, 13, 37$ we have that

$$2L_{-4}(s)L_q(s) = \zeta_Q(s) - \zeta_{Q'}(s).$$

Now apply KLF to the right hand side and the famous result of Dirichlet

$$\sqrt{q}L_q(1) = \log \tfrac{1}{2}(t + u\sqrt{q})$$

to the left hand side to get (14).

3. Proof of KLF

There are a number of proofs, starting with Kronecker's own. See [52] for a discussion. Here we will give a brief treatment of one of the most transparent ones. For $z \in \mathcal{H}$ and $\operatorname{Re} s > 1$ define $E(z, s)$ by

$$\zeta(2s)E(z, s) = \tfrac{1}{2} y^s \sideset{}{'}\sum_{m,n\in\mathbb{Z}} |m + nz|^{-2s}. \tag{15}$$

We may work with $\zeta(2s)E(z, s)$ to prove KLF since

$$|D|^{\frac{s}{2}} \zeta_Q(s) = 2^{s+1} \zeta(2s) E(z_Q, s). \tag{16}$$

The key input is the Fourier expansion of $\zeta(2s)E(z, s)$, which was found by Deuring [11] and Chowla-Selberg [8]. We simply state the result; an excellent source for its proof is [33]. It is convenient to state the result for the completion

$$E^*(z, s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z, s). \tag{17}$$

As usual set $\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$. Then we have the Fourier expansion

$$E^*(z,s) = \Lambda(2s)y^s + \Lambda(2-2s)y^{1-s} + 4y^{1/2}\sum_{n\geq 1} n^{\frac{1}{2}-s}\sigma_{2s-1}(n)K_{s-\frac{1}{2}}(2\pi ny)\cos(2\pi nx), \tag{18}$$

where $\sigma_s(n) = \sum_{d|n} d^s$ and $K_s(y)$ is the usual Bessel function. By making use of (16) and (17), the proof of KLF can be reduced to the following statement:

$$2E^*(z,s) = \frac{1}{s-1} + \gamma_0 - \log(4\pi y) - 4\log|\eta(z)| + O(s-1). \tag{19}$$

To obtain (19), first note that

$$K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}.$$

By (18) it follows that

$$2E^*(z,s) = \frac{1}{s-1} + \gamma_0 - \log(4\pi y) + \tfrac{1}{3}\pi y + 4\mathrm{Re}\Big(\sum_{n\geq 1} n^{-1}\sigma_1(n)q^n\Big) + O(s-1).$$

Now apply

$$\mathrm{Re}\Big(\sum_{n\geq 1} n^{-1}\sigma_1(n)q^n\Big) = \mathrm{Re}\Big(\sum_{m,n\geq 1} n^{-1}q^{mn}\Big) = -\Big(\log|\eta(z)| + \tfrac{\pi y}{12}\Big)$$

to finish the proof of (19), hence of KLF.

Remark: A quite different kind of proof was given by Shintani [43] using the Barnes double gamma function.

Expansions around $s = 0$

It turns out to often be advantageous to have expansions of KLF–type around $s = 0$. An added bonus of the Fourier expansion (18) is that it renders as obvious the analytic continuation and functional equation of $E^*(z,s)$. Using that $\Lambda(1-s) = \Lambda(s)$, $K_s(y) = K_{-s}(y)$ and

$$n^s\sigma_{-2s}(n) = n^{-s}\sigma_{2s}(n)$$

it follows from (18) that $s(s-1)E^*(z,s)$ is entire and that

$$E^*(1-s,z) = E^*(z,s). \tag{20}$$

This makes it easy to compute the Laurent expansion of $\zeta_Q(s)$ around $s = 0$, which no longer contains the mysterious Euler constant that occurs in the formulas around $s = 1$. By the functional equation (20) and (19) we have

$$2E^*(z,s) = -s^{-1} + \gamma_0 - \log(4\pi) + H(z) + O(s) \quad \text{hence} \tag{21}$$

$$\zeta_Q(s) = -1 - \Big(\log(8\pi^2|D|^{-\frac{1}{2}}) - H(z_Q)\Big)s + O(s^2). \tag{22}$$

There is a connection of $E(z,s)$ to the spectral theory of the Laplacian on $\Gamma\backslash\mathcal{H}$ hinted at by the occurrence of the Bessel function in the Fourier expansion (18), one that is somewhat miraculous with hindsight. In fact we can write

$$E(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}\ \gamma z)^s, \tag{23}$$

where Γ_∞ is the usual parabolic subgroup generated by $\pm\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Let

$$\Delta = y^2(\partial_x^2 + \partial_y^2) \tag{24}$$

be the hyperbolic Laplacian. Now $\Delta \mathrm{Im} z = s(s-1)\mathrm{Im} z$ and Δ commutes with linear fractional action. It follows that $E(z,s)$ is an eigenfunction of $-\Delta$:

$$-\Delta E(z,s) = s(1-s)E(z,s). \tag{25}$$

For $\mathrm{Re}\, s = \frac{1}{2}$ it happens that $E(z,s)$ gives the continuous part of the spectral decomposition of Δ on $\Gamma\backslash\mathcal{H}$. Consider the following expansion of $E(z,s)$ around $s=0$:

$$E(z,s) = H_0(z) - H_1(z)s + H_2(z)s^2 - \cdots. \tag{26}$$

From (21) and (17) we see that $H_0(z) = 1$ and $H_1 = H$. Furthermore, upon using (25) in (26), we also have that $\Delta H = 1$ and

$$\Delta H_n = H_{n-2} + H_{n-1},$$

for $n \geq 2$. In particular, H is bi-harmonic with respect to Δ. For related work see [31].

L-functions with genus characters

Perhaps the most important applications of Kronecker's limit formulas are to L-functions associated to quadratic fields and their abelian extensions. The first limit formula applies to abelian extensions that are unramified. In fact, we will only consider L-functions associated to unramified *quadratic* extensions, namely those with genus characters, for which there is already an extremely rich theory.

Suppose that $D \neq 1$ is a fundamental discriminant and $\mathbb{K} = \mathbb{Q}(\sqrt{D})$. Let $\sigma : \mathbb{K} \to \mathbb{K}$ generate the Galois group of $\mathbb{K}/\mathbb{Q}$ and for $\alpha \in \mathbb{K}$ let $N(\alpha) = \alpha\alpha^\sigma$. Let Cl_D^+ be the group of (narrow) fractional ideal classes in $\mathbb{K}$. Thus two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in the same class if there is $\alpha \in \mathbb{K}$ with $N(\alpha) > 0$ so that $\mathfrak{a} = (\alpha)\mathfrak{b}$. Let

$$h(D) = \#\mathrm{Cl}_D^+$$

be the class number and $w = w_D$ be the number of roots of unity in $\mathbb{K}$ so that $w = 2$ unless $D = -3, -4$ when $w = 6, 4$, respectively. If $D > 1$ let ϵ_D be the smallest unit of norm 1 that is > 1 in the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$.

Associated to an ideal class A is the partial zeta function

$$\zeta_\mathbb{K}(s,A) = \zeta(s,A) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals in A. Note that $\zeta(s,A) = \zeta(s,A^{-1})$. Dirichlet applied his geometric method to evaluate

$$\lim_{s\to 1^+} (s-1)|D|^{\frac{s}{2}}\zeta(s,A) = \begin{cases} \frac{2\pi}{w} & \text{if } D < 0 \\ \log \epsilon_D & \text{if } D > 1. \end{cases} \tag{27}$$

.

Given a character χ of Cl_D^+ we have the L-function

$$L(s,\chi) = \sum_{\mathfrak{a}\subset\mathcal{O}_\mathbb{K}} \chi(\mathfrak{a})N(\mathfrak{a})^{-s} = \sum_{A\in\mathrm{Cl}_D^+} \chi(A)\zeta(s,A). \tag{28}$$

A genus is an element of the group of genera, which is (isomorphic to) the quotient group

$$\mathrm{Gen}(\mathbb{K}) = \mathrm{Cl}_D^+/(\mathrm{Cl}_D^+)^2. \tag{29}$$

It is known that $\mathrm{Gen}(\mathbb{K}) \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(D)-1}$ so if G_D is a genus in Cl_D^+ then

$$\#G_D = 2^{1-\omega(D)} h(D), \tag{30}$$

where $\omega(D)$ is the number of distinct prime factors of D. The characters of $\mathrm{Gen}(\mathbb{K})$ descend from the real characters of Cl_D^+ and are in one to one correspondence with co-prime factorizations $D = d'd$ where d and d' are fundamental discriminants (including 1). Kronecker discovered a factorization of $L(s, \chi)$ for such a χ that corresponds to $D = d'd$:

$$L(s, \chi) = L_{d'}(s) L_d(s). \tag{31}$$

In particular, the Dedekind zeta function of $\mathbb{K}$ satisfies

$$\zeta_{\mathbb{K}}(s) = L(s, 1) = \zeta(s) L_D(s). \tag{32}$$

By (27) and well-known functional equations we get

$$L_D(0) = \frac{2h(D)}{w} \text{ if } D < 0 \text{ and } L_D'(0) = \tfrac{1}{2} h(D) \log \epsilon_D \text{ if } D > 1. \tag{33}$$

The Hurwitz zeta function

A useful tool to study Dirichlet L-functions at $s = 0$ is the Hurwitz zeta function, which is defined for $x > 0$ and $\mathrm{Re}(s) > 1$ by

$$\zeta(s, x) = \sum_{n \geq 0} (n + x)^{-s}.$$

For fixed $x > 0$ it has an analytic continuation in s to an entire function except for a simple pole at $s = 1$. It has the expansion due to Hurwitz [28] (see also p. 269 of [53]):

$$\zeta(x, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \Big(\sin \tfrac{\pi s}{2} \sum_{n \geq 1} \frac{\cos 2\pi n x}{n^{1-s}} + \cos \tfrac{\pi s}{2} \sum_{n \geq 1} \frac{\sin 2\pi n x}{n^{1-s}} \Big),$$

valid for $0 < x \leq 1$ and $\mathrm{Re}\, s < 0$. This can be used to find the Laurent expansion of $\zeta(x, s)$ around $s = 0$. In particular we have

$$\zeta(x, 0) = \tfrac{1}{2} - x,$$

and from Kummer's formula (10)

$$\partial_s \zeta(x, 0) = \log \big((2\pi)^{-\frac{1}{2}} \, \Gamma(x) \big).$$

Since for any D we have

$$L_D(s) = |D|^{-s} \sum_{n=1}^{|D|} \chi_D(n) \zeta(s, \tfrac{n}{|D|})$$

we immediately deduce the following for $D \neq 1$:

$$L_D(0) = -|D|^{-1} \sum_{n=1}^{|D|-1} n \chi_D(n) \quad \text{and} \quad L_D'(0) = -L_D(0) \log |D| + \sum_{n=1}^{|D|-1} \chi_D(n) \log \Gamma\big(\tfrac{n}{|D|}\big). \tag{34}$$

Also, for $D = 1$ we have $\zeta(0) = -\frac{1}{2}, \zeta'(0) = -\frac{1}{2} \log 2\pi$. If $D > 1$, we have that $L_D(0) = 0$ and

$$L_D'(0) = -\tfrac{1}{2} \sum_{n=1}^{D} \chi_D(n) \log \big(\sin \tfrac{n\pi}{D} \big), \tag{35}$$

by using the identity $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x}$ in the second formula of (34). Thus by (34), (35) and (33) we recover Dirichlet's results

$$h(D) = -\tfrac{w}{2|D|}\sum_{n=1}^{|D|-1} n\chi_D(n) \quad \text{and} \quad h(D)\log\epsilon_D = -\sum_{n=1}^{D}\chi_D(n)\log\left(\sin\tfrac{n\pi}{D}\right)$$

for $D < 0$ and $D > 1$, respectively. The function

$$R(x) = -\partial_s^2\zeta(x,0) \tag{36}$$

was studied by by Landau [32] and Ramanujan [37, Chapter 8] and applied to KLF by Deninger [10]. We have for $D \neq 1$

$$L_D''(0) = -L_D(0)\log^2|D| - 2L_D'(0)\log|D| - 2\sum_{n=1}^{|D|-1}\chi_D(n)R(\tfrac{n}{|D|}). \tag{37}$$

KLF applied in the imaginary quadratic case

Happily, KLF applies directly when $D < 0$. The examples we outlined above had $\mathbb{K}$ imaginary of class number 1 or 2. The general case when $D < 0$ and with a genus character is very similar. To start,

$$\zeta(s,A) = \tfrac{1}{w}\zeta_Q(s) \tag{38}$$

for some positive integral Q of discriminant D. Here A is represented by $\mathfrak{a} = \mathbb{Z} + z_Q\mathbb{Z}$. Set $z_A = z_Q$ for any such choice, for instance that $z_A \in \mathcal{F}$, the standard fundamental domain for Γ. This point z_A is often called a CM point, the CM short for "complex multiplication".

It follows from (16) and the duplication formula for the gamma function that

$$\Lambda(s,A) := |D|^{\frac{s}{2}}\pi^{-s}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{s+1}{2})\zeta(s,A) = \tfrac{4\sqrt{\pi}}{w}E^*(z_Q,s). \tag{39}$$

Thus $\zeta(s,A)$ is entire but for a simple pole at $s = 1$ and by (20) satisfies the functional equation

$$\Lambda(s,A) = \Lambda(1-s,A).$$

A direct application of KLF in the form (22) to (38) gives the next result.

Theorem 2 (Kronecker)**.** *For $D < 0$ we have*

$$\zeta(0,A) = -\tfrac{1}{w} \quad \text{and}$$
$$\zeta'(0,A) = -\tfrac{1}{w}\log\left(8\pi^2|D|^{-\frac{1}{2}}\right) + \tfrac{1}{w}H(z_A).$$

Given any character χ of Cl_D^+ we can use this to give a formula for $L'(0,\chi)$. In case χ is a genus character we obtain the following generalizations of the examples (11) and (14) from above.

Corollary 1. *Suppose that $D < 0$. Then for $w = w_D$ the following holds:*

$$\sum_{A\in\mathrm{Cl}_D^+} H(z_A) = \tfrac{1}{2}h(D)\log(16\pi^2|D|) - \tfrac{w}{2}\sum_{n=1}^{|D|-1}\chi_D(n)\log\Gamma\left(\tfrac{n}{|D|}\right) \qquad \text{(Lerch)}. \tag{40}$$

For $D = d'd$ with $d' > 1$ and $d < 0$ co-prime fundamental discriminants and χ the associated genus character we have

$$\sum_{A\in\mathrm{Cl}_D^+} \chi(A)H(z_A) = \tfrac{w_D}{w_d}h(d)h(d')\log\epsilon_{d'} \qquad \text{(Kronecker)}. \tag{41}$$

Chowla and Selberg [8] reproved (40) independently and applied it to show that a period of an elliptic curve defined over an algebraic number field with CM by an order in K of discriminant $D<0$ is an algebraic multiple of

$$\sqrt{\pi}\prod_{n=1}^{|D|-1}\Gamma\left(\tfrac{n}{|D|}\right)^{\frac{w\chi_D(n)}{4h(D)}}.$$

Note that (41) can be put in the form of an average:

$$\tfrac{1}{h(D)}\sum_{A\in\mathrm{Cl}_D^+}\chi(A)H(z_A)=\frac{(h(d)/w_d)(h(d')\log\epsilon_{d'})}{h(D)/w_D}. \tag{42}$$

Indefinite binary quadratic forms: Zagier's limit formula

From now on we assume that $D>0$. Since real quadratic field norms comes from indefinite binary quadratic forms, KLF does not apply *directly* to compute $\zeta(s,A)$ when $\mathbb{K}$ is real quadratic. To address this problem, Zagier [54] gave an analogue of KLF for zeta functions associated to certain indefinite forms, which we will state here without proof. Let now

$$Q(x,y)=ax^2+bxy+cy^2$$

be a real binary quadratic form with positive coefficients and positive discriminant $D=b^2-4ac$. Then the roots $w'<w$ of $Q(1,-x)=0$ are positive. Define

$$Z_Q(s)=\sum_{n\geq1}\sum_{m\geq0}Q(n,m)^{-s}.$$

Theorem 3 (Zagier's limit formula)**.** *The function $Z_Q(s)$ has an analytic continuation to the half-plane* $\mathrm{Re}s>\frac{1}{2}$ *with a simple pole at $s=1$ and*

$$\lim_{s\to1}\left(D^{\frac{s}{2}}Z_Q(s)-\frac{\frac12\log\frac{w}{w'}}{s-1}\right)=P(w,w'),$$

where $P(x,y)=\mathcal{F}(x)-\mathcal{F}(y)+\mathrm{Li}_2(\frac{y}{x})-\frac{\pi^2}{6}+(\log\frac{x}{y})(\gamma_0-\frac12\log(x-y)+\frac14\log\frac{x}{y})$ *and*

$$\mathcal{F}(x)=\sum_{n\geq1}\frac{\psi(nx)-\log(nx)}{n},$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ *is the digamma function.*

In fact, by [56, Satz 1, p.132] we have that $Z_Q(s)$ has an analytic continuation to $\mathrm{Re}\,s>-\frac12$ with a (possible) additional simple pole only at $s=\frac12$ and that

$$Z_Q(0)=\tfrac{1}{24}(\tfrac{b}{a}+\tfrac{b}{c}-6). \tag{43}$$

To proceed, we must relate $Z_Q(s)$ to $\zeta(s,A)$. Each $A\in\mathrm{Cl}_D^+$ contains fractional ideals of the form $w\mathbb{Z}+\mathbb{Z}\in A$ where $w\in\mathbb{K}$ is such that $w>w^\sigma$. Consider the *minus* (or backward) continued fraction of w:

$$w=[\![a_0,a_1,a_2,\ldots]\!]=a_0-\cfrac{1}{a_1-\cfrac{1}{a_2-\cfrac{1}{a_3-\cdots}}} \tag{44}$$

where $a_j\in\mathbb{Z}$ with $a_j\geq2$ for $j\geq1$. This continued fraction is eventually periodic and has a unique primitive cycle $(\!(n_1,\ldots,n_\ell)\!)$ of length ℓ, only defined up to cyclic permutations.

Different admissible choices of w lead to the same primitive cycle. The continued fraction is purely periodic precisely when w is reduced in the sense that

$$0 < w^\sigma < 1 < w$$

(see [54],[56]). The cycle $(\!(n_1, \dots, n_\ell)\!)$ characterizes A; it is a *complete* class invariant. The length $\ell = \ell_A$, which is also the number of distinct reduced w, is another invariant as is the sum

$$m = m_A = n_1 + \cdots + n_\ell. \tag{45}$$

It is convenient to define a third class invariant

$$\Psi(A) = m_A - 3\ell_A. \tag{46}$$

Note that the cycle of A^{-1} is given by that of A reversed:

$$(\!(n_\ell, \dots, n_1)\!)\,. \tag{47}$$

To see this observe that A^{-1} is represented by $(1/w^\sigma)\mathbb{Z} + \mathbb{Z}$ and by [56, p.128] the continued fraction of $1/w^\sigma$ has (47) as its cycle. Thus

$$\Psi(A^{-1}) = \Psi(A). \tag{48}$$

Similarly, using the relation between the minus continued fractions and regular simple continued fractions, it is shown in [27, p. 49] that

$$\Psi(A) = \ell_{AJ} - \ell_A, \tag{49}$$

where J denotes the class of the different $(\sqrt{D})$ of K. Hence we also have that

$$\Psi(A^{-1}J) = \Psi(AJ) = -\Psi(A),$$

using the conjugacy invariance of Ψ, (48) and that $J^2 = I$, where I is the principal class.

Let $w_1, w_2, \dots, w_\ell$ be the reduced values, which may be obtained from (44) and the cyclic permutations of the cycle. For each w_k we define the indefinite binary quadratic form

$$Q_k(x, y) = \tfrac{1}{w_k - w_k^\sigma}(y + xw_k)(y + xw_k^\sigma),$$

which has positive coefficients and discriminant one. Then Zagier gave the following important decomposition:

$$\zeta(s, A) = \sum_{k=1}^{\ell} Z_{Q_k}(s). \tag{50}$$

By evaluating the corresponding limit for each $Z_{Q_k}(s)$ we get a formula for the constant term in the Laurent expansion of $|D|^{\frac{s}{2}}\zeta(s, A)$ around $s = 1$ that involves a summation over the roots w_k and w_k^σ of the fixed function P from above.

Theorem 4 (Zagier)**.** *For $D > 0$*

$$\lim_{s\to 1}\left(D^{\frac{s}{2}}\zeta(s, A) - \frac{\log \epsilon_D}{s-1}\right) = \sum_{k=1}^{\ell} P(w_k, w_k^\sigma). \tag{51}$$

Remarks:

1. To get the residue in (51), Zagier showed directly that

$$\epsilon_D^2 = \prod_{k=1}^{\ell} \frac{w_k}{w_k^\sigma}. \tag{52}$$

In particular, this together with (51) gives an apparently non-geometric proof of Dirichlet's (27) when $D > 1$.

2. In a recent paper [49], Vlasenko and Zagier generalize (51) to $s = 2, 3, \dots$ and interpret that in terms of the cohomology of Γ.

Hecke's results

Zagier's paper followed earlier work of Hecke on the problem of extending Kronecker's ideas to real quadratic fields. Hecke's approach to KLF for a real quadratic field $\mathbb{K}$ was to integrate the definite version over appropriate cycles coming from the unit group of $\mathcal{O}_\mathbb{K}$. His method is a direct descendent of that used by Dirichlet to prove his class number formula.

A subtlety about real quadratic fields, which turns out to be crucial, is the possible existence of units with negative norm. As above we denote by I the principal class and by J the class of the different $(\sqrt{D})$ of K, which coincides with the class of principal ideals (α) where $N(\alpha) = \alpha\alpha^\sigma < 0$. Then Cl_D^+/J is the class group in the wide sense, which is trivial iff $\mathcal{O}_\mathbb{K}$ is a UFD. Clearly $J \neq I$ iff $\mathcal{O}_\mathbb{K}$ contains no unit of norm -1. In this case each wide ideal class is the union of two narrow classes, say A and JA. A sufficient condition for $J \neq I$ is that D is divisible by a prime $p \equiv 3 \pmod 4$.

For a fixed narrow ideal class $A \in \mathrm{Cl}_D^+$ and $\mathfrak{a} = w\mathbb{Z} + \mathbb{Z} \in A$ with $w > w^\sigma$, let $\mathcal{S}_w$ be the geodesic in $\mathcal{H}$ with endpoints w^σ and w. The modular closed geodesic $\mathcal{C}_A$ on $\Gamma\backslash\mathcal{H}$ is defined as follows. Define $\gamma_w = \pm \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, where $a, b, c, d \in \mathbb{Z}$ are determined by

$$\begin{aligned} \epsilon_D w &= aw + b \\ \epsilon_D &= cw + d, \end{aligned} \tag{53}$$

with ϵ_D our unit. Then γ_w is a primitive hyperbolic transformation in Γ with fixed points w^σ and w. Since

$$(cw + d)^{-2} = \epsilon_D^{-2} < 1,$$

we have that w is the attracting fixed point of γ_w. This induces on the geodesic $\mathcal{S}_w$ a clockwise orientation. Distinct $\mathfrak{a}$ and w for A induce Γ-conjugate transformations γ_w. If we choose some point z_0 on $\mathcal{S}_w$ then the directed arc on $\mathcal{S}_w$ from z_0 to $\gamma_w(z_0)$, when reduced modulo Γ, is the associated closed geodesic $\mathcal{C}_A$ on $\Gamma\backslash\mathcal{H}$. It is well-defined for the class A and gives rise to a unique set of oriented arcs (that could overlap) in $\mathcal{F}$. We also use $\mathcal{C}_A$ to denote this set of arcs.

It is not difficult to show that the closed geodesic $\mathcal{C}_{A^{-1}J}$ has the same image as $\mathcal{C}_A$ but with the opposite orientation. The arcs of $\mathcal{C}_A$ retrace back over themselves when $A^{-1}J = A$ or, equivalently, $A^2 = J$, i.e. J is in the principal genus. Sarnak [41] gave a nice account of these reciprocal geodesics.

It is possible to compute γ_w associated to a reduced w as above simply using the ordering of the cycle $(\!(n_1, \dots, n_\ell)\!)$ that corresponds to w. In fact, if as usual we set $T = \pm \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), S = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \Gamma$ then

$$\gamma_w = T^{n_1} S T^{n_2} S \cdots T^{n_\ell} S. \tag{54}$$

A nice treatment of this is given by S. Katok in [29, p.11.]. The class invariant $\Psi(A)$ can now be computed in terms of the entries in γ_w. The Rademacher symbol is defined for any $\gamma = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by

$$\Psi(\gamma) = \Phi(\gamma) - 3\,\mathrm{sign}(c(a+d)). \tag{55}$$

Here $\Phi(\gamma)$ is the Dedekind symbol given for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ by

$$\Phi(\gamma) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12\,\mathrm{sign}\, c \cdot s(a,c) & \text{if } c \neq 0, \end{cases} \tag{56}$$

where $s(a,c)$ is the Dedekind sum, defined for $\gcd(a,c) = 1$, $c \neq 0$ by

$$s(a,c) = \sum_{n=1}^{|c|} \left(\left(\frac{n}{c}\right)\right)\left(\left(\frac{na}{c}\right)\right).$$

As usual, $((x)) = 0$ if $x \in \mathbb{Z}$ and otherwise $((x)) = x - \lfloor x \rfloor - 1/2$. Rademacher showed that $\Psi(\gamma)$ is invariant under conjugation in Γ. We also have that

$$\Psi(A) = \Psi(\gamma_w). \tag{57}$$

This follows from a well known formula for $\Psi(\gamma)$ given by Rademacher [19] which, when applied to the expansion of γ_w in (54) gives $\Psi(A)$ as defined in (46).

Even and Odd characters. Say a character χ of Cl_D^+ is even if $\chi(J) = 1$ and odd if $\chi(J) = -1$. In the more general terminology of [20], the odd characters are norm class characters of norm-signature type. Clearly the even characters are precisely those that induce wide class characters. It can be checked that a genus character coming from a decomposition $D = d'd$ is odd if and only if d and d' are both negative.

In order to adapt (39) for the presence of an infinite groups of units, Hecke invented his famous trick of dividing out the action of the unit group on generators of principal ideals. For real quadratic fields this procedure amounts to either integrating the Eisenstein series with respect to arc length over the associated closed geodesic $\mathcal{C}_A$ or integrating its derivative over this geodesic. In case $I \neq J$ it is necessary to define

$$\zeta_\pm(s,A) = \tfrac{1}{2}\big(\zeta(s,A) \pm \zeta(s,JA)\big)$$

and their completions

$$\Lambda_+(s,A) = D^{\frac{s}{2}}\pi^{-s}\Gamma(\tfrac{s}{2})^2\zeta_+(s,A) \;\text{ and }\; \Lambda_-(s,A) = D^{\frac{s}{2}}\pi^{-s}\Gamma(\tfrac{s+1}{2})^2\zeta_-(s,A). \tag{58}$$

In general

$$L(s,\chi) = \sum_{A \in \mathrm{Cl}_D^+} \chi(A)\zeta_\pm(s,A)$$

where $\chi(J) = \pm 1$.

Hecke found the following integral representations (see [44, Chap. II § 5 pp. 114–149.]):

$$\Lambda_+(s,A) = \int_{\mathcal{C}_A} E^*(z,s)y^{-1}|dz| \tag{59}$$

$$\Lambda_-(s,A) = \int_{\mathcal{C}_A} i\,\partial_z E^*(z,s)\,dz. \tag{60}$$

These show, in particular, that $s(1-s)\Lambda_\pm(s,A)$ are entire and invariant under $s \mapsto 1-s$.

First we examine the even case. By (59) and (21) we derive the following theorem.

Theorem 5 (Hecke)**.** *For $D>1$ we have that $\zeta_+(0,A)=0$ and*

$$\zeta_+'(0,A) = -\tfrac{1}{8}\int_{\mathcal{C}_A} y^{-1}|dz| = -\tfrac{1}{4}\log\epsilon_D \tag{61}$$

$$\zeta_+''(0,A) = \tfrac{1}{4}\int_{\mathcal{C}_A}\big(H(z)+\tfrac{1}{2}\log(\tfrac{D}{(2\pi)^4})\big)y^{-1}|dz|. \tag{62}$$

Given any even character χ of Cl_D^+ we can use these to give a formulas for $L'(0,\chi)$ and $L''(0,\chi)$. In case χ is a genus character the resulting formulas imply the following. Recall the definition of $R(x)$ from (36) and (37).

Corollary 2. *For $D>1$*

$$\sum_{A\in\mathrm{Cl}_D^+}\int_{\mathcal{C}_A} y^{-1}|dz| = 2h(D)\log\epsilon_D \quad \text{and} \tag{63}$$

$$\sum_{A\in\mathrm{Cl}_D^+}\int_{\mathcal{C}_A} H(z)y^{-1}|dz| = (\log D + 2\log 2\pi)h(D)\log\epsilon_D + 4\sum_{n=1}^{D-1}\chi_D(n)R(\tfrac{n}{D}). \tag{64}$$

For $D=d'd$ with $d',d>1$ co-prime fundamental discriminants and χ the associated genus character,

$$\sum_{A\in\mathrm{Cl}_D^+}\chi(A)\int_{\mathcal{C}_A} H(z)y^{-1}|dz| = 2h(d)h(d')\log\epsilon_d\log\epsilon_{d'}. \tag{65}$$

Of course (63) is due to Dirichlet. The formulas (64) and (65) seem to have been first written down by Deninger [10] and Siegel [44, p.97.], respectively.

In response to Hecke's paper, Herglotz [25] managed to express the integral $\int_{\mathcal{C}_A} H(z)y^{-1}|dz|$ in terms of integrals of elementary functions, but in general the resulting formula is pretty complicated. Before becoming aware of Herglotz's papers, Zagier applied his more general formula (51) to $\zeta_+(s,A)$ to give another (equivalent) version. As an interesting application of his formula, Herglotz applied it to (65) in special cases to evaluate some elementary integrals that seem to defy other proofs. Recently Muzaffar and Williams [35] found another equivalent formulation and gave several more examples, including

$$\int_0^1 \frac{\log(1+x^{2+\sqrt{3}})}{1+x}dx = \frac{\pi^2}{12}(1-\sqrt{3}) + \log 2\log(1+\sqrt{3}).$$

Finally, note that (65) can be put in the form

$$\frac{1}{\sum_A\int_{\mathcal{C}_A} y^{-1}|dz|}\sum_{A\in\mathrm{Cl}_D^+}\chi(A)\int_{\mathcal{C}_A} H(z)y^{-1}|dz| = \frac{(h(d)\log\epsilon_d)(h(d')\log\epsilon_{d'})}{h(D)\log\epsilon_D}, \tag{66}$$

which is a real quadratic average analogous to (42).

Expansion of $\zeta_-(s,A)$ around $s=0$: the value $\zeta_-(0,A)$

From now on we consider only the remaining cases of $\zeta_-(s,A)$ and $L(s,\chi)$ with odd genus characters χ. In view of the second formula of (58) and the functional equation we wish to evaluate $\zeta_-(0,A)$ and $\zeta_-'(0,A)$.

As far as we know, of these only $\zeta_-(0,A)$ has been evaluated before. We review this here and consider $\zeta_-'(0,A)$ in the following section. There are several approaches to $\zeta_-(0,A)$. Probably the most direct is that of Zagier in [56], where it is evaluated in terms of the invariant Ψ of (46). Using (43), (49) and (50) it is elementary to deduce the following (see [56, Satz 2, p.132]):

Theorem 6. *For A a narrow ideal class in the real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$*

$$\zeta_-(0, A) = \tfrac{1}{12}\Psi(A).$$

Given any odd character χ of Cl_D^+ we can use this to give a formula for $L(0, \chi)$. In case χ is a genus character the resulting formula implies the following.

Corollary 3. *For $D = d'd > 1$ with $d', d < 0$ co-prime fundamental discriminants and χ the associated genus character,*

$$\sum_{A \in \mathrm{Cl}_D^+} \chi(A)\Psi(A) = \frac{48h(d)h(d')}{w_d w_{d'}}. \tag{67}$$

A nice special case is when $D = 4q$ where $q \equiv 3 \pmod 4$ is prime and $h(4q) = 2$, which is equivalent to $\mathbb{Z}[\sqrt{q}]$ being a UFD. This happens for the q of our first set of examples and, according to a well-known conjecture of Cohen and Lenstra [9], occurs for $> 75\%$ of all primes $q \equiv 3 \pmod 4$. For these $q > 3$, (67) gives

$$h(-q) = \tfrac{1}{3}\sum_{k=1}^{\ell} n_k - \ell$$

where $(\!(n_1, \ldots, n_\ell)\!)$ is the cycle of $\sqrt{q}$. Alternatively, using (49), we have

$$h(-q) = \tfrac{1}{3}(\ell' - \ell),$$

where ℓ' is the length of the cycle of $-\sqrt{q}$. For example, $\sqrt{163}$ has the cycle given by:

$$(\!(5, 2, 2, 4, 3, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2, 3, 4, 2, 2, 5, 26)\!)$$

so $h(-163) = \frac{1}{3} \cdot 108 - 35 = 1$. One may check that the cycle of $-\sqrt{163}$ has length 38. See [26] and [56] for more insight into these examples.

Given (57) we can use Hecke's representation (60) for another way to prove Theorem 6. The Fourier expansion of $E^*(z, s)$ from (18) yields Hecke's limit formula

$$\lim_{s \to 0} \partial_z E^*(z, s) = -\tfrac{\pi i}{12} E_2^*(z),$$

where $E_2^*(z) = E_2(z) - \frac{3}{\pi y}$ and

$$E_2(z) = 1 - 24\sum_{n \geq 1} \sigma_1(n) q^n.$$

Therefore by (58) and (60) we have

$$\zeta_-(0, A) = \tfrac{1}{12}\int_{\mathcal{C}_A} E_2^*(z)\, dz. \tag{68}$$

Now apply Dedekind's [10] evaluation of the transformation law for $\log \eta(z)$. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\log \eta(\gamma z) - \log \eta(z) = \tfrac{1}{4}\log(-(cz + d)^2) + \tfrac{\pi i}{12}\Phi(\gamma), \tag{69}$$

where $\Phi(\gamma)$ is given by the formula (56) and where we choose $\arg(-(cz + d)^2) \in (-\pi, \pi)$. Since

$$\tfrac{1}{12}E_2(z) = \tfrac{1}{\pi i}\tfrac{d}{dz}\log \eta(z),$$

a careful evaluation of the integral of $-\frac{1}{4\pi y}$ shows that

$$\int_{\mathcal{C}_A} E_2^*(z)\, dz = \Psi(\gamma_w).$$

This is in essence the first proof of Theorem 6 historically, modulo identity (57). Hecke [23, p. 416.] indicated how the evaluation of $\zeta_-(0, A)$ depends on the transformation formula for $\log \eta(z)$ (see also [22], [36], [44, p.134.] and [55]).

We propose yet another proof of Theorem 6, one that will adapt well to computing $\zeta'_-(0, A)$ in terms of an integral of $H(z)$. Let $j(z)$ be the usual modular j-function as before. Write $j' = \frac{1}{2\pi i}\frac{dj}{dz}$ and let $d\mu(z) = \frac{dxdy}{y^2}$ and $\mathcal{F}$ be the standard fundamental domain for Γ. The kernel function

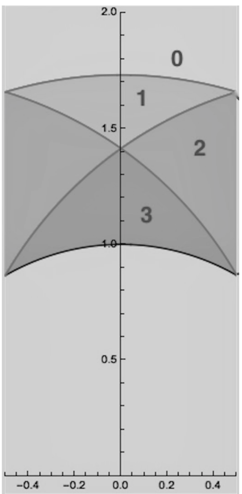
$$K(z, \tau) = \frac{j'(\tau)}{j(z) - j(\tau)}, \tag{70}$$

which is weight 0 in z and weight 2 in τ, shows up a lot in the theory of modular forms (see e.g. [1], [14]). It has the expansion, convergent for $\operatorname{Im} z < \operatorname{Im} \tau$,

$$K(z, \tau) = \sum_{m \geq 0} j_m(z) q_\tau^m \tag{71}$$

where $q_\tau = e(\tau)$ and $j_m(z) = q^{-m} + O(q)$ is weakly holomorphic.

FIGURE 3. The function $\nu_I(z)$ for $D = 12$



Define

$$\nu_A(z) = \int_{\mathcal{C}_A} \frac{j'(\tau)}{j(z) - j(\tau)}\, d\tau \tag{72}$$

This function is Γ-invariant and is zero for $\operatorname{Im} z$ sufficiently large. The value of $\nu_A(z)$ for z not on $\mathcal{C}_A$ is an integer that counts with signs the number of crossings that a path from $i\infty$ to z in $\mathcal{F}$ makes with $\mathcal{C}_A$. This is easily verified using the winding number and well-known properties of j.

We claim that Theorem 6 is equivalent to the statement that

$$\tfrac{3}{\pi} \int_{\mathcal{F}} \nu_A(z) d\mu(z) = \Psi(A). \tag{73}$$

This equivalence may be deduced from (68) and the following elegant formula, which is a consequence of Lemma 1 proven below and is obtained from it by taking $s \to 0$:

$$E_2^*(\tau) = \tfrac{3}{\pi} j'(\tau) \int_{\mathcal{F}} \frac{1}{j(z) - j(\tau)} d\mu(z). \tag{74}$$

Here is an example illustrating the formula (73). Take $D = 12 = (-3)(-4)$ as above with $h(12) = 2$. We have the reduced root $w_I = 2 + \sqrt{3} = \epsilon_{12}$ with $w\mathbb{Z} + \mathbb{Z} = \mathcal{O}_{\mathbb{K}}$ and with cycle $(\!(4)\!)$. Thus $\Psi(I) = 4 - 3 = 1$. In Figure 3 we have plotted $\mathcal{C}_I$, where the arcs are all oriented left to right and are given by the equations

$$y = \sqrt{3 - x^2} \quad \text{and} \quad y = \sqrt{3 - (x \pm 1)^2}.$$

The values of $\nu_I(z)$ are also shown. It can be checked that (73) holds in this case since

(75)

$$\tfrac{3}{\pi} \int_{\mathcal{F}} \nu_I(z) d\mu(z) = A(0) + A(1) + A(-1) = 1, \text{ where } A(u) = \tfrac{6}{\pi} \int_0^{1/2} \int_{\sqrt{1-x^2}}^{\sqrt{3-(x+u)^2}} d\mu(z).$$

In general, (73) was obtained in [5] for certain surfaces using topological arguments (Gauss-Bonnet) and adapted in [6] to $\Gamma\backslash\mathcal{H}$, but without using the analytic expression for $\nu_A(z)$ in (72). Partially motivated by their work and also by a desire to combine Zagier's approach using continued fractions with the more geometric methods of Dirichlet/Hecke, in [15] we constructed a hyperbolic surface $\mathcal{F}_A$ (an orbifold, actually) associated to A that is bounded by $\mathcal{C}_A$ and has area $\ell_A \pi$, also a consequence of the Gauss-Bonnet theorem. This surface is a partial cover of $\mathcal{F}$ with $m_A - \nu_A(z)$ points of $\mathcal{F}_A$ over $z \in \mathcal{F}$. In particular, we get a more analytic proof of (73) by combining this counting interpretation with (46).

The Rademacher symbol has many other geometric/algebraic/topological interpretations. See Hirzebruch's article [26] for relations to Hilbert modular surfaces. More recently, Ghys [18] gave the beautiful result that the Rademacher symbol gives the linking number of a modular knot (a lift of $\mathcal{C}_A$ to the unit tangent bundle) with a certain trefoil knot. See also [16].

Evaluation of $\zeta'_-(0, A)$: new results

We now turn to the problem of computing $\zeta'_-(0, A)$, Returning to Hecke's representation (60), we see that to evaluate $\zeta'_-(0, A)$ using it directly we must integrate $\partial_z H_2(z)$ from (26) over $\mathcal{C}_A$. Following the example of the third proof of Theorem 6 given above, we will instead integrate $H(z)$ against $\nu_A(z)$ over $\mathcal{F}$. The following result appears to be a new contribution to the classical theory of KLF.

Theorem 7. *For A a narrow ideal class in the real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{D})$, where $D = d'd$ with d and d' negative co-prime fundamental discriminants, we have*

$$\zeta_-(0, A) = \tfrac{1}{4\pi} \int_{\mathcal{F}} \nu_A(z) d\mu(z) \quad \text{and}$$

$$\zeta'_-(0, A) = -\tfrac{1}{4\pi} \int_{\mathcal{F}} \nu_A(z) \big(H(z) + 1 + \tfrac{1}{2} \log(\tfrac{D}{2^8 \pi^4}) \big) d\mu(z),$$

where $\nu_A(z)$ is defined in (72).

When applied to an odd genus character this yields an analogue of Lerch's formula (40).

Corollary 4. *Write $h = h(d)$ and $h' = h(d')$ and similarly $w = w_d$ and $w' = w_{d'}$. For $D = d'd$ with $d', d < 0$ co-prime fundamental discriminants and χ the associated genus character,*

$$\tfrac{3}{\pi}\sum_{A\in \mathrm{Cl}_D^+}\chi(A)\int_{\mathcal{F}}\nu_A(z)d\mu(z) = \tfrac{48hh'}{ww'} \quad \text{and}$$

$$\tfrac{3}{\pi}\sum_{A\in \mathrm{Cl}_D^+}\chi(A)\int_{\mathcal{F}}\nu_A(z)H(z)d\mu(z) = -24\Big(\tfrac{h'}{w'}\sum_{n=1}^{|d|-1}\chi_d(n)\log\Gamma\big(\tfrac{n}{|d|}\big) + \tfrac{h}{w}\sum_{n=1}^{|d'|-1}\chi_{d'}(n)\log\Gamma\big(\tfrac{n}{|d'|}\big)\Big)$$
$$+\tfrac{48hh'}{ww'}\big(\tfrac{1}{2}\log D - 1 + \log(16\pi^2)\big).$$

This corollary implies, after some computation, an identity that is somewhat analogous to (42) and (66) and is the main new result of this paper. It gives a surprising geometrical relationship between the CM points associated to the imaginary quadratic fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{d'})$ and the winding number functions $\nu_A(z)$ coming from the closed geodesics associated to the real quadratic field $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\sqrt{d'd})$.

Corollary 5. *Assumptions as above,*

$$\frac{\sum_{C\in \mathrm{Cl}_D^+}\chi(C)\int_{\mathcal{F}}\nu_C(z)H^*(z)d\mu(z)}{\sum_{C\in \mathrm{Cl}_D^+}\chi(C)\int_{\mathcal{F}}\nu_C(z)d\mu(z)} = \tfrac{1}{h(d)}\sum_{A\in \mathrm{Cl}_d^+}H^*(z_A) + \tfrac{1}{h(d')}\sum_{B\in \mathrm{Cl}_{d'}^+}H^*(z_B),$$

where $H^*(z) = H(z) - 1$.

In case d and d' are prime discriminants there are exactly two genera, one containing I and the other containing J. If $h(D) = 2$, Corollary 5 simplifies to

$$\tfrac{1}{\int_{\mathcal{F}}\nu_I(z)d\mu(z)}\int_{\mathcal{F}}\nu_I(z)H^*(z)d\mu(z) = \tfrac{1}{h(d)}\sum_{A\in \mathrm{Cl}_d^+}H^*(z_A) + \tfrac{1}{h(d')}\sum_{B\in \mathrm{Cl}_{d'}^+}H^*(z_B). \tag{76}$$

For example, when $D = 12$ we have numerically

$$\tfrac{3}{\pi}\int_{\mathcal{F}}H^*(z)\nu_I(z)d\mu(z) = B(0) + B(1) + B(-1) = 0.0882075\ldots,$$

where

$$B(u) = \tfrac{6}{\pi}\int_0^{1/2}\int_{\sqrt{1-x^2}}^{\sqrt{3-(x+u)^2}}H^*(z)d\mu(z).$$

By (76) and (75) this must equal

$$H^*(i) + H^*\big(\tfrac{1+\sqrt{-3}}{2}\big) = -2 - \log\Big(\frac{3\Gamma(\frac{1}{3})^6\Gamma(\frac{1}{4})^4}{512\pi^7}\Big),$$

as may be verified numerically.

Given (73), the proof of Theorem 7 comes down to Hecke's representation (60) and the following result.

Lemma 1. *For $\tau \in \mathrm{int}\,\mathcal{F}$ and $K(z,\tau)$ defined in (70) we have*

$$i\partial_\tau E^*(\tau,s) = \tfrac{s(s-1)}{2}\int_{\mathcal{F}}K(z,\tau)E^*(z,s)d\mu(z).$$

Proof. Note that for fixed $\tau \in \mathcal{F}$ and $y > \operatorname{Im}\tau + 1$ we have that $K(z,\tau) = O(e^{-2\pi y})$ where as usual $z = x + iy$. Clearly

$$4y^2 \partial_{\overline{z}} \partial_z = \Delta$$

for Δ from (24). Thus by (25) we need to show

$$g(\tau) = -\int_{\mathcal{F}} K(z,\tau) \partial_{\overline{z}}\, g(z) d\overline{z} dz,$$

after using that $d\overline{z}dz = 2i dx\, dy$ and writing $g(z) = \partial_z E^*(z,s)$ for fixed $s \neq 0, 1$.

For $\delta > 0$, let $D_\tau(\delta) = \{z \in \mathbb{C} : |z - \tau| < \delta\}$ and $\mathcal{F}(Y) = \{z \in \mathcal{F} : y \leq Y\}$. Since $K(z,\tau)$ is holomorphic, for $\tau \in \operatorname{int}\mathcal{F}$ we need to show

$$g(\tau) = -\lim_{\substack{Y\to\infty \\ \delta\to 0}} \int_{\mathcal{F}(Y)\setminus D_\tau(\delta)} \partial_{\overline{z}}\big(g(z)K(z,\tau)\big) d\overline{z} dz.$$

For $\tau \in \operatorname{int}\mathcal{F}$ choose Y large enough and δ small enough so we may apply Stokes' theorem to get

$$\int_{\mathcal{F}(Y)\setminus D_\tau(\delta)} \partial_{\overline{z}}\big(g(z)K(z,\tau)\big) d\overline{z} dz = \int_{\partial\mathcal{F}_Y} g(z)K(z,\tau)dz - \int_{\partial D_\tau(\delta)} g(z)K(z,\tau)dz.$$

Now $\lim_{Y\to\infty} \int_{\partial\mathcal{F}_Y} g(z)K(z,\tau)dz = 0$ and $\lim_{\delta\to 0} \int_{\partial D_\tau(\delta)} g(z)K(z,\tau)dz = g(\tau)$, giving the result.

$\square$

The main goal of our companion paper [15] was to study the distribution properties of the surfaces $\mathcal{F}_A$ when A is averaged over a genus. For this we needed to make use of integrals over $\mathcal{F}_A$ of Maass cusp forms as well as the Eisenstein series $E(z,s)$, but only for $\operatorname{Re} s = \frac{1}{2}$, since these are the eigenfunctions that occur in the spectral expansion of Δ. It is notable that $H(z)$, which comes from $s = 0$, is not integrable on $\mathcal{F}_A$ with respect to $d\mu$. This is one reason for our use here of $\nu_A(z)$ and the singular kernel $K(z,\tau)$ in Lemma 1, instead of simply integrating $H(z)$ over the surface $\mathcal{F}_A$.

4. Concluding remarks

We have only looked at the applications of KLF to L-functions for quadratic $\mathbb{K}$ with genus characters. Already here there are interesting applications to certain quadratic extensions of $\mathbb{K}$ giving class number relations, but this is just the beginning, even when one is only interested in abelian extensions of quadratic fields (see [44]). A lot of attention has been devoted to finding elaborations of the ideas originated by Kronecker with his limit formula to other L-functions and other values of s.

A rather different theme, one that we have emphasized without actually saying so, is the use of KLF to study averages of the height function H, like in (42), (66) and Corollary 5. The height function $H(\tau)$, thought of as being defined through KLF, gives the height of the torus $\mathbb{C}/L$, where $L = (\operatorname{Im}\tau)^{-1/2}(\mathbb{Z} + \tau\mathbb{Z})$. This definition has been generalized using spectral zeta functions of curves of genus greater than one. The corresponding higher dimensional versions of KLF and their applications to Riemann surfaces and in physics, pioneered in papers of Ray-Singer [38], Fay [17], D'Hoker–Phong [13] and Sarnak [39], [40], represent another major topic initiated by Kronecker's ideas.

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