THE KATOK-SARNAK FORMULA FOR HIGHER WEIGHTS

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Abstract. We prove a Katok-Sarnak type formula for Maass forms of even weight k and odd level $N \geqslant 1$ that includes and extends the results of Waldspurger, Kohnen-Zagier, Katok-Sarnak, Baruch-Mao, and Biró.

1. Introduction

The goal of this note is to prove an extension of the Katok-Sarnak formula along the lines given in [10] for the case of weight zero and non-square discriminant. Our main theorem, Theorem 1.4, includes, extends and reproves theorems of Waldspurger [33, 34], Kohnen-Zagier [20, 19], Katok-Sarnak [16], Baruch-Mao [3], and Biró [5].

These results relate the Fourier coefficients of half-integral weight modular forms and certain cycle integrals (integrals along closed geodesics) of their Shimura lift.

To explicitly state the above mentioned results and our theorem we will first introduce notation and normalizations for the automorphy factor, spectral parameters, Fourier expansions and cycle integrals that will be fixed throughout the paper.

1.1. Basic notation.

Definition 1.1. A *Maass cusp form* of weight $k \in 2\mathbf{Z}$ for the group $\Gamma = \Gamma_0(N)$ with $N \geqslant 1$ is a real-analytic function $\varphi : \mathbf{H} \to \mathbf{C}$ such that

(1) $\Delta_k\varphi = \lambda\varphi$ for a $\lambda \in \mathbf{C}$, where $\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + iky(\partial_x + i\partial_y)$ is the hyperbolic Laplace-Beltrami operator,

(2) $\varphi|_k\gamma = (cz+d)^{-k}\varphi\left(\frac{az+b}{cz+d}\right) = \varphi(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

(3) $\displaystyle\int_{\Gamma\backslash\mathbf{H}} |\varphi(z)|^2 \frac{dxdy}{y^2} < \infty$,

(4) $\displaystyle\int_0^1 (\varphi|_k\sigma_{\mathfrak{a}})(x+iy)dx = 0$ for all $y > 0$, where for each cusp $\mathfrak{a}$ of Γ the matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbf{R})$ is such that $\sigma_{\mathfrak{a}}.\mathfrak{a} = i\infty$.

We will write the eigenvalue of a Maass cusp form φ as $\lambda = (s - k/2)(1 - k/2 - s)$ for an $s \in \mathbf{C}$ and call s the *spectral point* of φ. Maass cusp forms with the same spectral point form a finite-dimensional vector-space, which we denote by $U_{k,N}(s)$.

The Laplacian Δ_k can be written either in terms of the ξ-operator

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \text{ where } \xi_k = 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}, \tag{1.1}$$

or in terms of raising and lowering operators

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$$\Delta_k = L_{k+2}R_k - k = R_{k-2}L_k, \tag{1.2}$$

where

$$L_k = 2iy^2\partial_{\overline{z}} \text{ and } R_k = \frac{k}{y} + 2i\partial_z. \tag{1.3}$$

Clearly, $\xi_k f = 0$ if and only if f is holomorphic and hence holomorphic cusp forms are Maass cusp forms.

By Stokes' theorem, for any $\varphi_1, \varphi_2 \in U_{k,N}(s)$ we have

$$\langle \Delta_k\varphi_1, \varphi_2\rangle = \langle \xi_k\varphi_1, \xi_k\varphi_2\rangle, \tag{1.4}$$

where

$$\langle \varphi_1, \varphi_2\rangle = \int_{\Gamma\backslash\mathbf{H}} \varphi_1(z)\overline{\varphi_2(z)}y^{k-2}dxdy. \tag{1.5}$$

Since $\langle\cdot,\cdot\rangle$ is Hermitian, the eigenvalue of a Maass cusp form is always real. Since $\langle \xi_k\varphi, \xi_k\varphi\rangle > 0$ if and only if $\xi_k\varphi$ is not identically zero, the space of holomorphic cusp forms $S_{k,N}$ is, in fact, equal to the space $U_{k,N}(k/2)$.

The eigenvalue λ being real means that the imaginary part of $(s - k/2)(1 - k/2 - s)$ must vanish for every spectral point $s \in \mathbf{C}$ with $U_k(s) \neq \{0\}$. This translates to $s = \frac{1}{2} + ir$ for an $r \in \mathbf{R}$ or to s being real. The non-zero eigenvalues of the latter form constitute the exceptional spectrum.

Similarly, we define half integral weight modular forms with the θ-multiplier. For $\kappa \in \frac{1}{2}\mathbf{Z}$ we write

$$\psi|_\kappa\gamma = j(\gamma, z)^{-2\kappa}\psi(\gamma z) \tag{1.6}$$

where $j(\gamma, z) = \frac{\theta(\gamma.z)}{\theta(z)}$. For $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(4N)$, the cocycle is given by

$$j(\gamma, z) = \varepsilon_a^{-1}\left(\frac{c}{a}\right)(cz + d)^{\frac{1}{2}}$$

with

$$\varepsilon_a = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 4, \\ i, & \text{if } a \equiv 3 \pmod 4, \end{cases}$$

and the square root is defined by the canonical branch of the complex logarithm.

The space $U_{\kappa,4N}(s)$ consist of functions that satisfy (1), (3) and (4) of Definition 1.1 (with κ instead of k) and where (2) is replaced by (1.6).

1.2. **The Shimura correspondence.** It is easily seen that any $\varphi \in U_{k,N}(s)$ has a Fourier-expansion given by

$$\varphi(z) = \sum_{n\neq 0} a_\varphi(n)(4\pi|n|y)^{-\frac{k}{2}}W_{\operatorname{sign}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y)e(nx), \; a_\varphi(n) \in \mathbf{C}. \tag{1.7}$$

Here $W_{\nu,\mu}(y)$ is the W-Whittaker function [4, Ch. 6.7], which is exponentially decaying as $y \to +\infty$, hence the same remains true for $\varphi(z)$.

There is a Hecke and Atkin-Lehner theory [1] on $U_{k,N}(s)$ for all $s \in \mathbf{C}$ as there is one on $S_{k,N} = U_{k,N}(k/2)$ and hence there exists a basis of simultaneous eigenforms for all Hecke operators $T_k(n)$ with $(n, N) = 1$. If φ is a newform, we have $a(1) \neq 0$, and we may normalize φ to have $a(1) = 1$; such a form will be called *Hecke-normalized*. The linear map $X : \varphi(z) \to y^k\varphi(-\overline{z})$ is a map $U_{k,N}(s) \to U_{-k,N}(s)$. Supposing that $k > 0$, we may first apply the lowering operators $L_{2-k} \circ L_{k-4} \circ \cdots \circ L_k$ and then apply X so that we get an involution $U_{k,N}(s) \to U_{k,N}(s)$, which commutes with the Laplacian Δ_k. Similarly, we use raising operators for weight $k < 0$. As the map is an involution, it has eigenvalue ± 1. We call Maass cusp forms with eigenvalue $+1$ even Maass forms (and forms with eigenvalue -1 odd forms). The Fourier coefficients of any even form $\varphi(z)$ of weight k satisfy

$$a_\varphi(-n) = \frac{\Gamma(s+k/2)}{\Gamma(s-k/2)} a_\varphi(n) \tag{1.8}$$

with $n > 0$; see [8, Sec. 4] for more details.

For convenience the Fourier coefficients of half integral weight Maass cusp form $\psi \in U_{\kappa,4N}(s)$ will be designated by $b_\psi(n)$, so that we have

$$\psi(z) = \sum_{n \neq 0} b_\psi(n)(4\pi|n|y)^{-\frac{\kappa}{2}} W_{\mathrm{sign}(n)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi|n|y)e(nx). \tag{1.9}$$

The celebrated work of Shimura [31] gives a Hecke theory and a correspondence between half integral weight forms and integral weight forms in the holomorphic case, and this extends to Maass cusp forms; see [12], [21], [29], and [25].

For k an even integer, $\psi \in U_{\frac{k+1}{2},4N}(s)$ a Hecke eigenform, and $(-1)^{\frac{k}{2}} d$ a fundamental discriminant, we define the *Shimura lift* of ψ as follows. For $m > 0$ let

$$\sum_{\substack{n|m,\\(n,N)=1}} n^{k/2-1} \left(\frac{(-1)^{\frac{k}{2}} d}{n}\right) b_\psi\left(\frac{m^2 d}{n^2}\right) = a(m) b_\psi(d). \tag{1.10}$$

and extend this to $m < 0$ by (1.8). By Hecke theory the $a(n)$ are well defined and we let

$$\mathrm{Shim}_d(\psi) = \sum_{n \neq 0} a(n)(4\pi|n|y)^{-\frac{k}{2}} W_{\mathrm{sign}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi|n|y)e(nx). \tag{1.11}$$

The function $\mathrm{Shim}_d(\psi)$ is a Maass cusp form of weight k [12]. This is the non-holomorphic analogue of Shimura's original result [31].

Generally, we will restrict Shim_d to the Kohnen plus space

$$U^+_{\frac{k+1}{2},4N}(s) = \left\{\psi(z) \in U_{\frac{k+1}{2},4N}(s) : b_\psi(n) \neq 0 \text{ only if } (-1)^{\frac{k}{2}} n \equiv 0, 1 \pmod 4\right\}. \tag{1.12}$$

when we have

$$\mathrm{Shim}_d : U^+_{\frac{k+1}{2},4N}(s) \to U_{k,N}(2s-1).$$

1.3. **The Katok-Sarnak formula.** It was first shown by Shintani [32] that we may view the Fourier coefficients of a half-integral weight Maass cusp form in terms of traces of its image under the Shimura lift (this is not a precise statement, as the Shimura correspondence is not generally known to be an isomorphism). This idea lead to formulas for the central value of certain automorphic L-functions, a question with a long and rich history, with many applications, see for example [6, 7, 23, 27, 28].

The Shimura-Shintani [31, 32] correspondence allows one to express the central L-value of an even weight holomorphic modular form in terms of Fourier coefficients of its half integral weight correspondent. Waldspurger [33, 34] was the first to establish such a relation which was then made explicit by Kohnen and Zagier [19].

The method applied by Shintani to prove his result was based on a theta-correspondence. In the setting of non-holomorphic modular forms, this idea was originally introduced by Maass [25] . The methods in [25] were later explicated and further developed by Katok and Sarnak in [16]. The main result of [16] is the analog[1] of the results of Waldspurger and Kohnen-Zagier in the case of weight zero Maass forms for $\mathrm{SL}_2(\mathbf{Z})$. To state the Katok-Sarnak formula we will briefly define the traces alluded above (see section 3.1 for further details.)

Let $\mathcal{Q}_{N,D}$ be the set of integral binary quadratic forms $Q(x,y) = [A,B,C] = Ax^2 + Bxy + Cy^2$ of discriminant $D > 0$ such that $N|A$, whereas for $D < 0$, we restrict $\mathcal{Q}_{N,D}$ to the set of positive definite binary forms $Q(x,y)$ with $N|A$. The forms $Q \in \mathcal{Q}_{N,D}$ are acted on as usual by Γ, resulting in finitely many classes $\Gamma\backslash\mathcal{Q}_{N,D}$.

For $Q \in \mathcal{Q}_{N,D}$ and $D < 0$, let z_Q be the root of $Q(z,1) = 0$ in $\mathbf{H}$ and for $D > 0$, let S_Q be the associated geodesic in $\mathbf{H}$ and $C_Q = \Gamma_Q\backslash S_Q$, where Γ_Q is the group of automorphs of Q. The geodesic associated to a binary quadratic form of square discriminant is non-compact.

Definition 1.2. Let d be a fundamental discriminant and $d' \equiv 0,1 \pmod 4$. For $dd' > 0$ the *twisted trace* of an automorphic function $\varphi : \mathbf{H} \to \mathbf{C}$ of weight $k \in 2\mathbf{Z}$ for Γ is defined by

$$\mathrm{Tr}_{d,d'}(\varphi) = \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,dd'}} \chi_d(Q) \int_{C_Q} \varphi(z) Q(z,1)^{k/2} \frac{\sqrt{dd'}dz}{Q(z,1)},$$

whenever the integral converges. The character χ_d is the genus character given by

$$\chi_d([A,B,C]) = \begin{cases} \left(\frac{d}{r}\right), & \text{if } (A,B,C,d) = 1,\ Q \text{ represents } r, \text{ and } (r,d) = 1; \\ 0, & \text{if } (A,B,C,d) > 1. \end{cases}$$

Note that when restricted to S_Q the holomorphic differential form $\sqrt{dd'}dz/Q(z,1)$ simply becomes arc length measure.

Definition 1.3. Let d be a fundamental discriminant and $d' \equiv 0,1 \pmod 4$. For $dd' < 0$, the *twisted trace* of an automorphic function f of weight $k \in 2\mathbf{Z}$ for Γ is defined for $k \geqslant 0$ by

$$\mathrm{Tr}_{d,d'}(f) = |dd'|^{\frac{k}{4}} \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,dd'}} \chi_d(Q)\omega_Q^{-1} L_k^k f(z_Q)$$

and for $k < 0$ by

$$\mathrm{Tr}_{d,d'}(f) = |dd'|^{\frac{k}{4}} \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,dd'}} \chi_d(Q)\omega_Q^{-1} R_k^{-k} f(z_Q)$$

Here $\omega_Q = |\Gamma_Q|$ and for $n \geqslant 1$, we set $L_k^{2n} = L_{k-2n}L_{k-2n+2}\cdots L_{k-2}L_k$ and $R_k^{2n} = R_{k+2n}R_{k+2n-2}\cdots R_{k+2}R_k$. The character χ_d is defined as in Definition 1.2.

Theorem (Katok-Sarnak [16])**.** *Let φ be an even Maas cusp form in $U = U_{0,1}$ then*

$$6\sqrt{|d|}\sum_{\psi} \frac{\overline{b_\psi(1)}b_\psi(d)}{\langle\psi,\psi\rangle} = \langle\varphi,\varphi\rangle^{-1}\cdot \begin{cases} 2\sqrt{\pi}\mathrm{Tr}_{d,1}(\varphi), & \text{if } d < 0, \\ \mathrm{Tr}_{d,1}(\varphi), & \text{if } d > 0. \end{cases} \tag{1.13}$$

where ψ runs over an orthogonal basis of the preimage of φ under Shim_1.

[1]For this reason these results are sometimes referred to in the literature as formulas of Waldspurger or Kohnen-Zagier type.

To see the relation of this result to the central L-values note that when $d = 1$ the sum in the $\mathrm{Tr}(\varphi)$ on the right hand side of (1.13) has only one form xy, and the corresponding cycle is the geodesic $[0, i\infty]$. This gives

$$6\sqrt{|d|}\sum_{\psi}\frac{\overline{b_\psi(1)}b_\psi(d)}{\langle\psi,\psi\rangle} = \frac{1}{\langle\varphi,\varphi>}\int_0^\infty \varphi(iy)\frac{dy}{y}.$$

Hence in this case the right hand side of (1.13) can be written in terms of the central L-value $L(\varphi, 1/2)$ of $L(\varphi, w) = \sum_{n=1}^{\infty} a_\varphi(n)/n^{w-1/2}$. As a corollary one obtains the following non-negativity result.

Theorem (Katok-Sarnak)**.** *If φ is a Hecke normalized even Maass form in U then*

$$L(\varphi, 1/2) \geqslant 0$$

1.4. **The main result of the paper.** The results of Katok-Sarnak are generalized to higher levels by Baruch-Mao [3] and Biró [5], using different methods. Our main theorem includes, extends and reproves theorems of Waldspurger [33, 34], Kohnen-Zagier [19, 20], Katok-Sarnak [16], Baruch-Mao [3] and Biró [5].

Theorem 1.4. *Let k be an even integer, $N \geqslant 1$ odd and $\varphi \in U_{k,N}(s)$ be an even normalized newform with* $\mathrm{Re}(s) > 0$*. Let d, d' be a pair of integers such that $(-1)^{\frac{k}{2}}d, (-1)^{\frac{k}{2}}d' \equiv 0, 1 \pmod 4$ and $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant. Then we have*
(1.14)

$$6(-1)^{\lfloor k/4\rfloor}\sqrt{|D|}\sum_{\mathrm{Shim}_d(\psi)=\varphi}\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle} = \langle\varphi,\varphi\rangle^{-1}\cdot\begin{cases} 2\sqrt{\pi}\,\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\varphi), & \text{if } dd' < 0,\\ 2^{k/2}\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\varphi), & \text{if } d, d' > 0,\\ 2^{1-k/2}\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\xi_k\varphi), & \text{if } d, d' < 0,\end{cases}$$

where $\sum_{\mathrm{Shim}_d(\psi)=\varphi}$ means that ψ runs over an orthogonal basis in $U^+_{k,N}\left(\frac{s}{2}+\frac{1}{4}\right)$ of the preimage of φ under Shim_d.

Remark 1.5. The results of [16, 3, 5] do not include the case of $d, d' < 0$. The extension to the case of both discriminants $d, d' < 0$ was first done in [10] only in the case of weight 0 and dd' not a square. Hence Theorem 1.4 generalizes and extends all the previous results of this type.

Remark 1.6. Note that if φ is a holomorphic cusp form of weight $k > 0$, then $L_k^k\varphi = 0$ and $\xi_k\varphi = 0$. Hence if at least one of d, d' is negative then both sides of the identity (1.14) are zero (as the negative Fourier coefficients of ψ vanish).

Remark 1.7. The Fourier coefficients in equation (1.14) are as in (1.9) and so agree with the usual Fourier coefficients in the holomorphic case. For holomorphic forms the case $d, d' > 0$ was first proved by Kohnen [20, Thm. 3], where the constant differs from ours by $6\sqrt{D}$. This comes from our normalization for the inner product which leads to a factor $|\Gamma_0(N) : \Gamma_0(4N)| = 6$, and that we have an additional factor of $\sqrt{D}$ in our definition of the trace to match the arc length integrals in [16].

The following Corollary is a generalization of a result of Baruch and Mao [3, Thm. 1.4] to higher weights.

Corollary 1.8. *Let k be an even non-negative integer, let $\varphi \in U_{k,N}(s)$ be an even normalized newform with spectral point $s \neq \frac{k}{2}$, and let d be a fundamental discriminant with*

$(-1)^{\frac{k}{2}}d > 0$ and $(d, N) = 1$. Suppose that for all $m\|N$, its eigenvalues w_m under the Atkin-Lehner involution W_m are equal to $\left(\frac{d}{m}\right)$. Then

$$6 \sum_{\mathrm{Shim}_d(\psi)=\varphi} \frac{|b_\psi(|d|)|^2}{\langle \psi, \psi \rangle} = \frac{2^{\nu(N)}|d|^{\frac{k-1}{2}}}{(4\pi)^{k/2}\sqrt{\pi}} \Gamma\left(\frac{s}{2} + \frac{k}{2}\right) \Gamma\left(\frac{1-s}{2} + \frac{k}{2}\right) \frac{L(\varphi, d, 1/2)}{\langle \varphi, \varphi \rangle},$$

where $\nu(N)$ is the number of distinct prime factors of N and where $L(\varphi, d, 1/2)$ is defined by (5.1).

Finally we should mention that there are other generalizations of the results of Waldspurger and Kohnen-Zagier. They are too numerous to list here but we mention the papers of Khuri-Makdisi [17] and Kojima [22, 24] for generalizations to the case of number fields.

1.5. **The method of proof.** In this paper we use spectral methods similar to the ones employed by Duke, Imamoḡlu, and Tóth [10]. Our main result, Theorem 1.4, relates traces of integral weight cusp forms to the products of half integral weight Fourier coefficients. The proof uses an idea that goes back to Selberg [30] (see also [13] for a spectacular application). Roughly speaking our method involves proving identities for cycle integrals of Poincaré series, built from some test function ϕ_s, that are dependent on the spectral parameter s. The cycle integrals are then expressible in terms of sums of terms of the form $K(m, n; c)F_s(mn/c^2)$, where $K(m, n; c)$ are certain Kloosterman sums. These expressions will have poles at spectral points that may be exploited to prove various identities. The function F_s is a complicated integral transform of ϕ_s and we replace them with their first order approximations which makes the arguments elementary. This is one of the main technique used in our proof. The other input we need involves the choice of the Poincaré series.

The main tool in [10] was Niebur Poincaré series. These series are eigenfunctions of the Laplacian and appear as Fourier coefficients of the resolvent kernel of Δ_k; see [11]. They have the disadvantage of being of exponential growth at the cusp and non-integrable over the non-compact geodesics corresponding to square discriminants.

To circumvent this problem, we work instead with Selberg Poincaré series, which do not have these convergence problems. The use of both Niebur and Selberg Poincaré series is based on the fact that they have poles at spectral points with residues giving cuspidal Maass forms. The Selberg Poincaré series are not eigenfunctions of the Laplacian but have better analytical properties.

1.6. **Outline of the paper.** The main steps in the proof of Theorem 1.4 and the structure of the paper is as follows.

In the next section we start reviewing some standard results from spectral theory. We use them to get to Maass cusp forms through the residues of Selberg and Niebur Poincaré series. The needed spectral theory results are reviewed in Propostions 2.3 and 2.6. By going one step further and looking at the Fourier coefficients of the Poincaré series one can write the residues of Kloosterman zeta function in terms of products of Fourier coefficients of cusp forms. This is the content of Proposition 2.5.

The next ingredient in the proof of Theorem 1.4, which is also the main technical result of the paper, is Theorem 3.1. This theorem which relates the traces of Selberg Poincaré series to Kloosterman zeta functions together with other results about the cycle integrals are given in Section 3.

In Section 4 we start by proving an averaged version of our main theorem in Proposition 4.2. This proposition follows by taking residues on both sides of the identities for traces of

Selberg Poincaré series given in Theorem 3.1 together with the Shimura relations (1.10) and the spectral result in Proposition 2.5. The proof of Theorem 1.4 is then finished by use of some linear algebra.

Finally in Section 5, the main theorem is applied in the case of square discriminants to relate L-values of integral weight forms to the squares of half integral weight coefficients.

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2. Poincaré series

In this section, we collect some facts about Selberg and Niebur Poincaré series. Most of these are well documented in the literature but this allows us to fix some normalizations.

2.1. The Niebur Poincaré series. Let κ be an integer or half an odd integer, $\Gamma = \Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N \right\}$$

and let Γ_∞ denote the stabilizer of $i\infty$.

The Poincaré series of the Niebur type for $m \neq 0$ is defined by

$$F_m^\kappa(z,s) = -\frac{\Gamma\left(s - \operatorname{sign}(m)\frac{\kappa}{2}\right)}{4\pi|m|\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (y^{-\frac{\kappa}{2}} M_{\operatorname{sign}(m)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi|m|y)e(mx)|_\kappa \gamma), \tag{2.1}$$

where $M_{\mu,\nu}(y)$ is the M-Whittaker function [4, Ch. 6.7]. Since [4, 6.1.2, 6.7.2]

$$M_{\mu,\nu}(y) = y^{\nu+1/2}(1 + O(y)),\ y \to 0, \tag{2.2}$$

the Poincaré series $F_m^\kappa(z,s)$ is absolutely convergent for $\operatorname{Re}(s) > 1$.

Remark 2.1. With our normalization, the Niebur Poincaré series $F_m^\kappa(z,s)$ is an eigenfunction of the Laplacian Δ_κ with eigenvalue $(s-\kappa/2)(1-\kappa/2-s)$. One should compare our definition with the normalization given in Fay [11], which is $y^{\frac{\kappa}{2}} F_m^\kappa(z,s)$ in our notation. More generally, an eigenfunction $f(z)$ with eigenvalue $s(1-s)$ of the Laplacian $-D_\kappa$ in Fay's paper yields an eigenfunction $y^{-\frac{\kappa}{2}} f(z)$ of Δ_κ with eigenvalue $(s-\kappa/2)(1-\kappa/2-s)$ and vice versa.

It follows from [11, Thm. 3.4] that the Niebur Poincaré series $F_m^\kappa(z,s)$ has the following Fourier expansion

Proposition 2.2. *Let m be a nonzero integer and* $\operatorname{Re}(s) > 1$. *We have*

$$\begin{aligned} F_m^\kappa(z,s) = &-\frac{\Gamma\left(s - \operatorname{sign}(m)\frac{\kappa}{2}\right)}{4\pi|m|\Gamma(2s)} y^{-\frac{\kappa}{2}} M_{\operatorname{sign}(m)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi|m|y)e(mx) \\ &- \sum_{n \in \mathbf{Z}} \frac{e^{-\pi i \frac{\kappa}{2}}}{2\sqrt{|mn|}} \frac{\Gamma\left(s - \operatorname{sign}(m)\frac{\kappa}{2}\right)}{\Gamma\left(s + \operatorname{sign}(n)\frac{\kappa}{2}\right)} L_{m,n}^\kappa(s) y^{-\frac{\kappa}{2}} W_{\operatorname{sign}(m)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi|n|y)e(nx), \end{aligned}$$

where

$$L_{m,n}^\kappa(s) = \begin{cases} \sum_{0<c\equiv 0(N)} \frac{K_\kappa(m,n;c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \operatorname{sign}(mn) > 0, \\ \sum_{0<c\equiv 0(N)} \frac{K_\kappa(m,n;c)}{c^{2s}} & \text{if } n = 0, \\ \sum_{0<c\equiv 0(N)} \frac{K_\kappa(m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } \operatorname{sign}(mn) < 0, \end{cases}$$

where

$$K_\kappa(m,n;c)=\begin{cases}\sum_{a(c)^*} e\left(\frac{ma+n\bar{a}}{c}\right) & \text{if } \kappa\in\mathbf{Z},\\ \sum_{a(c)^*} \varepsilon_a^{2\kappa}\left(\frac{c}{a}\right)^{2\kappa} e\left(\frac{ma+n\bar{a}}{c}\right) & \text{if } \kappa\in\frac{1}{2}\mathbf{Z},\end{cases}$$

is the Kloosterman sum, which runs over all a modulo c for which an integer $\bar{a}$ exists such that $a\bar{a}\equiv 1 \pmod c$, and the functions $I_\nu(x)$, and $J_\nu(x)$ are the Bessel-functions of the first kind.

From the theory of the resolvent kernel one knows that $F_m^\kappa(z,s)$ has an analytic continuation to all of $s\in\mathbf{C}$ as long as $(s-\kappa/2)(1-\kappa/2-s)$ does not lie in the spectrum of the operator Δ_κ and that $F_m^\kappa(z,s)$ has poles at spectral points of Maass cusp forms with the residue given in terms of their Fourier coefficients [11, Cor. 3.6]. More precisely, we have the following proposition.

Proposition 2.3. *Let $m\neq 0$. The Niebur Poincaré series $F_m^\kappa(z,s)$ has poles at the spectral points $s=\frac{1}{2}+ir$ and*

$$\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m^\kappa(z,s)=-(4\pi|m|)^{-k/2}\sum_\varphi \frac{\overline{a_\varphi(m)}}{\langle\varphi,\varphi\rangle}\varphi(z),$$

where the sum goes over a basis of Hecke eigenforms φ of the subspace $U_{\kappa,N}(s)$.

Proof. The proposition is an adaptation of Corollary 3.6 in Fay's paper [11]. There, the Fourier coefficients of the Niebur Poincaré series have poles at all s such that $s(1-s)$ lies in the discrete spectrum of Fay's normalization of the Laplacian $-D_\kappa$. The value of the residue then is equal to the Fourier coefficients of the reproducing kernel of the corresponding eigenspaces. Again, as in Remark 2.1, we note that the eigenfunctions $f(z)$ of $-D_\kappa$ are in one-to-one correspondence with the eigenfunctions $y^{-\frac{\kappa}{2}}f(z)$ of Δ_κ, i.e. the eigenspaces are isomorphic to the spaces $U_{\kappa,N}(s)$. See also the proof of [10, Prop. 3] for further clarification of the argument. □

For κ half an odd integer, we also need the projection of the Niebur Poincaré series onto the Kohnen plus space (1.12). We fix k to be an even integer, $N\geqslant 1$ odd, and $\Gamma=\Gamma_0(4N)$. For the Niebur Poincaré series $F_m^{\frac{k+1}{2}}(z,s)$, let $F^+_{m,\frac{k+1}{2}}$ be its projection onto the Kohnen plus space $U^+_{\frac{k+1}{2},4N}(s)$. The same argument as in the holomorphic case [20, p. 250-257], proves the following proposition.

Proposition 2.4. *Let m be a nonzero integer such that $(-1)^{\frac{k}{2}}m\equiv 0,1 \pmod 4$ and $\mathrm{Re}(s)>1$. We have*

$$\begin{aligned}F^+_{m,\frac{k+1}{2}}(z)&=-\frac{\Gamma\left(s-\mathrm{sign}(m)\frac{k+1}{4}\right)}{6\pi|m|\Gamma(2s)}y^{-\frac{k+1}{4}}M_{\mathrm{sign}(m)\frac{k+1}{4},s-\frac{1}{2}}(4\pi|m|y)e(mx)\\&\quad-\sum_{(-1)^{\frac{k}{2}}n\equiv 0,1(4)}\frac{2}{3}\left(\delta_{m,n}+\frac{(-1)^{\lfloor\frac{k}{4}+\frac{1}{2}\rfloor}}{\sqrt{2|mn|}}\frac{\Gamma\left(s-\mathrm{sign}(m)\frac{k+1}{4}\right)}{\Gamma\left(s+\mathrm{sign}(n)\frac{k+1}{4}\right)}L_{m,n}^{\frac{k+1}{2},+}(s)\right)\times\\&\quad y^{-\frac{k+1}{4}}W_{\mathrm{sign}(n)\frac{k+1}{4},s-\frac{1}{2}}(4\pi|n|y)e(nx),\end{aligned}$$

where

$$L^{\frac{k+1}{2},+}_{m,n}(s) = \begin{cases} \sum_{0<c\equiv 0(4N)} \frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \textit{if } \operatorname{sign}(mn)>0; \\ \sum_{0<c\equiv 0(4N)} \frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c^{2s}} & \textit{if } n=0; \\ \sum_{0<c\equiv 0(4N)} \frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \textit{if } \operatorname{sign}(mn)<0, \end{cases}$$

where

(2.3)

$$K^+_{\frac{k+1}{2}}(m,n;c) = (1-(-1)^{\frac{k}{2}}i)\left(1+\left(\frac{4}{c/4}\right)\right)\sum_{d(c)^*}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{\frac{k+1}{2}} e\left(\frac{md+n\overline{d}}{c}\right),\ 4|c.$$

Let $(-1)^{\frac{k}{2}}m \equiv 0,1 \pmod 4$ and $r \in \mathbf{R}$. Taking the projection onto the Kohnen plus space on both sides of Proposition 2.3, we see that

$$\operatorname{Res}_{s=\frac{1}{2}+\frac{ir}{2}}(2s-1)F^+_{m,\frac{k+1}{2}}(z,s) = -(4\pi|m|)^{-k/2}\sum_{\psi}\frac{\overline{b_\psi(m)}}{\langle\psi,\psi\rangle}\psi(z), \tag{2.4}$$

where the sum goes over a basis of Hecke eigenforms ψ of the subspace $U^+_{\frac{k+1}{2},4N}(s)$, as the projection operator commutes with the Hecke operators.

Next we look at the Fourier coefficients on both sides of (2.4). Let

$$Z_\kappa(m,n;s) = \sum_{0<c\equiv 0(N)}\frac{K_\kappa(m,n;c)}{c^{2s}} \tag{2.5}$$

be the Kloosterman Zeta function.

For κ an integer, the Kloosterman sum $K_\kappa(m,n;c) = K(m,n;c)$ does not depend on the weight. The best bound for $K(m,n;c)$ is the Weil bound, which states that

$$K(m,n;c) \ll c^{1/2+\varepsilon} \text{ for any } \varepsilon>0. \tag{2.6}$$

This bound, which is also valid (and, in fact, elementary) for κ half an integer, gives that $Z_\kappa(m,n;s)$ converges absolutely for $\operatorname{Re}(s) > \frac{3}{4}$ and is thus holomorphic in that domain.

The modified Kloosterman Zeta function

$$Z^+_{\frac{k+1}{2}}(m,n;s) = \sum_{0<c\equiv 0(4N)}\frac{K^+_{\frac{k+1}{2}}(m,n;c)}{c^{2s}} \tag{2.7}$$

gives the main term in the residue of $F^+_{m,\frac{k+1}{2}}(z,s)$.

Proposition 2.5. *Let $(-1)^{\frac{k}{2}}n \equiv 0,1 \pmod 4$, and $r \in \mathbf{R}$. Then the residue of the modified Kloosterman zeta function $Z^+_{\frac{k+1}{2}}(m,n;s)$ at $s=\frac{1}{2}+\frac{ir}{2}$ is equal to*

$$\frac{3}{2\sqrt{2}}(-1)^{\lfloor\frac{k}{4}+\frac{1}{2}\rfloor}2^{-ir-k}\pi^{-\frac{k+1}{2}-ir}|mn|^{1/2-\frac{k+1}{4}-\frac{ir}{2}}\frac{\Gamma(ir)\Gamma\left(\frac{1}{2}+\frac{ir}{2}+\operatorname{sign}(n)\frac{k+1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{ir}{2}-\operatorname{sign}(m)\frac{k+1}{4}\right)}\sum_{\psi}\overline{b_\psi(m)}b_\psi(n),$$

where ψ runs over an orthonormal basis $U^+_{\frac{k+1}{2}}\left(\frac{1}{2}+\frac{ir}{2}\right)$ of Hecke eigenforms.

Proof. Using the asymptotic expansions

$$J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right),\ I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) = \frac{(2\pi)^{2s-1}|mn|^{s-\frac{1}{2}}}{\Gamma(2s)c^{2s-1}} + O(c^{-2s-1}),$$

we see that

$$\mathrm{Res}_{s=\frac{1}{2}+\frac{ir}{2}}(2s-1)L_{m,n}^{\frac{k+1}{2},+}(s) = \mathrm{Res}_{s=\frac{1}{2}+\frac{ir}{2}}\frac{(2\pi)^{2s-1}|mn|^{s-\frac{1}{2}}}{\Gamma(2s-1)}Z_{\frac{k+1}{2}}^{+}(m,n;s) \text{ for } n \neq 0.$$

Comparing Fourier coefficients on both sides of (2.4) then gives the desired claim. □

2.2. **The Selberg Poincaré series.** For $m > 0$, we define the Selberg Poincaré series [30] by

$$P_m^\kappa(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(y^{s-\kappa/2}e(mz))|_\kappa\gamma, \ \mathrm{Re}(s) > 1. \tag{2.8}$$

The series in (2.8) converges absolutely for $\mathrm{Re}(s) > 1$ and uniformly in z for compact subsets of the upper half plane $\mathbf{H}$. It can be analytically continued in s to the whole complex plane.

As mentioned in the introduction, for $\mathrm{Re}(s) > 0$ the poles of $F_m^\kappa(z,s)$ and of $P_m^\kappa(z,s)$ agree and we have the following analog of Proposition 2.3; see [13].

Proposition 2.6. *Let $m > 0$. The Selberg Poincaré series $P_m^\kappa(z,s)$ has analytic continuation in s to $\mathrm{Re}(s) > 0$. Moreover, we have*

$$\mathrm{Res}_{s=\frac{1}{2}+ir}P_m^\kappa(z,s) = (4\pi m)^{1/2-ir-\kappa/2}\frac{\Gamma(2ir)}{\Gamma\left(\frac{1}{2}+ir-\frac{k}{2}\right)}\sum_\varphi\frac{\overline{a_\varphi(m)}}{\langle\varphi,\varphi\rangle}\varphi(z),$$

where the sum goes over a basis of Hecke eigenforms φ of the subspace $U_{\kappa,N}(s)$.

Proof. From (2.2) the function

$$\phi_s(y) = (4\pi|m|y)^s - M_{\mathrm{sign}(m)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|m|y)$$

is $O(y^{s+1})$ as y approaches 0, and so the Poincaré series

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(y^{-\kappa/2}\phi_s(y)e(mx))|_\kappa\gamma$$

converges for $\mathrm{Re}(s) > 0$. It is clear that this gives an holomorphic continuation of

$$P_m^\kappa(z,s) + \frac{(4\pi|m|)^{1-s}\Gamma(2s)}{\Gamma\left(s-\mathrm{sign}(m)\frac{\kappa}{2}\right)}F_m^\kappa(z,s) \ \text{ to } \mathrm{Re}(s) > 0$$

and hence of $P_m^\kappa(z,s)$ by the analytic continuation of $F_m^\kappa(z,s)$. Taking residues at $s = \frac{1}{2}+ir$, we thus get

$$\mathrm{Res}_{s=\frac{1}{2}+ir}P_m^\kappa(z,s) = -(4\pi m)^{\frac{1}{2}-ir}\frac{\Gamma(2ir)}{\Gamma\left(\frac{1}{2}+ir-\frac{k}{2}\right)}\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m^\kappa(z,s)$$

and the claim follows by Proposition 2.3. □

Remark 2.7. One can extend this approximation using the power series expansion of the function $y^{-\nu-1/2}M_{\mu,\nu}(y)$ which leads to a formal expression for the Niebur Poincaré series. Yoshida [36, Prop. 2] showed that this formal series converges.

In the case of weight $2-\kappa$ we want to write the residues of $P_m^{2-\kappa}(z,s)$ in terms of cusp forms in the space $U_{\kappa,N}(1/2+ir)$. To this end, we recall that for any function $f : \mathbf{H} \to \mathbf{C}$, the ξ_κ-operator satisfies

$$\xi_\kappa(f|_\kappa\gamma) = (\xi_\kappa f)|_{2-\kappa}\gamma \text{ for all } \gamma \in \Gamma. \tag{2.9}$$

A non-holomorphic Maass cusp form does not vanish under the ξ_κ-operator. In fact, for non-holomorphic Maass cusp forms the ξ_κ-operator is an isomorphism.

Lemma 2.8. *The operator ξ_κ is an isomorphism $U_{\kappa,N}(s) \to U_{2-\kappa,N}(s)$ for $s \neq \frac{\kappa}{2}, 1-\frac{\kappa}{2}$.*

Proof. Let $\varphi \in U_{\kappa,N}(s)$. With (1.1) we obatin

$$\Delta_{2-\kappa}\xi_\kappa\varphi = \xi_\kappa\Delta_\kappa\varphi = (s-\kappa/2)(1-\kappa/2-s)\xi_\kappa\varphi.$$

Moreover, by (2.9) the function $\xi_\kappa\varphi$ is invariant under the $|_{2-\kappa}$-operator and we can use L'Hoptial's rule to show that $\xi_\kappa\varphi$ vanishes at the cusps of Γ. For $s \neq \frac{\kappa}{2}, 1-\frac{\kappa}{2}$, the inverse of ξ_κ is given by $-\frac{1}{(s-\kappa/2)(1-\kappa/2-s)}\xi_{2-\kappa}$, hence ξ_κ is an isomorphism. □

Lemma 2.9 shows that in Proposition 2.6 the sum running over a basis of Hecke eigenforms of $U_{2-\kappa,N}(\frac{1}{2}+ir)$ can be replaced by a sum that runs over $\xi_\kappa\varphi$ for a basis of Hecke eigenforms φ of $U_{\kappa,N}(\frac{1}{2}+ir)$. Using the next lemma we can also write the Fourier coefficient $\overline{a_\varphi(m)}$, in terms of the coefficients of $\xi_\kappa\varphi$.

Lemma 2.9. *Let $\varphi \in U_{\kappa,N}(s)$ and denote its n-th Fourier coefficient by $a_\varphi(n)$. The n-th Fourier coefficient of $\xi_\kappa\varphi$ is given by $-(4\pi n)^{1-\kappa}\overline{a_\varphi(-n)}$ for $n>0$ and by $(s-\kappa/2)(1-\kappa/2-s)(4\pi|n|)^{1-k}\overline{a_\varphi(-n)}$ for $n<0$.*

Proof. Let $\varphi = \sum_{n\neq 0} a_\varphi(n)(4\pi|n|y)^{-\frac{\kappa}{2}}W_{\mathrm{sign}(n)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(nx)$. Write

$$f_n(x,y) = a_\varphi(n)(4\pi|n|y)^{-\frac{\kappa}{2}}W_{\mathrm{sign}(n)\frac{\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(nx).$$

A calculation using

$$W'_{\nu,\mu}(4\pi|n|y) = \frac{1}{y}\left(4\pi|n|y-\nu\right)W_{\nu,\mu}(4\pi|n|y) - \frac{1}{y}W_{\nu+1,\mu}(4\pi|n|y) \tag{2.10}$$

gives that for $n<0$

$$\begin{aligned}\xi_\kappa f_n(x,y) &= iy^\kappa\cdot\overline{-i(4\pi|n|)^{-\frac{\kappa}{2}}a_\varphi(n)y^{-\frac{\kappa}{2}-1}W_{-\frac{\kappa}{2}+1,s-\frac{1}{2}}(4\pi|n|y)e(nx)}\\ &= -(4\pi|n|)^{1-\kappa}\overline{a_\varphi(n)}(4\pi|n|y)^{\frac{\kappa}{2}-1}W_{\frac{2-\kappa}{2},s-\frac{1}{2}}(4\pi|n|y)e(-nx).\end{aligned}$$

For $n>0$, using (2.10) and

$$\begin{aligned}W_{\frac{\kappa}{2}+1,s-\frac{1}{2}}(4\pi ny) &= (4\pi ny-\kappa)\,W_{\frac{\kappa}{2},s-\frac{1}{2}}(4\pi ny)\\ &\quad+\frac{1}{4}\left(4(s-1/2)^2-(1-\kappa)^2\right)W_{\frac{\kappa}{2}-1,s-\frac{1}{2}}(4\pi ny),\end{aligned}$$

we obtain

$$\xi_\kappa(f_n(x,y)) = (s-\kappa/2)(1-\kappa/2-s)(4\pi n)^{1-\kappa}\overline{a_\varphi(n)}(4\pi|n|y)^{\frac{\kappa}{2}-1}W_{-\frac{2-\kappa}{2},s-\frac{1}{2}}(4\pi ny)e(-nx).$$

Thus, the $-n$-th coefficient of $\xi_\kappa\varphi \in U_{2-\kappa,N}(s)$ is equal to $(s-\kappa/2)(1-\kappa/2-s)\overline{a_\varphi(n)}$. □

The following proposition now follows easily from Proposition 2.6, Lemma 2.8 and Lemma 2.9.

Proposition 2.10. *Let $m>0$. The Selberg Poincaré series $P_m^{2-\kappa}(z,s)$ satisfies*

$$\mathrm{Res}_{s=\frac{1}{2}+ir}P_m^{2-\kappa}(z,s) = (4\pi m)^{1/2-ir-\frac{\kappa}{2}}\frac{\Gamma(2ir)}{(1/2+ir-\kappa/2)\Gamma\left(\frac{1}{2}+ir+\frac{\kappa}{2}\right)}\sum_\varphi\frac{\overline{-a_\varphi(-m)}}{\langle\varphi,\varphi\rangle}\xi_\kappa\varphi(z),$$

where the sum goes over a basis of Hecke eigenforms φ of the subspace $U_{\kappa,N}(1/2+ir)$.

3. Traces of the Selberg Poincaré Series

In this section, we will prove the following theorem which is the core technical result of the paper.

Theorem 3.1. *Let k be an even integer, let d, d' be such that $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant, $(-1)^{\frac{k}{2}}d' \equiv 0, 1 \pmod 4$, and suppose that* $\mathrm{Re}(s) > 0$. *Write $D = dd'$.*

(1) *If $D < 0$, we have*

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s)$$
$$= \frac{(-1)^{\frac{k}{2}}\Gamma(s)}{\Gamma(s-k/2)} 2^{s-1}|D|^{k/4+s/2} \sum_{\substack{n|m,\\(n,N)=1}} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) \frac{1}{n^s} Z^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2}, d'; \frac{s}{2}+\frac{1}{4}\right) + H_1(s),$$

where $H_1(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

(2) *For $D = dd' > 0$ with $d > 0$, we have*

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s)$$
$$= \frac{(-1)^{\frac{k}{2}}2\pi D^{k/4+s/2}\Gamma(s)}{\Gamma\left(\frac{s+1}{2}+\frac{k}{4}\right)\Gamma\left(\frac{s+1}{2}-\frac{k}{4}\right)} \sum_{\substack{n|m,\\(n,N)=1}} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) n^{-s} Z^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2}, d'; \frac{s}{2}+\frac{1}{4}\right) + H_2(s),$$

where $H_2(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

(3) *For $D = dd' > 0$ with $d < 0$, we have*

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^{2-k}(z,s)$$
$$= -\frac{(-1)^{\frac{k}{2}}2\pi D^{k/4+s/2}\Gamma(s)}{\Gamma\left(\frac{s}{2}+1-\frac{k}{4}\right)\Gamma\left(\frac{s}{2}+\frac{k}{4}\right)} \sum_{\substack{n|m,\\(n,N)=1}} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) n^{-s} Z^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2}, d'; \frac{s}{2}+\frac{1}{4}\right) + H_3(s),$$

where $H_3(s)$ is holomorphic for $\mathrm{Re}(s) > 0$.

3.1. **Binary quadratic forms and traces.** We recall some facts about binary quadratic forms. Let $\Gamma = \Gamma_0(N)$ for $N \geqslant 1$ odd and consider

$$\mathcal{Q}_{N,D} = \{[A,B,C] \,:\, B^2 - 4AC = D, N|A\}.$$

We identify $[A,B,C]$ with $Ax^2+Bxy+Cy^2$, so $\mathcal{Q}_{N,D}$ is the set of binary quadratic forms of discriminant D such that $N|Q(1,0)$. The subset of $Q \in \mathcal{Q}_{N,D}$ with $A > 0$ will be denoted by $\mathcal{Q}^+_{N,D}$. We use the convention $\mathcal{Q}_{N,D} = \mathcal{Q}^+_{N,D}$ for $D < 0$.

On $\mathcal{Q}_{N,D}$ there is a left-action of Γ, namely for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we set

$$(\gamma Q)(x,y) = Q(dx - by, ay - cx). \tag{3.1}$$

On several occasions we will need the following lemma whose proof is obvious.

Lemma 3.2. *The map*

$$\{(A,B) \,:\, N|A > 0, B\ (2A), B^2 \equiv D\ (4A)\} \to \Gamma_\infty\backslash\mathcal{Q}^+_{N,D},$$

sending (A,B) to $[A,B,(B^2-D)/4A]$ is a bijection.

For a quadratic form $Q \in \mathcal{Q}_{N,D}$ let

$$\Gamma_Q = \{\gamma \in \Gamma \,:\, \gamma Q = Q\}$$

denote the group of automorphs of Q.

If $D < 0$, the subgroup of automorphs is trivial, except when Q is equivalent to $A(x^2+y^2)$ or $A(x^2+xy+y^2)$. For those binary quadratic forms the group of automorphs has order 2 resp. 3. For $Q = [A, B, C] \in \mathcal{Q}_{N,D}$ we let

$$z_Q = \frac{-B + i\sqrt{|D|}}{2A} \in \mathbf{H},$$

so that $Q(z_Q, 1) = 0$. Note that $\gamma z_Q = z_{\gamma Q}$.

If $D > 0$, the group of automorphs of $[A, B, C]$ is generated by

$$\gamma_Q = \begin{pmatrix} \frac{u+Bt}{2} & -Ct \\ At & \frac{u-Bt}{2} \end{pmatrix},$$

where (u, t) is the smallest positive integer solutions of the Pell equation $t^2 - Du^2 = 4$ (the matrix γ_Q lies in $\Gamma_0(N)$ as A is divisible by N). So if $D > 0$ is not a square, the group of automorphs is infinite cyclic.

For $Q = [A, B, C] \in \mathcal{Q}_{N,D}$ let

$$S_Q = S_{[A,B,C]} = \{z \in \mathbf{H} : A z^2 + B\,\mathrm{Re}(z) + C = 0\}, \tag{3.2}$$

which for $A \neq 0$ is a semi-circle in the upper half plane $\mathbf{H}$. It is easy to see that

$$\gamma S_Q = S_{\gamma Q}$$

for any $\gamma \in \Gamma$. We orient the S_Q counterclockwise for $A > 0$ and clockwise for $A < 0$.

Let now $C_Q = \Gamma_Q \backslash S_Q$. One easily checks that the holomorphic differential form

$$d_Q z = \frac{\sqrt{D} dz}{Q(z, 1)}$$

is Γ_Q-invariant, and so for D not a square

$$\int_{C_Q} f(z) Q(z, 1)^{k/2} d_Q z$$

equals to an integral over a directed arc from z to $\gamma_Q z$ for any $z \in S_Q$.

For square D the group of automorphs is trivial since $(\pm 2, 0)$ is the only solution to the Pell equation in this case. It is also possible to give an explicit description of the classes in this case.

For an integer $m > 1$ with $m|N$ and $(m, \frac{N}{m}) = 1$, we write $m \| N$. For $m \| N$, pick $\beta, \delta \in \mathbf{Z}$ such that $\delta m^2 - N\beta = m$ and let

$$W_m = \frac{1}{\sqrt{m}} \begin{pmatrix} m & \beta \\ N & \delta m \end{pmatrix}. \tag{3.3}$$

The matrix W_m acting on a Maass cusp form is an Atkin-Lehner involution.

Lemma 3.3. *Let $d \geqslant 1$ be an integer. The set of binary quadratic forms Q in $\Gamma \backslash \mathcal{Q}_{N,d^2}$ is represented by $\{W_m.[0, d, \mu] : 0 \leqslant \mu < d, m \| N\}$.*

Proof. See Kohnen's [20, p. 243] or Biró's paper [5, p. 131]. □

For $D = d^2$ and $Q = [A, B, C] \in \mathcal{Q}_{N,D}$ the geodesic S_Q defined in (3.2) is still a semi-circle for $A \neq 0$, but for $Q = Q_\mu = [0, d, \mu]$ it is the half-line

$$S_{Q_\mu} = \{-\frac{\mu}{d} + it : t > 0\}.$$

In the case of Q_μ, the line S_{Q_μ} shall be directed from $-\frac{\mu}{d}$ to $i\infty$.

3.2. **Salie sums.** In Proposition 3.9, we rewrite the trace of $P_m^k(z,\phi)$ in terms of an infinite series of Salie sums, which are defined by

$$S_m(d,d';c) = \sum_{b\ (c),\ b^2\equiv dd'(c)} \chi_d\left(\left[\frac{c}{4}, b, \frac{b^2-dd'}{c}\right]\right) e\left(-\frac{2mb}{c}\right) \tag{3.4}$$

for c divisible by 4. The character χ_d in (3.4) is defined as in Definition 1.2. In particular,

$$\chi_d(-Q) = \operatorname{sign}(d)\chi_d(Q). \tag{3.5}$$

Note that for any $\varepsilon > 0$ the Salie sums are bounded in c by

$$S_m(d,d';c) \ll_\varepsilon c^\varepsilon \text{ as } c \to +\infty. \tag{3.6}$$

Kohnen [20, Prop. 5] showed that the Salie sum (3.4) is equal to a linear combination of the Kloosterman sums $K^+_{\frac{k+1}{2}}(m,n;c)$ defined in (2.3).

Proposition 3.4 (Kohnen)**.** *Let m, d, d' be non-zero integers and k an even integer. Suppose further that $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant, $c \equiv 0 \pmod 4$, and $(-1)^{\frac{k}{2}}d' \equiv 0,1 \pmod 4$. Then we have*

$$S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d';c) = \sum_{n|(m,c/4)} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)\sqrt{\frac{n}{c}}K^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2},d';\frac{c}{n}\right).$$

Remark 3.5. Together with the bound on Salie sums (3.6), this implies the Weil bound for the modified Kloosterman sum $K^+_{\frac{k+1}{2}}(d,d';c) \ll_\varepsilon c^{\frac{1}{2}+\varepsilon}$ as $c \to +\infty$ for all $\varepsilon > 0$.

3.3. **The proof of Theorem 3.1 for $D < 0$.**

Proposition 3.6. *Assume that $D = dd' < 0$, where $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant and $(-1)^{\frac{k}{2}}d' \equiv 0,1 \pmod 4$. Then*

$$\operatorname{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^0(z,s) = 2^{s-1}|D|^{s/2} \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d';c)}{c^s} e^{-4\pi m\frac{\sqrt{|D|}}{c}}.$$

Proof. This is a calculation:

$$\begin{aligned}
&\sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}} \chi_{(-1)^{\frac{k}{2}}d}(Q)\omega_Q^{-1}P_m^0(z_Q,s)\\
&= \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi_{(-1)^{\frac{k}{2}}d}(Q)\omega_Q^{-1}\operatorname{Im}(\gamma.z_Q)^s e(m\gamma.z_Q)\\
&= \sum_{Q\in\Gamma\backslash\mathcal{Q}_{N,D}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q} \chi_{(-1)^{\frac{k}{2}}d}(Q)\operatorname{Im}(z_{\gamma.Q})^s e(mz_{\gamma.Q})\\
&= \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{N,D}} \chi_{(-1)^{\frac{k}{2}}d}(Q)(\operatorname{Im}z_Q)^s e\left(mz_Q\right),
\end{aligned}$$

which by Lemma 3.2 yields the result. ☐

We will also need the following expressions for lowering resp. raising operators applied to $P_m^k(z,s)$.

Lemma 3.7.

$$L_k^k P_m^k(z,s) = \frac{(-1)^{\frac{k}{2}}\Gamma(s)}{\Gamma(s-k/2)} P_m^0(z,s), \text{ if } k>0, \text{ and} \tag{3.7}$$

$$R_k^k P_m^k(z,s) = (-1)^{\frac{k}{2}} \sum_{l=0}^{|k/2|} (4\pi m)^l \frac{\Gamma(s+l)}{\Gamma(s+l-k/2)} P_m^0(z,s+l) \text{ if } k<0. \tag{3.8}$$

Proof. We use the identities

$$L_{k'}(y^{s-k/2}e(mz)) = -(s-k'/2)y^{s+1-k/2}e(mz) \text{ for all } k' \in \mathbf{Z}$$

and

$$R_{k'}(y^{s-k'/2}e(mz)) = -(s-k'/2)y^{s+1-k'/2}e(mz) - 4\pi m y^{s+1-k'/2}e(mz) \text{ for all } k' \in \mathbf{Z}.$$

By applying the identities to (2.8), we get the claim for $\mathrm{Re}(s) > 1$. Uniqueness of the analytic continuation yields the result for all complex s. □

The proof of Theorem 3.1 for $D<0$. Assume that $k>0$. After applying (3.7), we see

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s) = \frac{(-1)^{\frac{k}{2}}|D|^{\frac{k}{4}}\Gamma(s)}{\Gamma(s-k/2)} \mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^0(z,s)$$

$$= \frac{(-1)^{\frac{k}{2}}\Gamma(s)}{\Gamma(s-k/2)} |D|^{k/4+s/2} \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d';c)}{c^s} e^{-4\pi m\frac{\sqrt{|D|}}{c}}.$$

Using the first order approximation $e^{-4\pi m\frac{\sqrt{|D|}}{c}} = 1 + O(1/c)$, we get

$$\sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d';c)}{c^s} e^{-4\pi m\frac{\sqrt{|D|}}{c}} = \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d';c)}{c^s} + H_1(s) \tag{3.9}$$

with $H_1(s)$ holomorphic for $\mathrm{Re}(s) > 0$ by (3.6). Applying Proposition 3.4 leads to the claim (1) of Theorem 3.1 in the case $k>0$.

For $k<0$ we arrive at (3.9) with the same argument as above, since for $l \geqslant 1$ the terms $P_m^0(z,s+l)$ in (3.8) are holomorphic in $\mathrm{Re}(s) > 0$. □

3.4. **Cycle integrals of Poincaré series.** We will treat a slightly more general family of Poincaré series. Let $\phi : \mathbf{R}_{>0} \to \mathbf{R}$ be a infinitely differentiable function such that $\phi(y) = O(y^{1-k/2+\varepsilon})$ as $y \to 0$ for any $\varepsilon > 0$. Then the Poincaré series

$$P_m^k(z,\phi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\phi(y)e(mx))\,|_k\gamma \tag{3.10}$$

converges for any integer m. The following lemma is is a preliminary step in evaluating cycle integrals of these Poincaré series.

Lemma 3.8. *Let k be an even integer, d a fundamental discriminant and $d' \equiv 0,1 \pmod 4$. Let $m>0$ be an integer.*

1. *If $\mathrm{sign}(d) \neq (-1)^{k/2}$ then $\mathrm{Tr}_{d,d'} P_m^k(z,\phi) = 0$.*
2. *If $\mathrm{sign}(d) = (-1)^{k/2}$ and $D = dd' > 0$ is not a square then*

$$\mathrm{Tr}_{d,d'} P_m^k(z,\phi) = 2 \sum_{\Gamma_\infty\backslash\mathcal{Q}_{N,D}^+} \chi_{(-1)^{\frac{k}{2}}d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^{\frac{k}{2}} d_Q z$$

(3) *If* $\mathrm{sign}(d) = (-1)^{k/2}$ *and* $d = d'$ *then*

$$\mathrm{Tr}_{d,d'} P_m^k(z,\phi) = 2 \sum_{\Gamma_\infty \backslash \mathcal{Q}_{N,D}^+} \chi_{(-1)^{\frac{k}{2}} d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^{\frac{k}{2}} d_Q z$$
$$+ \sum_{\substack{Q \in \Gamma_\infty \backslash \mathcal{Q}_{N,D}, \\ Q(1,0)=0}} \chi_{(-1)^{\frac{k}{2}} d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^{\frac{k}{2}} d_Q z.$$

Proof. The argument is standard. We only point out the need for the sign condition.

The group of automorphs Γ_Q acts freely on $\Gamma_\infty \backslash \Gamma$ on the right, and a direct calculation shows that

$$\begin{aligned} &\int_{C_Q} P_m^k(z,\phi) Q(z,1)^{\frac{k}{2}} d_Q z \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{C_Q} j(\gamma,z)^{-k} e(m\mathrm{Re}(\gamma z))\phi(\mathrm{Im}(\gamma z)) Q(z,1)^{\frac{k}{2}} d_Q z \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_Q} \sum_{\gamma \in \Gamma_Q} \int_{\gamma . C_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^{\frac{k}{2}} d_Q z. \end{aligned}$$

Therefore

$$\mathrm{Tr}_{d,d'} P_m^k(z,\phi) = \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{N,dd'}} \chi_{(-1)^{\frac{k}{2}} d}(Q) \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^{\frac{k}{2}} d_Q z.$$

Note that $dz_{-Q} = -d_Q z$ and that $S_{[A,B,C]}$ and $S_{[-A,-B,-C]}$ are the same semicircle with opposite orientations, therefore

$$\int_{S_{-Q}} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))(-Q(z,1))^{\frac{k}{2}} d_{-Q} z = (-1)^{k/2} \int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z)) Q(z,1)^{\frac{k}{2}} d_Q z$$

by (3.5). □

Because of the lemma above, we will be interested in factorizing D into factors whose sign is $(-1)^{k/2}$. As in the definition of the Kohnen plus space, we will build this into the notation, so that the distinction between $k \equiv 0$ or $2 \pmod 4$ becomes automatic.

Proposition 3.9. *Let k be an even integer, $d > 0$ be such that $(-1)^{\frac{k}{2}} d$ is a fundamental discriminant and $(-1)^{\frac{k}{2}} d' \equiv 0, 1 \pmod 4$ such that $D = dd' > 0$. Let $m > 0$ be an integer.*

(1) *If $D = dd'$ is not a square then*

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}} d, (-1)^{\frac{k}{2}} d'} P_m^k(z,\phi) = D^{k/4} \sum_{0 < c \equiv 0 (4N)} S_m((-1)^{\frac{k}{2}} d, (-1)^{\frac{k}{2}} d'; c) \Phi_m^k \left(\frac{2\sqrt{dd'}}{c} \right)$$

with

$$\Phi_m^k(t) = (it)^{\frac{k}{2}} \int_0^\pi e\left(mt\cos\theta\right) \phi(t\sin\theta)(\sin\theta)^{k/2-1} e^{i\theta k/2} d\theta. \tag{3.11}$$

(2) *If $d = d'$, then*

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d} P_m^k(z,\phi) = d^{k/2} \sum_{0<c\equiv 0(4N)} S_m((-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d;c)\Phi_m^k\left(\frac{2d}{c}\right) + 2(id)^{k/2}G_m^k(d)\mathcal{M}(\phi)(k/2)$$

where $\mathcal{M}(\phi)(s)$ is the Mellin transform of ϕ and

$$G_m^k(d) = \sum_{\mu(d)*} \left(\frac{(-1)^{k/2}d}{\mu}\right) e\left(-m\frac{\mu}{d}\right)$$

is a Gauss sum.

Proof. The first claim follows easily from Lemmas 3.2 and 3.8 after parametrizing the semicircle S_Q by $\theta \in (0,\pi)$ such that

$$z = -\frac{b}{2a} + \begin{cases} \frac{\sqrt{dd'}}{2a}e^{i\theta}, & \text{if } a > 0, \\ \frac{\sqrt{dd'}}{2a}e^{-i\theta}, & \text{if } a < 0, \end{cases}$$

where $Q = [a,b,c]$; see, e.g., [9].

To prove case (2), i.e. $D = d^2$, we use part (3) of Lemma 3.8. The sum

$$\sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{N,d^2},\\ Q(1,0)=0}} \chi_{(-1)^{\frac{k}{2}}d}(Q)\int_{S_Q} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^{\frac{k}{2}}d_Qz \tag{3.12}$$

runs over $\{\pm Q_\mu : 0 \leqslant \mu < d\}$ with $Q_\mu = [0,d,\mu]$. We have $\chi_{(-1)^{\frac{k}{2}}d}(\pm[0,d,\mu]) = \left(\frac{(-1)^{k/2}d}{\mu}\right)$ for $(\mu,d) = 1$. Thus, the sum (3.12) is equal to

$$2\sum_{(\mu,d)=1}\left(\frac{(-1)^{\frac{k}{2}}d}{\mu}\right)\int_{\{-\frac{\mu}{d}+it:t>0\}} e(m\mathrm{Re}(z))\phi(\mathrm{Im}(z))Q(z,1)^{\frac{k}{2}}d_Qz$$

$$= 2\sum_{(\mu,d)=1}\left(\frac{(-1)^{\frac{k}{2}}d}{\mu}\right)e\left(-m\frac{\mu}{d}\right)\int_0^\infty \phi(t)(id)^{\frac{k}{2}}t^{k/2}\frac{dt}{t} = 2G_m^k(d)(id)^{\frac{k}{2}}\mathcal{M}(\phi)(k/2).$$

□

3.5. **The proof of Theorem 3.1 for $D > 0$.** We now consider $\phi(y) = y^{s-k/2}e^{-2\pi my}$ in (3.10), so that $P_m^k(z,\phi) = P_m^k(z,s)$. Except for the holomorphic case $s = k/2$ (Propositon 3.11), the evaluation of the transform

$$\int_0^\pi e\left(mt\cos\theta\right)\phi(t\sin\theta)(\sin\theta)^{k/2-1}e^{i\theta k/2}d\theta$$

is bypassed by the next lemma which allows us to express the main term contributing to the residue of each pole of $P_m^k(z,s)$ along the line $\mathrm{Re}(s) = \frac{1}{2}$.

Lemma 3.10. *For $\mathrm{Re}(s) > 0$, we have*

$$\Phi_m^k(t) = c_k(s)t^s + O(t^{s+1}), \text{ as } t \to 0^+,$$

where

$$c_k(s) = (-1)^{k/2}\frac{\pi\Gamma(s)}{2^{s-1}\Gamma\left(\frac{s+1}{2}+\frac{k}{4}\right)\Gamma\left(\frac{s+1}{2}-\frac{k}{4}\right)}.$$

Proof. Using $e(mt\cos\theta) = 1 + O(mt\cos\theta)$ and $e^{-2\pi mt\sin\theta} = 1 + O(mt\sin\theta)$, we get

$$\Phi_m^k(t) = i^{\frac{k}{2}} t^s \left(\int_0^\pi (\sin\theta)^{s-1} e^{ik\theta/2} d\theta + O(t) \right).$$

The proof is finished with the integral formula (see [14, p. 485, 3.892(1)])

$$\int_0^\pi e^{i\beta x} (\sin x)^{\nu-1} dx = \frac{\pi e^{i\pi\beta/2}\Gamma(\nu)}{2^{\nu-1}\Gamma\left(\frac{\nu+\beta+1}{2}\right)\Gamma\left(\frac{\nu-\beta+1}{2}\right)}, \ \mathrm{Re}(\nu) > 0,$$

with $\nu = s$ and $\beta = \frac{k}{2}$. □

With Lemma 3.10, we are set to prove Theorem 3.1.

Proof of Theorem 3.1 for $D > 0$. To prove (2), we use Proposition 3.9 and Lemma 3.10 to write

(3.13)

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s) = c_k(s) D^{k/4} \sum_{0<c\equiv 0(4N)} S_m((-1)^{\frac{k}{2}}d, (-1)^{\frac{k}{2}}d'; c) \left(\frac{2\sqrt{dd'}}{c}\right)^s + H_2(s),$$

where $H_2(s)$ is a holomorphic function on $\mathrm{Re}(s) > 0$.

Note that in the case of $d = d'$, the Mellin transform term $(id)^{k/2} G_m^k(d)\mathcal{M}(\phi)(k/2)$ is holomorphic for $\mathrm{Re}(s) > 0$ and is part of $H_2(s)$.

With Proposition 3.4, we may rewrite equation (3.13) as

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s) = c_k(s) D^{k/4} \sum_{n|m} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) \left(\frac{2\sqrt{dd'}}{n}\right)^s \sum_{0<c\equiv 0(4N)} \frac{K^+_{\frac{k+1}{2}}\left(\frac{m^2 d}{n^2}, d'; c\right)}{c^{s+1/2}} + H_2(s)$$

by substituting $c \to nc$.

To prove (3) suppose that $d, d' < 0$. By Proposition 3.9 for weight $2-k$ and $-d, -d' > 0$ and Lemma 3.10 we get

(3.14)

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^{2-k}(z,s) = c_{2-k}(s) \sum_{0<c\equiv 0(4N)} S_m((-1)^{\frac{k}{2}}d, (-1)^{\frac{k}{2}}d'; c) \left(\frac{2\sqrt{dd'}}{c}\right)^s + H_3(s),$$

where $H_3(s)$ is a holomorphic function on $\mathrm{Re}(s) > 0$. Once again, Proposition 3.4 gives the claim. □

3.6. **The holomorphic case.** Evaluating $\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z,s)$ at the spectral point $s = \frac{k}{2} > 1$ reduces to considering $\phi(y) = e^{-2\pi my}$ in (3.10) so that $P_m^k(z,\phi) = P_m^k(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(mz)|_k\gamma$, the holomorphic Poincaré series of weight k. For completeness we give the result in this case.

Proposition 3.11. *Suppose that $k > 2$ and $d, d' > 0$ with $d \neq d'$ such that $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant and $(-1)^{\frac{k}{2}}d' \equiv 0, 1 \pmod 4$. We have*

$$\begin{aligned}
&\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z) \\
&= (-1)^{\frac{k}{2}} \frac{\Gamma(k/2) D^{\frac{k}{4}+\frac{3}{4}}}{2^{\frac{k}{2}-\frac{3}{2}} m^{\frac{k-1}{2}} \pi^{\frac{k}{2}-1}} \sum_{0<c\equiv 0(4N)} \frac{S_m((-1)^{\frac{k}{2}}d, (-1)^{\frac{k}{2}}d'; c)}{\sqrt{c}} J_{\frac{k-1}{2}}\left(\frac{4\pi m\sqrt{D}}{c}\right) \\
&= (-1)^{\frac{k}{2}} \frac{\Gamma(k/2) D^{\frac{k}{4}+\frac{3}{4}}}{2^{\frac{k}{2}-\frac{3}{2}} m^{\frac{k-1}{2}} \pi^{\frac{k}{2}-1}} \sum_{n|m} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) \sum_{0<c\equiv 0(4N)} \frac{1}{c} K^+_{\frac{k+1}{2}}\left(\frac{m^2 d}{n^2}, d'; c\right) J_{\frac{k-1}{2}}\left(\frac{4\pi mn\sqrt{D}}{c}\right).
\end{aligned}$$

If $d = d'$, we get the additional term $(id)^{\frac{k}{2}}(4\pi m)^{-\frac{k}{2}}\Gamma(k/2)$ on the right hand side.

Proof. Suppose that $D = dd'$ is not a square. By Lemma 3.8, we get

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z) = 2 \sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}_{N,D}^+} \chi_{(-1)^{\frac{k}{2}}d}(Q) \int_{S_Q} e(mz)Q(z,1)^{\frac{k}{2}} d_Q z.$$

After the change of variable $z \to \frac{\sqrt{D}z-B}{2A}$ for $Q = [A, B, C]$, one needs to evaluate

$$-\int_{-1}^{1} e\left(m\frac{\sqrt{D}}{2A}z\right)(z^2-1)^{\frac{k}{2}-1}dz = \frac{(-1)^{\frac{k}{2}}2^{\frac{k-1}{2}}\Gamma(k/2)}{\pi^{\frac{k}{2}-1}}\left(\frac{\sqrt{D}m}{A}\right)^{-\frac{k}{2}+\frac{1}{2}} J_{\frac{k-1}{2}}\left(\frac{\pi m\sqrt{D}}{A}\right),$$

which readily yields the claim for D not being a square. The case when D is a square can be dealt with in the same fashion as in the proof of Proposition 3.9. □

Remark 3.12. Proposition 3.11 is also valid in weight $k = 2$ when $P_m^2(z)$ is defined by Hecke's convergence trick.

4. The Proof of the Generalized Katok-Sarnak Formula

The goal of this section is to prove Theorem 1.4. Let $m > 0$ and let d, d' be integers such that $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant and $(-1)^{\frac{k}{2}}d' \equiv 0, 1 \pmod 4$. To get to the traces of Maass cusp forms we will use Proposition 2.6, and Theorem 3.1 but we start by noting the shift in the spectral point to $\frac{s}{2} + \frac{1}{4}$ on the right hand side of Theorem 3.1; see [10, p. 982].

Lemma 4.1. *For integers m, n, we have*

$$\mathrm{Res}_{s=\frac{1}{2}+ir} Z^+_{\frac{k+1}{2}}\left(m, n; \frac{s}{2}+\frac{1}{4}\right) = 4\ \mathrm{Res}_{s=\frac{1}{2}+\frac{ir}{2}} Z^+_{\frac{k+1}{2}}(m, n; s).$$

Next, we calculate the residue of the twisted trace of the Selberg Poincaré series explicitly in terms of half integral weight coefficients. To this end, we use Theorem 3.1, Lemma 4.1, and Proposition 2.5 for $dd' < 0$ and obtain

(4.1)
$$\begin{aligned}
&\operatorname*{Res}_{s=\frac{1}{2}+ir} \mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'} P_m^k(z, s) \\
&= \frac{|D|^{\frac{k+1}{4}+\frac{ir}{2}}\Gamma\left(\frac{1}{2}+ir\right)}{2^{-1/2-ir}\Gamma\left(\frac{1}{2}+ir-\frac{k}{2}\right)} \sum_{\substack{n|m,\\(n,N)=1}} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)\frac{1}{n^{1/2+ir}} \operatorname*{Res}_{s=\frac{1}{2}+\frac{ir}{2}} Z^+_{\frac{k+1}{2}}\left(\frac{m^2}{n^2}d, d'; s\right) \\
&= \frac{6(-1)^{\lfloor k/4\rfloor}\sqrt{|D|}(4\pi m)^{1-ir-\frac{k+1}{2}}\Gamma(2ir)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}+ir-\frac{k}{2}\right)} \sum_{\substack{n|m,\\(n,N)=1}} \left(\frac{(-1)^{\frac{k}{2}}d}{n}\right) n^{k/2-1} \overline{\sum_{\psi} b_\psi\left(\frac{m^2d}{n^2}\right)} b_\psi(d')
\end{aligned}$$

with ψ running over an orthonormal basis of $U^+_{\frac{k+1}{2},4N}\left(\frac{1}{2}+\frac{ir}{2}\right)$.

Similarly for $d, d' > 0$, we have

(4.2)
$$\begin{aligned}&\operatorname*{Res}_{s=\frac{1}{2}+ir}\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}P_m^k(z,s)\\&=\frac{(-1)^{k/2}8\pi D^{\frac{k+1}{4}+\frac{ir}{2}}\Gamma\left(\frac{1}{2}+ir\right)}{\Gamma\left(\frac{ir}{2}+\frac{k+3}{4}\right)\Gamma\left(\frac{ir}{2}-\frac{k-3}{4}\right)}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)\frac{1}{n^{\frac{1}{2}+ir}}\operatorname*{Res}_{s=\frac{1}{2}+\frac{ir}{2}}Z^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2},d';s\right)\\&=\frac{6(-1)^{\lfloor k/4\rfloor}\sqrt{D}(4\pi m)^{1-ir-\frac{k+1}{2}}\Gamma(2ir)}{2^{\frac{k}{2}}\Gamma\left(\frac{1}{2}+ir-\frac{k}{2}\right)}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)n^{k/2-1}\sum_{\psi}\overline{b_\psi\left(\frac{m^2d}{n^2}\right)}b_\psi\left(d'\right).\end{aligned}$$

Finally, for both $d, d' < 0$, since $\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}P_m^k(z,s)$ vanishes, we look at the trace of $P_m^{2-k}(z,s)$ to get some information about $\sum_\psi \overline{b_\psi\left(\frac{m^2d}{n^2}\right)}b_\psi\left(d'\right)$ for $\psi \in U^+_{\frac{k+1}{2},4N}\left(\frac{1}{2}+\frac{ir}{2}\right)$. We then have

(4.3)
$$\begin{aligned}&\operatorname*{Res}_{s=\frac{1}{2}+ir}\mathrm{Tr}_{(-1)^{\frac{2-k}{2}}(-d),(-1)^{\frac{2-k}{2}}(-d')}P_m^{2-k}(z,s)\\&=-\frac{(-1)^{\frac{k}{2}}8\pi D^{\frac{k+1}{4}+\frac{ir}{2}}\Gamma\left(\frac{1}{2}+ir\right)}{\Gamma\left(\frac{ir}{2}-\frac{k-5}{4}\right)\Gamma\left(\frac{ir}{2}+\frac{k+1}{4}\right)}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)\frac{1}{n^{\frac{1}{2}+ir}}\operatorname*{Res}_{s=\frac{1}{2}+\frac{ir}{2}}Z^+_{\frac{k+1}{2}}\left(\frac{m^2d}{n^2},d';s\right)\\&=6(-1)^{\lfloor k/4\rfloor}\frac{\sqrt{D}(4\pi m)^{1-ir-\frac{k+1}{2}}}{\left(\frac{1}{2}+ir-\frac{k}{2}\right)}\frac{\Gamma(2ir)2^{k/2-1}}{\Gamma\left(\frac{1}{2}+ir+\frac{k}{2}\right)}\sum_{\substack{n|m,\\(n,N)=1}}\left(\frac{(-1)^{\frac{k}{2}}d}{n}\right)n^{k/2-1}\sum_{\psi}\overline{b_\psi\left(\frac{m^2d}{n^2}\right)}b_\psi\left(d'\right),\end{aligned}$$

where ψ runs over an orthonormal basis of $U^+_{\frac{k+1}{2},4N}\left(\frac{1}{2}+\frac{ir}{2}\right)$.

On the other hand, we can use Proposition 2.6 in (4.1), (4.2) and Proposition 2.10 in (4.3) to calculate the residue of the traces of the Poincaré series. When combined with the Shimura relation (1.10), this proves the following Proposition.

Proposition 4.2. *Let d, d' be integers such that $(-1)^{\frac{k}{2}}d$ is a fundamental discriminant and $(-1)^{\frac{k}{2}}d' \equiv 0, 1 \pmod 4$. We have*

(1) *for $dd' < 0$:*
$$6(-1)^{\lfloor k/4\rfloor}\sqrt{|dd'|}\sum_\psi\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle}\mathrm{Shim}_d(\psi)(z)=2\sqrt{\pi}\sum_\varphi\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\varphi)\frac{\varphi(z)}{\langle\varphi,\varphi\rangle},$$

(2) *for $d, d' > 0$:*
$$6(-1)^{\lfloor k/4\rfloor}\sqrt{dd'}\sum_\psi\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle}\mathrm{Shim}_d(\psi)(z)=2^{k/2}\sum_\varphi\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\varphi)\frac{\varphi(z)}{\langle\varphi,\varphi\rangle},$$

(3) *for $d, d' < 0$:*
$$6(-1)^{\lfloor k/4\rfloor}\sqrt{dd'}\sum_\psi\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle}\mathrm{Shim}_d(\psi)(z)=2^{1-k/2}\sum_\varphi\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\xi_k\varphi)\frac{\varphi(z)}{\langle\varphi,\varphi\rangle},$$

where ψ and φ run over a basis of normalized Hecke eigenforms of $U^+_{\frac{k+1}{2},4N}\left(\frac{1}{2}+\frac{ir}{2}\right)$ and $U_{k,N}\left(\frac{1}{2}+ir\right)$ respectively.

Note that one does not need to know in advance that $\mathrm{Shim}_d(\psi)$ as defined for $\psi \in U^+_{\frac{k+1}{2},4N}(s)$ in (1.11) is a Maass cusp form. Similarly to Biro's work [5, p. 129] our approach establishes this fact as a byproduct of Proposition 4.2.

Proof of Theorem 1.4. We now collect everything to prove our generalization of the Katok-Sarnak formula in Theorem 1.4.

For any normalized newform $\psi \in U^+_{\frac{k+1}{2},4N}\left(\frac{s}{2}+\frac{1}{4}\right)$, we have $\mathrm{Shim}_d(\psi) \in U_{k,N}(s)$.

Since the Hecke operators commute with the Shimura lift, the Maass cusp form $\mathrm{Shim}_d(\psi)$ has the same eigenvalue as an integral weight Maass form $\varphi \in U_{k,N}$. If φ is a normalized even newform, we must have $\mathrm{Shim}_d(\psi)(z) = \varphi(z)$ by matching their first Fourier coefficient. Hence, after applying the projection onto the space $U^{\mathrm{new},+}_{\frac{k+1}{2},4N}\left(\frac{s}{2}+\frac{1}{4}\right)$, we may write the equations in Proposition 4.2 as for $d, d' > 0$ or $dd' < 0$

(4.4)

$$6(-1)^{\lfloor k/4\rfloor}\sqrt{|dd'|}\sum_{\varphi}\sum_{\mathrm{Shim}_d(\psi)=\varphi}\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle}\varphi = \sum_{\varphi}\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\varphi)\frac{\varphi}{\langle\varphi,\varphi\rangle}\begin{cases}2\sqrt{\pi}, & dd' < 0,\\ 2^{k/2}, & d,d' > 0,\end{cases}$$

and for $d, d' < 0$

$$6(-1)^{\lfloor k/4\rfloor}\sqrt{dd'}\sum_{\varphi}\sum_{\mathrm{Shim}_d(\psi)=\varphi}\frac{\overline{b_\psi(d)}b_\psi(d')}{\langle\psi,\psi\rangle}\varphi = \sum_{\varphi}2^{1-k/2}\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(\xi_k\varphi)\frac{\varphi}{\langle\varphi,\varphi\rangle}, \tag{4.5}$$

where ψ runs over a basis of normalized newforms of $U^{\mathrm{new},+}_{\frac{k+1}{2},4N}\left(\frac{s}{2}+\frac{1}{4}\right)$.

Since the φ's form a basis, this proves Theorem 1.4 in the non-holomorphic case.

To show the theorem in the holomorphic case, i.e. when φ has spectral point $s = \frac{k}{2} > 1$, let $P^k_m(z) = P^k_m(z, k/2)$ be the holomorphic Poincaré series of weight k. Assume that $d, d' > 0$; the other cases are trivial. We have

$$\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(P^k_m(z)) = \sum_{f\in S_k}\langle P^k_m, f\rangle\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(f)$$

and

$$P^{\frac{k+1}{2},+}_m(z) = \sum_{g\in S^+_k}\langle P^{\frac{k+1}{2}}_m, g\rangle g(z). \tag{4.6}$$

Using Proposition 2.4 at $s = \frac{k+1}{2}$ we may match the Fourier coefficients in (4.6) with the expression for $\mathrm{Tr}_{(-1)^{\frac{k}{2}}d,(-1)^{\frac{k}{2}}d'}(P^k_m)$ in section 3.6. Finally, as the inner products of Poincaré series are equal to Fourier coefficients of holomorphic modular forms, the same argument as in the non-holomorphic case yields the claim. □

5. Fourier Coefficients of Maass Cusp Forms as L-Values

The coefficients of half-integral weight forms are closely related to the values of L-functions. In this section, we investigate the central value of the twisted L-function

$$L(\varphi, d, w) = \sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{a_\varphi(n)}{n^{w+(k-1)/2}},\ (d, N) = 1 \tag{5.1}$$

for a normalized newform $\varphi \in U_{k,N}(s)$.

Note that for $k > 0$ and $\varphi \in U_{2-k,N}(s)$ even and normalized, we have

$$L(\xi_{2-k}\varphi, d, w) = -(4\pi)^{k-1}\frac{\Gamma(\overline{s}+k/2)}{\Gamma(\overline{s}-k/2)}\overline{L(\varphi, d, \overline{w})}$$

by Lemma 2.9 and (1.8). Hence, we may restrict to non-negative weights, as the twisted L-functions of Maass cusp forms of negative weights reduce to that case.

We can get the L-function for an even normalized Maass form of weight $k \geqslant 0$ from Theorem 1.4 through a Mellin transform. The following integral was first solved by Duke, Friedlander, and Iwaniec [8, Lemma 8.2] (there's a typo in the orginial version, see Young's paper [35, Sec. 12] for the correct evaluation). We present an alternative proof in our special case, which has the advantage of making the evaluation of the Mellin transform explicit instead of relying on a recursively defined polynomial.

Lemma 5.1. *Let $k \geqslant 0$ be an even integer and $s \neq \pm\frac{k}{2}$. The Mellin-transform*

$$\Psi_k(s) = \int_0^\infty \left(W_{\frac{k}{2},s-\frac{1}{2}}(y) + (-1)^{k/2}\frac{\Gamma(s+k/2)}{\Gamma(s-k/2)}W_{-\frac{k}{2},s-\frac{1}{2}}(y)\right)\frac{dy}{y} \tag{5.2}$$

takes the values

$$\Psi_k(s) = \frac{1}{2^{k/2}\sqrt{\pi}}\Gamma\left(\frac{s}{2}+\frac{k}{2}\right)\Gamma\left(\frac{1-s}{2}+\frac{k}{2}\right).$$

Proof. Let us write

$$\Psi_k^{\pm}(s) = \int_0^\infty \left(W_{\frac{k}{2},s-\frac{1}{2}}(y) \pm \frac{\Gamma(s+k/2)}{\Gamma(s-k/2)}W_{-\frac{k}{2},s-\frac{1}{2}}(y)\right)\frac{dy}{y}.$$

With this notation we have

$$\Psi_k(s) = \begin{cases}\Psi_k^+(s), & k \equiv 0 \pmod 4,\\ \Psi_k^-(s), & k \equiv 2 \pmod 4.\end{cases}$$

Consider the generating series $\sum_{k=0}^\infty \Psi_{2n}^+(s)\frac{x^n}{n!}$. We use the identities (see [26, Sec. 7.3.4])

$$e^{\frac{xy}{2(1+x)}}W_{0,s-\frac{1}{2}}\left(\frac{y}{1+x}\right) = \sum_{n=0}^\infty W_{n,s-\frac{1}{2}}(y)\frac{x^n}{n!}$$

and

$$e^{-\frac{xy}{2(1+x)}}W_{0,s-\frac{1}{2}}\left(\frac{y}{1+x}\right) = \sum_{n=0}^\infty \frac{\Gamma(s+n)}{\Gamma(s-n)}W_{-n,s-\frac{1}{2}}(y)\frac{x^n}{n!}.$$

With these the generating series becomes

$$\begin{aligned}\sum_{n=0}^\infty \Psi_{2n}^+(s)\frac{x^n}{n!} &= \int_0^\infty \sum_{n=0}^\infty \left(W_{n,s-\frac{1}{2}}(y) + \frac{\Gamma(s+n)}{\Gamma(s-n)}W_{-n,s-\frac{1}{2}}(y)\right)\frac{x^n}{n!}\frac{dy}{y}\\ &= \int_0^\infty W_{0,s-\frac{1}{2}}(y/(1+x))\left(e^{\frac{xy}{2(1+x)}} + e^{-\frac{xy}{2(1+x)}}\right)\frac{dy}{y}.\end{aligned}$$

This gives

$$\begin{aligned}\sum_{n=0}^\infty \Psi_{2n}^+(s)\frac{x^n}{n!} &= \int_0^\infty W_{0,s-\frac{1}{2}}(y)\left(e^{\frac{xy}{2}} + e^{-\frac{xy}{2}}\right)\frac{dy}{y}\\ &= 2\sum_{n=0}^\infty \frac{x^{2n}}{(2n)!\,2^{2n}}\int_0^\infty y^{2n-1}W_{0,s-\frac{1}{2}}(y)dy.\end{aligned}$$

We also see here that $\Psi_k^+(s) = 0$ for $k \equiv 2 \pmod 4$.

The Whittaker function $W_{0,s-\frac{1}{2}}(y)$ is a modified Bessel-function of second kind $K_{s-\frac{1}{2}}$, namely

$$W_{0,s-\frac{1}{2}}(y) = 2\sqrt{\frac{y}{\pi}}K_{s-\frac{1}{2}}(2y).$$

The integral of the K-Bessel function is well-known and evaluates to

$$\begin{aligned}\int_0^\infty y^{2n-1}W_{0,s-\frac{1}{2}}(y) &= \frac{2}{\sqrt{\pi}}2^{-2n-\frac{1}{2}}\int_0^\infty y^{2n+\frac{1}{2}-1}K_{s-\frac{1}{2}}(y)dy \\ &= \frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}+n\right)\Gamma\left(\frac{1-s}{2}+n\right).\end{aligned}$$

Hence, for $k \equiv 0 \pmod 4$, we get the desired result. The same argument for the generating series $\sum_{n=0}^\infty \Psi_{2n}^-(s)\frac{x^n}{n!}$ yields the result for the case $k \equiv 2 \pmod 4$. □

Remark 5.2. Our proof also shows that

$$\int_0^\infty \left(W_{\frac{k}{2},s-\frac{1}{2}}(y) - (-1)^{k/2}\frac{\Gamma(s+k/2)}{\Gamma(s-k/2)}W_{-\frac{k}{2},s-\frac{1}{2}}(y)\right)\frac{dy}{y} = 0$$

for all $k \geqslant 0$.

The following Corollary is a generalization of a result of Baruch and Mao [3, Thm. 1.4] to higher weights.

Corollary 5.3. *Let k be an even non-negative integer, let $\varphi \in U_{k,N}(s)$ be an even normalized newform with spectral point $s \neq \frac{k}{2}$, and let d be a fundamental discriminant with $(-1)^{\frac{k}{2}}d > 0$ and $(d,N) = 1$. Suppose that for all $m\|N$, its eigenvalues w_m under the Atkin-Lehner involution W_m are equal to $\left(\frac{d}{m}\right)$. Let*

$$\psi = \sum_{n\neq 0} b_\psi(n)(4\pi|n|y)^{-\frac{k+1}{4}}W_{\mathrm{sign}(n)\frac{k+1}{4},\frac{s}{2}-\frac{1}{4}}(4\pi|n|y)e(nx)$$

be a Maass cusp form of weight $\frac{k+1}{2}$ such that $\mathrm{Shim}_d(\psi) = \varphi$ and $b_\psi(n) \neq 0$ only if $(-1)^{\frac{k}{2}}n \equiv 0,1 \pmod 4$. Then:

$$6\sum_{\mathrm{Shim}_d(\psi)=\varphi}\frac{|b_\psi(|d|)|^2}{\langle\psi,\psi\rangle} = \frac{2^{\nu(N)}|d|^{\frac{k-1}{2}}}{(4\pi)^{k/2}\sqrt{\pi}}\Gamma\left(\frac{s}{2}+\frac{k}{2}\right)\Gamma\left(\frac{1-s}{2}+\frac{k}{2}\right)\frac{L(\varphi,d,1/2)}{\langle\varphi,\varphi\rangle},$$

where $\nu(N)$ is the number of distinct prime factors of N.

Proof. Let $d = d'$ in Theorem 1.4 with $(-1)^{k/2}d > 0$. We then have

$$6d(-1)^{\lfloor k/4\rfloor}\sum_{\mathrm{Shim}_d(\psi)=\varphi}|b_\psi(|d|)|^2 = 2^{k/2}\mathrm{Tr}_{d,d}(\varphi).$$

We may now evaluate the trace on the right hand side as

$$\mathrm{Tr}_{d,d}(\varphi) = \sum_{\mu(d)}\sum_{m\|D}\chi_d(W_m.[0,d,\mu])\int_{W_m.C_{Q_\mu}}\varphi(z)(W_m.Q_\mu)(z,1)^{\frac{k}{2}}dz_{W_m.Q_\mu}$$

by Lemma 3.3 and since $\chi_d(W_m.[0,d,\mu]) = \left(\frac{d}{\mu}\right)\left(\frac{d}{m}\right)$ we have

$$\mathrm{Tr}_{d,d}(\varphi) = \sum_{\mu(d)}\sum_{m\|N}\left(\frac{d}{\mu}\right)\left(\frac{d}{m}\right) w_m \int_{C_{Q_\mu}} \varphi(z)(Q_\mu)(z,1)^{\frac{k}{2}} dz_{Q_\mu} =$$
$$2^{\nu(N)}\sum_{\mu(d)}\left(\frac{d}{\mu}\right)\int_{C_{Q_\mu}} \varphi(z)(Q_\mu)(z,1)^{\frac{k}{2}} dz_{Q_\mu}.$$

Now the Fourier expansion of ϕ leads to

$$\sum_{\mu(d)}\left(\frac{d}{\mu}\right)\int_{C_{Q_\mu}} \varphi(z)(Q_\mu)(z,1)^{\frac{k}{2}} dz_{Q_\mu} =$$
$$d^{\frac{k}{2}-1}|d|i^{\frac{k}{2}}\sum_{\mu(d)}\left(\frac{d}{\mu}\right)\int_0^\infty \sum_{n\neq 0}\frac{a_\varphi(n)}{(4\pi|n|)^{k/2}} W_{\mathrm{sign}(n)\frac{k}{2},\tilde{s}-\frac{1}{2}}(4\pi|n|t)e^{-2\pi i n\frac{\mu}{d}}\frac{dt}{t} =$$
$$\frac{d^{\frac{k}{2}-1}|d|}{(4\pi)^{k/2}} i^{\frac{k}{2}}\left(\frac{d}{-1}\right)^{1/2}\sum_{n\neq 0}\left(\frac{d}{n}\right)\frac{a_\varphi(n)}{|n|^{k/2}}\int_0^\infty W_{\mathrm{sign}(n)\frac{k}{2},\tilde{s}-\frac{1}{2}}(t)\frac{dt}{t}.$$

Here we used the evaluation of the Gauss sum

$$\sum_{\mu(d)}\left(\frac{d}{\mu}\right) e\left(-n\frac{\mu}{d}\right) = \sqrt{|d|}\left(\frac{d}{-1}\right)^{1/2}\left(\frac{d}{n}\right).$$

Finally the the condition $(-1)^{k/2}d > 0$ gives $\left(\frac{d}{-n}\right) = (-1)^{k/2}\left(\frac{d}{|n|}\right)$ for $n > 0$. Combining this with the fact that $a_\varphi(-n) = \frac{\Gamma(s+k/2)}{\Gamma(s-k/2)}a_\varphi(n)$ for $n > 0$, if $\tilde{s} \neq k/2$ gives

$$\mathrm{Tr}_{d,d}(\varphi) = 2^{\nu(N)}\frac{|d|^{\frac{k+1}{2}}(-1)^{\lfloor k/4\rfloor}}{(4\pi)^{k/2}}\Psi_k(s)\sum_{n>0}\frac{a_\varphi(n)}{n^{k/2}}\left(\frac{d}{n}\right)$$

where the Mellin transform $\Psi_k(s)$ is defined in (5.2). The evaluation in Lemma 5.1 then finishes the proof except when $s = \frac{k}{2}$. Since in that case $a_\varphi(n) = 0$ for $n < 0$, one sees easily that $\Psi_k(k/2) = 2^{\frac{k}{2}}\Gamma(k/2)$, which recovers Kohnen's result in the holomorphic case [20, Cor. 1]. ☐

Remark 5.4. The proof of Corollary 5.3 together with Remark 5.2 shows that $\mathrm{Tr}_{d,d}(\varphi) = 0$ for φ being an odd Maass cusp form (as the Fourier coefficients of an odd Maass cusp form φ satisfy $a_\varphi(-n) = -\frac{\Gamma(s+k/2)}{\Gamma(s-k/2)}a_\varphi(n)$ for $n > 0$).

Finally, Corollary 5.3 can be used to prove nonnegativity of twisted L-functions of Maass cusp forms at the central value.

Corollary 5.5. *Let k be an even non-negative integer, let $\varphi \in U_{k,N}(s)$ be an even normalized newform, and let d be a fundamental discriminant such that $(-1)^{k/2}d > 0$ and $(d,N) = 1$. Then $L(\varphi, d, 1/2) \geqslant 0$.*

Proof. This follows from Corollary 1.8 and $\Gamma(x+iy)\Gamma(x-iy) = |\Gamma(x+iy)|^2$. ☐

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