Lagrangian Hofer metric

Reference: Hofer's metric on the space of diameters
M. Khanevsky, 2009

(M,ω) – symplectic manifold

$\mathcal{L}$ – a collection of Lagrangians in M

d_H – Lagrangian Hofer metric on $\mathcal{L}$

Question: Does $(\mathcal{L}, d_H)$ have finite or infinite diameter?

$$\sup_{L,L' \in \mathcal{L}} d_H(L,L') \overset{?}{<} \infty$$

Khanevsky: $M =$ disk, cylinder, 2-sphere
(∞) (∞) $(?)$

Background in Symplectic Geometry

(M,ω) symplectic manifold: M smooth manifold

$\omega \in \Omega^2(M)$ closed and non-degenerate

	D^2	S^2
manifold	$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ (open)	$S^2 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ (compact)
symplectic structure	$\omega = \frac{1}{\pi}\, dx \wedge dy$	$\omega = \frac{1}{4\pi}\,\Omega$ $\Omega_x(\xi,\eta) = \langle \eta, x \times \xi \rangle$ $(x\in S^2, \ \xi,\eta \in T_x S^2 \subset \mathbb{R}^3)$ 

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Lagrangians $L \subset M^{2n}$ are n-dimensional smooth submanifolds with

$$\omega\big|_{TL} = 0$$

i.e. $\forall p \in L, \forall \xi, \eta \in T_pL \subset T_pM: \ \omega(\xi, \eta) = 0.$

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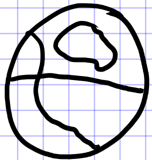
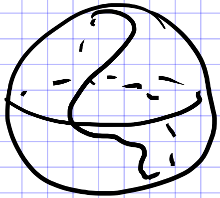
Lagrangians	1-dimensional submanifolds $\omega\big|_{TL}=0$ automatic 	1-dimensional submanifolds 

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Given a smooth function $H: \mathbb{R} \times M \longrightarrow \mathbb{R}$ (<u>Hamiltonian</u>), consider the <u>Hamiltonian vector fields</u>

$$\omega(X_t^H(p), \xi) = -(dH_t)_p(\xi) \quad \text{for } p \in M, \ \xi \in T_pM,$$

$$H_t(p) = H(t,p).$$

Hamiltonian vector fields

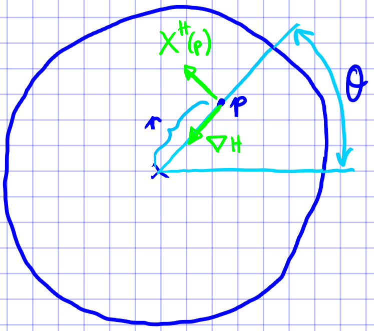
Think of X_t^H as "orthogonal" to ∇H_t: In $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$

$$X_t^H = J(\nabla H_t) \quad \text{where} \quad J = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$ in basis $x_1, y_1, \dots, x_n, y_n$.

Example $D \subset \mathbb{R}^2$

$H_t(r, \theta) := -\frac{r^2}{2} + \frac{1}{2}$ on $\{r \leq 1-\varepsilon\}$ (independent of t)

$H_t(r, \theta) = 0$ on $\{r > 1-\frac{\varepsilon}{2}\}$



$\Rightarrow X_t^H(p) = 2 \cdot r \cdot \hat{t}$, where $\hat{t}$ is the unit vector tangential to p pointing counter-clockwise direction.

if $p \in \{r \leq 1-\varepsilon\}$

Hamiltonian flow

Study the flow of X_t^H : Consider solutions $\varphi_t^H : M \longrightarrow M$ of

$$\begin{cases} \frac{d}{dt}\varphi_t^H(x) = X_t^H(\varphi_t(x)) & \text{for all } t\in\mathbb{R},\ x\in M \\ \varphi_0^H(x) = x & \text{for all } x\in M \end{cases}$$

Assume that H is compactly supported.

$\rightsquigarrow$ Unique solutions existing for all times.

$$\begin{cases} x\in M \\ t \longmapsto \varphi_t^H(x) = \gamma(t) \\ \quad \dot\gamma(t) = X_t^H(\gamma(t)) \\ \quad \gamma(0) = x \end{cases}$$

Fact $(\varphi_t^H)^*\omega = \omega$ and φ_t sends Lagrangians to Lagrangians.

Lagrangian Hofer metric

Let $L_0, L_1 \subset M$ be Lagrangians, that are Hamiltonian isotopic,

i.e. $\exists\, H : \mathbb{R} \times M \longrightarrow \mathbb{R}$ so that $\varphi_1^H(L_0) = L_1$.

Lagrangian Hofer metric:

$$d_H(L_0, L_1) = \inf \left\{ \int_0^1 \left(\max_M H_t - \min_M H_t \right) dt \;\middle|\; H : \mathbb{R}\times M \to \mathbb{R},\ \varphi_1^H(L_0) = L_1 \right\}$$

This is often a metric on

$$\mathcal{L}(L_0) = \{ L \subset M \mid L \text{ Hamiltonian isotopic to } L_0 \}.$$

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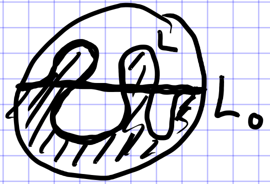
Lagrangians	1-dimensional submanifolds	1-dimensional submanifolds
$\mathcal{L}$	$L_0 = \{y=0\} \subset D^2$ $\mathcal{L}(L_0) = \{L \text{ Ham. isotopic to } L_0\}$ $= \{$all equators, coinciding with L_0 near $\partial D\}$  $L = \varphi_1^H(L_0)$	$L_0 = \{z=0\} \subset S^2$ $\mathcal{L}(L_0) = \{L \text{ Ham. isotopic to } L_0\}$ $= \{$all equators in $S^2\}$ 

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Sketch of proof for $D^2 \subset \mathbb{R}^2$

Prop There exists a non-trivial homogeneous quasimorphism

$$r : \mathrm{Ham}(D^2) \longrightarrow \mathbb{R}$$

so that

(1) $$|r(\phi)| \leq C \cdot \|\phi\|_H$$

(2) $$r(\phi) = 0 \text{ if } \phi(L_0) = L_0 .$$

$$\mathrm{Ham}(D^2) = \{\text{Hamiltonian diffeo's}\} = \{\varphi_t^H : D^2 \to D^2 \mid H \text{ Hamilt.}\}$$
$$= \{\text{area-pres diffeos isotopic to id}\} = \{\text{area-pres diffeos}\}$$

$$\|\phi\|_H = \inf\left\{ \int_0^1 \max H_t - \min H_t \, dt \;\middle|\; H : \phi_1^H = \phi \right\}$$

$$d(L_0, \phi(L_0)) = \inf\left\{ \|\psi\|_H \;\middle|\; \psi(L_0) = \phi(L_0) \right\}$$

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* If $\phi, \phi' \in \mathrm{Ham}(D)$ with $\phi(L_0) = \phi'(L_0) = L'$, one has

$$|r(\phi) - r(\phi')| \leq \mathrm{defect}(r).$$

$$\phi \phi'^{-1}(L_0) = L_0 \Rightarrow r(\phi \phi'^{-1}) = 0$$

$$\Rightarrow |r(\phi) - r(\phi')| = |r(\phi) + r(\phi'^{-1}) - r(\phi\phi'^{-1})| \leq \mathrm{defect}(r)$$

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* $d_H(L_0, L') \ge \frac{1}{C}|r(\phi_0)| - \frac{D}{C}$ for any ϕ_0 with $\phi_0(L_0) = L'$

$\|$

$$\inf\{\underbrace{\|\phi\|_H}_{\overset{(1)}{\ge} \frac{1}{C}|r(\phi)|} \mid \underline{\phi(L_0) = L'}\} \ge \frac{1}{C}\left(r(\phi_0) - D\right)$$

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* $d_H(L_0, \phi_0^n(L_0)) \geq \frac{1}{C} |r(\phi_0^n)| - \frac{D}{C}$ $= \frac{1}{C} r(\phi_0) \cdot n - \frac{D}{C}$

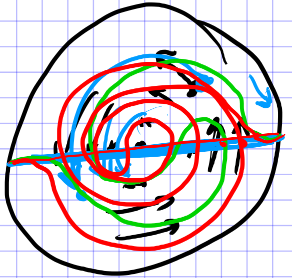
$$\xrightarrow{n \to \infty} +\infty$$

if $r(\phi_0) \neq 0$.

Two Remarks: $d_H(L_0, \varphi_t^H(L_0)) = \inf\{ \int_0^1 \max K_t - \min K_t \, dt \mid K : \Phi_1^K(L_0) = \varphi_t^H(L_0) \}$

1) The Hamiltonian $H : D^2 \longrightarrow \mathbb{R}$ with $H(r,\theta) = -\frac{r^2}{2} + \frac{1}{2}$ on $\{r < 1-\varepsilon\}$ induces $\varphi_0 := \varphi_1^H \in \mathrm{Ham}(D^2)$ which satisfies

$$r(\varphi_0) \neq 0.$$



Thus $d(L_0, \varphi_0^n(L_0)) \xrightarrow{n \to \infty} \infty$.

2) Construction of $r : \mathrm{Ham}(D^2) \longrightarrow \mathbb{R}$:

(a) $\gamma : \mathrm{Ham}(S^2) \longrightarrow \mathbb{R}$ homogeneous "Calabi" quasimorphism, coming from Floer theory (spectral numbers) [Entov-Polterovich, 2002]

(b) $D^2 \hookrightarrow S^2 \rightsquigarrow$

$$\begin{tikzcd} \mathrm{Ham}(D^2) \arrow[r, hook] \arrow[rr, bend right, "\gamma_D"'] & \mathrm{Ham}(S^2) \arrow[r, "\gamma"] & \mathbb{R} \end{tikzcd}$$

$\mathrm{Cal}_{D^2}(\Phi_1^H)$
$= \int\left(\int H_t \omega\right) dt$

and $\mathrm{Cal}_{D^2} : \mathrm{Ham}(D^2) \longrightarrow \mathbb{R}$ homomorphism.

Then $r := \mathrm{Cal}_{D^2} - \gamma_D$ satisfies the proposition.

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