Lagrangian Cobordisms

§1 Definition

§2 Lagrangian suspension

§3 Surgery cobordism

(§4 The Mak-Wu cobordism)

Lagrangian Cobordisms

§1 - Definition

(M, ω) closed symplectic manifold.

$L_1, \dots, L_{k_-}, L_1', \dots, L_{k_+}' \subset M$ closed Lagrangian submanifolds

A <u>Lagrangian cobordism</u> $V : (L_j') \rightsquigarrow (L_i)$ is a

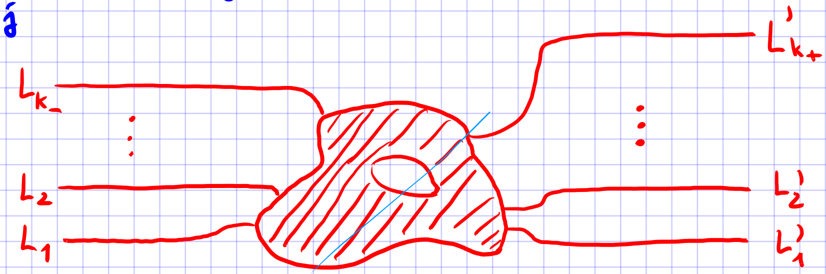
Lagrangian submanifold $V \subset (\mathbb{R}^2 \times M, dt \wedge ds + \omega)$ such that

$$V \cap \pi_{\mathbb{R}^2}^{-1}((-\infty, 0] \times \mathbb{R}) = \coprod_i (-\infty, 0] \times \{i\} \times L_i,$$

$$V \cap \pi_{\mathbb{R}^2}^{-1}([1, +\infty) \times \mathbb{R}) = \coprod_j [1, +\infty) \times \{j\} \times L_j'$$

$V \cap \pi_{\mathbb{R}^2}^{-1}([0,1] \times \mathbb{R})$ is compact

<u>Picture of $\pi_{\mathbb{R}^2}(V)$:</u>



§2 - Lagrangian Suspension

$H: \mathbb{R} \times M \longrightarrow \mathbb{R}$ Hamiltonian function with

$$H_t = H(t, -) = 0 \quad \text{for} \quad t \notin (0,1).$$

Its flow:

$$\phi_t^H : M \longrightarrow M \quad , \quad t \in [0,1] \, , \quad \phi_0^H = id$$

ie.

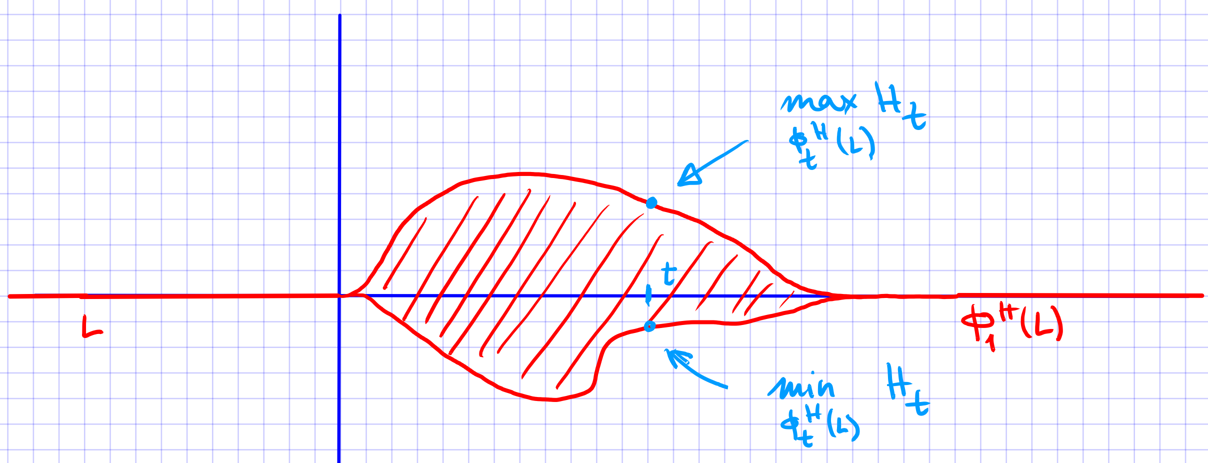
$$\frac{d}{dt} \phi_t^H(x) = X_t^H(\phi_t^H(x)) \, , \quad \omega(X_t^H(\cdot), \cdot) = - dH_t(\cdot) \, .$$

$L \subset M$ Lagrangian

Then

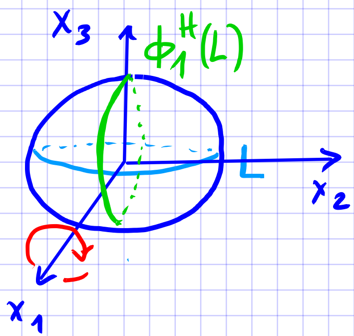
$$\mathcal{S}_{L,H} = \{ (t, \; H_t(\phi_t^H(x)), \phi_t^H(x)) \mid t \in \mathbb{R}, x \in L \} \subset \mathbb{R}^2 \times M$$

is a Lagrangian cobordism $\phi_1^H(L) \rightsquigarrow L$.



$$\mathcal{S}_{L,H} = \{ (t, H_t(\phi_t^H(x)), \phi_t^H(x)) \mid t \in \mathbb{R}, x \in L \} \subset \mathbb{R}^2 \times M$$

<u>Example</u> $M = S^2 \subset \mathbb{R}^3$ unit sphere, $L = \{x_3 = 0\} \subset S^2$

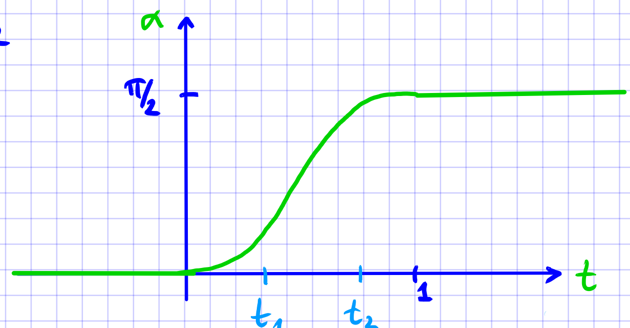


<u>Idea</u>: Rotate $90°$ around x-axis.

$$H_t(x_1, x_2, x_3) = \alpha'(t)\, x_1$$

$$\phi_t^H(\underline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \end{pmatrix}$$

where



$\mathcal{S}_{L,H}$ is a Lagrangian cobordism from L to $\phi_1^H(L)$.

$$\mathcal{S}_{L,H} = \{ (t,\ H_t(\phi_t^H(x)),\ \phi_t^H(x)) \mid t\in\mathbb{R},\ x\in L \} \subset \mathbb{R}^2\times M$$

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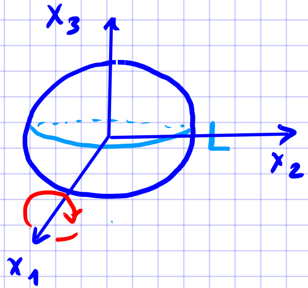

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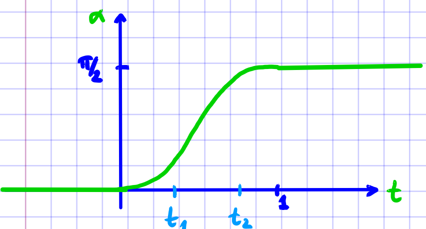
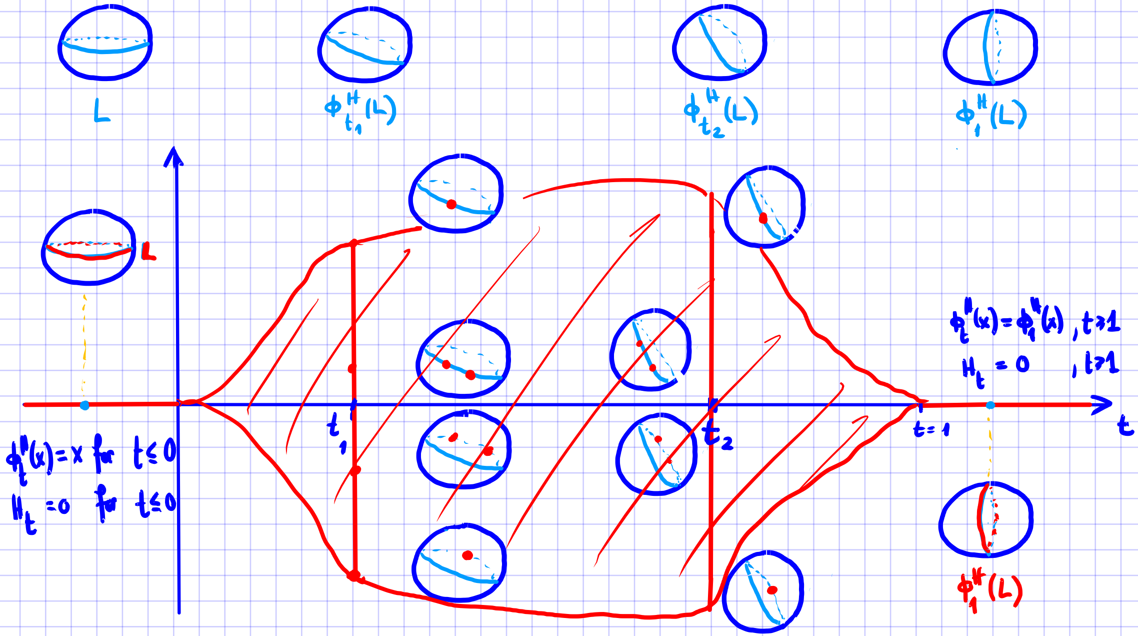
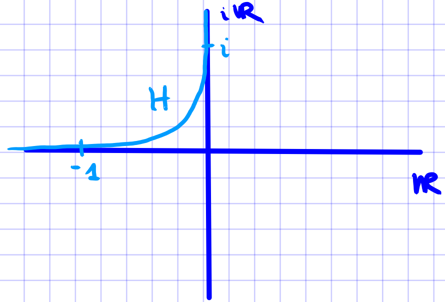




§3 – Surgery cobordism



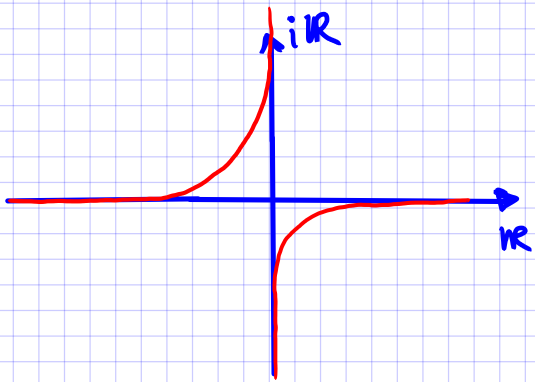
$\mathbb{R}^n, i\mathbb{R}^n \subset \mathbb{C}^n$,

Curve $H \subset \mathbb{C}$ parametrized by $a(t) + ib(t)$

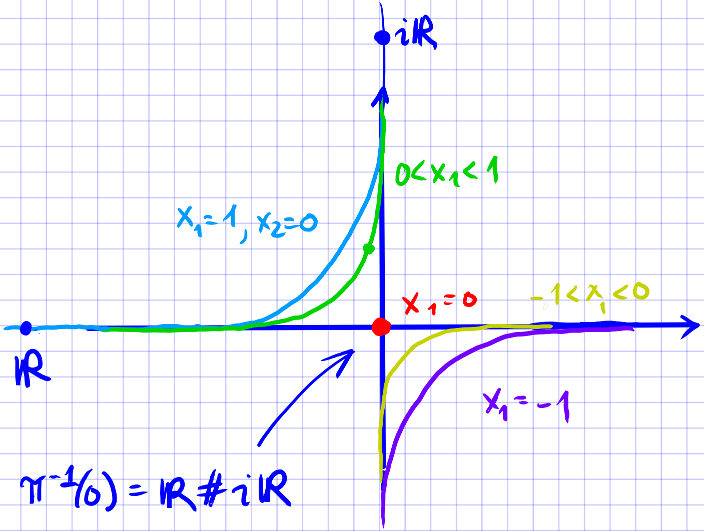
$$\mathbb{R}^n \# i\mathbb{R}^n := \left\{ \left((a(t)+ib(t))x_1, \dots, (a(t)+ib(t))x_n\right) \,\middle|\, t \in \mathbb{R}, \sum_{i=1}^n x_i^2 = 1 \right\} \subset \mathbb{C}^n$$

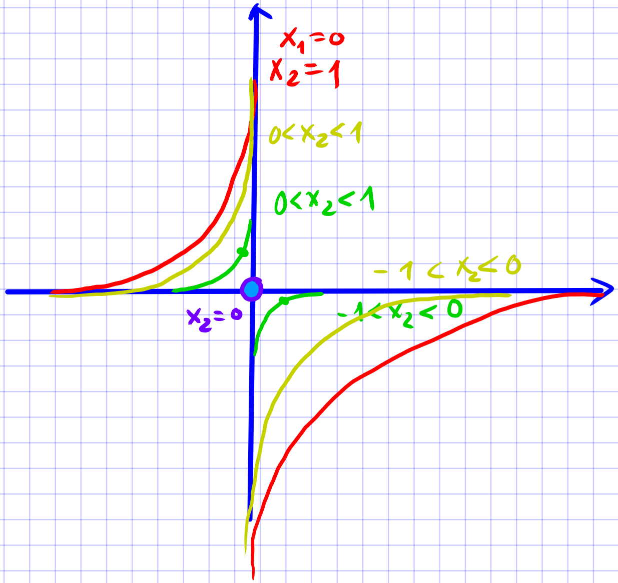
This is a Lagrangian submanifold. (Polterovich, 1990)

<u>$n=1$</u>: $\mathbb{R} \# i\mathbb{R} = \{ a(t) + ib(t) \mid t \in \mathbb{R} \}$
$\cup \{ -(a(t) + ib(t)) \mid t \in \mathbb{R} \}$

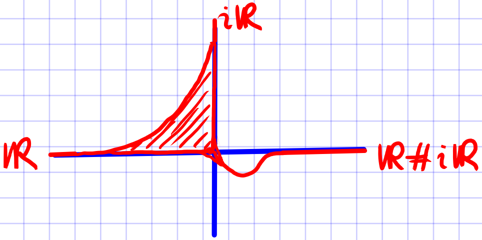


$\underline{n=2:}$ $\quad \mathbb{R}^2 \# i\mathbb{R}^2 = \{ ((a(t)+ib(t))\,x_1,\ (a(t)+ib(t))\,x_2) \mid t\in\mathbb{R},\ x_1^2+x_2^2=1 \} \subset \mathbb{C}^4$

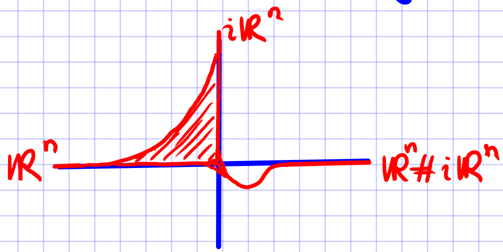




$\Longrightarrow$ Taking half of $\mathbb{R}^2 \# i\mathbb{R}^2$ and perturbing the corner gives a cobordism with ends $\mathbb{R}$, $i\mathbb{R}$ and $\mathbb{R} \# i\mathbb{R}$.

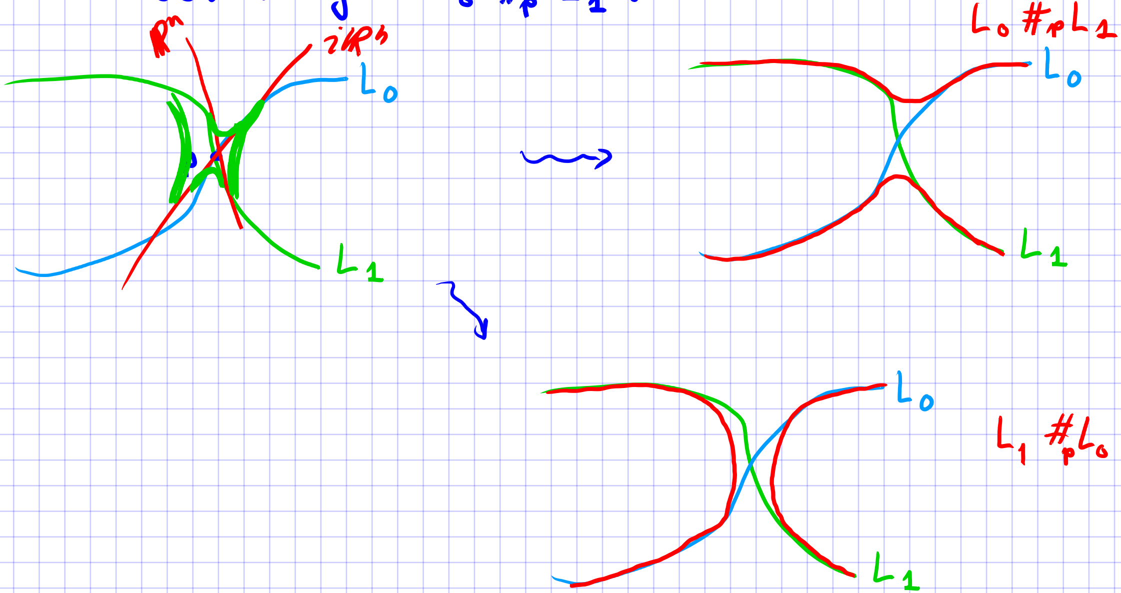


Similarly, for $n \geq 1$, $\mathbb{R}^{n+1} \# i\mathbb{R}^{n+1}$ yields a cobordism



"Surgery cobordism" (Biran-Cornea 2012)

For general Lagrangians $L_0, L_1 \subseteq M$ intersecting transversely in $p \in L_0 \cap L_1$ glue in the local model to get $L_0 \#_p L_1$.



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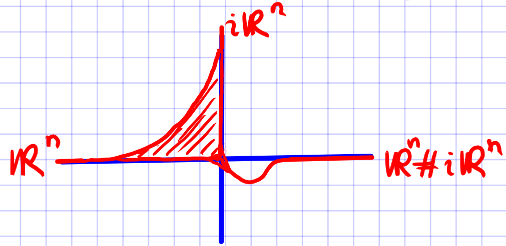

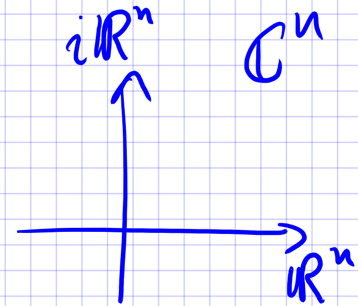

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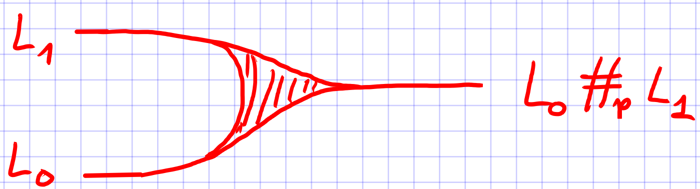
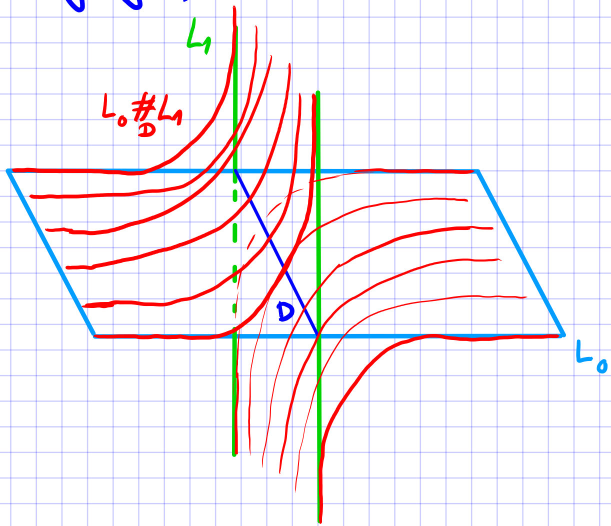
There is a Lagrangian cobordism:

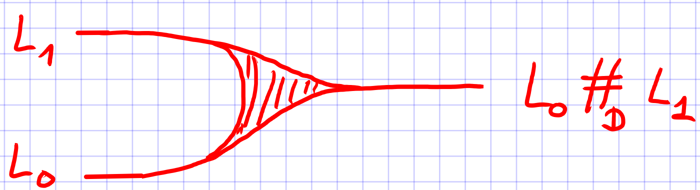


§4 - The Mak-Wu cobordism

There is a version of surgery for clean intersections:



Surgery for $\mathbb{R} \times L_0$ and $i\mathbb{R} \times L_1$ along $\{0\} \times D$ yields



Mak-Wu cobordism (Mak-Wu, 2018)

$S \subset M$ Lagrangian sphere $\qquad \Delta = \{(x,x) \mid x \in M\} \subset M \times M^-$

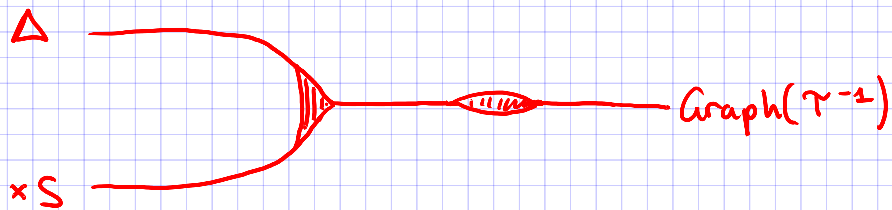
$\tau : M \longrightarrow M$ Dehn twist along S. $\qquad \Delta_S = \{(x,x) \mid x \in S\} \subset \Delta$

Then

$$(S \times S) \#_{\Delta_S} \Delta \subset (M \times M, \omega \oplus -\omega)$$

is Hamiltonian isotopic to Graph(τ^{-1}).

Corollary



Lagrangian cobordism $\subseteq \mathbb{R}^2 \times M \times M^-$.