Uniqueness of Embeddings of the Affine Line into Algebraic Groups

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Introduction

We study (algebraic) embeddings $X \to Y$ of varieties over the complex numbers $\mathbb{C}$ up to (algebraic) automorphisms of Y. We say that two closed (algebraic) embeddings $f, g\colon X \to Y$ are *equivalent* if there exists an automorphism $\varphi\colon Y \to Y$ such that $\varphi \circ f = g$.

Problem

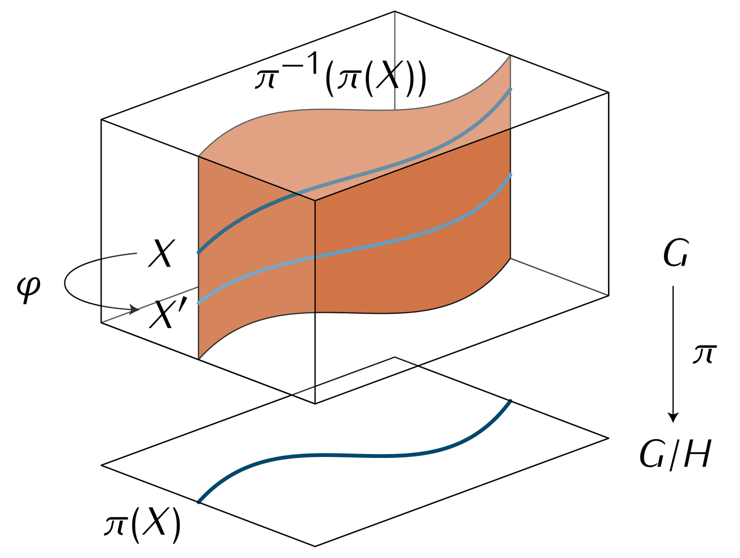
For varieties X and Y, describe the equivalence classes of closed embeddings $X \to Y$.

We consider embeddings of the affine line $\mathbb{C}$ into varieties Y that arise as underlying varieties of *(affine) algebraic groups*.

Tools

Moving Tool. Let X be a curve in an algebraic group G that is isomorphic to $\mathbb{C}$. The following is the main tool to move X in G via an automorphism of G.

Let $H \subseteq G$ be a closed subgroup such that G/H is quasi-affine and let $\pi\colon G \to G/H$ be the quotient map. If π restricts to an embedding on X and if X' is another section of $\pi^{-1}(\pi(X)) \to \pi(X)$, then there exists an automorphism φ of G that preserves π and maps X onto X'.



Main Generic Quotient Result. In order to use our moving tool, we need results which enable us to quotient an algebraic group G by closed subgroups H such that the quotient map $G \to G/H$ restricts to a closed embedding on a fixed curve in G. Our main result in this direction is the following.

If G is simple and of rank at least two, and if H is a closed unipotent subgroup, then for any curve $X \subseteq G$ that is isomorphic to $\mathbb{C}$ there exists an automorphism φ of G such that for generic $g \in G$ the quotient map $\pi_g\colon G \to G/gHg^{-1}$ restricts to an embedding on $\varphi(X)$:

$$\begin{array}{ccc} \varphi(X) & \subseteq & G \\ \cong\downarrow & & \downarrow \pi_g \\ \pi_g(\varphi(X)) & \subseteq & G/gHg^{-1} \end{array} \qquad \text{for generic } g \in G.$$

Summary

Uniqueness (up to automorphisms) of embeddings of $\mathbb{C}$ into different varieties:

	dim 2	dim 3	dim 4	dim 5	···
Affine space	✓	?	✓	✓	···
Algebraic group without non-trivial characters	✓	?	✓	✓	···
Smooth irreducible affine flexible variety	✗	?	?	?	···
Smooth irreducible contractible affine variety	✓	✗	✗	✗	···

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Main Result

Theorem

Let G be a connected affine algebraic group. Then two embeddings of the affine line $\mathbb{C}$ into G are equivalent provided that G is not isomorphic as a variety to a product of a torus $(\mathbb{C}^*)^k$ and one of the three varieties $\mathbb{C}^3$, $\mathrm{SL}_2(\mathbb{C})$, and $\mathrm{PSL}_2(\mathbb{C})$.

In particular, $\mathbb{C}$ embeds uniquely (up to automorphisms) into algebraic groups without non-trivial characters of dimension different than 3. The special case, when $G = SL_n$ for $n \geq 3$ is done by the second author [Sta15].

Background and Further Question

Embedding problems in affine algebraic geometry are most classically considered for $Y = \mathbb{C}^n$. We recall what is known about uniqueness of embeddings of $\mathbb{C}$ into $\mathbb{C}^n$. If $n = 2$, all embeddings are equivalent by the Abhyankar-Moh-Suzuki Theorem [AM75, Suz74]. For $n \geq 4$, again all embeddings are equivalent by the work of Srinivas [Sri91], where he in particular shows that smooth affine varieties of dimension d embed uniquely into $\mathbb{C}^n$ whenever $n \geq 2d + 2$. The case $n = 3$ remains open [Kra96] and seems to be very hard.
For a different point of view we consider the notion of flexible varieties as studied by various authors in [AFK+13]. Flexible varieties can be seen as generalization of algebraic groups without non-trivial characters. Smooth irreducible affine flexible varieties of dimension ≥ 2 have the property that all embeddings of a fixed finite set are equivalent [AFK+13]. Our main result states that in most algebraic groups even all embeddings of $\mathbb{C}$ are equivalent. The following question is natural in light of our main result.

Question

Let Y be a smooth irreducible affine flexible variety of dimension at least four. Are all embeddings of $\mathbb{C}$ into Y equivalent?

Outline of the Proof

Let G be an algebraic group and let $X \subseteq G$ be a curve that is isomorphic to $\mathbb{C}$. The proof of our main result divides up into four steps

- Reduce to the case when G is simple and of rank at least two. We fix then a maximal parabolic subgroup P in G. Furthermore, we denote by E the inverse image of the unique Schubert curve in the flag variety G/P under the quotient map $G \to G/P$.
- One can move X into E via an automorphism of G. This is the key step in our proof.
- If $X \subseteq E$, then there exists an automorphism ψ of G such that $\psi(X)$ is a unipotent subgroup of G.
- All embeddings of $\mathbb{C}$ into G with a unipotent image are equivalent.

The Key Step: Moving X into E

Let P^- be an opposite parabolic subgroup to P and denote by $\pi\colon G \to G/R_u(P^-)$ the quotient map with respect to the unipotent radical of P^-. We establish, that the restriction of π to E is a locally trivial $\mathbb{C}$-bundle and $\pi(E)$ is a big open subset of $G/R_u(P^-)$, i.e. the complement is a closed subset of codimension at least two in $G/R_u(P^-)$. One can move X into E via the following steps.

- Using our main generic quotient result, we can achieve that π restricts to an embedding on X.
- Using that $\pi(E)$ is a big open subset of $G/R_u(P^-)$ and the G-equivariancy of π, we can move X into $\pi^{-1}(\pi(E))$ and π restricts still to an embedding on X.
- Since $E \to \pi(E)$ is a locally trivial $\mathbb{C}$-bundle, it has a section $X' \subseteq E$ over $\pi(X) \cong \mathbb{C}$. Therefore, we can move X into X' with our moving tool.