Recent highlights in low-dimensional topology

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We present three highlights, one from each of the years 2019, 2020, and 2021.

2020: Topology input yields Euclidean geometry result. An n-gone in the Euclidean plane $\mathbb{R}^2$ is said to *inscribe* in a Jordan curve $\Gamma \subset \mathbb{R}^2$ if there exists an orientation-preserving similarity of $\mathbb{R}^2$ that maps the vertices of the n-gone into Γ. A quadrilateral (i.e. a four-gone) in $\mathbb{R}^2$ is called *cyclic* if its vertices lie on a circle. The following result characterizes the quadrilaterals that inscribe in all smooth Jordan curves.

Theorem 1 (Greene-Lobb, 2020 [GL20]). *Every cyclic quadrilateral inscribes in every smooth Jordan curve in the Euclidean plane.*

The surprising (symplectic) topology input in Greene and Lobb's proof of Theorem 1 is the fact that there do not exist embedded Lagrangian tori in $(\mathbb{R}^4, \omega_{\text{std}})$ with minimum Maslov number 4 [Vit90, Pol91], where ω_{std} denotes the symplectic form $dx \wedge dy + dz \wedge dw$. We describe three stepping stones towards Theorem 1.

Firstly, in 2018, Hugelmeyer [Hug21] discovered a strategy of proof that allows to recover Schnirelman's result that squares inscribe in all smooth Jordan curves using the following knot theory input. The $T(4,5)$ torus knot in the three-sphere $\mathbb{S}^3 = \partial\mathbb{D}^4$ is not the boundary of an embedded smooth Möbius band in the four-ball $\mathbb{D}^4$. In fact, this is only implicit in Hugelmeyer's work (see [FG20] for details); instead, Hugelmeyer proved the following new result: rectangles with aspect ratio $\sqrt{3}$ inscribe in every smooth Jordan curve. For this he used that another knot, the $T(5,6)$ torus knot, is not the boundary of an embedded smooth Möbius band in $\mathbb{D}^4$ [Hug18]. Secondly, in 2019, Hugelmeyer followed up by showing that for every smooth Jordan curve "a third" of all rectangles inscribe [Hug21]. Thirdly, building on ideas from Hugelmeyer's follow-up, but crucially employing a symplectic topology perspective, Greene and Lobb showed that all rectangles inscribe in all smooth Jordan curves [GL21]. For this they employ that there do not exist embedded Lagrangian Klein bottles in $(\mathbb{R}^4, \omega_{\text{std}})$. The proof of Theorem 1 can be understood as an improvement on their argument for this result.

2019: Porting $\mathrm{Diff}^+(S_g)/\mathrm{Diff}_0(S_g)$ technology to $\mathrm{Diff}_0(S_g)$. The identity component of the group of C^∞-diffeomorphisms of a compact smooth manifold M, denoted by $\mathrm{Diff}_0(M)$, is perfect [Mat71, Mat74, Thu74]. In fact, results from [BIP08, Tsu08, Tsu12] imply that, for every closed and oriented manifold M that is diffeomorphic to a sphere or has dimension two or four, the group $\mathrm{Diff}_0(M)$ is uniformly perfect: there exist an $N \in \mathbb{N}$ such that every element can be written as a product of at most N commutators. In contrast, for the smooth, oriented, and closed surfaces S_g of genus $g \geq 1$ one has the following striking result.

Theorem 2 (Bowden-Hensel-Webb, 2019 [BHW19]). *For $g \geq 1$, the space of homogeneous quasimorphisms on $\mathrm{Diff}_0(S_g)$ is (uncountably) infinite dimensional.*

Theorem 2 relates to uniform perfectness as follows. A short calculation shows, that, if G is a group for which there exists of a homogeneous quasimorphism $f\colon G \to \mathbb{R}$ that is not constantly 0, then G is not uniformly perfect. Hence, Theorem 2 implies that $\mathrm{Diff}_0(S_g)$ is not uniformly perfect.

For the proof of Theorem 2, the authors proceed in analogy to an idea that can be used to show that the mapping class group $\mathrm{MCG} := \mathrm{Diff}^+(S_g)/\mathrm{Diff}_0(S_g)$ for $g \geq 3$ has many homogeneous quasimorphisms and hence, while being perfect, is not uniformly perfect (originally proven in [EK01]). Here is a terse account of this idea for MCG. Set $\mathcal{C} := \{[K] \mid K \text{ is an essential simple closed curve in } S_g\}$, where $[K]$ denotes the isotopy class of K. The group MCG acts on $\mathcal{C}$ via $\mathrm{MCG} \times \mathcal{C} \to \mathcal{C}, ([\phi],[K]) \mapsto [\phi(K)]$. This action allows to construct many homogeneous quasimorphism on MCG, using the following celebrated fact. Equipped with the curve graph metric[1], $\mathcal{C}$ is a Gromov-hyperbolic metric space [MM99].

Bowden, Hensel, and Webb fearlessly consider the following "large" analogue of $\mathcal{C}$: the set $\mathcal{C}^\dagger := \{K \mid K \text{ is an essential simple closed curve in } S_g\}$ with a similarly defined metric (simply dropping equivalence classes in the definition). Guided by analogy to the MCG setup, they show that $\mathcal{C}^\dagger$ is Gromov-hyperbolic and they construct many homogeneous quasimorphisms on $\mathrm{Diff}_0(S_g)$ using the action $\mathrm{Diff}_0(S_g) \times \mathcal{C}^\dagger \to \mathcal{C}^\dagger, (\phi, K) \mapsto \phi(K)$.

2021: A space version of the light bulb theorem for all dimensions. In this section results are only described in vague terms. In particular, information about orientations and framings is suppressed.

The (folklore) light bulb theorem says that all smooth embeddings of the interval $\mathbb{D}^1$ in $\mathbb{S}^2 \times \mathbb{D}^1$ with boundary $\{p\} \times (\partial\mathbb{D}^1)$ are isotopic rel boundary. Recent developments are Gabai's 4D light bulb theorem (same statement with $\mathbb{D}^1$ replaced by $\mathbb{D}^2$) and further results concerning the fourth dimension [Gab20, Sch20, ST19].

An elegant perspective allows to put all of this in a "spacified" context. Informally, the following result says that for $1 \leq k \leq d$ the space of embeddings of the k-disk $\mathbb{D}^k$ into a smooth oriented d-dimensional manifold M with prescribed boundary $s\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, denoted by $\mathrm{Emb}_\partial(\mathbb{D}^k, M)$, is homotopy equivalent to a certain path space of embeddings of the $(k-1)$-disk into a d-dimensional manifold, if s has a *geometrically dual sphere* G, i.e. $\mathbb{S}^{d-k} \cong G \subseteq \partial M$ and $|s(\mathbb{S}^{k-1}) \pitchfork G| = 1$.

Theorem 3 (Kosanović-Teichner, 2021 [KT21])**.** *Let $G \subseteq \partial M$ be a geometrically dual sphere for s, and set M_G to be the result of attaching a $(d-k+1)$-handle to M along G. Then* $\mathrm{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)$.

Without further describing the path space $\Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)$, we note that in the case of $k = 1$ and $d \geq 3$, one finds $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)) \cong \pi_1(M_G) \cong \pi_1(M)$. This recovers the light bulb theorem, since $\pi_0(\mathrm{Emb}_\partial(\mathbb{D}^1, M)) \cong \pi_1(M) = \{1\}$ for $M = \mathbb{S}^2 \times \mathbb{D}^1$. In case of $k = 2$ and $d = 4$, Kosanović and Teichner explicitly describe $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^1, M_G))$ using a so-called Dax invariant. This π_0-calculation amounts to a generalization of all prior light bulb theorems in 4D due

[1]The metric is the one induced from the graph with vertices $\mathcal{C}$ and one edge (of length 1) between $[K]$ and $[L]$ for all disjoint, non-isotopic, and essential simple closed curves K and L.

to the bijection between $\pi_0(\mathrm{Emb}_\partial(\mathbb{D}^2, M))$ and $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^1, M_G))$ provided by Theorem 3. In general, the homotopy type of embedding spaces (and loop spaces thereof) are easier to understand the larger the codimension $d-k$ is. The striking point of Theorem 3 is that, in the presence of dual spheres, the homotopy type of the embedding space of interest can be understood via the homotopy type of an embedding space with larger codimension.

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