Diss. ETH No. 17560

Absolute Irreducibility of the Residual Representation and Adelic Openness in generic characteristic for Drinfeld modules

A dissertation submitted to the
ETH ZÜRICH

for the degree of
Doctor of Sciences

presented by
EGON PAUL RÜTSCHE
Dipl. Math. ETH Zürich
born February 3, 1980
citizen of Kirchberg SG

accepted on the recommendation of
Prof. Dr. Richard Pink, examiner
Prof. Dr. Gebhard Böckle, co-examiner

2007

Acknowledgement

I want to thank Prof. Dr. Richard Pink, the supervisor of my thesis. I have benefited a lot from his knowledge, his ideas and suggestions. He has been an enormous help for all the questions and problems that arose during the construction of this thesis. He has been a perfect helmsman through the sometimes treacherous waters of my PhD-time.

Next, I want to thank Prof. Dr. Gebhard Böckle for giving me many hints and remarks and for careful reading of my thesis.

I also want to thank Thomas Huber and Nicolas Stalder for many helpful discussions and remarks on several points of my work.

Finally, I want to thank all those who supported me during my PhD-time in whatever way.

Contents

Summary

Let p be a prime number, and let q be a power of p. Let F be a finitely generated field of transcendence degree 1 over the finite field $\mathbb{F}_q$ of q elements, and let K be a finitely generated extension of $\mathbb{F}_q$. Fix a place ∞ of F, and denote by A the ring of all elements of F which are integral outside ∞.

In this thesis we study the images of Galois representations associated to Drinfeld modules.

To present the two main results, let $\varphi : A \longrightarrow K\{\tau\}$ be a Drinfeld A-module over K of rank r. For every prime $\mathfrak{p}$ of A we have a continuous Galois representation

$$\rho_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big)$$

on the $\mathfrak{p}$-adic Tate module of φ where G_K denotes the absolute Galois group of K. By reduction modulo $\mathfrak{p}$ we get *the residual representation*

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{\kappa_{\mathfrak{p}}}\big(\varphi[\mathfrak{p}](K^{sep})\big)$$

where $\kappa_{\mathfrak{p}}$ denotes the residue field at $\mathfrak{p}$. The natural question is to ask how large the image of this representation is. Our first main result shows that the image of $\overline{\rho_{\mathfrak{p}}}$ is typically quite large.

Theorem. *Let φ be a Drinfeld A-module over K. Assume that $\mathrm{End}_K(\varphi) = A$. Then the residual representation is absolutely irreducible for almost all primes $\mathfrak{p}$ of A.*

Next, if φ is of generic characteristic, we consider *the adelic representation*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f)$$

where $\mathbb{A}_F^f$ denotes the ring of finite adeles of F. The natural question again is to ask how large the image of this representation is. It has been conjectured that the image of this representation is open under suitable hypotheses, i.e., it is essentially as large as possible.

Our second main result proves this conjecture.

Theorem. *Let* φ *be a Drinfeld* A*-module over* K *of generic characteristic. Assume that* $\mathrm{End}_{\bar{K}}(\varphi) = A$*. Then the image of the adelic representation is open.*

Zusammenfassung

Sei p eine Primzahl, und sei q eine Potenz von p. Sei F ein endlich erzeugter Körper vom Transzendenzgrad 1 über dem endlichen Körper $\mathbb{F}_q$ mit q Elementen, und sei K eine endlich erzeugte Erweiterung von $\mathbb{F}_q$. Fixiere eine Stelle ∞ von F, und bezeichne mit A den Ring aller Elemente von F, welche ausserhalb ∞ ganz sind.

In dieser Arbeit studieren wir die Bilder von Galois Darstellungen, welche zu Drinfeld Moduln gehören.

Um die beiden Hauptresultate zu präsentieren, sei $\varphi : A \longrightarrow K\{\tau\}$ ein Drinfeld A-Modul über K vom Rang r. Für jedes Primideal $\mathfrak{p}$ von A haben wir eine stetige Galois Darstellung

$$\rho_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big)$$

auf dem $\mathfrak{p}$-adischen Tate Modul von φ, wobei G_K die absolute Galois Gruppe von K bezeichnet. Durch Reduktion modulo $\mathfrak{p}$ erhalten wir *die residuelle Darstellung*

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{\kappa_{\mathfrak{p}}}\big(\varphi[\mathfrak{p}](K^{sep})\big),$$

wobei $\kappa_{\mathfrak{p}}$ den Restklassenkörper bei $\mathfrak{p}$ bezeichnet. Es drängt sich die Frage auf, wie gross das Bild dieser Darstellung ist. Unser erstes Hauptresultat zeigt, dass das Bild von $\overline{\rho_{\mathfrak{p}}}$ typischerweise ziemlich gross ist.

Theorem. *Sei φ ein Drinfeld A-Modul über K. Nehme an, dass $\mathrm{End}_K(\varphi) = A$ ist. Dann ist die residuelle Darstellung absolut irreduzibel für fast alle Primideale $\mathfrak{p}$ von A.*

Dann betrachten wir für φ von generischer Charakteristik *die adelische Dartsellung*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f),$$

wobei $\mathbb{A}_F^f$ den Ring der endlichen Adelen von F bezeichnet. Die natürliche Frage ist wieder die nach der Grösse des Bildes der Darstellung. Es wurde vermutet, dass das Bild dieser Darstellung unter geeigneten Bedingungen offen ist, i.e., es ist im Wesentlichen so gross wie möglich.

Unser zweites Hauptresultat beweist dies.

Theorem. *Sei φ ein Drinfeld A-Modul über K von generischer Charakteristik. Nehme an, dass $\mathrm{End}_{\bar{K}}(\varphi) = A$ ist. Dann ist das Bild der adelischen Darstellung offen.*

Introduction

Notation

The following notation will be used throughout the whole thesis.

For any commutative ring R and any natural number n, we denote by $\mathrm{M}_n(R)$ the ring of $n \times n$-matrices with entries in R.

For any field L, we denote by $\bar{L}$ a fixed algebraic closure of L and by L^{sep} the separable closure of L in $\bar{L}$. By $\mathrm{G}_L := \mathrm{Gal}(L^{sep}/L)$ we denote the absolute Galois group of L.

Let p be a prime number, and let q be a power of p. Let F be a finitely generated field of transcendence degree 1 over the finite field $\mathbb{F}_q$, and let K be a finitely generated extension of $\mathbb{F}_q$.
Fix a place ∞ of F, and denote by A the ring of all elements of F which are integral outside ∞.

If K has transcendence degree 1 over its prime field, we denote by $\mathfrak{P}, \mathfrak{Q}, \ldots,$ places of K.
By $\mathfrak{p}, \mathfrak{q}, \ldots,$ we denote primes of A.
For any prime $\mathfrak{p}$ of A, we denote the residue field at $\mathfrak{p}$ by $\kappa_{\mathfrak{p}}$, and for any place $\mathfrak{P}$ of K, we denote the residue field at $\mathfrak{P}$ by $k_{\mathfrak{P}}$.

We assume that K is an A-field, i.e., it is endowed with a ring homomorphism

$$\iota : A \longrightarrow K.$$

The kernel of ι is called the characteristic of K. The field K is said to have generic characteristic if ι is injective, and special characteristic if $\mathfrak{p}_0 := \ker(\iota)$ is a nonzero prime of A. We denote by

$$D : \mathrm{End}_K(\mathbb{G}_a) = K\{\tau\} \longrightarrow K$$

the derivative at 0, i.e., if $f = \sum_{i=0}^{n} a_i \tau^i \in K\{\tau\}$, the derivative of f is given by $Df = a_0$. By φ we will always denote a Drinfeld A-module over K of rank r. The characteristic of K is also called the characteristic of φ.

Outline of the thesis

In this thesis, we study the Galois representations associated to Drinfeld modules. The aim is to describe their images qualitatively. If one regards Drinfeld modules as function field analogues of elliptic curves, it is not surprising that our main theorems hold. Serre proved the adelic openness for elliptic curves without complex multiplication in [29]. We will proceed along his lines. Actually, elliptic curves can best be compared with Drinfeld modules of rank 2. Since we want to prove our results for Drinfeld modules of arbitrary rank, we can not exactly take the same route as Serre did because some results he uses simply do not hold for higher rank. The residual representation will play a major role throughout the text.
The thesis is divided into three chapters. The first chapter presents some background material, the subsequent chapters each present one of the two main results.

Chapter 1. We assume that the reader is familiar with the basic notions of Drinfeld modules. In Section 1.1 we collect several results on the Galois representations $\rho_\mathfrak{p} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(A_\mathfrak{p})$ and $\overline{\rho_\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_\mathfrak{p})$. One of the most important ingredients from Chapter 1 is the openness of the image of the Galois representation for a finite set of primes of A. This result due to Pink can be found in [20].

In Section 1.2 we analyze the action of inertia groups on torsion points. The first result can be found in Pink and Traulsen [23] and states that the restriction of $\rho_\mathfrak{p}$ to the inertia group at any place not lying above $\mathfrak{p}$ is unipotent. It then remains to consider the action of the inertia group at a place $\mathfrak{P}$ on the $\mathfrak{p}$-torsion module $\varphi[\mathfrak{p}]$ for $\mathfrak{P}$ above $\mathfrak{p}$. To analyze it, we follow Serre [29]. We introduce the notion of fundamental characters, which will play a very important role. Using these, the action of inertia can be described. This is a major step for proving the absolute irreducibility.

Section 1.3 contains a class field theoretical result. Again, we follow Serre's approach. We introduce certain algebraic groups, for which we state and prove a result on interpolation of some characters.

Chapter 2. The second chapter contains the first main result of this thesis, the absolute irreducibility of the residual representation.

Theorem. *Let φ be a Drinfeld A-module over K. Assume that $\mathrm{End}_K(\varphi) = A$. Then the residual representation*

$$\overline{\rho_\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_\mathfrak{p})$$

is absolutely irreducible for almost all primes $\mathfrak{p}$ of A.

The proof of this result is divided into three sections. In Section 2.1 we prove it in the case where the endomorphisms of φ over $\bar{K}$ are scalar and where K is a finite extension of F. Pink and Traulsen proved the analogous result for Drinfeld modules of special characteristic in [23, Theorem 3.1]. Therefore we assume that φ has generic characteristic. We will give an indirect proof, assuming that the residual representation is not absolutely irreducible for infinitely many primes. First, we consider the determinant of the residual representation and describe its ramification. For this we can use the results from Chapter 1. Translating the situation into class field theoretical terms allows us to compare our character to characters of some algebraic groups. Using this we are able to construct an algebraic relation contradicting one of Pink's results.

In Section 2.2 we still assume that φ has generic characteristic, but we allow the Drinfeld module to have arbitrary endomorphism ring. Of course, the residual representation will no longer be absolutely irreducible. The best possible result is to give a description of the image of the group ring under the Galois representation. We will give this description without proving the details, since one can use exactly the same argument as Pink and Traulsen did in [23].

In Section 2.3 we present the proof of the general case. We do this by proving the general case of the result on the image of the group ring. The absolute irreducibility then is an immediate consequence. The proof will be given by reduction to the case where K is a finite extension F. This will be done using a similar argument as Pink did in [20]

Chapter 3. The third chapter deals with the adelic openness in generic characteristic. Section 3.1 gives some preparatory results on subgroups of matrix groups and on algebraic groups which will be important in the subsequent sections.

In Section 3.2 we show that the residual representation is surjective for almost all primes $\mathfrak{p}$ of A. The major steps are as follows. First, by our result from Chapter 1, we know that the image of the tame inertia group under the residual representation is quite big. Using this together with the absolute irreducibility from Chapter 2, we will show that all conjugates of the image of the tame inertia group generate $\mathrm{GL}_r(\kappa_\mathfrak{p})$. Finally, by using a result from Section 3.1, we can show that the residual representation is surjective for almost all primes $\mathfrak{p}$ of A.

Section 3.3 contains the proof of the adelic openness in generic characteristic for the case that K is a finite extension of F. We deduce the result from the surjectivity of the residual representation for almost all primes of A. The argument is very similar to the one in Gardeyn [10]. By Pink's result [20], we can discard a finite set of primes of A. For the remaining primes, we first prove a result on subgroups of $\mathrm{GL}_r(\kappa_\mathfrak{p})$. Together with a statement on the size of a certain ramification index, we can then show that $\mathrm{GL}_r(A_\mathfrak{p})$, as a factor of $\mathrm{GL}_r(\mathbb{A}_F^f)$, is contained in the image of the adelic representation. This implies that the image of the adelic representation is open.

In Section 3.4 we give a specialisation result whose proof is based on an argument from Pink [18] and uses a result from Pink [20].

Finally, Section 3.5 contains the second main result of this thesis.

Theorem. *Let φ be a Drinfeld A-module over K with generic characteristic. Assume that* $\mathrm{End}_{\bar{K}}(\varphi) = A$*. Then the image of the adelic representation*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f)$$

is open.

The proof is a reduction to the case of a finite extension using the specialisation result from Section 3.4.

The adelic openness means that the image of the adelic representation is as big as possible up to commensurability. This is the best result possible and gives a complete answer to the qualitative question what the image of the Galois representation looks like.

Chapter 1

General Results

In this chapter, we introduce some general results on Drinfeld modules which will be used in the following chapters. We assume that the reader is familiar with the basic notions of Drinfeld modules. These can be found for example in Drinfeld [6], Deligne and Husemöller [5], Hayes [12] or Goss [11, Chapter 4].

In the first section, we list some important results concerning Galois representations on Tate modules. These are due to Pink [20], [21], Taguchi [32], [33], [35] and Tamagawa [37].

In the second section, some results on the action of inertia groups on torsion points of Drinfeld modules are stated. The first is due to Pink and Traulsen [23]. The second we develop here. Following Serre's results from [29] for elliptic curves, we introduce the notion of fundamental characters and prove some results for these.

In Section 1.3, two algebraic groups are constructed in the same way as Serre did in [27] and [29]. Then we state a result which will allow us in the next chapter to compare certain characters.

1.1 Galois Representations associated to Drinfeld modules

Consider a Drinfeld A-module $\varphi : A \to K\{\tau\},\ a \mapsto \varphi_a$ over K of rank r and (any) characteristic $\mathfrak{p}_0$. For any ideal $\mathfrak{a}$ of A, denote by

$$\varphi[\mathfrak{a}] := \bigcap_{a\in\mathfrak{a}} \mathrm{Ker}(\varphi_a : \mathbb{G}_{a,K} \longrightarrow \mathbb{G}_{a,K}),$$

which is an intersection of closed subschemes of $\mathbb{G}_{a,K}$, and by

$$\varphi[\mathfrak{a}](K^{sep}) := \{x \in K^{sep} \mid \forall a \in \mathfrak{a} : \varphi_a(x) = 0\}$$

the module of $\mathfrak{a}$-torsion of φ. If $\mathfrak{a}$ does not divide the characteristic of φ, by Lang's theorem, this is a free $A/\mathfrak{a}$-module of rank r. For any prime $\mathfrak{p}$ of A different from the characteristic of φ, the $\mathfrak{p}$-adic Tate module

$$T_{\mathfrak{p}}(\varphi) := \lim_{\leftarrow} \varphi[\mathfrak{p}^n](K^{sep})$$

of φ is a free $A_{\mathfrak{p}}$-module of rank r where $A_{\mathfrak{p}}$ denotes the completion of A at $\mathfrak{p}$. We denote by $V_{\mathfrak{p}}(\varphi) := T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}$ the rational Tate module of φ.

For all $\mathfrak{p} \neq \mathfrak{p}_0$, there is a continuous Galois representation on the Tate module

$$\rho_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{A_{\mathfrak{p}}} \big(T_{\mathfrak{p}}(\varphi)\big) \cong \mathrm{GL}_r(A_{\mathfrak{p}}).$$

By definition, the reduction modulo $\mathfrak{p}$ is a continuous Galois representation on the module of $\mathfrak{p}$-torsion

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{Aut}_{\kappa_{\mathfrak{p}}} \big(\varphi[\mathfrak{p}](K^{sep})\big) \cong \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

over the residue field $\kappa_{\mathfrak{p}} := A/\mathfrak{p}$. We call it the residual representation.

Our first stated result is the semisimplicity conjecture.

Theorem 1.1.1 (Semisimplicity conjecture). *Let φ be a Drinfeld A-module over K and $\mathfrak{p}$ a prime of A different from the characteristic of φ. Then the $F_{\mathfrak{p}}[\mathrm{G}_K]$-module $V_{\mathfrak{p}}(\varphi)$ is semisimple.*

Proof. For the case where K is of transcendence degree 1 over $\mathbb{F}_q$, see Taguchi [32, Theorem 0.1] in special characteristic and Taguchi [33, Theorem 0.1] in generic characteristic. For the general case, see Pink [20, Theorem 1.4]. □

The next result has been proven independently by Taguchi and Tamagawa.

Theorem 1.1.2 (Tate conjecture for Drinfeld modules). *Let φ_1 and φ_2 be two Drinfeld A-modules over K of the same characteristic. Then for all primes $\mathfrak{p}$ of A different from the characteristic of K, the natural map*

$$\mathrm{Hom}_K(\varphi_1, \varphi_2) \otimes_A A_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{A_{\mathfrak{p}}[\mathrm{G}_K]} \big(T_{\mathfrak{p}}(\varphi_1), T_{\mathfrak{p}}(\varphi_2)\big)$$

is an isomorphism.

Proof. See Taguchi [36] or Tamagawa [37]. □

Combining these two theorems, on gets another important result which parallels one we are going to prove in the next chapter.

Theorem 1.1.3. *Let φ be a Drinfeld A-module over K, and assume that $\mathrm{End}_K(\varphi) = A$. Then the representation*

$$\mathrm{G}_K \longrightarrow \mathrm{Aut}_{A_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi)\big)$$

is absolutely irreducible for all primes $\mathfrak{p}$ of A different from the characteristic of φ.

Proof. Combine Theorem 1.1.1 and Theorem 1.1.2. □

We are now coming to a result which will be very useful later on. It is about the size of the image of the Galois group under the representation on the Tate module. Serre proved that for an elliptic curve without complex multiplication, the image of the Galois group is as big as possible.

Pink studied the analogous problem for Drinfeld modules.

Theorem 1.1.4. *Let φ be a Drinfeld A-module over K of generic characteristic and assume that $\mathrm{End}_{\bar{K}}(\varphi) = A$. Then for any finite set Λ of primes of A the image of the homomorphism*

$$\mathrm{G}_K \longrightarrow \prod_{\lambda\in\Lambda} \mathrm{GL}_r(F_\lambda)$$

is open.

Proof. See Pink [20]. □

Theorem 1.1.5. *Let φ be a Drinfeld A-module over K of special characteristic and assume that $\mathrm{End}_{\bar{K}}(\varphi) = A$. Then the image of $\rho_\mathfrak{p}$ is Zariski dense in $\mathrm{GL}_{r,F_\mathfrak{p}}$ for all primes $\mathfrak{p}$ of A different from the characteristic of φ.*

Proof. See Pink [21]. □

Theorem 1.1.6 (Isogeny conjecture for Drinfeld modules)**.** *Let φ be a Drinfeld A-module over K. Assume that K is of transcendence degree 1. Then up to K-isomorphism, there are only finitely many Drinfeld A-modules φ' over K for which there exists a K-isogeny $\varphi \to \varphi'$ of degree not divisible by $\mathfrak{p}_0$.*

Proof. See Taguchi [32] in special characteristic and Taguchi [35] in generic characteristic. □

In [23], Pink and Traulsen reformulated Theorem 1.1.6 into a result on Galois invariant submodules.

Proposition 1.1.7. *Let φ be a Drinfeld A-module over K. Assume that K is of transcendence degree 1. Then for almost all primes $\mathfrak{p}$ of A and all natural numbers $n > 0$, every G_K-invariant $A/\mathfrak{p}^n$-submodule of $\varphi[\mathfrak{p}^n](K^{sep})$ has the form $\alpha\big(\varphi[\mathfrak{p}^n](K^{sep})\big)$ for some $\alpha \in \mathrm{End}_K(\varphi)$.*

Proof. See [23, Proposition 2.3]. □

Corollary 1.1.8. *Let φ be a Drinfeld A-module over K. Assume that K is of transcendence degree* 1 *and that* $\mathrm{End}_K(\varphi) = A$. *Then the representation $\overline{\rho_{\mathfrak{p}}}$ is irreducible for almost all primes $\mathfrak{p}$ of A.*

Proof. Set $n = 1$ in Proposition 1.1.7. □

1.2 Action of inertia groups on torsion points

In this section we give three results which will be useful in the next chapter. The first result gives us some information on the characteristic polynomial of certain Frobenius elements. It is for Drinfeld modules of both generic and special characteristic. Throughout the section we assume that K is of transcendence degree 1.

Proposition 1.2.1. *Let φ be a Drinfeld A-module over K of arbitrary characteristic, and let $\mathfrak{P}$ be a place of K where φ has good reduction. Then for every prime $\mathfrak{p}$ of A different from the characteristic of φ and not lying below $\mathfrak{P}$, the representation $\rho_\mathfrak{p}$ is unramified at $\mathfrak{P}$, and the characteristic polynomial of $\rho_\mathfrak{p}(\mathrm{Frob}_\mathfrak{P})$ has coefficients in A and is independent of $\mathfrak{p}$.*

Proof. See Goss [11, Theorem 4.12.12 (2)]. □

For the remainder of this section, we assume that φ is of generic characteristic. The next result has been proven by Pink and Traulsen in [23] for Drinfeld modules in special characteristic. The proof also works in generic characteristic and is omitted here.

Proposition 1.2.2. *Let φ be a Drinfeld A-module over K of generic characteristic. After replacing K by a suitable finite extension, for all primes $\mathfrak{p}$ of A and all places $\mathfrak{P}$ of K not lying above $\mathfrak{p}$, the restriction of $\rho_\mathfrak{p}$ to the inertia group at $\mathfrak{P}$ is unipotent.*

Proof. See Pink and Traulsen [23, Proposition 2.7]. □

Our next result will give us information on the action of the inertia group of a place $\mathfrak{P}$ of K on $\varphi[\mathfrak{p}]$ if $\mathfrak{p}$ lies below $\mathfrak{P}$. To achieve this goal, we need to introduce *fundamental characters*. We then can prove a result very similar to one of Serre's in [29, §1].

Remark. On the next pages we will have to analyze the restriction of characters to inertia groups of K. If $\mathfrak{P}$ is a place of K we have to choose a place $\bar{\mathfrak{P}}$ of $\bar{K}$ above $\mathfrak{P}$ in order to talk about an inertia group. If we took another place above $\mathfrak{P}$ then the different inertia groups are conjugated. Since our characters have abelian image, it does not matter which place above $\mathfrak{P}$ we choose. We therefore fix one and write $\mathrm{I}_\mathfrak{P}$ and $\mathrm{I}_\mathfrak{P}^t$ for the inertia group and the tame inertia group of K at the place $\mathfrak{P}$, respectively.

Fundamental characters. Fix a place $\mathfrak{P}$ of K, a place $\bar{\mathfrak{P}}$ of $\bar{K}$, and denote by $v_\mathfrak{P}$ be the according normalized valuation on the completion $K_\mathfrak{P}$ as well as its extension to $\bar{K}_{\bar{\mathfrak{P}}}$. Denote the respective residue fields by $k_{\mathfrak{P}}$ and $k_{\bar{\mathfrak{P}}}$. The field $k_{\bar{\mathfrak{P}}}$ is an algebraic closure of $k_{\mathfrak{P}}$. Denote by $K_{\mathfrak{P}}^{nr}$ the maximal unramified extension of $K_{\mathfrak{P}}$ inside $\bar{K}_{\bar{\mathfrak{P}}}$, and by $K_{\mathfrak{P}}^{t}$ the maximal tamely ramified extension of $K_{\mathfrak{P}}$ inside $\bar{K}_{\bar{\mathfrak{P}}}$. The tame inertia group $\mathrm{I}_{\mathfrak{P}}^{t}$ is $\mathrm{Gal}(K_{\mathfrak{P}}^{t}/K_{\mathfrak{P}}^{nr})$. Let π be a uniformizer at $\mathfrak{P}$. Let λ be a finite extension of $k_{\mathfrak{P}}$ inside $k_{\bar{\mathfrak{P}}}$, and let π_λ be any nonzero solution in $K_{\mathfrak{P}}^{t}$ of the equation $X^{|\lambda|} - \pi X = 0$.

Definition. The *fundamental character of* λ is the homomorphism

$$\zeta_\lambda : \mathrm{I}_{\mathfrak{P}} \longrightarrow \lambda^*, \sigma \mapsto \sigma(\pi_\lambda)/\pi_\lambda \mod \pi.$$

Remark. For any other uniformizer π' and any nonzero solution π'_λ of the equation $X^{|\lambda|} - \pi' X = 0$, the elements π_λ and π'_λ have the same valuation and therefore differ by a unit $u \in K_{\mathfrak{P}}^{t}$. The value $\zeta_\lambda(\sigma)$ then changes by $\sigma(u)/u$, which is congruent to 1 modulo $\mathfrak{P}$ because σ acts trivially on the residue field. Therefore ζ_λ is independent of the choices of π and π_λ. Moreover, it factors through the tame inertia group $\mathrm{I}_{\mathfrak{P}}^{t}$ because $\pi_\lambda \in K_{\mathfrak{P}}^{t}$.

The fundamental characters form a projective system with respect to the norm maps, i.e., if λ' is a finite extension of λ inside $k_{\bar{\mathfrak{P}}}$, then we have the following equality

$$\zeta_\lambda = \mathrm{N}_{\lambda'/\lambda} \circ \zeta_{\lambda'}$$

where $\mathrm{N}_{\lambda'/\lambda} : \lambda' \to \lambda$ is the Norm map. Fundamental characters will be important in analyzing the action of the inertia group at a place $\mathfrak{P}$ of K on $\varphi[\mathfrak{p}]$ for a prime $\mathfrak{p}$ of A lying below $\mathfrak{P}$.

Fix a place $\mathfrak{p}$ of F, a place $\mathfrak{P}$ of K above $\mathfrak{p}$, a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$ and a place $\bar{\mathfrak{P}}$ of $\bar{K}$ above $\mathfrak{P}$. Let $q_{\mathfrak{p}}$ denote the cardinality of $\kappa_{\mathfrak{p}}$, the residue field of F at the place $\mathfrak{p}$. For any power m of p denote by k_m be the subfield of $k_{\bar{\mathfrak{P}}}$ with m elements.

Assume that φ has good reduction at $\mathfrak{P}$, and let $h_{\mathfrak{P}}$ be the height of the reduced Drinfeld module. The connected-étale decomposition of $\varphi[\mathfrak{p}]$ gives an exact sequence of group schemes over $\operatorname{Spec} \mathcal{O}_{K_{\mathfrak{P}}}$

$$0 \longrightarrow \varphi[\mathfrak{p}]^0 \longrightarrow \varphi[\mathfrak{p}] \longrightarrow \varphi[\mathfrak{p}]^{et} \longrightarrow 0.$$

The set $\varphi[\mathfrak{p}]^0(K^{sep})$ is an $h_{\mathfrak{P}}$-dimensional $\kappa_{\mathfrak{p}}$ vector space. The following result is an analogue of Proposition 9 in Serre's paper [29]. The analogue of Corollary 1.2.4 for τ-sheaves has been proven by Gardeyn in [10]. Abbreviate ${q_{\mathfrak{p}}}^{h_{\mathfrak{P}}}$ by n.

Proposition 1.2.3. *Assume that the extension $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is unramified and that φ has good reduction at $\mathfrak{P}$. Then the following properties hold.*

(i) The inertia group $\mathrm{I}_{\mathfrak{P}}$ acts trivially on $\varphi[\mathfrak{p}]^{et}(K^{sep})$.

(ii) The action of the wild inertia group at $\mathfrak{P}$ on $\varphi[\mathfrak{p}]^0(K^{sep})$ is trivial.

(iii) The $\kappa_{\mathfrak{p}}$ vector space structure of $\varphi[\mathfrak{p}]^0(K^{sep})$ extends uniquely to a one dimensional k_n vector space structure such that the action of $\mathrm{I}^t_{\mathfrak{P}}$ on $\varphi[\mathfrak{p}]^0(K^{sep})$ is given by the fundamental character ζ_{k_n}.

Proof. Assertion (i) follows immediately from the definition of an etale group scheme. To prove (ii), define

$$\begin{aligned} \alpha &:= 1/(n-1), \\ u_\alpha &:= \{x \in K^{sep}_{\mathfrak{P}} \mid v_{\mathfrak{P}}(x) \geq \alpha\}, \\ u'_\alpha &:= \{x \in K^{sep}_{\mathfrak{P}} \mid v_{\mathfrak{P}}(x) > \alpha\}, \text{ and} \\ V_\alpha &:= u_\alpha/u'_\alpha. \end{aligned}$$

Let π_n be a nonzero solution of the equation $X^n - \pi X = 0$. The set V_α is a one dimensional $k_{\bar{\mathfrak{P}}}$ vector space and isomorphic as G_K-module to $\pi_n k_{\bar{\mathfrak{P}}}$. By definition of the fundamental character ζ_{k_n}, the action of $\mathrm{I}_{\mathfrak{P}}$ on V_α factors through $\mathrm{I}^t_{\mathfrak{P}}$, and the wild inertia group at $\mathfrak{P}$ acts trivially.

We claim that for every non-zero element $s \in \varphi[\mathfrak{p}]^0(K^{sep})$ we have $v_{\mathfrak{P}}(s) = \alpha$. This can be done by considering an appropriate Newton polygon. For this, let $a \in A$ be a function with a zero of order one at $\mathfrak{p}$. Then $(a) = \mathfrak{p}\mathcal{I}$ for an ideal $\mathcal{I}$ of A which is prime to $\mathfrak{p}$. Then we have

$$\varphi[a] = \varphi[\mathfrak{p}] \oplus \varphi[\mathcal{I}].$$

Since $\varphi[\mathcal{I}]$ is étale, we get

$$\varphi[a]^0 = \varphi[\mathfrak{p}]^0$$

as group schemes over $\operatorname{Spec} \mathcal{O}_{K_{\mathfrak{P}}}$. The polynomial φ_a is given by

$$\varphi_a = \sum_{i=0}^{r \deg(\mathfrak{p})} \varphi_{a,i} \tau^i.$$

The Drinfeld module φ has good reduction at $\mathfrak{P}$. For the valuations of the coefficients, we thus get, with $i_0 := h_{\mathfrak{P}} \deg(\mathfrak{p})$,

$$\begin{array}{rcl} v_{\mathfrak{P}}(\varphi_{a,0}) & = & v_{\mathfrak{P}}\big(\iota(a)\big) = 1, \\ v_{\mathfrak{P}}(\varphi_{a,i}) & \geq & 1 \text{ for } 0 < i < i_0, \\ v_{\mathfrak{P}}(\varphi_{a,i_0}) & = & 0, \\ v_{\mathfrak{P}}(\varphi_{a,i}) & \geq & 0 \text{ for } i > i_0. \end{array}$$

The fact that $\varphi[\mathfrak{p}]^0(K^{sep})$ is an $h_{\mathfrak{P}}$-dimensional $\kappa_{\mathfrak{p}}$ vector space and the above observations imply that $(1,1)$ and $({q_{\mathfrak{p}}}^{h_{\mathfrak{P}}}, 0)$ are vertices of the Newton polygon of $\varphi_a(x) = 0$. Therefore every non-zero element $s \in \varphi[\mathfrak{p}]^0(K^{sep})$ satisfies

$$v_{\mathfrak{P}}(s) = \alpha,$$

whence $s \in u_\alpha$. Because $\varphi[\mathfrak{p}]^\circ(K^{sep})$ is a group under addition, the natural projection $u_\alpha \twoheadrightarrow V_\alpha$ thus induces an injective G_K-equivariant homomorphism

$$\varphi[\mathfrak{p}]^0(K^{sep}) \hookrightarrow V_\alpha.$$

Let V be its image. Since the above homomorphism is G_K-equivariant, the wild inertia group at $\mathfrak{P}$ acts trivially on V, and the action of the tame inertia group at $\mathfrak{P}$ and is given by ζ_{k_n}. Therefore V is invariant under multiplication by k_n^*. From $|V| = |k_n|$, we deduce that V is a one dimensional k_n vector space and $\varphi[\mathfrak{p}]^\circ(K^{sep}) \cong V_\alpha$ as $\kappa_{\mathfrak{p}}[\mathrm{G}_K]$-modules. This implies (ii).

It remains to show that this vector space structure is an extension of the previously given $\kappa_{\mathfrak{p}}$ vector space structure on $\varphi[\mathfrak{p}]^0(K^{sep})$. For this, let $\bar{b} \in \kappa_{\mathfrak{p}}$, and let b be an element of A whose residue class in $\kappa_{\mathfrak{p}}$ is equal to $\bar{b}$. Then the action of $\bar{b}$ on $x \in \varphi[\mathfrak{p}]^0(K^{sep})$ is given by

$$\varphi_b(x).$$

The element $\bar{b}$ induces the element $\iota(b) \mod \mathfrak{P} \in k_{\mathfrak{P}}$ and thus acts in a second way on x through $\iota(b)x \mod v_{\mathfrak{P}}(.) > \alpha$. We have to show that these two actions coincide on $\varphi[\mathfrak{p}]^0(K^{sep})$. By definition of V_α this is equivalent to showing that

$$v_{\mathfrak{P}}\big(\varphi_b(x) - \iota(b)x\big) > \alpha.$$

Since φ is a Drinfeld A-module over K, we have $\varphi_b(x) = \iota(b)x + \sum_{i=1}^{r\deg(b)} b_i x^{q^i}$. Therefore we get

$$\varphi_b(x) - \iota(b)x = \sum_{i=1}^{r\deg(b)} b_i x^{q^i}.$$

Since φ has good reduction at $\mathfrak{P}$, we know that $v_{\mathfrak{P}}(b_i) \geq 0$ for $i = 1, \ldots, r\deg(b)$. By definition of V_α we have $v_{\mathfrak{P}}(x) = \alpha > 0$. We thus get $v_{\mathfrak{P}}(b_i x^{q^i}) = v_{\mathfrak{P}}(b_i) + q^i v_{\mathfrak{P}}(x) > \alpha$, for $i = 1, \ldots, r\deg(b)$, and therefore $v_{\mathfrak{P}}\big(\varphi_b(x) - \iota(b)x\big) > \alpha$.

The $\kappa_{\mathfrak{p}}$ vector space structure on $\varphi[\mathfrak{p}]^0(K^{sep})$ therefore extends uniquely to a one dimensional k_n vector space structure, which shows that the action of $\mathrm{I}_{\mathfrak{P}}^t$ on V is given by the fundamental character ζ_{k_n}. □

Consider $\varphi[\mathfrak{p}]^0(K^{sep}) \otimes_{k_{\mathfrak{P}}} k_{\bar{\mathfrak{P}}}$. This is an $h_{\mathfrak{P}}$-dimensional $k_{\bar{\mathfrak{P}}}$ vector space. The action of the tame inertia group $\mathrm{I}_{\mathfrak{P}}^t$ on it can be brought into diagonal form and is therefore given by a set of $h_{\mathfrak{P}}$ characters

$$\psi_i : \mathrm{I}_{\mathfrak{P}}^t \longrightarrow k_{\bar{\mathfrak{P}}}^*,\ i = 1, \ldots, h_{\mathfrak{P}}.$$

Define $\Sigma_{h_{\mathfrak{P}}} := \mathrm{Hom}_{k_{\mathfrak{P}}}(k_n, k_{\bar{\mathfrak{P}}})$.

Corollary 1.2.4. *Assume that $K_{\mathfrak{P}}/F_{\mathfrak{p}}$ is unramified. Then the set of characters $\{\psi_i\}_i$ is given by $\{\overline{\sigma} \circ \zeta_{k_n}\}_{\overline{\sigma} \in \Sigma_{h_{\mathfrak{P}}}}$.*

Proof. By Proposition 1.2.3, the representation of $\mathrm{I}_{\mathfrak{P}}^t$ over $k_{\mathfrak{P}}$ is given by ζ_{k_n}. By the representation theory of finite groups, the representation $\zeta_{k_n} \otimes_{k_{\mathfrak{P}}} k_{\bar{\mathfrak{P}}}$ is given by

$$\bigoplus_{\overline{\sigma} \in \Sigma_{h_{\mathfrak{P}}}} \overline{\sigma} \circ \zeta_{k_n},$$

as desired. □

1.3 An interpolation result from class field theory

In this section, we introduce two algebraic groups in the same way as Serre did in [27, Chapter II] and [29, §3]. Then we analyze their character groups and indicate how to construct a system of $\mathfrak{p}$-adic representations out of a character of $\mathbb{S}$.

Remark. Serre's construction is somewhat more general. He uses the notion of modulus of support or, equivalently, conductors. Our construction will only give us strictly compatible systems with trivial conductor. This is sufficient for the application in the next chapter.

Define

$$U := \prod_{\mathfrak{P}\nmid\infty} \mathcal{O}_{\mathfrak{P}}^* \times \prod_{\infty'|\infty} K_{\infty'}^* \subset \mathbb{A}_K^*,$$

and

$$C := \mathbb{A}_K^*/K^*U.$$

Then C is a finite abelian group and sits in the exact sequence

$$1 \longrightarrow K^*/(K^* \cap U) \longrightarrow \mathbb{A}_K^*/U \longrightarrow C \longrightarrow 1.$$

Consider the Weil restriction $\mathrm{Res}_F^K(\mathbb{G}_{m,K})$ of the multiplicative group over K to F. By definition, its points over a F-algebra B are given by

$$\mathrm{Res}_F^K(\mathbb{G}_{m,K})(B) := (B \otimes_F K)^*.$$

Let $\overline{K^* \cap U}$ be the Zariski closure of $K^* \cap U$ in $\mathrm{Res}_F^K(\mathbb{G}_{m,K})$ and consider the quotient

$$\mathbb{T} := \mathrm{Res}_F^K(\mathbb{G}_{m,K})/\overline{K^* \cap U}.$$

Let $\mathbb{S}$ be the push-out of $\mathbb{T}$ and $\mathbb{A}_K^*/U$ over $K^*/(K^* \cap U)$. This is an algebraic group with the universal property that, for any algebraic group $\mathbb{S}'$ over F together with homomorphisms $\mathbb{T} \to \mathbb{S}'$ and $\mathbb{A}_K^*/U \to \mathbb{S}'(F)$ such that the following diagram

$$\begin{array}{ccc} K^*/(K^* \cap U) & \longrightarrow & \mathbb{A}_K^*/U \\ \downarrow & & \downarrow \\ \mathbb{T}(F) & \longrightarrow & \mathbb{S}'(F) \end{array}$$

commutes, there exists a unique homomorphism $\mathbb{S} \to \mathbb{S}'$ through which the maps $\mathbb{T} \to \mathbb{S}'$ and $\mathbb{A}_K^*/U \to \mathbb{S}'(F)$ factor. A more explicit construction of the algebraic group $\mathbb{S}$ can be found in Serre [27, Chapter II]. The definitions of $\mathbb{T}$ and $\mathbb{S}$ give us a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K^*/(K^* \cap U) \arrow[r] \arrow[d] & \mathbb{A}_K^*/U \arrow[r] \arrow[d, "\gamma'"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \mathbb{T}(F) \arrow[r] & \mathbb{S}(F) \arrow[r] & C \arrow[r] & 1.
\end{tikzcd}$$

Denote by $\gamma : \mathbb{A}_K^* \to \mathbb{S}(F)$ the compositie of γ' with $\mathbb{A}_K^* \to \mathbb{A}_K^*/U$. Let $\mathfrak{p}$ be any prime of A, and fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$. Define

$$U^{\mathfrak{p}} := \prod_{\mathfrak{Q} \nmid \{\mathfrak{p},\ \infty\}} \mathcal{O}_{\mathfrak{Q}}^* \times \prod_{\infty' | \infty} K_{\infty'}^* \subset \mathbb{A}_K^*,$$

$$K_{\mathfrak{p}} := \prod_{\mathfrak{P} | \mathfrak{p}} K_{\mathfrak{P}},$$

and

$$\mathcal{O}_{\mathfrak{p}} := \prod_{\mathfrak{P} | \mathfrak{p}} \mathcal{O}_{\mathfrak{P}}.$$

The composite of γ with $\mathbb{S}(F) \to \mathbb{S}(F_{\mathfrak{p}})$ is the continuous homomorphism

$$\gamma_{\mathfrak{p}} : \mathbb{A}_K^* \longrightarrow \mathbb{S}(F_{\mathfrak{p}}).$$

We know that

$$\mathrm{Res}_F^K(\mathbb{G}_{m,K})(F_{\mathfrak{p}}) = (F_{\mathfrak{p}} \otimes_F K)^* = K_{\mathfrak{p}}^*.$$

We thus can consider $\mathrm{Res}_F^K(\mathbb{G}_{m,K})(F_{\mathfrak{p}})$ as a direct factor of $\mathbb{A}_K^*$. Taking the projection onto this factor we get a continuous homomorphism

$$\delta_{\mathfrak{p}} : \mathbb{A}_K^* \to K_{\mathfrak{p}}^* = \mathrm{Res}_F^K(\mathbb{G}_{m,K})(F_{\mathfrak{p}}) \to \mathbb{T}(F_{\mathfrak{p}}) \to \mathbb{S}(F_{\mathfrak{p}}).$$

It follows from the commutativity of the above diagram that $\gamma_{\mathfrak{p}}|_{K^*} = \delta_{\mathfrak{p}}|_{K^*}$. Therefore the continuous homomorphism

$$\gamma_{\mathfrak{p}}\delta_{\mathfrak{p}}^{-1} : \mathbb{A}_K^* \longrightarrow \mathbb{S}(F_{\mathfrak{p}})$$

is trivial on K^*. Since both $\gamma_{\mathfrak{p}}$ and $\delta_{\mathfrak{p}}$ are trivial on $U^{\mathfrak{p}}$, the continuous homomorphism $\gamma_{\mathfrak{p}}\delta_{\mathfrak{p}}^{-1}$ factors through a continuous homomorphism

$$\varepsilon_{\mathfrak{p}} : \mathbb{A}_K^*/\overline{K^*U^{\mathfrak{p}}} \longrightarrow \mathbb{S}(F_{\mathfrak{p}}),$$

where $\overline{K^*U^{\mathfrak{p}}}$ is the closure of $K^*U^{\mathfrak{p}}$ in $\mathbb{A}_K^*$.

Characters of $\mathbb{T}$ and $\mathbb{S}$. Define $\Sigma := \mathrm{Hom}_F(K, \bar{F})$. Every $\sigma \in \Sigma$ extends to a homomorphism $\bar{F} \otimes_F K \to \bar{F}$ and thus gives a character $[\sigma] : \mathrm{Res}_F^K(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,\bar{F}}$ of $\mathrm{Res}_F^K(\mathbb{G}_{m,K})$. These $[\sigma]$ form a $\mathbb{Z}$-basis of the character group $\mathrm{X}(\mathrm{Res}_F^K(\mathbb{G}_{m,K}))$. Since $\mathbb{T} = \mathrm{Res}_F^K(\mathbb{G}_{m,K})/\overline{K^* \cap U}$, the character group of $\mathbb{T}$ is given by

$$\mathrm{X}(\mathbb{T}) = \left\{ \prod_{\sigma \in \Sigma} \sigma^{n_\sigma} \,\middle|\, \prod_{\sigma} \sigma(x)^{n_\sigma} = 1 \text{ for all } x \in K^* \cap U \right\}.$$

For the character groups of C, $\mathbb{T}$, and $\mathbb{S}$ we have the exact sequence

$$1 \longrightarrow \mathrm{X}(C) \longrightarrow \mathrm{X}(\mathbb{S}) \longrightarrow \mathrm{X}(\mathbb{T}) \longrightarrow 1$$

where $\mathrm{X}(C)$ is the finite group $\mathrm{Hom}(C, \bar{F}^*)$. Any character μ of $\mathbb{T}$ can be extended to a character θ of $\mathbb{S}$ in $|C|$ ways.

Let a character θ of the algebraic group $\mathbb{S}$ be given. It induces a continuous homomorphism $\mathbb{S}(F_{\mathfrak{p}}) \to \bar{F}_{\bar{\mathfrak{p}}}^*$. Its composite with $\varepsilon_{\mathfrak{p}}$ is a continuous homomorphism

$$\theta_{\bar{\mathfrak{p}}} : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*.$$

Since $\mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}}$ is compact, the image of $\theta_{\bar{\mathfrak{p}}}$ lies in $\mathcal{O}_{\bar{F}_{\bar{\mathfrak{p}}}}^*$. Therefore we can reduce it mod $\bar{\mathfrak{p}}$ and get

$$\bar{\theta}_{\bar{\mathfrak{p}}} : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

The Artin reciprocity map of global class field theory induces a continuous isomorphism

$$\omega : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$$

where $K^{ab,\mathfrak{p}}$ is the maximal abelian extension of K which splits completely at primes ∞' above ∞ and is unramified at places not lying above $\mathfrak{p}$. If we compose the homomorphisms $\theta_{\bar{\mathfrak{p}}}$ and $\bar{\theta}_{\bar{\mathfrak{p}}}$ with the inverse of ω we obtain continuous representations

$$\theta_{\bar{\mathfrak{p}}} : \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*,$$

and

$$\bar{\theta}_{\bar{\mathfrak{p}}} : \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

By Serre [27, Chapter II], the $\theta_{\bar{\mathfrak{p}}}$ form a system of strictly compatible $\mathfrak{p}$-adic representations.

Interpolation of characters. We will now see how to construct a character of $\mathbb{S}$ out of a certain system of $\mathfrak{p}$-adic characters.

For this, let S be an infinite set of primes of A. For any $\mathfrak{p} \in S$, fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$ and consider a continuous homomorphism

$$\eta_{\bar{\mathfrak{p}}} : \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Every $\sigma \in \Sigma$ extends uniquely to a homomorphism

$$\sigma_{\mathfrak{p}} : K_{\mathfrak{p}}^* \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*,$$

and this homomorphism is equal to 1 on all factors $K_{\mathfrak{P}}^*$ of $K_{\mathfrak{p}}^*$ except the one for which $\mathfrak{P} = \sigma^{-1}(\bar{\mathfrak{p}})$.

Proposition 1.3.1. *Assume that there exist integers $n(\sigma, \mathfrak{p})_{\sigma\in\Sigma,\mathfrak{p}\in S}$ whose absolute values are bounded and such that for all $\mathfrak{p} \in S$ and all $x \in \mathcal{O}_{\mathfrak{p}}^*$ we have*

$$\eta_{\bar{\mathfrak{p}}}(x) = \left(\prod_{\sigma\in\Sigma} \sigma_{\mathfrak{p}}^{n(\sigma,\mathfrak{p})}(x^{-1}) \mod \bar{\mathfrak{p}}\right).$$

Then there exist $\theta \in X(\mathbb{S})$ and an infinite subset S' of S such that for all $\mathfrak{p} \in S'$ we have

$$\bar{\theta}_{\bar{\mathfrak{p}}} = \eta_{\bar{\mathfrak{p}}}.$$

Proof. Since the values of the $n(\sigma, \mathfrak{p})$ are bounded and Σ is finite, there exists an infinite subset S'' of S such that for all $\mathfrak{p} \in S''$ the value $n(\sigma, \mathfrak{p})$ is independent of $\mathfrak{p}$. Denote this value by n_σ. Define $\alpha := \prod_{\sigma\in\Sigma} \sigma_{\mathfrak{p}}^{n_\sigma}$. This is a character of $\mathrm{Res}_F^K(\mathbb{G}_{m,K})$.

Take any $x \in K^* \cap U$. We know that $\eta_{\bar{\mathfrak{p}}}(x) = 1$ and, by assumption, that $\eta_{\bar{\mathfrak{p}}}(x) \equiv \alpha(x^{-1}) \mod \bar{\mathfrak{p}}$. Therefore we have $\alpha(x) \equiv 1 \mod \bar{\mathfrak{p}}$ for all $\mathfrak{p} \in S''$. Since S'' is infinite, we get the equality $\alpha(x) = 1$. This implies that $\alpha \in \mathrm{X}(\mathbb{T})$.

Abbreviate $n := |C|$. Extend α to a character $\theta' \in \mathrm{X}(\mathbb{S})$. Then for any $\mathfrak{p} \in S''$, the character

$$\beta_{\bar{\mathfrak{p}}} := \eta_{\bar{\mathfrak{p}}} \bar{\theta}'^{-1}_{\bar{\mathfrak{p}}} : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*$$

factors through C. Therefore it takes values in the group of n-th roots of unity $\mu_{\bar{\mathfrak{p}},n}$ of $\kappa_{\bar{\mathfrak{p}}}$. Let μ_n be the group of n-th roots of unity of $\bar{F}$. Then, for any prime $\mathfrak{p} \in S''$, the reduction map $\mu_n \longrightarrow \mu_{\bar{\mathfrak{p}},n}$ is an isomorphism. Therefore we can consider $\beta_{\bar{\mathfrak{p}}}$ as homomorphism into μ_n and thus as an element of $\mathrm{X}(C)$. The character group $\mathrm{X}(C)$ of C is finite. So there exist $\beta \in \mathrm{X}(C)$ and an infinite subset S' of S'' such that for all $\mathfrak{p} \in S'$ we have $\beta_{\bar{\mathfrak{p}}} = \beta$. Define θ as the product of θ' and the image of β in $X(\mathbb{S})$. For all $\mathfrak{p} \in S'$ we then have $\bar{\theta}_{\bar{\mathfrak{p}}} = \eta_{\bar{\mathfrak{p}}}$. □

Chapter 2

Absolute Irreducibility of the Residual Representation

In this chapter we prove the absolute irreducibility of the residual representation. We are doing this in three steps, each of them being a section.

In Section 2.1 we assume that φ is of generic characteristic, that the extension K/F is finite and that the endomorphism ring of φ over $\bar{K}$ is A. We give an indirect proof. If the residual representation $\overline{\rho_{\mathfrak{p}}}$ is irreducible, but not absolutely irreducible, it can be considered as a representation of some smaller dimension over an extension of the residue field $\kappa_{\mathfrak{p}}$ at $\mathfrak{p}$. Its determinant over that extension is an abelian character $\overline{\chi}_{\mathfrak{p}}$. We can extend it to a character $\overline{\chi}_{\bar{\mathfrak{p}}}$ with values in $\kappa_{\bar{\mathfrak{p}}}$. We then consider the restriction of this character to the inertia group of any place of K lying above $\mathfrak{p}$. Using Proposition 1.2.3 we can show that this restriction is equal to a certain fundamental character. Next, we translate our setting into a class field theoretical one using the result from Section 1.3. Having done this, we consider characteristic polynomials of Frobenius elements and show that a certain resultant vanishes $\mod \mathfrak{p}$ for any prime $\mathfrak{p}$ of A where the residual representation is not absolutely irreducible. If this happens for infinitely many primes $\mathfrak{p}$ of A, the congruence relations give an equality which yields an algebraic relation for $\rho_{\mathfrak{p}}(\mathrm{G}_K)$. By Theorem 1.1.4 we know that this image is Zariski dense in $\mathrm{GL}_{r,F_{\mathfrak{p}}}$ which allows us to construct the desired contradiction.

Section 2.2 deals with the case of a larger endomorphism ring. We can no longer expect the residual representation to be absolutely irreducible. Instead, we describe the image of the group ring $A_{\mathfrak{p}}[\mathrm{G}_K]$ in the endomorphism ring of the Tate module for almost all primes $\mathfrak{p}$ of A. The section will be quite short since the results are the same as in the paper by Pink and Traulsen [23, Section 4]. All arguments also work in generic characteristic.

Finally, in Section 2.3, we prove the general case, i.e., we do not make any assumptions on the characteristic of φ, and the field K is finitely generated over $\mathbb{F}_q$. We prove this by reduction to the case of transcendence degree 1 which is already proven in Section 2.2. The argument is very similar to the one in Pink [20, Theorem 1.4].

2.1 The case $\mathrm{End}_{\bar{K}}(\varphi) = A$ and $[K : F] < \infty$

The aim of this section is to prove the following result.

Theorem 2.1.1. *Let φ be a Drinfeld A-module over K of generic characteristic. Assume that $\mathrm{End}_{\bar{K}}(\varphi) = A$ and that K is a finite extension of F. Then the residual representation*

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

is absolutely irreducible for almost all primes $\mathfrak{p}$ of A.

Remarks. 1. The analogous result for Drinfeld A-modules of special characteristic was proven by Pink and Traulsen in [23, Theorem 3.1].

2. We need the assumption $\mathrm{End}_{\bar{K}}(\varphi) = A$ because we use Theorem 1.1.4 at the end of this section.

By Corollary 1.1.8 we know that $\overline{\rho_{\mathfrak{p}}}$ is irreducible for almost all primes $\mathfrak{p}$ of A. By the Lemma of Schur, for these primes the ring $\mathrm{End}_{\kappa_{\mathfrak{p}}}(\overline{\rho_{\mathfrak{p}}})$ is a finite dimensional division algebra over $\kappa_{\mathfrak{p}}$. Since $\kappa_{\mathfrak{p}}$ is finite, every finite dimensional division algebra over $\kappa_{\mathfrak{p}}$ is a commutative field. The ring $\mathrm{End}_{\kappa_{\mathfrak{p}}}(\overline{\rho_{\mathfrak{p}}})$ is thus a finite field extension of the residue field $\kappa_{\mathfrak{p}}$ of some degree $s_{\mathfrak{p}}$. Denote this field extension by $\lambda_{\mathfrak{p}}$. Since $r = \dim_{\kappa_{\mathfrak{p}}}(\overline{\rho_{\mathfrak{p}}}) = \dim_{\lambda_{\mathfrak{p}}}(\overline{\rho_{\mathfrak{p}}})[\lambda_{\mathfrak{p}} : \kappa_{\mathfrak{p}}] = \dim_{\lambda_{\mathfrak{p}}}(\overline{\rho_{\mathfrak{p}}})s_{\mathfrak{p}}$, the integer $s_{\mathfrak{p}}$ must divide the rank r of φ. Setting $t_{\mathfrak{p}} := r/s_{\mathfrak{p}}$ we see that the residual representation $\overline{\rho_{\mathfrak{p}}}$ factors through $\mathrm{GL}_{t_{\mathfrak{p}}}(\lambda_{\mathfrak{p}}) \subset \mathrm{GL}_r(\kappa_{\mathfrak{p}})$.

To prove Theorem 2.1.1 we have to show that $s_{\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. If not, some value of $s_{\mathfrak{p}} > 1$ must occur infinitely often. To give an indirect proof, we make the following assumption.

Assumption 2.1.2. *There exist integers $s > 1$ and t with $st = r$ and an infinite set S of primes of A such that for all $\mathfrak{p} \in S$ the residual representation $\overline{\rho_{\mathfrak{p}}}$ factors through $\mathrm{GL}_t(\lambda_{\mathfrak{p}})$ where $\lambda_{\mathfrak{p}}$ is a field extension of $\kappa_{\mathfrak{p}}$ of degree s.*

For $\mathfrak{p} \in S$ we can consider $\overline{\rho_{\mathfrak{p}}}$ as a homomorphism $\mathrm{G}_K \longrightarrow \mathrm{GL}_t(\lambda_{\mathfrak{p}})$. If we compose $\overline{\rho_{\mathfrak{p}}}$ with the determinant

$$\det_{\lambda_{\mathfrak{p}}} : \mathrm{GL}_t(\lambda_{\mathfrak{p}}) \longrightarrow \lambda_{\mathfrak{p}}^*,$$

we get a character

$$\overline{\chi}_{\mathfrak{p}} := \det_{\lambda_{\mathfrak{p}}} \circ \overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \lambda_{\mathfrak{p}}^*.$$

Remarks. 1. Since S is infinite, we can remove finitely many primes, and S is still infinite. We remove those primes of S where φ has bad reduction and those which are ramified in K. Then

- *φ has good reduction at all places of K lying above places in S*, and
- *all $\mathfrak{p} \in S$ are unramified in K.*

2. It is enough to prove Theorem 2.1.1 for an open subgroup of G_K . This allows us to replace K by a finite extension. We replace K by a finite extension such that the restriction of $\overline{\rho_{\mathfrak{p}}}$ to any inertia group of a place not lying above $\mathfrak{p}$ is unipotent, which is possible by Proposition 1.2.2. Next, enlarge K such that the lattices at the places above ∞ become K-rational. Again, only a finite extension is needed. Then the following two properties hold:

- *for all $\mathfrak{p} \in S$ and for all places $\mathfrak{P}$ of K not lying above $\mathfrak{p}$ we have* $\overline{\chi}_{\mathfrak{p}}|_{\mathrm{I}_{\mathfrak{P}}} = 1$, and
- *for all places ∞' of K lying above ∞ we have* $\overline{\chi}_{\mathfrak{p}}|_{\mathrm{D}_{\infty'}} = 1$.

3. By replacing K by a finite extension as above, we only have to deal with characters whose prime to $\mathfrak{p}$ conductor is 1 and which totally decompose above ∞.

Let $\mathfrak{p}$ be any prime in S. Fix a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$. The residue field $\kappa_{\bar{\mathfrak{p}}}$ at $\bar{\mathfrak{p}}$ is an algebraic closure of $\kappa_{\mathfrak{p}}$. Denote by $\kappa_{\mathfrak{p}}^{[s]}$ the extension of $\kappa_{\mathfrak{p}}$ of degree s inside $\kappa_{\bar{\mathfrak{p}}}$.

Choose an embedding $\beta_{\mathfrak{p}} : \lambda_{\mathfrak{p}} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$. Composing $\overline{\chi}_{\mathfrak{p}}$ with $\beta_{\mathfrak{p}}$ gives a character

$$\overline{\chi}_{\bar{\mathfrak{p}}} := \beta_{\mathfrak{p}} \circ \overline{\chi}_{\mathfrak{p}} : \mathrm{G}_K \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Define $\Sigma := \mathrm{Hom}_F(K, \bar{F})$. Let $\mathfrak{P}$ be any place of K above $\mathfrak{p}$. Then the set $\Sigma_{\mathfrak{P}} := \{\sigma \in \Sigma \mid \mathfrak{P} = \sigma^{-1}(\bar{\mathfrak{p}})\}$ is non-empty. Any $\sigma \in \Sigma_{\mathfrak{P}}$ induces an embedding $k_{\mathfrak{P}} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$. Let $k_{q_{\mathfrak{p}}{}^s}$ be the field with $q_{\mathfrak{p}}^s$ elements inside $k_{\mathfrak{P}}$.

Lemma 2.1.3. *Let $\mathfrak{p}$ be any prime in S and $\mathfrak{P}$ a place of K above $\mathfrak{p}$. Then the following properties hold.*

(i) We have $s \mid [k_{\mathfrak{P}} : \kappa_{\mathfrak{p}}]$, and so any $\sigma \in \Sigma_{\mathfrak{P}}$ induces an embedding $\bar{\sigma} : k_{q_{\mathfrak{p}}^s} \hookrightarrow \kappa_{\bar{\mathfrak{p}}}$.

(ii) There exists an element $\sigma \in \Sigma_{\mathfrak{P}}$ such that

$$\overline{\chi}_{\bar{\mathfrak{p}}}|_{\mathrm{I}_{\mathfrak{P}}} = \bar{\sigma} \circ \zeta_{k_{q_{\mathfrak{p}}^s}}.$$

Proof. Let $\mathfrak{P}$ be any place of K above $\mathfrak{p}$. The Drinfeld module φ has good reduction at $\mathfrak{P}$. We thus have an exact sequence of $\kappa_{\mathfrak{p}}$ vector spaces

$$0 \longrightarrow \varphi[\mathfrak{p}]^0(K^{sep}) \longrightarrow \varphi[\mathfrak{p}](K^{sep}) \longrightarrow \varphi[\mathfrak{p}]^{et}(K^{sep}) \longrightarrow 0.$$

Let $h_{\mathfrak{P}}$ denote the height of the reduced Drinfeld module and abbreviate $q_{\mathfrak{p}} := |\kappa_{\mathfrak{p}}|$ and $n := q_{\mathfrak{p}}^{h_{\mathfrak{P}}}$. By Proposition 1.2.3, the group $\mathrm{I}_{\mathfrak{P}}$ acts tri-vially on $\varphi[\mathfrak{p}]^{et}(K^{sep})$ and, because of the above remarks, it has no coinvariants on $\varphi[\mathfrak{p}]^0(K^{sep})$. Thus the group of $\mathrm{I}_{\mathfrak{P}}$-coinvariants of $\varphi[\mathfrak{p}](K^{sep})$ is $\varphi[\mathfrak{p}]^{et}(K^{sep})$. Since the representation factors through $\mathrm{GL}_t(\lambda_{\mathfrak{p}})$, it follows that the above exact sequence is a sequence of $\lambda_{\mathfrak{p}}$ vector spaces. Thus s must divide $h_{\mathfrak{P}}$.

Moreover, the determinant over $\lambda_{\mathfrak{p}}$ of the representation $\overline{\rho_{\mathfrak{p}}}|_{\mathrm{I}_{\mathfrak{P}}}$ is equal to the determinant of the subrepresentation on $\varphi[\mathfrak{p}]^0(K^{sep})$. By Proposition 1.2.3 (ii) we know that the $\kappa_{\mathfrak{p}}$ vector space structure of $\varphi[\mathfrak{p}]^0(K^{sep})$ extends to a one dimensional k_n vector space structure such that the action of $\mathrm{I}_{\mathfrak{P}}$ on it is given by the fundamental character $\zeta_{k_n} : \mathrm{I}_{\mathfrak{P}} \to k_n^*$. The action of $\lambda_{\mathfrak{p}}$ induces an embedding $\lambda_{\mathfrak{p}} \hookrightarrow k_n$ and therefore an identification $\lambda_{\mathfrak{p}} \cong k_{q_{\mathfrak{p}}^s}$ over $\kappa_{\mathfrak{p}}$. Via this identification, the determinant over $\lambda_{\mathfrak{p}}$ of an element $x \in k_n^*$ is the norm $\mathrm{N}_{k_n/\lambda_{\mathfrak{p}}}(x) \in \lambda_{\mathfrak{p}}^*$. It thus follows that $\det_{\lambda_{\mathfrak{p}}} \circ \overline{\rho_{\mathfrak{p}}}|_{\mathrm{I}_{\mathfrak{P}}}$ is the fundamental character $\zeta_{k_{q_{\mathfrak{p}}^s}} : \mathrm{I}_{\mathfrak{P}} \to \lambda_{\mathfrak{p}}^*$.

Therefore $\zeta_{k_{q_{\mathfrak{p}}^s}}$ extends to an abelian character of G_K. The fundamental character is equivariant under conjugation by $\mathrm{G}_{K_{\mathfrak{P}}}$. Since it is also surjective, we get that $\mathrm{G}_{K_{\mathfrak{P}}}$ acts trivially on $\lambda_{\mathfrak{p}}^*$. Therefore $\lambda_{\mathfrak{p}}$ is contained in the residue field $k_{\mathfrak{P}}$, and so s divides $[k_{\mathfrak{P}}/\kappa_{\mathfrak{p}}]$, proving (i).,

To prove (ii), note that the embedding $\beta_{\mathfrak{p}}$ induces an isomorphism

$$\beta_{\mathfrak{p},s} : \lambda_{\mathfrak{p}} \cong \kappa_{\mathfrak{p}}^{[s]}.$$

Every $\sigma \in \Sigma_{\mathfrak{P}}$ induces a $\bar{\sigma}$ and therefore an isomorphism

$$\bar{\sigma}_s : k_{q_{\mathfrak{p}}{}^s} \cong \kappa_{\mathfrak{p}}^{[s]}.$$

Take the $\sigma \in \Sigma_{\mathfrak{P}}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{I}_{\mathfrak{P}}^t \arrow[r, "\overline{\chi}_{\mathfrak{p}}"] \arrow[d, equal] & \lambda_{\mathfrak{p}}^* \arrow[r, "\beta_{\mathfrak{p},s}"] & \kappa_{\mathfrak{p}}^{[s]*} \arrow[d, equal] \\
\mathrm{I}_{\mathfrak{P}}^t \arrow[r, "\zeta_{k_{q_{\mathfrak{p}}{}^s}}"] & k_{q_{\mathfrak{p}}{}^s} \arrow[r, "\bar{\sigma}_s"] & \kappa_{\mathfrak{p}}^{[s]*}.
\end{tikzcd}$$

For this σ we get $\overline{\chi}_{\bar{\mathfrak{p}}}|_{\mathrm{I}_{\mathfrak{P}}} = \bar{\sigma} \circ \zeta_{k_{q_{\mathfrak{p}}{}^s}}$. □

Translation into a class field theoretical setting. To get more information on the ramification of the character $\overline{\chi}_{\bar{\mathfrak{p}}}$ we use some elements of class field theory. We will get a new character and some information on the ramification of it. We use the same notation as in Section 1.3.

Since the characters $\overline{\chi}_{\mathfrak{p}}$ and $\overline{\chi}_{\bar{\mathfrak{p}}}$ are abelian, unramified at places not lying above $\mathfrak{p}$, and trivial if restricted to the decomposition group at any place ∞' of K lying above ∞, they factor through $\mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$. Therefore we can compose them with the Artin reciprocity map

$$\omega : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$$

and get new characters

$$\overline{\psi}_{\mathfrak{p}} := \overline{\chi}_{\mathfrak{p}} \circ \omega : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \lambda_{\mathfrak{p}}^*,$$

and

$$\overline{\psi}_{\bar{\mathfrak{p}}} := \overline{\chi}_{\bar{\mathfrak{p}}} \circ \omega : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Lemma 2.1.4. *For any $\mathfrak{p} \in S$ there exist $n(\sigma, \mathfrak{p}) \in \{0, 1\}$ such that for any $x \in \mathcal{O}_{\mathfrak{p}}^*$*

$$\overline{\psi}_{\bar{\mathfrak{p}}}(x) = \left(\prod_{\sigma \in \Sigma} \sigma_{\mathfrak{p}}^{n(\sigma,\mathfrak{p})}(x^{-1}) \mod \bar{\mathfrak{p}} \right).$$

Proof. Let $\mathfrak{p}$ be any prime in S and $\mathfrak{P}$ a place of K lying above $\mathfrak{p}$. Then for any $\sigma \in \Sigma_{\mathfrak{P}}$ as in Lemma 2.1.3 (ii), we get

$$\overline{\chi}_{\bar{\mathfrak{p}}}|_{\mathrm{I}_{\mathfrak{P}}} = \bar{\sigma} \circ \zeta_{k_{q_{\mathfrak{p}}^s}} = \bar{\sigma} \circ \mathrm{N}_{k_{\mathfrak{P}}/k_{q_{\mathfrak{p}}^s}} \circ \zeta_{k_{\mathfrak{P}}}.$$

Since the norm is the product of all Galois conjugates, and $\mathfrak{P}$ is unramified over $\mathfrak{p}$, the latter is equal to

$$\prod_{\sigma' \in \Sigma'_{\mathfrak{P}}} \bar{\sigma}' \circ \zeta_{k_{\mathfrak{P}}}$$

where $\Sigma'_{\mathfrak{P}} := \{\sigma' \in \Sigma_{\mathfrak{P}} : \sigma'|_{k_{q_{\mathfrak{p}}^s}} = \sigma|_{k_{q_{\mathfrak{p}}^s}}\}$.

If we compose the fundamental character $\zeta_{k_{\mathfrak{P}}}$ with the inverse of the local norm residue symbol, $\omega_{\mathfrak{P}} : K_{\mathfrak{P}}^* \to \mathrm{G}_{K_{\mathfrak{P}}}$, we get

$$\zeta_{k_{\mathfrak{P}}} \circ \omega_{\mathfrak{P}} : \mathcal{O}_{\mathfrak{P}}^* \longrightarrow k_{\mathfrak{P}}^*, \; x \mapsto x^{-1} \mod \mathfrak{P}.$$

Therefore the above equality is equivalent to

$$\overline{\psi}_{\bar{\mathfrak{p}}}(x) \equiv \prod_{\sigma' \in \Sigma'_{\mathfrak{P}}} \sigma'(x^{-1}) \mod \bar{\mathfrak{p}}$$

for all $x \in \mathcal{O}_{\mathfrak{P}}^*$. Set $n(\sigma', \mathfrak{p}) := 1$ whenever $\sigma' \in \Sigma'_{\mathfrak{P}}$ for some $\mathfrak{P}$ above $\mathfrak{p}$, and 0 otherwise. Because of $\mathcal{O}_{\mathfrak{p}}^* = \prod_{\mathfrak{P}|\mathfrak{p}} \mathcal{O}_{\mathfrak{P}}^*$, we then have for all $x = (x_{\mathfrak{P}}) \in \mathcal{O}_{\mathfrak{p}}^*$

$$\begin{aligned} \overline{\psi}_{\bar{\mathfrak{p}}}(x) &\equiv \prod_{\mathfrak{P}|\mathfrak{p}} \prod_{\sigma' \in \Sigma'_{\mathfrak{P}}} \sigma'(x_{\mathfrak{P}}^{-1}) \mod \bar{\mathfrak{p}} \\ &= \prod_{\sigma \in \Sigma} \sigma_{\mathfrak{p}}(x^{-1})^{n(\sigma,\mathfrak{p})}. \end{aligned}$$

□

Comparison of characters. We can now use Lemma 2.1.4 to construct a character of $\mathbb{S}$ such that the induced $\mathfrak{p}$-adic representation reduced $\bmod \bar{\mathfrak{p}}$ will coincide with $\overline{\psi}_{\bar{\mathfrak{p}}}$.

Lemma 2.1.5. *There exist $\theta \in \mathrm{X}(\mathbb{S})$ and an infinite subset S' of S such that for all $\mathfrak{p} \in S'$ we have*

$$\bar{\theta}_{\bar{\mathfrak{p}}} = \overline{\psi}_{\bar{\mathfrak{p}}}.$$

Proof. By Lemma 2.1.4 we know that for all $\mathfrak{p} \in S$ there exist $n(\sigma, \mathfrak{p}) \in \{0, 1\}$ such that

$$\overline{\psi}_{\bar{\mathfrak{p}}}(x) = \left(\prod_{\sigma \in \Sigma} \sigma_{\mathfrak{p}}^{n(\sigma,\mathfrak{p})}(x^{-1}) \mod \bar{\mathfrak{p}} \right) \text{ for all } x \in \mathcal{O}_{\mathfrak{p}}^*.$$

The exponents are bounded by 1. This implies that the characters $\overline{\psi}_{\bar{\mathfrak{p}}}$ for $\mathfrak{p} \in S$ satisfy the assumptions of Proposition 1.3.1. Therefore there exist $\theta \in X(\mathbb{S})$ and an infinite subset S' of S such that for all $\mathfrak{p} \in S'$ we have $\bar{\theta}_{\bar{\mathfrak{p}}} = \overline{\psi}_{\bar{\mathfrak{p}}}$. □

If we compose the homomorphisms $\theta_{\bar{\mathfrak{p}}}$ and $\bar{\theta}_{\bar{\mathfrak{p}}}$ with the Artin reciprocity map $\omega : \mathbb{A}_K^*/\overline{K^* U^{\mathfrak{p}}} \longrightarrow \mathrm{Gal}(K^{ab,\mathfrak{p}}/K)$ we can consider them as continuous representations

$$\theta_{\bar{\mathfrak{p}}} : \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \longrightarrow \bar{F}_{\bar{\mathfrak{p}}}^*,$$

and

$$\bar{\theta}_{\bar{\mathfrak{p}}} : \mathrm{Gal}(K^{ab,\mathfrak{p}}/K) \longrightarrow \kappa_{\bar{\mathfrak{p}}}^*.$$

Replace S by S'. For all $\mathfrak{p} \in S$ we then get by Lemma 2.1.5 the equality

$$\bar{\theta}_{\bar{\mathfrak{p}}} = \overline{\chi}_{\bar{\mathfrak{p}}}.$$

Construction of an algebraic relation. Let n be an integer, and let $f(T) := \prod_{i=1}^n (T-\alpha_i) = \sum_{i=0}^n \beta_i T^i$ be any monic polynomial of degree n. For any integer $m \leq n$ define

$$f^{(m)}(T) := \prod_I \left(T - \prod_{i\in I} \alpha_i \right)$$

where the outer product ranges over all subsets I of $\{1,\ldots,n\}$ of cardinality m. The coefficients of $f^{(m)}(T)$ are symmetric polynomials in the α_i and are therefore polynomials in $\beta_1,\ldots,\beta_n$ with coefficients in $\mathbb{Z}$. The above construction can thus be applied to any monic polynomial with coefficients in any commutative ring. If f is the characteristic polynomial of a linear map M, then $f^{(m)}$ is the characteristic polynomial of $\bigwedge^m M$. We have $f^{(m)}(\alpha) = 0$ if and only if f has m zeros with product α.

Fix a place $\mathfrak{Q}$ of K where φ has good reduction. Let $\mathfrak{p}$ be any prime of A not lying below $\mathfrak{Q}$. Denote by $f_{\mathfrak{Q}}$ the characteristic polynomial of $\rho_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$. By 1.2.1 it has coefficients in A and is independent of $\mathfrak{p}$.
Denote by $\bar{f}_{\mathfrak{Q},\mathfrak{p}}$ the characteristic polynomial of $\overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}}) \in \mathrm{GL}_r(\kappa_{\mathfrak{p}})$, and by $\bar{g}_{\mathfrak{Q},\mathfrak{p}}$ the characteristic polynomial of $\overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}}) \in \mathrm{GL}_t(\lambda_{\mathfrak{p}})$. We have

$$\bar{f}_{\mathfrak{Q},\mathfrak{p}} \equiv f_{\mathfrak{Q}} \mod \mathfrak{p},$$

and

$$\bar{f}_{\mathfrak{Q},\mathfrak{p}} = \mathrm{N}_{\lambda_{\mathfrak{p}}/\kappa_{\mathfrak{p}}}\, \bar{g}_{\mathfrak{Q},\mathfrak{p}}.$$

The fact that the $\theta_{\bar{\mathfrak{p}}}$ form a system of strictly compatible $\mathfrak{p}$-adic representations means that $u_{\mathfrak{Q}} := \theta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}})$ lies in $\bar{F}^*$ and is independent of $\mathfrak{p}$. It is integral outside ∞ and the places lying below $\mathfrak{Q}$.

Lemma 2.1.6. *For all places $\mathfrak{Q}$ of K where φ has good reduction we have*

$$f_{\mathfrak{Q}}^{(t)}(u_{\mathfrak{Q}}) = 0.$$

Proof. Fix a place $\mathfrak{Q}$ of K where φ has good reduction. Let $\mathfrak{p} \in S$ be a prime not lying below $\mathfrak{Q}$. By Lemma 2.1.5 we have

$$\overline{\chi}_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}}) = \bar{\theta}_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}}).$$

This implies that the product of the t zeros of $\bar{g}_{\mathfrak{Q},\mathfrak{p}}$ is equal to $\bar{\theta}_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}})$, which shows that $\bar{f}^{(t)}_{\mathfrak{Q},\mathfrak{p}}\big(\bar{\theta}_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}})\big) = 0$. Since $\bar{f}^{(t)}_{\mathfrak{Q},\mathfrak{p}} \equiv f^{(t)}_{\mathfrak{Q}} \mod \mathfrak{p}$ and $\bar{\theta}_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}}) \equiv u_{\mathfrak{Q}} \mod \mathfrak{p}$, we get $f^{(t)}_{\mathfrak{Q}}(u_{\mathfrak{Q}}) \equiv 0 \mod \mathfrak{p}$. This happens for infinitely many $\mathfrak{p} \in S$. Therefore we get $f^{(t)}_{\mathfrak{Q}}(u_{\mathfrak{Q}}) = 0$. $\square$

Conclusion. We can now prove Theorem 2.1.1 using the above results.

Proof of Theorem 2.1.1. Fix a prime $\mathfrak{p}$ of A. Consider the representation

$$\rho_{\mathfrak{p}} \times \theta_{\bar{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_{r,\bar{F_{\bar{\mathfrak{p}}}}} \times \mathrm{GL}_{1,\bar{F_{\bar{\mathfrak{p}}}}}$$

and denote by $\Gamma_{\mathfrak{p}}$ its image. Consider the morphism

$$\nu : \mathrm{GL}_r \times \mathrm{GL}_1 \to \mathbb{A}^1,\ (g,h) \mapsto \det(\Lambda^t g - h 1_{\binom{r}{t}}).$$

By Lemma 2.1.6 we know that $\nu\big(\rho_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}}), \theta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{Q}})\big) = 0$ for all places $\mathfrak{Q}$ of K with $\mathfrak{Q} \nmid \mathfrak{p}, \mathfrak{Q} \nmid \infty$, and where φ has good reduction. Since these $\mathrm{Frob}_{\mathfrak{Q}}$ form a dense subset, we get

$$\nu|_{\Gamma_{\mathfrak{p}}} = 0.$$

Let $\Gamma_{\mathfrak{p}}^{\mathrm{der}}$ be the commutator subgroup of $\Gamma_{\mathfrak{p}}$. Then we have

$$\Gamma_{\mathfrak{p}}^{\mathrm{der}} \subset \mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1.$$

By [23, Lemma 3.7] we know that the commutator morphism

$$[\_\,,\_] : \mathrm{GL}_r \times \mathrm{GL}_r \to \mathrm{SL}_r$$

is dominant. Together with Theorem 1.1.4 we see that the projection of $\Gamma_{\mathfrak{p}}^{\mathrm{der}}$ to the first factor lies Zariski dense in $\mathrm{SL}_{r,F_{\mathfrak{p}}}$. Note that in order to use Theorem 1.1.4, we need $\mathrm{End}_{\bar{K}}(\varphi) = A$. This was assumed at the beginning of this section. Since ν is an algebraic morphism, it follows that ν vanishes on $\mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1$.

But we have

$$\nu(\begin{pmatrix} \alpha & & & \\ & \ddots & & \\ & & \alpha & \\ & & & \alpha^{-r+1} \end{pmatrix}, 1) = (\alpha^t - 1)^{\binom{r-1}{t}} (\alpha^{t-r} - 1)^{\binom{r-1}{t-1}}.$$

We assumed that $s > 1$, which implies $t < r$. Therefore the restriction of ν to $\mathrm{SL}_{r,F_{\mathfrak{p}}} \times 1$ is non-constant. This is a contradiction, and so Assumption 2.1.2 is false, as desired. $\square$

2.2 The image of the group ring in the case $[K : F] < \infty$

In this section, we assume that φ is a Drinfeld A-module over K of generic characteristic and arbitrary endomorphism ring $E := \mathrm{End}_K(\varphi)$. Let $B_{\mathfrak{p}}$ be the image of the natural homomorphism

$$A_{\mathfrak{p}}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big).$$

For all primes $\mathfrak{p}$ of A the natural homomorphism

$$E_{\mathfrak{p}} := E \otimes_A A_{\mathfrak{p}} \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\big(T_{\mathfrak{p}}(\varphi)\big)$$

is injective by Pink and Traulsen [23, Proposition 4.1], and by Theorem 1.1.2 its image is equal to the commutant of $B_{\mathfrak{p}}$. Denote by c the rank of E as A-module, and define $d := r/c$. Since φ has generic characteristic, the ring E is commutative, and the number d is an integer. Define $E_{\mathfrak{p}} := E \otimes_A A_{\mathfrak{p}}$.

We can now state the result on the image of the group ring. It is analogous to Theorem B in Pink and Traulsen [23]. The only difference in our case is that the endomorphism ring E is always commutative since φ has generic characteristic.

Theorem 2.2.1. *For almost all primes $\mathfrak{p}$ of A we have $B_{\mathfrak{p}} \cong \mathrm{M}_d(E_{\mathfrak{p}})$.*

Proof. Since E is commutative, its center is E as well. All arguments of the proof by Pink and Traulsen also work in generic characteristic with the center Z of E replaced by E. The only missing part is the absolute irreducibility of the residual representation in the case where $\mathrm{End}_{\bar{K}}(\varphi) = A$ which has been proven in the previous section. $\square$

2.3 The general case

In this section we prove the absolute irreducibility of the residual representation for a Drinfeld module of arbitrary characteristic over a general finitely generated field K over $\mathbb{F}_q$. We will first generalise Theorem 2.2.1. For this, denote the center of $E = \mathrm{End}_K(\varphi)$ by Z. Define $c := [Z : A]$, $e^2 := [E : Z]$, and $d := r/ce$. The number d is an integer. Define $Z_\mathfrak{p} := Z \otimes_A A_\mathfrak{p}$. If φ has generic characteristic, then $E = Z$ and $e = 1$. As in the previous section, let $B_\mathfrak{p}$ denote the image of $A_\mathfrak{p}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi)\big)$, and define $E_\mathfrak{p} := E \otimes_A A_\mathfrak{p}$.

Theorem 2.3.1. *Let φ be a Drinfeld A-module over K. Then for almost all primes $\mathfrak{p}$ of A we have $E_\mathfrak{p} \cong \mathrm{M}_e(Z_\mathfrak{p})$ and $B_\mathfrak{p} \cong \mathrm{M}_d(Z_\mathfrak{p})$.*

We will prove Theorem 2.3.1 by reducing it to the case of transcendence degree 1. We use a similar argument as Pink did in [20]. Let X be a model of K of finite type over $\mathrm{Spec}(\mathbb{F}_p)$ such that φ defines a family of Drinfeld A-modules of rank r over X and such that $\mathrm{End}_K(\varphi)$ acts on the whole family of Drinfeld A-modules over X. For any point $x \in X$, we then get a Drinfeld A-module φ_x of rank r over the residue field k_x at x. Its characteristic is the image λ_x of x under the morphism $X \longrightarrow \mathrm{Spec}(A)$.

Let $\bar{x}$ be a geometric point of X over x such that $k_{\bar{x}} = k_x^{sep}$. The morphisms $\mathrm{Spec}(K) \hookrightarrow X \hookleftarrow x$ induce homomorphisms of the étale fundamental groups

$$\mathrm{G}_K \twoheadrightarrow \pi_1^{et}(X, \bar{x}) \leftarrow \pi_1^{et}(x, \bar{x}) = \mathrm{G}_{k_x}\,.$$

For any prime $\mathfrak{p} \neq \lambda_x$ of A, the specialisation map induces an isomorphism

$$V_\mathfrak{p}(\varphi) \longrightarrow V_\mathfrak{p}(\varphi_x).$$

This isomorphism is equivariant under the above étale fundamental groups. Moreover, since $\mathrm{End}_K(\varphi)$ acts faithfully on the Tate module $V_\mathfrak{p}(\varphi_x)$, we obtain a natural embedding $\mathrm{End}_K(\varphi) \hookrightarrow \mathrm{End}_{k_x}(\varphi_x)$. Let $\mathfrak{p}_0$ denote the characteristic of φ.

Proposition 2.3.2. *Assume that $K/\mathbb{F}_p$ has transcendence degree at least 1. Then there exists a point $x \in X$ such that the following properties hold.*

(i) *k_x has transcendence degree 1 over $\mathbb{F}_p$.*

(ii) *x lies over $\mathfrak{p}_0$.*

(iii) $\mathrm{End}_K(\varphi)$ *has finite index in* $\mathrm{End}_{k_x}(\varphi_x)$.

Proof. Let $\mathfrak{p}$ be any prime of A different from $\mathfrak{p}_0$. Denote by $\Gamma_\mathfrak{p}$ the image of G_K under the representation $\rho_\mathfrak{p} : \mathrm{G}_K \to \mathrm{GL}_r(A_\mathfrak{p})$. By Pink [20, Lemma 1.5], there exists an open normal subgroup $\Gamma_1 \subset \Gamma_\mathfrak{p}$ such that for any subgroup $\Delta \subset \Gamma_\mathfrak{p}$ with $\Delta\Gamma_1 = \Gamma_\mathfrak{p}$ we have $F_\mathfrak{p}\Delta = F_\mathfrak{p}\Gamma_\mathfrak{p}$ as subalgebras of $\mathrm{M}_r(F_\mathfrak{p})$. Let K' be the corresponding finite Galois extension of K, and let X' be the normalization of X in K'. Denote by π the morphism $X' \to X$.

By Pink [20, Lemma 1.6], there exists a point $x \in X$ satisfying (i) and (ii), and such that $\pi^{-1}(x) \subset X'$ is irreducible. Denote by $\Delta_\mathfrak{p}$ the image of G_{k_x} in the representation on $V_\mathfrak{p}(\varphi_x)$. Since $\mathfrak{p} \neq \lambda_x$, we have $V_\mathfrak{p}(\varphi_x) \cong V_\mathfrak{p}(\varphi)$, turning $\Delta_\mathfrak{p}$ into a subgroup of $\Gamma_\mathfrak{p}$. From the irreducibility of $\pi^{-1}(x)$ we get $\mathrm{Gal}(k_{\pi^{-1}(x)}/k_x) \cong \mathrm{Gal}(K'/K)$, and so $\Delta_\mathfrak{p}\Gamma_1 = \Gamma_\mathfrak{p}$, and therefore $F_\mathfrak{p}\Delta_\mathfrak{p} = F_\mathfrak{p}\Gamma_\mathfrak{p}$ by the above explanation. Therefore the images of the two natural homomorphisms $F_\mathfrak{p}[\mathrm{G}_{k_x}] \longrightarrow \mathrm{End}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi_x)\big) \cong \mathrm{End}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi)\big)$ and $F_\mathfrak{p}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{F_\mathfrak{p}}\big(V_\mathfrak{p}(\varphi)\big)$ coincide and thus also their commutants, which by Theorem 1.1.2 means that $\mathrm{End}_K(\varphi) \otimes_A F_\mathfrak{p} = \mathrm{End}_{k_x}(\varphi_x) \otimes_A F_\mathfrak{p}$. The structure theorem for finitely generated modules over Dedekind rings implies that $\mathrm{End}_K(\varphi)$ has finite index in $\mathrm{End}_{k_x}(\varphi_x)$. □

Proof of Theorem 2.3.1. Let x be a point of X as in Proposition 2.3.2. Denote the center of $E' := \mathrm{End}_{k_x}(\varphi_x)$ by Z'. By Proposition 2.3.2 we know that E has finite index in E'. Therefore we have $E_\mathfrak{p} \cong E'_\mathfrak{p}$ and $Z'_\mathfrak{p} \cong Z_\mathfrak{p}$ for almost all primes $\mathfrak{p}$ of A, and the invariants c, d, and e are the same for both tuples E, Z and E', Z'. Let $\bar{x}$ be a geometric point of X over x such that $k_{\bar{x}} = k_x^{sep}$.

Let $B'_\mathfrak{p}$ denote the image of the natural homomorphism

$$A_\mathfrak{p}[\mathrm{G}_{k_x}] \longrightarrow \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi_x)\big).$$

Since φ_x is a Drinfeld A-module over a finitely generated field of transcendence degree 1 over $\mathbb{F}_p$, Theorem 2.2.1 in generic characteristic and Theorem B of [23] in special characteristic imply that

$$B'_\mathfrak{p} \cong \mathrm{M}_d(Z'_\mathfrak{p}) \quad \text{and} \quad E'_\mathfrak{p} \cong \mathrm{M}_e(Z'_\mathfrak{p})$$

for almost all primes $\mathfrak{p}$ of A. Since $E_\mathfrak{p} \cong E'_\mathfrak{p}$ and $Z_\mathfrak{p} \cong Z'_\mathfrak{p}$ for almost all $\mathfrak{p}$, we get

$$E_\mathfrak{p} \cong \mathrm{M}_e(Z_\mathfrak{p})$$

for almost all primes $\mathfrak{p}$ of A.

We have $T_\mathfrak{p}(\varphi_x) \cong T_\mathfrak{p}(\varphi)$ for all primes $\mathfrak{p}$ of A different from λ_x, and this isomorphism is equivariant under the above étale fundamental groups. Thus the image of

$$A_\mathfrak{p}[\pi_1^{et}(X, \bar{x})] \longrightarrow \mathrm{End}_{A_\mathfrak{p}}\big(T_\mathfrak{p}(\varphi)\big)$$

is equal to $\mathrm{M}_d(Z_{\mathfrak{p}})$ for almost all primes $\mathfrak{p}$ of A. Since the action of G_K on $T_{\mathfrak{p}}(\varphi)$ factors through $\pi_1^{et}(X, \bar{x})$, we have

$$B_{\mathfrak{p}} \cong \mathrm{M}_d(Z_{\mathfrak{p}})$$

for almost all primes $\mathfrak{p}$ of A. $\square$

We can now prove the general case of the absolute irreducibility of the residual representation for a Drinfeld module with arbitrary characteristic over a finitely generated field K.

Theorem 2.3.3 (Absolute irreducibility of the residual representation)**.** *Let φ be a Drinfeld A-module over K. Assume that* $\mathrm{End}_K(\varphi) = A$. *Then the residual representation*

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

is absolutely irreducible for almost all primes $\mathfrak{p}$ of A.

Proof. By definition, we get $\overline{\rho_{\mathfrak{p}}}$ from $\rho_{\mathfrak{p}}$ by reduction $\bmod \bar{\mathfrak{p}}$. According to Bourbaki [2, §13, Proposition 5], it is therefore enough if we prove that the natural homomorphism

$$A_{\mathfrak{p}}[\mathrm{G}_K] \longrightarrow \mathrm{End}_{A_{\mathfrak{p}}}\left(T_{\mathfrak{p}}(\varphi)\right)$$

is surjective for almost all primes $\mathfrak{p}$ of A. This follows from Theorem 2.3.1 by setting $E = A$. $\square$

Chapter 3

Adelic Openness in generic characteristic

In this chapter, we prove that the image of the adelic representation associated to φ is open if φ is of generic characteristic.

In Section 3.1 we prove some general results for matrix groups and algebraic groups. These will be important in the subsequent sections.

In Section 3.2 we show that the residual representation is surjective for almost all primes of A if $\mathrm{End}_{\bar{K}}(\varphi) = A$ and φ is of generic characteristic and K is a finite extension of F. There we will need the absolute irreducibility of the residual representation from Chapter 2 and the results on the image of the inertia group from Chapter 1. Since the image of the tame inertia group is a torus and the residual representation is absolutely irreducible, the image of the residual representation is already quite large.

We show in Section 3.3 that the image of the adelic representation is open if K is a finite extension of F. We will show that this follows from the surjectivity of the residual representation for almost all primes of A. The argument is very similar to the one in Gardeyn [10, Chapter 3].

In Section 3.4 we prove a specialisation result. The proof is based on an argument from Pink [18] and uses a result from Pink [20].

Finally, in Section 3.5, we prove the adelic openness in generic characteristic in the general case, i.e., where K is a finitely generated extension of F. This will be done by reduction to the case of a finite extension of F. For doing this, we will use the specialisation result from Section 3.4.

Throughout this Chapter we assume that φ is a Drinfeld A-module over K with generic characteristic. As usual, we denote the rank of φ by r.

3.1 Preparatory results

In this section we prove some results on matrix groups and on algebraic groups. The first is on additive subgroups of the matrix group $\mathrm{M}_n(k)$.

Proposition 3.1.1. *Let n be any natural number, let k be a field with $|k| \geq 4$, and let H be an additive subgroup of $\mathrm{M}_n(k)$. Assume that H is invariant under conjugation by $\mathrm{GL}_n(k)$. Then either H is contained in the set of scalar matrices or H contains the set of matrices with trace 0.*

Proof. Let $T = \mathbb{G}_m^n$ denote the full diagonal torus. Its character group is given by $\mathrm{X}(T) = \mathbb{Z}^n$. Let $e_i, i = 1, \ldots, n$, be the standard basis of $\mathbb{Z}^n$. The torus T acts on $\mathfrak{gl}_n$ by conjugation, and thus we consider $\mathfrak{gl}_n$ as representation of T. Its weights are given by $e_i - e_j, i \neq j$, with multiplicity 1 and 0 with multiplicity n. The weight space W_0 of weight 0 is the group of diagonal matrices. The weight space $W_{i,j}$ of weight $e_i - e_j$ is the group of matrices with only zero entries except, possibly, for the position (i, j). We thus can decompose $\mathrm{M}_n(k)$ as

$$\mathrm{M}_n(k) = W_0 \oplus_{i,j} W_{i,j}.$$

Since $|k| \geq 4$, we have $|k^*| \geq 3$. Thus any two distinct weights of the form $e_i - e_j$ remain distinct on restriction to $T(k)$. Therefore we can decompose H as

$$H = (H \cap W_0) \oplus \bigoplus_{i,j} (H \cap W_{i,j}) \,.$$

Each $W_{i,j}$ is a k-vector space of dimension 1, and $T(k)$ acts on it through a surjective homomorphism $T(k) \twoheadrightarrow k^*$. Therefore $H \cap W_{i,j}$ is either 0 or equal to $W_{i,j}$. The permutation group S_n is a subgroup of $\mathrm{GL}_n(k)$ and permutes the weights $e_i - e_j$ transitively. Since H is invariant under conjugation by $\mathrm{GL}_n(k)$, we find that either all $H \cap W_{i,j} = 0$ or all $H \cap W_{i,j} = W_{i,j}$. In other words either H is contained in the set of diagonal matrices or H contains all $W_{i,j}$, which is the set of matrices with 0 on the diagonal.

If H is contained in the set of diagonal matrices, we take an element h of H. Denote its diagonal entries by $h_1, \ldots, h_n$. Let $i \neq j$. Denote by $u \in \mathrm{GL}_n(k)$ the matrix with entry 1 on the diagonal and in the (i, j)-entry and 0 elsewhere. Then the matrix uhu^{-1} has entry $h_i - h_j$ at the position (i, j). But this entry has to be 0 because $uhu^{-1} \in H$. We then get that $h_i = h_j$. This can be done for any pair (i, j), which shows that H is contained in the set of scalar matrices.

If H contains the set of matrices with 0 on the diagonal, we consider the trace form. It is given by $\langle A, B\rangle := \mathrm{tr}(AB)$. Denote by $H^\perp$ the orthogonal complement of H with respect to the trace form. The inclusion for orthogonal complements is reversed. Therefore $H^\perp$ is contained in the orthogonal complement of the set of matrices with 0 on the diagonal. This orthogonal complement is the group of diagonal matrices. By the above observation and the assumptions of the Proposition, we get that $H^\perp$ is contained in the set of scalar matrices. Therefore H contains the orthogonal complement of the scalar matrices, which are the matrices with trace 0. $\square$

The next two results are on subgroups of $\mathrm{GL}_n(k)$.

Proposition 3.1.2. *Let n be any natural number, let k a finite field, and let H be a normal subgroup of $\mathrm{GL}_n(k)$ containing a non scalar matrix. Assume that $(n, |k|)$ is different from $(2, 2)$ and $(2, 3)$. Then we have*

$$\mathrm{SL}_n(k) \subset H.$$

Proof. For any non-scalar element $h \in \mathrm{GL}_n(k)$, there exists an element $g \in \mathrm{GL}_n(k)$ such that the commutator $ghg^{-1}h^{-1}$ is again non-scalar. Thus H contains a non-scalar element of $\mathrm{SL}_n(k)$. In particular, we have $n \geq 2$, and H does not lie in the center $Z\big(\mathrm{SL}_n(k)\big)$ of $\mathrm{SL}_n(k)$. By Huppert [14], the group $\mathrm{SL}_n(k)/Z\big(\mathrm{SL}_n(k)\big)$ is simple, except for the cases where $(n, |k|) = (2, 2)$ or $(n, |k|) = (2, 3)$. By assumption, we are not in any of these two cases. Therefore we get that $HZ\big(\mathrm{SL}_n(k)\big) = \mathrm{SL}_n(k)$. Since $\mathrm{SL}_n(k)$ is perfect by Bass, Milnor and Serre [1, Corollary 4.3] or Rose [26], we get that $\mathrm{SL}_n(k) \subset H$. $\square$

The next result is on subgroups of $\mathrm{SL}_n(k)$.

Proposition 3.1.3. *Let n be any natural number, let c a constant, let k a finite field, and let H be a subgroup of $\mathrm{SL}_n(k)$ of index c. Assume that $(n, |k|)$ is different from $(2, 2)$ and $(2, 3)$ and that $c!n < |\mathrm{SL}_n(k)|$. Then*

$$H = \mathrm{SL}_n(k).$$

Proof. Denote $\mathrm{SL}_n(k)$ by G. The subgroup $N_G(H)$ of G has index at most c in G. Consider the group

$$N := \bigcap_{g \in G} gHg^{-1} = \bigcap_{x \in G/N_G(H)} xHx^{-1}.$$

The group H acts on $X := \{gH |\ g \in G\}$ through multiplication on the left. This corresponds to a homomorphism from G to the symmetric group S_X on X which, by assumption, is isomorphic to a subgroup of the symmetric group S_c on c elements.

The kernel of this homomorphism is N. Therefore N has index at most $c!$ in G. Only those scalar matrices with n-th roots of unity as diagonal entries lie in G. Therefore there are at most n scalar matrices in G. Since $c!n < |\,\mathrm{SL}_n(k)|$, the group N thus contains a non scalar element. Moreover, it is normal in $\mathrm{GL}_n(k)$. By Proposition 3.1.2, we therefore get $N = G$ and thus $H = G$. □

The next two results are on fibers of morphisms.

Proposition 3.1.4. *Let X be an irreducible algebraic variety over a field L, let G be an irreducible algebraic group over L, and let $f : X \longrightarrow G$ be a dominant morphism. Define $f^n : X^n \longrightarrow G$ by $(x_1, \ldots, x_n) \mapsto f(x_1) \cdot \ldots \cdot f(x_n)$, and denote by d and e the dimensions of G and X, respectively. Then for $n \geq d$ the fibers of f^n have dimension at most $ne - d$.*

Proof. Since f is dominant, there exists an open dense subset U of X such that all fibers of $f|_U$ have dimension $e - d$. We first consider the restriction of f^n to $X^{i-1} \times U \times X^{n-i}$ for any $1 \leq i \leq n$. We can write this restriction as the composite of morphisms

$$X^{i-1} \times U \times X^{n-i} \xrightarrow{\alpha} X^{i-1} \times G \times X^{n-i} \xrightarrow{\beta} X^{i-1} \times G \times X^{n-i} \xrightarrow{\gamma} G$$

where

$$\begin{aligned}
\alpha(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, f(x_i), x_{i+1}, \ldots, x_n), \\
\beta(x_1, \ldots, x_{i-1}, g, x_{i+1}, \ldots, x_n) &= \\
&\quad (x_1, \ldots, x_{i-1},\ f(x_1) \ldots f(x_{i-1}) g f(x_{i+1}) \ldots f(x_n),\ x_{i+1}, \ldots, x_n), \\
\gamma(x_1, \ldots, x_{i-1}, g, x_{i+1}, \ldots, x_n) &= g.
\end{aligned}$$

Here α has fiber dimension $e - d$, the morphism β is an isomorphism, and γ has fiber dimension $(n-1)e$. Thus all fibers of $f^n|_{X^i \times U \times X^{n-i-1}}$ have dimension at most $e - d + (n-1)e = ne - d$. Varying i, we get that all fibers of $f^n|_{X^n \setminus (X \setminus U)^n}$ have dimension at most $ne - d$.

On the other hand, all fibers of $f^n|_{(X \setminus U)^n}$ have dimension at most $\dim\big((X \setminus U)^n\big) \leq n(e-1)$. Since $n \geq d$, this is at most $ne - d$.

We have

$$X^n = (X \setminus U)^n \amalg \bigcup_{i=0}^{n-1} (X^i \times U \times X^{n-i-1}).$$

Therefore all fibers of f^n have dimension at most $ne - d$. □

Proposition 3.1.5. *Let X and Y be affine schemes of finite type over* $\operatorname{Spec}\mathbb{Z}$, *and let $f : X \longrightarrow Y$ be a morphism of finite type. Then there exists a constant c, depending only on X, Y and f, such that for any finite field k and any $y \in Y$*

$$|f^{-1}(y)(k)| \leq c|k|^{\dim(f^{-1}(y))}.$$

Proof. We induct on $\dim(Y)$. Since X and Y both have only finitely many irreducible components, we can assume that both X and Y are irreducible.

For points $y \notin f(X)$ of Y, there is nothing to prove. Therefore we can replace Y by the Zariski closure of $f(X)$ in Y, and assume that the morphism f is dominant.

If $\dim(Y) = 0$, we have $Y = \{\eta\}$ and $f^{-1}(\eta) = X$. Since X is affine we can use Noether normalisation to get a finite morphism $X \longrightarrow \mathbb{A}^n$, where n is the dimension of X. Let d_0 be its degree. Then we have $|f^{-1}(\eta)(k)| \leq d_0|k|^n$. Thus the proposition is true for $\dim(Y) = 0$ with constant d_0.

Assume that the proposition is true for $\dim(Y) < e$ with constant c'. Assume $\dim(Y) = e$, and let η be the generic point of Y. By Noether normalisation there exists a finite morphism $f^{-1}(\eta) \longrightarrow \mathbb{A}_\eta^{\dim(f^{-1}(\eta))}$ of degree, say, d. This finite morphism extends to an open neighbourhood V of η in Y. We thus get a quasifinite morphism $f^{-1}(V) \longrightarrow \mathbb{A}_V^{\dim(f^{-1}(\eta))}$ of degree at most d where $\dim(f^{-1}(\eta))$ is the constant fiber dimension of $f|_{f^{-1}(V)}$. For all $y \in V$, we thus get

$$|f^{-1}(y)(k)| \leq n|k|^{\dim(f^{-1}(\eta))}.$$

Therefore the proposition is true for all $y \in V$ with constant d. Let Y' be the complement of V. It is closed and therefore we have $\dim(Y') < e$. By assumption the proposition is true for all $y \in Y'$ with constant c'. Define $c := \max\{c', d\}$. Then the proposition is true for all $y \in Y$ with constant c. $\square$

3.2 Surjectivity of the residual representation

Throughout this section, we assume that K is a finite extension of F. We prove the following result.

Proposition 3.2.1. *Let φ be a Drinfeld A-module over K of generic characteristic. Assume that $\mathrm{End}_{\bar{K}}(\varphi) = A$ and that K is a finite extension of F. Then the residual representation*

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}})$$

is surjective for almost all primes $\mathfrak{p}$ of A.

Remark. Later in this section we have to show that a certain representation is unramified. For doing this, we need that φ has semistable reduction. Therefore we already now replace K by a finite extension such that φ has semistable reduction everywhere. Next, we replace K by a finite extension such that the decomposition group at a place of bad reduction acts trivially on the lattice at that place. Finally, we replace K by a finite extension such that the lattices at places above ∞ become K-rational.

By Theorem 2.3.3 the residual representation is absolutely irreducible for almost all primes of A. We have to show that the residual representation is surjective for almost all primes of A. We can therefore restrict ourselves to primes $\mathfrak{p}$ of A

- where the residual representation is absolutely irreducible,
- which lie below places where φ has good reduction, and
- for which we have $|\kappa_{\mathfrak{p}}| \geq 4$.

Consider such a prime $\mathfrak{p}$ of A, a place $\mathfrak{P}$ of K above $\mathfrak{p}$, a place $\bar{\mathfrak{p}}$ of $\bar{F}$ above $\mathfrak{p}$ and a place $\bar{\mathfrak{P}}$ of $\bar{K}$ above $\mathfrak{P}$. Denote by $\Gamma_{\mathfrak{p}}$ the image of the residual representation

$$\overline{\rho_{\mathfrak{p}}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\kappa_{\mathfrak{p}}).$$

Denote by $q_{\mathfrak{p}}$ the cardinality of $\kappa_{\mathfrak{p}}$. By assumption φ has good reduction at $\mathfrak{P}$. Denote by $h_{\mathfrak{P}}$ the height of the reduced Drinfeld module. Define $n := q_{\mathfrak{p}}^{h_{\mathfrak{P}}}$, and denote by k_n the subfield of $k_{\bar{\mathfrak{P}}}$ with n elements. For the inertia group $\mathrm{I}_{\mathfrak{P}}$ at $\mathfrak{P}$ we have an exact sequence

$$1 \longrightarrow \mathrm{I}_{\mathfrak{P}}^p \longrightarrow \mathrm{I}_{\mathfrak{P}} \longrightarrow \mathrm{I}_{\mathfrak{P}}^t \longrightarrow 1$$

where $\mathrm{I}_{\mathfrak{P}}^p$ and $\mathrm{I}_{\mathfrak{P}}^t$ denote the wild inertia group and tame inertia group, respectively. Fix a section $\mathrm{I}_{\mathfrak{P}}^t \longrightarrow \mathrm{I}_{\mathfrak{P}}$. By Proposition 1.2.3 we know that the image under $\overline{\rho_{\mathfrak{p}}}$ of the inertia group at $\mathfrak{P}$ is up to conjugation given by

$$\left(\begin{array}{c|c} k_n^* & * \\ \hline 0 & 1 \end{array}\right) \subset \Gamma_{\mathfrak{p}},$$

and the image of $\mathrm{I}^t_{\mathfrak{P}}$ thus is

$$\left(\begin{array}{c|c} k_n^* & 0 \\ \hline 0 & 1 \end{array}\right) \subset \Gamma_{\mathfrak{p}}.$$

Since $q_{\mathfrak{p}} \geq 4$, its centraliser in $\mathrm{GL}_{r,\kappa_{\mathfrak{p}}}$ looks like

$$\left(\begin{array}{c|c} T_{\mathfrak{P}} & 0 \\ \hline 0 & \mathrm{GL}_{(r-h_{\mathfrak{P}}),\kappa_{\mathfrak{p}}} \end{array}\right)$$

for a torus $T_{\mathfrak{P}}$ over $\kappa_{\mathfrak{p}}$ with $T_{\mathfrak{P}}(\kappa_{\mathfrak{p}}) = k_n^*$. The torus $T_{\mathfrak{P}}$ is the Weil restriction $\mathrm{Res}^{k_n}_{\kappa_{\mathfrak{p}}} \mathbb{G}_{m,k_n}$ and thus of dimension $h_{\mathfrak{P}}$.

Let $H_{\mathfrak{p}}$ be the algebraic subgroup of $\mathrm{GL}_{r,\kappa_{\mathfrak{p}}}$ generated by $\Gamma_{\mathfrak{p}}$ and $T_{\mathfrak{P}}$ for all $\mathfrak{P} \mid \mathfrak{p}$. Since $\Gamma_{\mathfrak{p}}$ acts absolutely irreducibly on $\kappa_{\mathfrak{p}}^r$, so does $H_{\mathfrak{p}}$. Let $\tilde{H}_{\mathfrak{p}}$ be the algebraic group generated by $\{\gamma T_{\mathfrak{P}} \gamma^{-1} \mid \mathfrak{P} \mid \mathfrak{p} \text{ and } \gamma \in \Gamma_{\mathfrak{p}}\}$. Then $\tilde{H}_{\mathfrak{p}}$ is contained in the identity component $H_{\mathfrak{p}}^\circ$ of $H_{\mathfrak{p}}$. In fact, since $H_{\mathfrak{p}} = \Gamma_{\mathfrak{p}} \tilde{H}_{\mathfrak{p}}$, the quotient $H_{\mathfrak{p}} / \tilde{H}_{\mathfrak{p}} \cong \Gamma_{\mathfrak{p}} / (\Gamma_{\mathfrak{p}} \cap \tilde{H}_{\mathfrak{p}})$ is finite, and thus $H_{\mathfrak{p}}^\circ = \tilde{H}_{\mathfrak{p}}$.

Lemma 3.2.2. *There exist a natural number $s_{\mathfrak{p}}$, elements $\gamma_1, \ldots, \gamma_{s_{\mathfrak{p}}} \in \Gamma_{\mathfrak{p}}$, and an $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$-irreducible vector space $W \subset {\kappa_{\bar{\mathfrak{p}}}}^r$ such that*

$$\kappa_{\bar{\mathfrak{p}}}^r = \gamma_1 W \oplus \ldots \oplus \gamma_{s_{\mathfrak{p}}} W.$$

Proof. Abbreviate $V := \kappa_{\bar{\mathfrak{p}}}^r$. Let W be a nontrivial $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$-invariant subspace of V of minimal dimension. Since $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$ is normalised by $\Gamma_{\mathfrak{p}}$, the vector space γW is also $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$-invariant for all $\gamma \in \Gamma_{\mathfrak{p}}$. The vector space $\sum_{\gamma \in \Gamma_{\mathfrak{p}}} \gamma W$ is $\Gamma_{\mathfrak{p}}$-invariant and therefore, by the irreducibility of V, equal to V. Since each γW is irreducible over $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$, we can choose $\gamma_1, \ldots, \gamma_{s_{\mathfrak{p}}} \in \Gamma_{\mathfrak{p}}$ such that

$$\gamma_1 W \oplus \ldots \oplus \gamma_{s_{\mathfrak{p}}} W = V.$$

$\square$

We fix a decomposition of $\kappa_{\bar{\mathfrak{p}}}^r$ as in Lemma 3.2.2. Then the subgroup of $\mathrm{GL}_{r,\kappa_{\bar{\mathfrak{p}}}}$ which acts on each summand separately is isomorphic to $\mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}}$ where $t_{\mathfrak{p}}$ denotes the dimension of W. The subgroup of $\mathrm{GL}_{r,\kappa_{\bar{\mathfrak{p}}}}$ which maps each summand to some, possibly other, summand, is then isomorphic to $\mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}} \rtimes S_{s_{\mathfrak{p}}}$.

Lemma 3.2.3. *We have*

$$H_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}} \subset \mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}} \rtimes S_{s_{\mathfrak{p}}}.$$

Proof. Define $W_i := \gamma_i W$ for $i = 1, \ldots, s_{\mathfrak{P}}$. We then have

$$\kappa_{\bar{\mathfrak{p}}}^r = \bigoplus_{i=1}^{s_{\mathfrak{p}}} W_i, \text{ and } H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}} \subset \mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}}.$$

There exists a basis of $\kappa_{\bar{\mathfrak{p}}}^r$ with respect to which

$$T_{\mathfrak{P},\kappa_{\bar{\mathfrak{p}}}} = \left(\begin{array}{ccc|ccc} * & & & & & \\ & \ddots & & & & \\ & & * & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) \cong \mathbb{G}^{h_{\mathfrak{P}}}_{m,\kappa_{\bar{\mathfrak{p}}}}$$

where the upper left block consists of diagonal $h_{\mathfrak{P}} \times h_{\mathfrak{P}}$-matrices. Define

$$\mu_1 : \mathbb{G}_{m,\kappa_{\bar{\mathfrak{p}}}} \longrightarrow T_{\mathfrak{P},\kappa_{\bar{\mathfrak{p}}}},\ t \mapsto \begin{pmatrix} t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

This is a cocharacter of $T_{\mathfrak{P},\kappa_{\bar{\mathfrak{p}}}}$ which on the given representation has weight 1 with multiplicity 1 and weight 0 with multiplicity $r-1$. Without loss of generality we can assume that μ_1 has its nontrivial weight on W_1 and weight zero on all other W_i. Since $T_{\mathfrak{P},\kappa_{\bar{\mathfrak{p}}}} \subset H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$, it follows that, as an $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$-representation, the space W_1 is not isomorphic to W_i for any $i \neq 1$. By conjugation, we get that any two of the W_i are non-isomorphic $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$-representations. This shows that the decomposition $\kappa_{\bar{\mathfrak{p}}}^r = \bigoplus_{i=1}^{s_{\mathfrak{p}}} W_i$ is the isotypical decomposition under $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$. Therefore it is normalised by $H_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$ and we have

$$H_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}} \subset \mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}} \rtimes S_{s_{\mathfrak{p}}}.$$

□

Define $\alpha_{\mathfrak{p}}$ as the composition of the following homomorphisms

$$\mathrm{G}_K \longrightarrow H_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}} \subset \mathrm{GL}^{s_{\mathfrak{p}}}_{t_{\mathfrak{p}},\kappa_{\bar{\mathfrak{p}}}} \rtimes S_{s_{\mathfrak{p}}} \twoheadrightarrow S_{s_{\mathfrak{p}}}.$$

Lemma 3.2.4. *The homomorphism $\alpha_{\mathfrak{p}}$ is unramified at all places of K lying above $\mathfrak{p}$.*

Proof. Fix a place $\mathfrak{P}$ of K lying above $\mathfrak{p}$. By assumption, the Drinfeld module φ has good reduction at $\mathfrak{P}$. Therefore the image under the residual representation of the inertia group $\mathrm{I}_{\mathfrak{P}}$ at $\mathfrak{P}$ looks like

$$\left(\begin{array}{c|c} k_n^* & \overline{\rho_{\mathfrak{p}}}(\mathrm{I}_{\mathfrak{P}}^p) \\ \hline 0 & 1 \end{array}\right) = T_{\mathfrak{P}}(\kappa_{\mathfrak{p}}) \ltimes \left(\begin{array}{c|c} 1 & \overline{\rho_{\mathfrak{p}}}(\mathrm{I}_{\mathfrak{P}}^p) \\ \hline 0 & 1 \end{array}\right).$$

Therefore the image under $\alpha_{\mathfrak{p}}$ of the tame inertia group at $\mathfrak{P}$ is trivial. The restriction of $\alpha_{\mathfrak{p}}$ to the wild inertia group at $\mathfrak{P}$ thus factors through the coinvariants of $\overline{\rho_{\mathfrak{p}}}(\mathrm{I}_{\mathfrak{P}}^p)$ under the image of the tame inertia group. Since $|k_n| = q_{\mathfrak{p}}^{h_{\mathfrak{P}}} \geq q_{\mathfrak{p}} \geq 4$, these coinvariants are trivial, as can be seen from the above semidirect product. Therefore the image under $\alpha_{\mathfrak{p}}$ of $\mathrm{I}_{\mathfrak{P}}^p$ is trivial, and the homomorphism $\alpha_{\mathfrak{p}}$ is unramified at $\mathfrak{P}$. □

Lemma 3.2.5. *For almost all primes $\mathfrak{p}$ of A, the homomorphism $\alpha_{\mathfrak{p}}$ is unramified at all places of K where φ has bad reduction.*

Proof. Fix a place $\mathfrak{Q}$ of K where φ has bad reduction. At the beginning of this section we replaced K by a finite extension such that φ has semistable reduction everywhere. Let $(\psi, \Lambda_{\mathfrak{Q}})$ be the Tate uniformisation of φ at $\mathfrak{Q}$. Then, ψ is a Drinfeld A-module over $K_{\mathfrak{Q}}$ of some rank $r' < r$, and $\Lambda_{\mathfrak{Q}}$ is, via ψ, an A-lattice in $K_{\mathfrak{Q}}^{sep}$ of rank $r - r'$.

For any prime $\mathfrak{p}$ of A with $\mathfrak{p} \nmid \mathfrak{Q}$ we have an exact sequence

$$0 \longrightarrow \psi[\mathfrak{p}](K^{sep}) \longrightarrow \varphi[\mathfrak{p}](K^{sep}) \longrightarrow \Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}} \longrightarrow 0.$$

The inertia group $\mathrm{I}_{\mathfrak{Q}}$ acts trivially on the first and the third term by hypothesis on K. Therefore its image under $\overline{\rho_{\mathfrak{p}}}$ lies in

$$\left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array}\right) \cong \mathrm{Hom}\left(\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}, \psi[\mathfrak{p}](K^{sep})\right).$$

The group $\alpha_{\mathfrak{p}}(\mathrm{I}_{\mathfrak{Q}})$ is normalised by $\alpha_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$. Since the order of $\alpha_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{Q}})$ divides $s_{\mathfrak{p}}!$, and hence also $r!$, the homomorphism $\mathrm{I}_{\mathfrak{Q}} \twoheadrightarrow \alpha_{\mathfrak{p}}(\mathrm{I}_{\mathfrak{Q}})$ factors through

$$\begin{tikzcd} \mathrm{I}_{\mathfrak{Q}} \arrow[rr, two heads] \arrow[dr] & & \alpha_{\mathfrak{p}}(\mathrm{I}_{\mathfrak{Q}}) \\ & \mathrm{Hom}\left(\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}, \psi[\mathfrak{p}](K^{sep})\right)_{\mathrm{Frob}_{\mathfrak{Q}}^{r!}}. \arrow[ur, two heads] & \end{tikzcd}$$

The action of $\mathrm{Frob}_{\mathfrak{Q}}^{r!}$ on $\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}$ is trivial. Therefore we have

$$\mathrm{Hom}\left(\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}, \psi[\mathfrak{p}](K^{sep})\right)_{\mathrm{Frob}_{\mathfrak{Q}}^{r!}} = 0$$

if and only if $\psi[\mathfrak{p}](K^{sep})_{\mathrm{Frob}_{\mathfrak{Q}}^{r!}} = 0$.

Denote by $f_{\mathfrak{Q}}$ the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{Q}}^{r!}$ on the Tate module of ψ at $\mathfrak{p}$. It is independent of $\mathfrak{p}$, and its coefficients lie in A. By purity, every eigenvalue of $\mathrm{Frob}_{\mathfrak{Q}}$ has valuation < 0 at ∞. Thus 1 is not an eigenvalue of $\mathrm{Frob}_{\mathfrak{Q}}^{r!}$, and so $f_{\mathfrak{Q}}(1) \in A \setminus \{0\}$. If $\mathfrak{p} \nmid f_{\mathfrak{Q}}(1)$, then no eigenvalue is congruent to 1 modulo a prime lying above $\mathfrak{p}$, and so

$$\mathrm{Hom}\left(\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}, \psi[\mathfrak{p}](K^{sep})\right)_{\mathrm{Frob}_{\mathfrak{Q}}^{r!}} = 0.$$

We thus get $\mathrm{Hom}\left(\Lambda_{\mathfrak{Q}}/\mathfrak{p}\Lambda_{\mathfrak{Q}}, \psi[\mathfrak{p}](K^{sep})\right)_{\mathrm{Frob}_{\mathfrak{Q}}^{r!}} = 0$ for almost all primes $\mathfrak{p}$ of A. Therefore, for these primes, the image under $\alpha_{\mathfrak{p}}$ of $\mathrm{I}_{\mathfrak{Q}}$ is trivial. There are only finitely many places $\mathfrak{Q}$ of K where φ has bad reduction. Therefore, there are only finitely many possibilities for the characteristic polynomial of the $r!$-th power of Frobenius at these places. Thus, for almost all primes $\mathfrak{p}$ of A, the homomorphism $\alpha_{\mathfrak{p}}$ is unramified at all places of K where φ has bad reduction.

□

Corollary 3.2.6. *For almost all primes $\mathfrak{p}$ of A, the homomorphism $\alpha_{\mathfrak{p}}$ is unramified everywhere and totally split at places ∞' above ∞.*

Proof. At the beginning of this section, we replaced K by a finite extension such that the action of the decomposition group at any place lying above ∞ is trivial. In particular, it is unramified at all places lying above ∞. The action of the inertia group at all places not lying above ∞ and where φ has good reduction is trivial. Therefore, for all primes $\mathfrak{p}$ of A, the homomorphism $\alpha_{\mathfrak{p}}$ is unramified at these places. Moreover, by Lemma 3.2.4, it is unramified at all places lying above $\mathfrak{p}$. By Lemma 3.2.5, for almost all primes $\mathfrak{p}$ of A, it is unramified at all places where φ has bad reduction. For these primes, the homomorphism $\alpha_{\mathfrak{p}}$ is unramified everywhere. □

Lemma 3.2.7. *For almost all primes $\mathfrak{p}$ of A, we have $s_{\mathfrak{p}} = 1$.*

Proof. Let $\mathfrak{p}$ a prime of A such that $\alpha_{\mathfrak{p}}$ is unramified everywhere, and let $K^{(\mathfrak{p})}$ the field fixed by the kernel of the homomorphism $\alpha_{\mathfrak{p}}$. By Corollary 3.2.6 it is unramified over K. Moreover, its degree $[K^{(\mathfrak{p})} : K] \leq s_{\mathfrak{p}}! \leq r!$ is bounded independently of $\mathfrak{p}$. By Goss [11, Theorem 8.23.5], a function field analogue of the Hermite-Minkowski Theorem about unramified extensions, there are only finitely many possibilities for $K^{(\mathfrak{p})}$. Therefore their compositum K' is a finite extension of K. The homomorphism

$$\alpha_{\mathfrak{p}} : \mathrm{G}_{K'} \longrightarrow S_{s_{\mathfrak{p}}}$$

is trivial for almost all primes $\mathfrak{p}$ of A.

Let $\mathfrak{p}$ be any such prime of A. If $s_\mathfrak{p} > 1$, then the residual representation $\overline{\rho_\mathfrak{p}}$ for the Drinfeld module φ considered as Drinfeld A-module over K' is not absolutely irreducible since the commutant of its image is too big. By Theorem 2.3.3 this can only happen for finitely many primes. Therefore we get $s_\mathfrak{p} = 1$ for almost all primes $\mathfrak{p}$ of A where $\alpha_\mathfrak{p}$ is unramified everywhere, which, by Corollary 3.2.6, are almost all primes $\mathfrak{p}$ of A. $\square$

Proposition 3.2.8. *For almost all primes* $\mathfrak{p}$ *of* A, *we have*

$$H^\circ_{\mathfrak{p},\kappa_\mathfrak{p}} = H_{\mathfrak{p},\kappa_\mathfrak{p}} = \mathrm{GL}_{r,\kappa_\mathfrak{p}} .$$

Proof. Lemma 3.2.7 implies that $H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}}$ acts irreducibly on ${\kappa_{\bar{\mathfrak{p}}}}^r$. Moreover, as has been explained in the proof of Lemma 3.2.3, it has a cocharacter with weight 1 with multiplicity 1 and weight 0 with multiplicity $r-1$. By Pink [20, Proposition A.3] we get

$$H^\circ_{\mathfrak{p},\kappa_{\bar{\mathfrak{p}}}} = \mathrm{GL}_{r,\kappa_{\bar{\mathfrak{p}}}},$$

and thus

$$H^\circ_{\mathfrak{p},\kappa_\mathfrak{p}} = \mathrm{GL}_{r,\kappa_\mathfrak{p}}$$

because $H^\circ_{\mathfrak{p},\kappa_\mathfrak{p}}$ is an algebraic subgroup of $\mathrm{GL}_{r,\kappa_\mathfrak{p}}$. Since $H^\circ_{\mathfrak{p},\kappa_\mathfrak{p}} \subset H_{\mathfrak{p},\kappa_\mathfrak{p}} \subset \mathrm{GL}_{r,\kappa_\mathfrak{p}}$, both inclusions are equalities. $\square$

The following result will be needed in the subsequent one to assert that a certain constant is independent of $\mathfrak{p}$.

Lemma 3.2.9. *There exists a scheme* Z *of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *and a closed subscheme* $\mathcal{T} \subset \mathrm{GL}_r \times Z$ *over* Z, *such that for almost all primes* $\mathfrak{p}$ *of* A, *any place* $\mathfrak{P} \mid \mathfrak{p}$ *of* K, *and any element* $\gamma \in \Gamma_\mathfrak{p}$, *there exists a point* $z \in Z(\kappa_\mathfrak{p})$ *such that* $\mathcal{T}_z = \gamma T_\mathfrak{P} \gamma^{-1}$.

Proof. Define

$$\begin{aligned} Z &:= \mathrm{GL}_r \times (\mathbb{A}^r)^{r-1}, \text{ and} \\ \mathcal{T} &:= \{(t, g, v_1, \dots, v_{r-1}) \mid tg = gt \text{ and } \forall i : tv_i = v_i\} \subset \mathrm{GL}_r \times Z. \end{aligned}$$

Then Z is a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$, and $\mathcal{T}$ is a closed subscheme of $\mathrm{GL}_r \times Z$. Let $\mathfrak{p}$ be a prime of A which is unramified in K and such that $\mid \kappa_\mathfrak{p} \mid \geq 4$ and such that φ has good reduction at all places of K lying above $\mathfrak{p}$. Take any $\mathfrak{P} \mid \mathfrak{p}$ and $\gamma \in \Gamma_\mathfrak{p}$. Let t be a generator of $T_\mathfrak{P}(\kappa_\mathfrak{p}) = k_n^*$, and let $w_1, \dots, w_{r-1} \in \kappa_\mathfrak{p}^r$ be generators of the space of invariants of $T_\mathfrak{P}$. Then

$$\mathrm{Cent}_{\mathrm{GL}_{r,\kappa_\mathfrak{p}}}(t) = \left(\begin{array}{c|c} T_\mathfrak{P} & 0 \\ \hline 0 & * \end{array}\right)$$

and
$$\mathrm{Stab}_{\mathrm{GL}_{r,\kappa_\mathfrak{p}}}(w_1) \cap \ldots \cap \mathrm{Stab}_{\mathrm{GL}_{r,\kappa_\mathfrak{p}}}(w_{r-1}) = \left(\begin{array}{c|c} T_\mathfrak{P} & 0 \\ \hline * & 1 \end{array}\right),$$
and their intersection is $T_\mathfrak{P}$. Conjugating by γ we deduce that the fiber $\mathcal{T}_z$ of $\mathcal{T}$ above $z = (\gamma t\gamma^{-1}, \gamma w_1, \ldots, \gamma w_{r-1})$ is $\gamma T_\mathfrak{P}\gamma^{-1}$. $\square$

Lemma 3.2.10. *There exists a constant c depending only on r such that for almost all primes $\mathfrak{p}$ of A*
$$[\mathrm{GL}_r(\kappa_\mathfrak{p}) : \Gamma_\mathfrak{p}] \leq c.$$

Proof. Let any prime $\mathfrak{p}$ as in Proposition 3.2.8. Then $\mathrm{GL}_{r,\kappa_\mathfrak{p}}$ is generated by the connected algebraic subgroups $\gamma T_\mathfrak{P}\gamma^{-1}$ for all $\mathfrak{P} \mid \mathfrak{p}$ and $\gamma \in \Gamma_\mathfrak{p}$. By Humphreys [13, Proposition 7.5] it follows that the morphism
$$f_\mathfrak{p} :\ X_\mathfrak{p} := \bigtimes_{i=1}^{m} \gamma_i T_{\mathfrak{P}_i}\gamma_i^{-1} \longrightarrow \mathrm{GL}_{r,\kappa_\mathfrak{p}},\ (t_1, \ldots, t_m) \mapsto t_1 \cdots t_m$$
is dominant for a suitable choice of m and $\mathfrak{P}_i \mid \mathfrak{p}$ and $\gamma_i \in \Gamma_\mathfrak{p}$. Since $\dim(\mathrm{GL}_{r,\kappa_\mathfrak{p}}) = r^2$, we can obtain this with $m = r^2$. In particular, we can assume that m is independent of $\mathfrak{p}$.

By Proposition 3.1.4 the fibers of
$$X_\mathfrak{p}^{r^2} \longrightarrow \mathrm{GL}_{r,\kappa_\mathfrak{p}},\ (x_1, \ldots, x_{r^2}) \mapsto f_\mathfrak{p}(x_1) \cdots f_\mathfrak{p}(x_{r^2})$$
have dimension at most $\dim(X_\mathfrak{p}^{r^2}) - \dim(\mathrm{GL}_{r,\kappa_\mathfrak{p}})$. We replace $X_\mathfrak{p}$ by $X_\mathfrak{p}^{r^2}$ and m by mr^2, which is still independent of $\mathfrak{p}$. Then with $e_\mathfrak{p} := \dim(X_\mathfrak{p})$ all fibers of $f_\mathfrak{p}$ have dimension at most $e_\mathfrak{p} - r^2$.

Let Z and $\mathcal{T} \subset \mathrm{GL}_r \times Z$ be as in Lemma 3.2.9. Then for every $1 \leq i \leq m$ we can choose a point $z_i \in Z(\kappa_\mathfrak{p})$ such that $\mathcal{T}_{z_i} = \gamma_i T_{\mathfrak{P}_i}\gamma_i^{-1}$. Denote the two projections by $\varepsilon :\ \mathcal{T} \to \mathrm{GL}_r$ and $\pi :\ \mathcal{T} \to Z$ and consider the morphism
$$f :\ \mathcal{T}^m \longrightarrow \mathrm{GL}_r \times Z^m,\ (t_1, \ldots, t_m) \mapsto \big(\varepsilon(t_1) \cdot \ldots \cdot \varepsilon(t_m), \pi(t_1), \ldots, \pi(t_m)\big).$$
The construction implies that f induces the morphism $f_\mathfrak{p}$ in the fiber above the point $(z_1, \ldots, z_m) \in Z^m(\kappa_\mathfrak{p})$. Denote the cardinality of $\kappa_\mathfrak{p}$ by $q_\mathfrak{p}$. Since f is independent of $\mathfrak{p}$, by Proposition 3.1.5 there exists a constant c_1 independent of $\mathfrak{p}$ such that for all $g \in \mathrm{GL}_r(\kappa_\mathfrak{p})$ we have
$$|f_\mathfrak{p}^{-1}(g)(\kappa_\mathfrak{p})| \leq c_1 q_\mathfrak{p}^{\dim(f_\mathfrak{p}^{-1}(g))} \leq c_1 q_\mathfrak{p}^{e_\mathfrak{p} - r^2}.$$

On the other hand, we have $|T_{\mathfrak{P}_i}(\kappa_\mathfrak{p})| = q_\mathfrak{p}^{h_{\mathfrak{P}_i}} - 1$, and hence

$$\begin{aligned} |X_\mathfrak{p}(\kappa_\mathfrak{p})| &= \prod_{i=1}^{m} (q_\mathfrak{p}^{h_{\mathfrak{P}_i}} - 1) \ \geq \ \prod_{i=1}^{m} \tfrac{1}{2}\, q_\mathfrak{p}^{h_{\mathfrak{P}_i}} \\ &= 2^{-m} q_\mathfrak{p}^{\sum h_{\mathfrak{P}_i}} \ = \ 2^{-m} q_\mathfrak{p}^{e_\mathfrak{p}}. \end{aligned}$$

Since $f_\mathfrak{p}\big(X_\mathfrak{p}(\kappa_\mathfrak{p})\big) \subset \Gamma_\mathfrak{p}$, we get

$$|\Gamma_\mathfrak{p}| \geq |f_\mathfrak{p}\big(X_\mathfrak{p}(\kappa_\mathfrak{p})\big)| \geq \frac{|X_\mathfrak{p}(\kappa_\mathfrak{p})|}{c_1 q_\mathfrak{p}^{e_\mathfrak{p} - r^2}} \geq \frac{2^{-m} q_\mathfrak{p}^{e_\mathfrak{p}}}{c_1 q_\mathfrak{p}^{e_\mathfrak{p} - r^2}} = \frac{q_\mathfrak{p}^{r^2}}{2^m c_1}.$$

It follows that

$$[\mathrm{GL}_r(\kappa_\mathfrak{p}) : \Gamma_\mathfrak{p}] = \frac{\prod_{i=0}^{r-1}(q_\mathfrak{p}^r - q_\mathfrak{p}^i)}{|\Gamma_\mathfrak{p}|} \leq 2^m c_1 \frac{\prod_{i=0}^{r-1}(q_\mathfrak{p}^r - q_\mathfrak{p}^i)}{q_\mathfrak{p}^{r^2}} \leq 2^m c_1.$$

Thus the lemma holds with $c := 2^m c_1$. □

Lemma 3.2.11. *For almost all primes* $\mathfrak{p}$ *of* A *we have*

$$\Gamma_\mathfrak{p} = \mathrm{GL}_r(\kappa_\mathfrak{p}).$$

Proof. By Lemma 3.2.10 there exists a constant c such that $[\mathrm{GL}_r(\kappa_\mathfrak{p}) : \Gamma_\mathfrak{p}] \leq c$ for almost all primes $\mathfrak{p}$ of A. It is therefore enough if we consider only these primes. We have

$$[\mathrm{SL}_r(\kappa_\mathfrak{p}) : \Gamma_\mathfrak{p} \cap \mathrm{SL}_r(\kappa_\mathfrak{p})] \leq c$$

for these primes. The constant c is independent of $\mathfrak{p}$ and we have $|\,\mathrm{SL}_r(\kappa_\mathfrak{p})| > c!r$ for almost all of these primes. By Proposition 3.1.3 we get $\Gamma_\mathfrak{p} \cap \mathrm{SL}_r(\kappa_\mathfrak{p}) = \mathrm{SL}_r(\kappa_\mathfrak{p})$ for almost all primes $\mathfrak{p}$ of A.

Since $T_\mathfrak{P}(\kappa_\mathfrak{p}) \subset \Gamma_\mathfrak{p}$ and $\det : T_\mathfrak{P}(\kappa_\mathfrak{p}) \cong k_n^* \longrightarrow \kappa_\mathfrak{p}^*$ is the norm map, which is surjective, the determinant map $\det : \Gamma_\mathfrak{p} \longrightarrow \kappa_\mathfrak{p}^*$ is surjective. We thus get $\Gamma_\mathfrak{p} = \mathrm{GL}_r(\kappa_\mathfrak{p})$ for almost all primes $\mathfrak{p}$ of A. □

Lemma 3.2.11 proves Proposition 3.2.1.

3.3 The case $[K : F] < \infty$

Throughout this section we assume that K is a finite extension of F. Let $\mathbb{A}_F^f$ be the ring of finite adeles of F. Consider the adelic representation

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \prod_{\mathfrak{p} \neq \infty} \mathrm{GL}_r(A_\mathfrak{p}) \subset \mathrm{GL}_r(\mathbb{A}_F^f).$$

Denote its image by Γ. The aim of this section is to prove the following result.

Theorem 3.3.1. *Let φ be a Drinfeld A-module over K with generic characteristic. Assume that $\mathrm{End}_{\bar{K}}(\varphi) = A$ and that K is a finite extension of F. Then the image of the adelic representation*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f)$$

is open.

We show that Theorem 3.3.1 follows from Proposition 3.2.1, the surjectivity of the residual representation for almost all primes of A. Using Proposition 3.1.1, we will first prove a result on subgroups of $\mathrm{GL}_r(A_\mathfrak{p})$. For this, we need to consider the congruence filtration defined below. We then have to consider all factors in order to get a description of the adelic image. We prove that for almost primes $\mathfrak{p}$ of A, the factor corresponding to $\mathrm{GL}_r(A_\mathfrak{p})$ is contained in Γ. We again use the images of inertia groups.

Congruence filtration of $\mathrm{GL}_r(A_\mathfrak{p})$. Fix a place $\mathfrak{p}$ of A, and let π be a uniformizer at $\mathfrak{p}$. Define

$$\begin{aligned} G_\mathfrak{p}^0 &:= \mathrm{GL}_r(A_\mathfrak{p}) \times \prod_{\mathfrak{q} \neq \mathfrak{p}} \{1\} \subset \mathrm{GL}_r(\mathbb{A}_F^f), \text{ and} \\ G_\mathfrak{p}^i &:= 1 + \pi^i \,\mathrm{M}_r(A_\mathfrak{p}). \end{aligned}$$

The i-th subquotient of the congruence filtration is given by

$$G_\mathfrak{p}^{[i]} := G_\mathfrak{p}^i / G_\mathfrak{p}^{i+1}.$$

Note that we have an isomorphism

$$v_0 : \mathrm{GL}_r(\kappa_\mathfrak{p}) \longrightarrow G_\mathfrak{p}^{[0]},$$

and for any $i \geq 1$ an isomorphism

$$v_i : \mathrm{M}_r(\kappa_\mathfrak{p}) \longrightarrow G_\mathfrak{p}^{[i]}, \; y \mapsto 1 + \pi^i y \mod G_\mathfrak{p}^{i+1}.$$

For any subgroup H of $\mathrm{GL}_r(A_\mathfrak{p})$, we define

$$\begin{aligned} H^i &:= H \cap G_\mathfrak{p}^i, \text{ and} \\ H^{[i]} &:= H^i/H^{i+1}. \end{aligned}$$

Proposition 3.3.2. *Let H be a closed subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$. Assume that $|\kappa_\mathfrak{p}| \geq 4$, $\det(H) = \mathrm{GL}_1(A_\mathfrak{p})$, $H^{[0]} = \mathrm{GL}_r(\kappa_\mathfrak{p})$, and that $H^{[1]}$ contains a non scalar matrix. Then we have*

$$H = \mathrm{GL}_r(A_\mathfrak{p}).$$

Proof. For $i \geq 1$, the conjugation actions

$$\begin{aligned} \mathrm{GL}_r(\kappa_\mathfrak{p}) \times \mathrm{M}_r(\kappa_\mathfrak{p}) &\longrightarrow \mathrm{M}_r(\kappa_\mathfrak{p}), \ (g,h) \mapsto g^{-1}hg, \text{ and} \\ G_\mathfrak{p}^{[0]} \times G_\mathfrak{p}^{[i]} &\longrightarrow G_\mathfrak{p}^{[i]}, \qquad (g,g') \mapsto g^{-1}g'g \end{aligned}$$

fit into the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_r(\kappa_\mathfrak{p}) \times \mathrm{M}_r(\kappa_\mathfrak{p}) \arrow[r] \arrow[d, "\cong"', "v_0 \times v_i"] & \mathrm{M}_r(\kappa_\mathfrak{p}) \arrow[d, "\cong"', "v_i"] \\ G_\mathfrak{p}^{[0]} \times G_\mathfrak{p}^{[i]} \arrow[r] & G_\mathfrak{p}^{[i]}. \end{tikzcd}$$

Via v_i we can identify $H^{[i]}$ with a subgroup of $\mathrm{M}_r(\kappa_\mathfrak{p})$. Since the conjugation action of H on $H^{[i]}$ factors through $H^{[0]} = \mathrm{GL}_r(\kappa_\mathfrak{p})$, the group $H^{[i]}$ is closed under conjugation by $\mathrm{GL}_r(\kappa_\mathfrak{p})$.

By assumption we know that $H^{[1]}$ contains a non scalar matrix. By Proposition 3.1.1 we therefore get that $H^{[1]}$ contains the matrices with trace 0. Consider the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^1/H^2 \arrow[r] \arrow[d, "\det"] & H/H^2 \arrow[r] \arrow[d] & \mathrm{GL}_r(\kappa_\mathfrak{p}) \arrow[r] \arrow[d, "\det"] & 0 \\ 0 \arrow[r] & (1+\pi A_\mathfrak{p}/\pi^2)^* \arrow[r] & (A_\mathfrak{p}/\mathfrak{p}^2)^* \arrow[r] & \kappa_\mathfrak{p}^* \arrow[r] & 0. \end{tikzcd}$$

The right vertical map is surjective with kernel equal to $\mathrm{SL}_r(\kappa_\mathfrak{p})$. By assumption, the middle vertical map is surjective as well. By the snake lemma, we thus get a surjective homomorphism from $\mathrm{SL}_r(\kappa_\mathfrak{p})$ onto the cokernel of the left vertical map. This cokernel is abelian. On the other hand, since $|\kappa_\mathfrak{p}| \geq 4$, the group $\mathrm{SL}_r(\kappa_\mathfrak{p})$ is perfect, and thus has no nontrivial such quotient. This implies that the determinant of H^1/H^2 is surjective. In other words the composite trace map

$$H^{[1]} \hookrightarrow \mathrm{M}_r(\kappa_\mathfrak{p}) \xrightarrow{tr} \kappa_\mathfrak{p}$$

is surjective. We thus get

$$H^{[1]} = \mathrm{M}_r(\kappa_\mathfrak{p}) = G_\mathfrak{p}^{[1]}.$$

Since $\det(H) = \mathrm{GL}_1(A_\mathfrak{p})$, in order to prove the proposition, it is enough to show that the commutator subgroup H' of H is $\mathrm{SL}_r(A_\mathfrak{p})$. Therefore it suffices to show that $H'^{[i]} = \mathrm{SL}_r(A_\mathfrak{p})^{[i]} = \mathfrak{sl}_r(\kappa_\mathfrak{p})$ for all $i \geq 0$.

For $i = 0$ this follows from $H'^{[0]} = (H^{[0]})' = \mathrm{GL}_r(\kappa_\mathfrak{p})' = \mathfrak{sl}_r(\kappa_\mathfrak{p})$. For $i = 1$, consider the commutator map

$$\begin{array}{rcl} H \times H^1 & \longrightarrow & H^1, \\ (g,h) & \mapsto & ghg^{-1}h^{-1}. \end{array}$$

Under v_0 and v_1, it induces the map

$$\begin{array}{rcl} H^{[0]} \times H^{[1]} & \longrightarrow & H'^{[1]}, \\ (g,h) & \mapsto & ghg^{-1} - h. \end{array}$$

Since $H^{[0]} = \mathrm{GL}_r(\kappa_\mathfrak{p})$ and $H^{[1]} = \mathrm{M}_r(\kappa_\mathfrak{p})$, we get

$$H'^{[1]} = \mathfrak{sl}_r(\kappa_\mathfrak{p}).$$

Assume now that $H'^{[i]} = \mathfrak{sl}_r(\kappa_\mathfrak{p})$ for some $i \geq 1$. The maps

$$\begin{array}{rcll} \mathrm{M}_r(\kappa_\mathfrak{p}) \times \mathrm{M}_r(\kappa_\mathfrak{p}) & \longrightarrow & \mathrm{M}_r(\kappa_\mathfrak{p}), & (h,h') \mapsto [h,h'] := hh' - h'h, \text{ and} \\ G_\mathfrak{p}^{[1]} \times G_\mathfrak{p}^{[i]} & \longrightarrow & G_\mathfrak{p}^{[i+1]}, & (g,g') \mapsto gg'g^{-1}g'^{-1} \end{array}$$

fit into the commutative diagram

$$\begin{tikzcd} \mathrm{M}_r(\kappa_\mathfrak{p}) \times \mathrm{M}_r(\kappa_\mathfrak{p}) \arrow[r] \arrow[d, "v_1 \times v_i"] & \mathrm{M}_r(\kappa_\mathfrak{p}) \arrow[d, "v_{i+1}"] \\ G_\mathfrak{p}^{[1]} \times G_\mathfrak{p}^{[i]} \arrow[r] & G_\mathfrak{p}^{[i+1]}. \end{tikzcd}$$

By Pink [19, Proposition 1.2], the group generated by $[\mathrm{M}_r(\kappa_\mathfrak{p}), \mathfrak{sl}_r(\kappa_\mathfrak{p})]$ is all of $\mathfrak{sl}_r(\kappa_\mathfrak{p})$. We thus get $H'^{[i+1]} = \mathfrak{sl}_r(\kappa_\mathfrak{p})$. □

Let Λ be the set of primes $\mathfrak{p}$ of A which satisfy any of the conditions below:
- $\mathfrak{p}$ lies below a place of K where φ has bad reduction,
- $|\kappa_\mathfrak{p}| < 4$,
- $\mathfrak{p}$ is ramified in K,
- $\overline{\rho_\mathfrak{p}}$ is not surjective,
- $\det(\Gamma)$ does not contain $\mathrm{GL}_1(A_\mathfrak{p}) \times \prod_{\mathfrak{p}' \neq \mathfrak{p}} \{1\}$.

The first three of these subsets are finite. By Proposition 3.2.1, the fourth one is finite. By Goss [11, Theorem 7.7.1] or Hayes [12], the last one is finite as well. Therefore Λ is a finite set.

Proposition 3.3.3. *For all $\mathfrak{p} \notin \Lambda$ and all places $\mathfrak{P}$ of K lying above $\mathfrak{p}$, the ramification index of the extension $K_{\mathfrak{P}}\big(\varphi[\mathfrak{p}^2](K^{sep})\big)/K_{\mathfrak{P}}\big(\varphi[\mathfrak{p}](K^{sep})\big)$ is at least $|\kappa_{\mathfrak{p}}|^{h_{\mathfrak{P}}}$.*

Proof. Denote by $v_{\mathfrak{P}}$ the normalized valuation of $K_{\mathfrak{P}}$ and by $q_{\mathfrak{p}}$ the cardinality of $\kappa_{\mathfrak{p}}$. Take any element $s \in \varphi[\mathfrak{p}](K^{sep})$ with $v_{\mathfrak{P}}(s) > 0$. Let $a \in A$ be a function with a zero of order 1 at $\mathfrak{p}$. We have shown in the proof of Proposition 1.2.3 that $v_{\mathfrak{P}}(s) = \alpha := 1/(q_{\mathfrak{p}}^{h_{\mathfrak{P}}} - 1)$ and that

$$\varphi[a]^\circ = \varphi[\mathfrak{p}]^\circ$$

as group schemes over $\operatorname{Spec} \mathcal{O}_{K_{\mathfrak{P}}}$. The polynomial φ_a is given by

$$\varphi_a = \sum_{i=0}^{r \deg(\mathfrak{p})} \varphi_{a,i} \tau^i.$$

For the valuations of the coefficients, we get, with $i_0 := h_{\mathfrak{P}} \deg(\mathfrak{p})$,

$$\begin{aligned} v_{\mathfrak{P}}(\varphi_{a,0}) &= 1, \\ v_{\mathfrak{P}}(\varphi_{a,i}) &\geq 1 \text{ for } 0 < i < i_0, \\ v_{\mathfrak{P}}(\varphi_{a,i_0}) &= 0, \text{ and} \\ v_{\mathfrak{P}}(\varphi_{a,i}) &\geq 0 \text{ for } i > i_0. \end{aligned}$$

This implies that $(0, \alpha)$ and $(q_{\mathfrak{p}}^{h_{\mathfrak{P}}}, 0)$ are vertices of the Newton polygon of the polynomial $\varphi_a(x) - s$. We thus can fix a zero s' of this polynomial with valuation

$$v_{\mathfrak{P}}(s') = \alpha / q_{\mathfrak{p}}^{h_{\mathfrak{P}}} > 0.$$

Since

$$\varphi[a^2]^\circ = \varphi[a\mathfrak{p}]^\circ = \varphi[\mathfrak{p}^2]^\circ,$$

we get $s' \in \varphi[\mathfrak{p}^2](K^{sep}) \backslash \varphi[\mathfrak{p}](K^{sep})$. Moreover, the ramification index of $K_{\mathfrak{P}}(s, s')/K_{\mathfrak{P}}(s)$ is $q_{\mathfrak{p}}^{h_{\mathfrak{P}}}$.

Abbreviate

$$\begin{array}{rcl}
L & := & K_{\mathfrak{P}}^{nr}, \\
L_1 & := & L\big(\varphi[\mathfrak{p}](K^{sep})\big), \\
L_2 & := & L\big(\varphi[\mathfrak{p}^2](K^{sep})\big), \\
L_1^\circ & := & L\big(\varphi[\mathfrak{p}]^\circ(K^{sep})\big), \\
L_2^\circ & := & L\big(\varphi[\mathfrak{p}^2]^\circ(K^{sep})\big), \\
\tilde{L}_1 & := & L(s), \text{ and} \\
\tilde{L}_2 & := & L(s').
\end{array}$$

The following picture illustrates the relative positions of these fields to each other

$$\begin{tikzcd}
& & L_2 \arrow[dd, dash] \\
& L_2^\circ \arrow[ur, dash] \arrow[dd, dash] & \\
\tilde{L}_2 \arrow[ur, dash] \arrow[dd, dash] & & L_1 \\
& L_1^\circ \arrow[ur, dash] & \\
\tilde{L}_1 \arrow[ur, dash] \arrow[d, dash] & & \\
L. & &
\end{tikzcd}$$

The extension L_1°/L is Galois. Via the action of A via φ, the element s' generates L_1°, and we get

$$\tilde{L}_1 = L_1^\circ.$$

Therefore the extension $\tilde{L}_1/L$ is Galois as well. For any conjugate $\sigma(s')$ of s' we have

$$\varphi_a\big(\sigma(s') - s'\big) = \sigma\big(\varphi_a(s')\big) - \varphi_a(s') = 0,$$

and hence $\sigma(s') - s' \in \varphi[\mathfrak{p}]^\circ(K^{sep})$. We thus get $\sigma(s') - s' \in \tilde{L}_1 = L_1^\circ$. Therefore the extension $\tilde{L}_2/L_1^\circ$ is Galois. Since $[\tilde{L}_2 : \tilde{L}_1] = q_{\mathfrak{p}}^{h_{\mathfrak{P}}}$ and $\tilde{L}_2/\tilde{L}_1$ is totally ramified, in order to prove the proposition, it is enough to show that the extensions L_2°/L_1° and L_1/L_1° are linearly disjoint, which is asserted by the following Lemma. $\square$

Lemma 3.3.4. *The extensions L_2°/L_1° and L_1/L_1° are linearly disjoint.*

Proof. Consider the bilinear map

$$\begin{array}{rcl}
\mathrm{Gal}(L_2^\circ/L_1^\circ) \times \varphi[\mathfrak{p}^2]^\circ(K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep}) & \longrightarrow & \varphi[\mathfrak{p}]^\circ(K^{sep}), \\
(\sigma, t) & \mapsto & (\sigma - 1)t.
\end{array}$$

It induces an injective group homomorphism

$$\mathrm{Gal}(L_2^\circ/L_1^\circ) \hookrightarrow \mathrm{Hom}_A\left(\varphi[\mathfrak{p}^2]^\circ(K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep}), \varphi[\mathfrak{p}]^\circ(K^{sep})\right).$$

Similarly, we have an injective group homomorphism

$$\mathrm{Gal}(L_1/L_1^\circ) \hookrightarrow \mathrm{Hom}_A\left(\varphi[\mathfrak{p}](K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep}), \varphi[\mathfrak{p}]^\circ(K^{sep})\right).$$

These two homomorphisms are equivariant under $\mathrm{Gal}(L_2/L)$, which acts through the tame inertia group $\mathrm{I}_{\mathfrak{P}}^t$. The linear disjointness of the field extensions will follow as soon as we know that the two groups of homomorphisms have no nontrivial isomorphic subquotients as $\mathrm{I}_{\mathfrak{P}}^t$-representations.

We give an explicit description of the action of $\mathrm{I}_{\mathfrak{P}}^t$ on the two groups of homomorphisms. Denote by k_n the extension of $\kappa_{\mathfrak{p}}$ of degree $q_{\mathfrak{p}}^{h_{\mathfrak{P}}}$ inside a fixed algebraic closure. The fundamental character ζ_{k_n} maps $\mathrm{I}_{\mathfrak{P}}^t$ surjectively to k_n^*. Both $\varphi[\mathfrak{p}]^\circ(K^{sep})$ and $\varphi[\mathfrak{p}^2]^\circ(K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep})$ are k_n vector spaces of dimension 1, and the group $\mathrm{I}_{\mathfrak{P}}^t$ acts on them through ζ_{k_n} and scalar multiplication by k_n^*. The quotient $\varphi[\mathfrak{p}](K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep})$ is equal to $\varphi[\mathfrak{p}]^{et}(K^{sep})$, and thus $\mathrm{I}_{\mathfrak{P}}^t$ acts trivially on it. Therefore we get the following identifications of $\mathrm{I}_{\mathfrak{P}}^t$-representations

$$\begin{aligned}
\mathrm{Hom}_A\left(\varphi[\mathfrak{p}^2]^\circ(K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep}), \varphi[\mathfrak{p}]^\circ(K^{sep})\right) &\cong k_n \otimes_{\kappa_{\mathfrak{p}}} k_n^\vee, \text{ and} \\
\mathrm{Hom}_A\left(\varphi[\mathfrak{p}](K^{sep})/\varphi[\mathfrak{p}]^\circ(K^{sep}), \varphi[\mathfrak{p}]^\circ(K^{sep})\right) &\cong \bigoplus_{i=1}^{r-h_{\mathfrak{P}}} k_n,
\end{aligned}$$

where $k_n^\vee$ denotes the dual of k_n. Therefore we must show that $k_n \otimes_{\kappa_{\mathfrak{p}}} k_n^\vee$ and k_n have no nontrivial isomorphic subquotients as representations of k_n^* over $\mathbb{F}_p$, where p denotes the characteristic of $\kappa_{\mathfrak{p}}$.

Denote by q the cardinality of $\kappa_{\mathfrak{p}}$. On k_n, the action of $t \in k_n^*$ is given by multiplication by t. Thus the representation $k_n \otimes_{\mathbb{F}_p} \bar{k}_n$ over $\bar{k}_n$ consists of the irreducible characters

$$k_n^* \longrightarrow \bar{k}_n^*,\ t \mapsto t^{p^m}$$

for all $m \in \mathbb{Z}$. We can identify $k_n \otimes_{\kappa_{\mathfrak{p}}} k_n^\vee$ as k_n^*-representation with $\oplus_{i=1}^{h_{\mathfrak{P}}} k_n$, where the action of $t \in k_n^*$ on the i-th summand is given by multiplication by t^{1-q^i}. Thus the representation $k_n \otimes_{\kappa_{\mathfrak{p}}} k_n^\vee \otimes_{\mathbb{F}_p} \bar{k}_n$ over $\bar{k}_n$ consists of the irreducible characters

$$k_n^* \longrightarrow \bar{k}_n^*,\ t \mapsto t^{(1-q^i)p^j}$$

for all $j \in \mathbb{Z}$.

We must show that no two such characters of the respective kinds are equal. They are equal if and only if

$$t^{(1-q^i)p^j} = t^{p^m}, \text{ for all } t \in k_n^*.$$

This is equivalent to

$$(1-q^i)p^j - p^m \equiv 0 \mod q^{h_\mathfrak{P}} - 1.$$

Since p is invertible modulo $q^{h_\mathfrak{P}} - 1$, the congruence relation is equivalent to

$$q^i - 1 \equiv -p^{m-j} \mod q^{h_\mathfrak{P}} - 1.$$

Since $q-1$ divides both $q^i - 1$ and $q^{h_\mathfrak{P}} - 1$, it also divides their greatest common divisor. But $q - 1 = |\kappa_\mathfrak{p}| - 1 > 1$ by the choice of Λ, and $q-1$ is relatively prime to p, which implies that the congruence relation cannot hold. □

Lemma 3.3.5. *For all $\mathfrak{p} \notin \Lambda$ we have*

$$\mathrm{GL}_r(A_\mathfrak{p}) \times \prod_{\mathfrak{p}' \neq \mathfrak{p}} \{1\} \subset \Gamma.$$

Proof. Fix a prime $\mathfrak{p} \notin \Lambda$. Identify $G_\mathfrak{p} := \mathrm{GL}_r(A_\mathfrak{p})$ with $\mathrm{GL}_r(A_\mathfrak{p}) \times \prod_{\mathfrak{p}' \neq \mathfrak{p}} \{1\}$, and define

$$H_\mathfrak{p} := \Gamma \cap G_\mathfrak{p}.$$

We have to show that

$$H_\mathfrak{p} = G_\mathfrak{p}.$$

If $r = 1$, we have $\mathrm{GL}_1(A_\mathfrak{p}) \times \prod_{\mathfrak{p}' \neq \mathfrak{p}} \{1\} \subset \Gamma$ for all $\mathfrak{p} \notin \Lambda$ by the choice of Λ, and the result follows.

For $r \geq 2$, we need to verify the assumptions of Proposition 3.3.2. The choice of Λ implies $|\kappa_\mathfrak{p}| \geq 4$. The Tate module of the maximal exterior power of φ is isomorphic to the Tate module of a Drinfeld module of rank 1. This implies that $\det(H_\mathfrak{p}) = \mathrm{GL}_1(A_\mathfrak{p})$.

Next, we need to show that $H_\mathfrak{p}^{[0]} = \mathrm{GL}_r(\kappa_\mathfrak{p})$. Let $\mathfrak{P}$ be a place of K lying above $\mathfrak{p}$. The image of the inertia group at $\mathfrak{P}$ under the adelic representation is trivial in any factor except the one at $\mathfrak{p}$. Hence it lies in $H_\mathfrak{p}$. By the choice of Λ, the Drinfeld module φ has good reduction at $\mathfrak{P}$. Let $h_\mathfrak{P}$ be the height of the reduced Drinfeld module. The connected-étale decomposition of $\varphi[\mathfrak{p}]$ gives an exact sequence

$$0 \longrightarrow \varphi[\mathfrak{p}]^0 \longrightarrow \varphi[\mathfrak{p}] \longrightarrow \varphi[\mathfrak{p}]^{et} \longrightarrow 0.$$

The set $\varphi[\mathfrak{p}]^0(K^{sep})$ is an $h_{\mathfrak{P}}$-dimensional $\kappa_{\mathfrak{p}}$ vector space. The inertia group at $\mathfrak{P}$ acts trivially on $\varphi[\mathfrak{p}]^{et}(K^{sep})$. By Proposition 1.2.3 we know that the wild inertia group at $\mathfrak{P}$ acts trivially on $\varphi[\mathfrak{p}]^0(K^{sep})$ and that the action of $\mathrm{I}_{\mathfrak{P}}^t$ is given by the fundamental character. Thus, if $h_{\mathfrak{P}} < r$, any diagonal matrix $h' \in \mathrm{GL}_{h_{\mathfrak{P}}}(\kappa_{\mathfrak{p}})$ different from the identity matrix gives a non scalar element

$$h = \left(\begin{array}{c|c} h' & * \\ \hline 0 & 1 \end{array}\right) \in H_{\mathfrak{p}}^{[0]}.$$

If $h_{\mathfrak{P}} = r$, any $h \in k_{q_{\mathfrak{p}}^r}^* \setminus \kappa_{\mathfrak{p}}^*$ is a non scalar element of $H_{\mathfrak{p}}^{[0]}$.

On the other hand, the group Γ acts through conjugation on $H_{\mathfrak{p}}$. The projection of Γ on the factor $\mathrm{GL}_r(\kappa_{\mathfrak{p}})$ is surjective by Proposition 3.2.1, which implies that the group $H_{\mathfrak{p}}^{[0]}$ is closed under conjugation by $\mathrm{GL}_r(\kappa_{\mathfrak{p}})$. Since it contains the non scalar element h, we get $\mathrm{SL}_r(\kappa_{\mathfrak{p}}) \subset H_{\mathfrak{p}}^{[0]}$ by Proposition 3.1.2. From $\det\left(\overline{\rho_{\mathfrak{p}}}(\mathrm{I}_{\mathfrak{P}}^t)\right) = {\kappa_{\mathfrak{p}}}^*$, we get $\det(H_{\mathfrak{p}}^{[0]}) = {\kappa_{\mathfrak{p}}}^*$, which implies $H_{\mathfrak{p}}^{[0]} = \mathrm{GL}_r(\kappa_{\mathfrak{p}}) = G_{\mathfrak{p}}^{[0]}$.

In order to apply Proposition 3.3.2, it remains to show that $H_{\mathfrak{p}}^{[1]}$ contains a non scalar matrix. We will find such a matrix in the subgroup

$$\left(\rho_{\mathfrak{p}}(\mathrm{I}_{\mathfrak{P}})\right)^{[1]} \subset H_{\mathfrak{p}}^{[1]} \subset \mathrm{M}_r(\kappa_{\mathfrak{p}}).$$

By Proposition 3.3.3, it has at least $|\kappa_{\mathfrak{p}}|^{h_{\mathfrak{P}}}$ elements. If $h_{\mathfrak{P}} > 1$, it thus contains a non scalar matrix. If $h_{\mathfrak{P}} = 1 < r$ then, for an appropriate basis, the subgroup consists of block matrices of the form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & 0 \end{array}\right),$$

where the upper left entry lies in $\kappa_{\mathfrak{p}}^*$. All these matrices are non scalar.

Thus, $H_{\mathfrak{p}}^{[1]}$ contains a non scalar matrix in any case. Now we can apply Proposition 3.3.2 and get $H_{\mathfrak{p}} = G_{\mathfrak{p}}$. $\square$

We can now prove the adelic openness in generic characteristic for the case where K is a finite extension of F.

Proof of Theorem 3.3.1. Lemma 3.3.5 implies that

$$\prod_{\mathfrak{p} \notin \Lambda} \mathrm{GL}_r(A_{\mathfrak{p}}) \subset \Gamma.$$

Therefore Γ is the inverse image of its image under the projection

$$\prod_{\mathfrak{p}} \mathrm{GL}_r(A_{\mathfrak{p}}) \longrightarrow \prod_{\mathfrak{p} \in \Lambda} \mathrm{GL}_r(A_{\mathfrak{p}}).$$

By Theorem 1.1.4, the image of

$$\mathrm{G}_K \longrightarrow \prod_{\mathfrak{p}\in\Lambda} \mathrm{GL}_r(F_{\mathfrak{p}})$$

is open. Therefore we get that Γ is open in $\prod_{\mathfrak{p}} \mathrm{GL}_r(A_{\mathfrak{p}})$. $\square$

3.4 Specialisation

In this section, we prove the following result which will be used in the next section. We use the same notation as in Section 2.3.

Proposition 3.4.1. *Let φ be a Drinfeld A-module over K of generic characteristic. Assume that* $\mathrm{End}_{\bar{K}}(\varphi) = A$*. Then there exists a point $x \in X$ such that k_x is a finite extension of F and*

$$\mathrm{End}_{\bar{k}_x}(\varphi_x) = A.$$

To prove Proposition 3.4.1, we first need to prove some other results. Fix a prime $\mathfrak{p}$ of A. Define

$$\begin{aligned} C_n &:= \{\gamma \in \mathrm{GL}_r(A_\mathfrak{p}) |\ \gamma \equiv 1 \mod \mathfrak{p}^n\}, \text{ and} \\ C_n^1 &:= C_n \cap \mathrm{SL}_r(A_\mathfrak{p}). \end{aligned}$$

For any two natural numbers n and l with $n \geq l$, we have the following natural group isomorphism

$$\begin{aligned} \log_{n,l} : C_n/C_{n+l} &\longrightarrow \mathfrak{gl}_r(\mathfrak{p}^n/\mathfrak{p}^{n+l}), \\ 1 + \mathfrak{p}^n M \mod \mathfrak{p}^{n+l} &\mapsto \mathfrak{p}^n M \mod \mathfrak{p}^{n+l}. \end{aligned}$$

As explained in Pink [18], this can be considered as a logarithm truncated after the first order term. In the same way, the inverse isomorphism is an exponential map truncated after the first order term. We call it $\exp_{n,l}$.

Lemma 3.4.2. *For any natural numbers $n, n' \geq l \geq 1$, the following properties hold.*

(i) The commutator $C_n \times C_n \longrightarrow C_n$, $(a, b) \mapsto aba^{-1}b^{-1}$ induces a bimultiplicative map

$$\begin{aligned} \{\ ,\ \}^- : C_n/C_{n+l} \quad \times \quad & C_{n'}/C_{n'+l} \longrightarrow C_{n+n'}/C_{n+n'+l}, \\ (\bar{a}, \bar{b}) \quad \mapsto \quad & \overline{aba^{-1}b^{-1}}. \end{aligned}$$

(ii) The Lie bracket $\mathfrak{gl}_r(\mathfrak{p}^n A_\mathfrak{p}) \times \mathfrak{gl}_r(\mathfrak{p}^n A_\mathfrak{p}) \longrightarrow \mathfrak{gl}_r(\mathfrak{p}^n A_\mathfrak{p})$ induces a bilinear map

$$\begin{aligned} [\ ,\ ]^- : \mathfrak{gl}_r(\mathfrak{p}^n A_\mathfrak{p})/\mathfrak{gl}_r(\mathfrak{p}^n/\mathfrak{p}^{n+l}) \times \mathfrak{gl}_r(\mathfrak{p}^{n'}/\mathfrak{p}^{n'+l}) &\longrightarrow \mathfrak{gl}_r(\mathfrak{p}^{n+n'}/\mathfrak{p}^{n+n'+l}), \\ (\bar{u}, \bar{v}) &\mapsto \overline{uv - vu}. \end{aligned}$$

(iii) We have

$$\log_{n+n',l}\left(\{\bar{a}, \bar{b}\}^-\right) = [\log_{n,l}(\bar{a}), \log_{n',l}(\bar{b})]^-.$$

Proof. Let $a = 1 + u \in C_n$, and let $b = 1 + v \in C_{n'}$. Then, their inverses are given by the geometric series

$$\begin{aligned} a^{-1} &= 1 - u + u^2 - +..., \text{ and} \\ b^{-1} &= 1 - v + v^2 - +.... \end{aligned}$$

We have

$$aba^{-1} = aa^{-1} + ava^{-1} = (1 + v) + (uv - vu) + T,$$

where T is an expression of degree at least 2 in u and degree at least 1 in v. We thus have $T \in \mathfrak{p}^{2n+n'} A_\mathfrak{p} \subset \mathfrak{p}^{n+n'+l} A_\mathfrak{p}$. We get

$$\begin{aligned} aba^{-1}b^{-1} &= (1 + v)(1 + v)^{-1} + (uv - vu)(1 + v)^{-1} + T(1 + v)^{-1} \\ &= 1 + (uv - vu) + T' + T(1 + v)^{-1}, \end{aligned}$$

where T' is an expression of degree at least 2 in v and degree at least 1 in u. We then get

$$aba^{-1}b^{-1} \equiv 1 + (uv - vu) \mod \mathfrak{p}^{n+n'+l}.$$

Assertion (i) follows.

Assertion (ii) is obvious. Assertion (iii) follows by the above computation, since we have

$$aba^{-1}b^{-1} \equiv 1 + (uv - vu) = 1 + [u, v] \mod \mathfrak{p}^{n+n'+l}.$$

□

Next consider a closed subgroup Δ of $\mathrm{GL}_r(A_\mathfrak{p})$. Define

$$\begin{aligned} \Delta_n &:= \Delta \cap C_n, \text{ and} \\ \Delta_n^1 &:= \Delta \cap C_n^1. \end{aligned}$$

Lemma 3.4.3. *Let n, n', and l be natural numbers with $n \geq l, n' \geq l$. Assume that $\Delta_n/\Delta_{n+l} = C_n/C_{n+l}$ and that $C^1_{n'}/C^1_{n'+l} \subset \Delta_{n'}/\Delta_{n'+l}$. Then we have*

$$\Delta^1_{n+n'}/\Delta^1_{n+n'+l} = C^1_{n+n'}/C^1_{n+n'+l}.$$

Proof. By Lemma 3.4.2, we have the following commutative diagram

$$\begin{tikzcd} C_n/C_{n+l} \times C^1_{n'}/C^1_{n'+l} \arrow[r, "\{\ ,\ \}^-"] \arrow[d] & C^1_{n+n'}/C^1_{n+n'+l} \arrow[d] \\ \mathfrak{gl}_r(\mathfrak{p}^n/\mathfrak{p}^{n+l}) \times \mathfrak{sl}_r(\mathfrak{p}^{n'}/\mathfrak{p}^{n'+l}) \arrow[r] & \mathfrak{sl}_r(\mathfrak{p}^{n+n'}/\mathfrak{p}^{n+n'+l}), \end{tikzcd}$$

where the lower horizontal arrow is given by $[\ ,\ ]^-$. As explained above, the vertical arrows are isomorphisms.

By Pink [19, Proposition 1.2], we have

$$[\mathfrak{gl}_r, \mathfrak{sl}_r] = \mathfrak{sl}_r,$$

which means that the subgroup of $\mathfrak{sl}_r$ generated by $[\mathfrak{gl}_r, \mathfrak{sl}_r]$ is equal to $\mathfrak{sl}_r$. Therefore we get that the group generated by

$$\exp_{n+n'+l}\left([\log_{n+l} C_n/C_{n+l}, \log_{n'+l} C^1_{n'}/C^1_{n'+l}]\right)$$

is equal to $C^1_{n+n'}/C^1_{n+n'+l}$.

By assumption, we have that $\Delta_n/\Delta_{n+l} = C_n/C_{n+l}$ and that $C^1_{n'}/C^1_{n'+l} \subset \Delta_{n'}/\Delta_{n'+l}$. Since Δ_m is a group for any natural number m and thus closed under the multiplicative commutator map, we get $\Delta^1_{n+n'}/\Delta^1_{n+n'+l} = C^1_{n+n'}/C^1_{n+n'+l}$. $\square$

Proposition 3.4.4. *Assume that there exists a natural number $n_0 > 0$ such that $\Delta_{n_0}/\Delta_{2n_0} = C_{n_0}/C_{2n_0}$. Then we have*

$$C^1_{n_0} \subset \Delta_{n_0}.$$

Proof. We have to show that $C^1_{n_0} = \Delta^1_{n_0}$. Since Δ is a closed subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$, it is enough to show that $\Delta_{in_0}/\Delta_{(i+1)n_0} = C_{in_0}/C_{(i+1)n_0}$ for every $i \geq 1$, because we then can pass to the limit.

We do induction on i. By assumption we have $\Delta_{n_0}/\Delta_{2n_0} = C_{n_0}/C_{2n_0}$, and thus $\Delta^1_{n_0}/\Delta^1_{2n_0} = C^1_{n_0}/C^1_{2n_0}$, proving the desired equality in the case $i = 1$. Assume that the equality holds for all $i \leq i_0$. By Lemma 3.4.3 it then also holds for $i = i_0 + 1$, which proves the induction step. $\square$

Proof of Proposition 3.4.1. If K is of transcendence degree 1 over $\mathbb{F}_q$, there is nothing to prove.

Assume that the transcendence degree of K over $\mathbb{F}_q$ is at least 2. Denote by Γ the image of G_K in the representation on the Tate module of φ at $\mathfrak{p}$. Since Γ is an open subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$ by Theorem 1.1.4, there exists a natural number $n_0 > 0$ such that $C_{n_0} \subset \Gamma$. Let K' be the finite Galois extension of K such that $\mathrm{Gal}(K'/K) = \Gamma/C_{2n_0}$, and let X' be the normalization of X in K'. Denote by π the morphism $X' \to X$.

By Pink [20, Lemma 1.6], there exists a point $x \in X$ such that k_x is a finite extension of F and $\pi^{-1}(x) \subset X'$ is irreducible. Denote by Δ the image of G_{k_x} in the representation on the Tate module of φ_x at $\mathfrak{p}$. This is a closed subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$. Since φ_x does not have characteristic $\mathfrak{p}$, the specialisation map identifies the Tate modules of φ and φ_x, turning Δ into a subgroup of Γ. Since $\pi^{-1}(x)$ is irreducible, we get $\mathrm{Gal}(k_{\pi^{-1}(x)}/k_x) \cong \mathrm{Gal}(K'/K)$, and so $\Delta C_{2n_0} = \Gamma$, and therefore $\Delta_{n_0} C_{2n_0} = C_{n_0}$. Therefore we have

$$\Delta_{n_0}/\Delta_{2n_0} = C_{n_0}/C_{2n_0}.$$

Proposition 3.4.4 implies

$$C^1_{n_0} \subset \Delta_{n_0}.$$

This shows that Δ contains an open subgroup of $\mathrm{SL}_r(A_\mathfrak{p})$. By Goss [11, Theorem 7.7.1], the image of Δ under the determinant is open in $\mathrm{GL}_1(A_\mathfrak{p})$. This then implies that Δ is an open subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$.

All endomorphisms of the Drinfeld module φ_x are defined over some finite separable extension k'_x of k_x. This extension corresponds to an open subgroup of $\Delta_\mathfrak{p}$, which by the above is again open in $\mathrm{GL}_r(A_\mathfrak{p})$. By the Tate conjecture, it follows that $\mathrm{End}_{\bar{k}_x}(\varphi_x) = \mathrm{End}_{k'_x}(\varphi_x) = A$. □

3.5 The general case

We now prove the adelic openness in generic characteristic in the general case, i.e., where K is a finitely generated extension of F.

Theorem 3.5.1 (Adelic openness in generic characteristic)**.** *Let φ be a Drinfeld A-module over K of generic characteristic. Assume that* $\mathrm{End}_{\bar{K}}(\varphi) = A$. *Then the image of the adelic representation*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \mathrm{GL}_r(\mathbb{A}_F^f)$$

is open.

We will prove Theorem 3.5.1 by reducing it to the case of a finite extension of F.

Proof. If K is of transcendence degree 1 over $\mathbb{F}_q$, the result follows from Theorem 3.3.1.

Assume that the transcendence degree of K over $\mathbb{F}_q$ is at least 2. Let x be a point of X as in Proposition 3.4.1. We can apply Theorem 3.3.1 to the Drinfeld module φ_x to get that the image of the adelic representation associated to φ_x is open in $\mathrm{GL}_r(\mathbb{A}_F^f)$. Since the Tate modules of φ and φ_x are isomorphic, this image is a subgroup of the image of the adelic representation associated to φ. Thus the latter is open in $\mathrm{GL}_r(\mathbb{A}_F^f)$ as well. □

If the endomorphism ring of φ is bigger than A, we can no longer expect the image of the adelic representation to be open in $\mathrm{GL}_r(\mathbb{A}_F^f)$. Since the endomorphism ring of φ acts on the Tate module and commutes with the $\mathfrak{p}$-adic representation, the image of G_K lies in the centraliser $\mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}\big(\mathrm{End}_K(\varphi)\big)$. We get that the image of the adelic representation is open in the product of the centralisers.

Theorem 3.5.2. *Let φ be a Drinfeld A-module over K of generic characteristic. Assume that* $\mathrm{End}_{\bar{K}}(\varphi) = \mathrm{End}_K(\varphi)$. *Then the image of the homomorphism*

$$\rho_{\mathrm{ad}} : \mathrm{G}_K \longrightarrow \prod_{\mathfrak{p}} \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}\big(\mathrm{End}_K(\varphi)\big)$$

is open.

Proof. The result can be deduced from Theorem 3.5.1. The argument is exactly the same as in Pink [20]. □

Bibliography

[1] H. Bass, J. Milnor, J.-P. Serre: *Solution of the congruence subgroup problem for* $\mathrm{SL}_n(n \geq 3)$ *and* $\mathrm{Sp}_{2n}(n \geq 2)$, Publications mathématiques de l' I.H.É.S., tome 33, 1967, 59-137

[2] N. Bourbaki: *Algèbre, Chapitre 8: Modules et anneaux semi-simples, Eléments de mathématiques XXIII*, Hermann, 1958

[3] G. Cornell, J. H. Silverman: *Arithmetic Geometry*, Springer, New York etc., 1986

[4] C. W. Curtis, I. Reiner: *Representation theroy of finite groups and associative algebras*, John Wiley & Sons, London, 1962

[5] P. Deligne, D. Husemöller: *Survey of Drinfeld modules*, Contemp. Math. 67, 1987, 25-91

[6] V. G. Drinfeld: *Elliptic modules* (Russian), Mat. Sbornik 94, 1974, 594-627, translated in Math. USSR Sbornik 23, 1974, 561-592

[7] V. G. Drinfeld: *Elliptic modules II* (Russian), Mat. Sbornik 102, 1977, 182-194, translated in Math. USSR Sbornik 31, 1977, 159-170

[8] F. Gardeyn: *Image of Galois on the torsion of a rank* 2 *Drinfeld module*, unpublished notes, 1997, 17 p.

[9] F. Gardeyn: *Openness of the Galois image for* τ*-modules of dimension* 1, Journal of Number Theory 102, 2003, 306-338

[10] F. Gardeyn: t*-Motives and Galois Representations*, Diss. Universiteit Gent, 2001

[11] D. Goss: *Basic Structures of Function Field Arithmetic*, Springer Verlag, Berlin etc., 1998

[12] D. R. Hayes: *Explicit calss field theory in global function fields*, Studies in Algebra and Number Theory, Adv. Math. Suppl. Stud. 6, 1979, 173-217

[13] J. E. Humphreys: *Linear Algebraic Groups*, Springer, New York etc., 1981

[14] B. Huppert: *Endliche Gruppen I*, Springer Verlag, Berlin etc., 1967

[15] M. Larsen, R. Pink: *Finite Subgroups of Algebraic Groups*, Preprint (61 p.), September 1998

[16] J. Neukirch: *Algebraische Zahlentheorie*, Springer Verlag, Berlin etc., 2002

[17] M. V. Nori: *On subgroups of* $\mathrm{GL}_n(\mathbb{F}_p)$, Invent. Math. 88, 1987, 257-275

[18] R. Pink: *Compact subgroups of linear algebraic groups*, J. Algebra 206 No. 2, 1998, 438-504

[19] R. Pink: *Strong approximation for Zariski dense subgroups over arbitrary global fields*, Comment. Math. Helv. 75 No. 4, 2000, 608-643

[20] R. Pink: *The Mumford-Tate conjecture for Drinfeld modules*, Publ. RIMS Kyoto University 33 No. 3, 1997, 393-425

[21] R. Pink: *The Galois Representation Associated to a Drinfeld Module in Special Characteristic, I: Zariski Density*, J. Number Theory 116 no. 2, 2006, 324-347

[22] R. Pink: *The Galois Representation Associated to a Drinfeld Module in Special Characteristic, II: Openness*, J. Number Theory 116 no. 2, 2006, 348-372

[23] R.Pink, M. Traulsen: *The Galois Representation Associated to a Drinfeld Module in Special Characteristic, III: Image of the Group Ring*, J. Number Theory 116 no. 2, 2006, 373-395

[24] M. Raynaud: *Schemas en groupes de type* $(p, \ldots, p)$, Bull. Soc. Math. France 102, 1974, 241-280

[25] I. Reiner: *Maximal Orders*, Clarendon Press, London, 2003

[26] J. S. Rose: *A Course on Group Theory*, Dover, New York, 1994

[27] J.-P. Serre: *Abelian ℓ-adic representations and elliptic curves*, Benjamin, New York, 1968

[28] J.-P. Serre: *Corps locaux*, Hermann, Paris, 1968

[29] J.-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. 15, 1972, 259-331

[30] J. H. Silverman: *The Arithmetic of Elliptic Curves*,Springer, New York etc., 1986

[31] T. A. Springer: *Linear Algebraic Groups*, 2en edition, Birkhäuser, Boston, 1998

[32] Y. Taguchi: *Semisimplicity of the Galois representations attached to Drinfeld modules over fields of "finite characteristics"*, Duke Math. J. 62, 1991, 593-599

[33] Y. Taguchi: *Semisimplicity of the Galois representations attached to Drinfeld modules over fields of "infinite characteristics"*, J. Number Theory 44, 1993, 292-314

[34] Y. Taguchi: *The Tate conjecture for $t-$motives*, Proc. Amer. Math. Soc. 123 No. 11, 1995, 3285-3287

[35] Y. Taguchi: *Finiteness of an isogeny class of Drinfeld modules*, J. Number Theory 74, 1999, 337-348

[36] Y. Taguchi: *On $\phi-$modules*, J. Number Theory 60, 1996, 124-141

[37] A. Tamagawa: *The Tate conjecture for A-premotives*, Preprint, 1994

[38] J. Tate: *p-Divisible Groups*, Proceedings of a Conference on Local Fields, Driebergen, Springer-Verlag, Berlin etc., 1967

[39] M. Traulsen: *Galois representations associated to Drinfeld modules in special characteristic and the isogeny conjecture for t-motives*, Diss. ETH Zurich, 2003

Curriculum Vitae

3. 2. 1980	Born in Appenzell, Switzerland
1987-1993	Primarschule Meistersrüte, Appenzell
1983-1999	Gymnasium St. Antonius Appenzell
1999	Matura Typus B
1999-2004	Studies in Mathematics at ETH Zürich. Diplom thesis *Über das Reduktionsverhalten von Punkten auf Abelschen Varietäten* under the supervision of Prof. Dr. Richard Pink
2004	Diplom in mathematics at the ETH Zürich
2004-2007	Ph.D. studies in mathematics at the ETH Zürich under the supervision of Prof. Dr. Richard Pink, teaching assistant at the Department of Mathematics of the ETH Zürich