Adelic Openness For Drinfeld Modules In Special Characteristic

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Abstract

This thesis deals with Galois representations associated to Drinfeld modules in special characteristic. Our main goal is to determine the best possible results about the openness of the image of the adelic Galois represenation.

Let K be a finitely generated field over a finite field κ of arbitrary transcendence degree and set $G_K^{\mathrm{geom}} := \mathrm{Gal}(K^{\mathrm{sep}}/K\overline{\kappa})$. Let φ be a Drinfeld A-module of rank r over K of special characteristic $\mathfrak{p}_0$ and let F denote the quotient field of A. The essential case boils down to proving the following statement: If the endomorphism ring D of φ over an algebraic closure of K is an order in a central simple algebra over F that does not grow when restricting φ to infinite subrings of A, then the intersection of the image of G_K^{geom} in the adelic representation with $\prod_{\mathfrak{p}\neq\mathfrak{p}_0} \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_r(A_\mathfrak{p})}(D \otimes_A A_\mathfrak{p})$ is open in both groups.

In closing we deduce from this the openness result for arbitrary Drinfeld modules in special characteristic.

Résumé

Cette thèse parle de représentations galoisiennes associées aux modules de Drinfeld en caractéristique spéciale. Notre but principal est de déterminer les meilleurs résultats possibles concernant l'ouverture de l'image de la représentation galoisienne adélique.

Soit K un corps finiment engendré sur un corps fini κ de degré de transcendence arbitraire et écrivons $G_K^{\mathrm{geom}} := \mathrm{Gal}(K^{\mathrm{sep}}/K\overline{\kappa})$. Soit φ un A-module de Drinfeld sur K de rang r en caractéristique spéciale $\mathfrak{p}_0$, et soit F le corps de fractions de A. Le cas essentiel revient à prouver l'affirmation suivante: si l'anneau des endomorphismes D de φ sur une clôture algébrique de K est un ordre dans une algèbre centrale simple sur F qui ne s'agrandit pas quand on restreint φ aux sous-anneaux infinis de A, alors l'intersection de l'image de G_K^{geom} dans la représentation adélique et de $\prod_{\mathfrak{p}\neq\mathfrak{p}_0} \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_r(A_\mathfrak{p})}(D \otimes_A A_\mathfrak{p})$ est ouverte dans les deux groupes.

Pour finir nous en déduisons le résultat concernant l'ouverture de l'image adélique pour un module de Drinfeld arbitraire en caractéristique spéciale.

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Contents

CHAPTER 1

Introduction

1.1. Notation

Let $\mathbb{F}_q$ be a finite field with q elements and of characteristic p. Let F be a finitely generated field of transcendence degree 1 over its constant field $\mathbb{F}_q$. Let A be the ring of elements of F which are regular outside a fixed place ∞ of F. Let K be another finitely generated field over $\mathbb{F}_q$ of arbitrary transcendence degree. Then the ring of $\mathbb{F}_q$-linear endomorphisms of the additive algebraic group over K is the non-commutative polynomial ring in one variable $K\{\tau\}$, where τ represents the endomorphism $u \mapsto u^q$ and satisfies the commutation relation $\tau u = u^q\tau$ for all $u \in K$. Consider a Drinfeld A-module

$$\varphi : A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K}) \cong K\{\tau\},\ a \mapsto \varphi_a$$

of rank $r \geq 1$ over K. In what follows we assume that φ has special characteristic. This means that the kernel $\mathfrak{p}_0$ of the homomorphism $A \to K$ determined by the lowest coefficient of φ is non-zero and therefore a maximal ideal of A. For the general theory of Drinfeld modules the reader can for example consult Drinfeld [**Dri74**], Deligne and Husemöller [**DH87**], Hayes [**Hay79**] or Goss [**Gos96**].

Inside a fixed algebraic closure $\overline{K}$ of K we let K^{sep} denote the separable closure of K. For any non-zero ideal $\mathfrak{a}$ of A we let

$$\varphi[\mathfrak{a}] := \{x \in \overline{K} \mid \forall a \in \mathfrak{a} : \varphi_a(x) = 0\}$$

denote the module of $\mathfrak{a}$-torsion of φ. If $\mathfrak{p}_0 \nmid \mathfrak{a}$, then its points are defined over K^{sep} and form a free $A/\mathfrak{a}$-module of rank r. For any prime $\mathfrak{p}$ of A let $A_\mathfrak{p}$ and $F_\mathfrak{p}$ denote the completions of A and F at $\mathfrak{p}$, respectively. For $\mathfrak{p} \neq \mathfrak{p}_0$, the $\mathfrak{p}$-adic Tate module $T_\mathfrak{p}(\varphi) := \varprojlim \varphi[\mathfrak{p}^n]$ is a free $A_\mathfrak{p}$-module of rank r, on which there is a natural action of the absolute Galois group G_K of K. This action commutes with the action of $\mathrm{End}_K(\varphi)$ on $T_\mathfrak{p}(\varphi)$. It was proved independently by Taguchi [**Tag95**] and Tamagawa [**Tam94a**], [**Tam94b**], [**Tam95**] that the natural homomorphism

$$\mathrm{End}_K(\varphi) \otimes_A A_\mathfrak{p} \longrightarrow \mathrm{End}_{A_\mathfrak{p},\mathrm{Gal}(K^{\mathrm{sep}}/K)}(T_\mathfrak{p}(\varphi)) \tag{1.1}$$

is an isomorphism. This yields a continuous representation

$$\rho_\mathfrak{p} : G_K \longrightarrow \mathrm{Cent}_{\mathrm{Aut}_{A_\mathfrak{p}}(T_\mathfrak{p}(\varphi))}(\mathrm{End}_K(\varphi) \otimes_A A_\mathfrak{p}) \cong \mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_K(\varphi) \otimes_A A_\mathfrak{p}).$$

We denote its image in $\mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(\mathrm{End}_K(\varphi) \otimes_A A_\mathfrak{p})$ by $\Gamma_\mathfrak{p}$.

Let κ denote the constant field of K and $\overline{\kappa}$ its algebraic closure in K^{sep}. Then $\mathrm{Gal}(\overline{\kappa}/\kappa)$ is the free pro-cyclic group topologically generated by the element Frob_κ

which acts on $\overline{\kappa}$ by $u \mapsto u^{|\kappa|}$. Writing $G_K^{\text{geom}} := \text{Gal}(K^{\text{sep}}/K\overline{\kappa})$, we have a natural short exact sequence

$$1 \longrightarrow G_K^{\text{geom}} \longrightarrow G_K \longrightarrow \text{Gal}(\overline{\kappa}/\kappa) \longrightarrow 1.$$

We are ultimately interested in the image of G_K^{geom} under $\rho_{\mathfrak{p}}$, which we denote by $\Gamma_{\mathfrak{p}}^{\text{geom}}$. By construction this is a closed normal subgroup of $\Gamma_{\mathfrak{p}}$ and the quotient is pro-cyclic.

Let P be a finite set of places $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F. We set $T_P(\varphi) := \oplus_{\mathfrak{p}\in P} T_{\mathfrak{p}}(\varphi)$, which is a free module over $A_P := \oplus_{\mathfrak{p}\in P} A_{\mathfrak{p}}$ of rank r. We denote the image of the combined representation

$$\rho_P : G_K \longrightarrow \text{Cent}_{\text{Aut}_{A_P}(T_P(\varphi))}(\text{End}_K(\varphi) \otimes_A A_P) \cong \text{Cent}_{\text{GL}_r(A_P)}(\text{End}_K(\varphi) \otimes_A A_P).$$

by Γ_P and the image of G_K^{geom} under ρ_P by Γ_P^{geom}.

For a place $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F with residue field $k_{\mathfrak{p}}$ we consider the residual representation

$$\overline{\rho_{\mathfrak{p}}} : G_K \longrightarrow \text{Cent}_{\text{Aut}_{k_{\mathfrak{p}}}(\varphi[\mathfrak{p}])}(\text{End}_K(\varphi) \otimes_A k_{\mathfrak{p}}) \cong \text{Cent}_{\text{GL}_r(k_{\mathfrak{p}})}(\text{End}_K(\varphi) \otimes_A k_{\mathfrak{p}}).$$

The name comes from the fact that this representation is nothing more than the reduction of $\rho_{\mathfrak{p}}$ modulo $\mathfrak{p}$.

For $n \geq 2$ we denote the reduction of $\rho_{\mathfrak{p}}$ modulo $\mathfrak{p}^n$ by $\rho_{\mathfrak{p},n}$.

Let $\mathbb{A}_F^{(\mathfrak{p}_0,\infty)}$ denote the ring of adeles of F outside of $\mathfrak{p}_0$ and ∞. We also consider the adelic representation

$$\rho_{\text{ad}} : G_K \longrightarrow \text{Cent}_{\text{GL}_r(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(\text{End}_K(\varphi) \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)}).$$

At last, we introduce some non-standard terminology that will be in use throughout this work: we say that a Drinfeld module $\varphi : A \to K\{\tau\}$ has minimal endomorphism ring if $\text{End}_{\overline{K}}(\varphi) = A$.

1.2. Main result

Let $\varphi : A \to K\{\tau\}$ be a Drinfeld module of rank r of special characteristic $\mathfrak{p}_0$. The aim of the present work is to describe the image of the adelic Galois representation up to commensurability. Pink [**Pin06b**] has shown that for all primes $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ the image of $\Gamma_{\mathfrak{p}}^{\text{geom}}$ under the determinant is finite; thus the subgroup $\det(\Gamma_{\mathfrak{p}}) \subset A_{\mathfrak{p}}^*$ is essentially pro-cyclic and therefore cannot be open. It follows that we cannot expect the image of G_K to be open in the adelic representation and the central question becomes describing the image of G_K^{geom} under ρ_{ad}.

Let $D := \text{End}_{\overline{K}}(\varphi)$, let Z denote the center of $D \otimes_A F$ and let us write

$$\dim_Z D \otimes_A F = d^2 \quad \text{and} \quad [Z/F] = e.$$

We know that there exists a finite separable extension K' of K such that all endomorphisms contained in D are already defined over K'; since we are only interested in the image of the Galois representation up to commensurability, we may thus assume that all endomorphisms of φ are defined over K. In this case we can select a maximal commutative subring $\hat{A}$ of D and pass to the corresponding Drinfeld module $\hat{\varphi} : \hat{A} \to K\{\tau\}$, which has rank $r' := r/de$ and satisfies $\mathrm{End}_{\overline{K}}(\hat{\varphi}) = \hat{A}$. The image of the adelic Galois representation associated to φ can be obtained as a projection of the image of the adelic Galois representation associated to $\hat{\varphi}$; thus we can reduce ourselves to the case of Drinfeld modules with minimal endomorphism ring.

In generic characteristic the case of such Drinfeld modules can be treated in a uniform way. However, in our setting a new phenomenon can occur that we need to take into account, namely the fact that Drinfeld modules in special characteristic can have non-commutative endomorphism rings. As a consequence, it is possible that if we restrict $\hat{\varphi}$ to a subring $\hat{B}$ of $\hat{A}$, then the endomorphism ring of the Drinfeld $\hat{B}$-module thus obtained is larger than the one we started out with. A natural question to ask then is whether the endomorphism ring can grow indefinitely if we undertake a series of successive restrictions, or whether the process stabilizes after a finite number of steps. This question was answered by Pink in [**Pin06b**] and it turns out that both cases can occur:

On the one hand, if $\hat{\varphi}$ is isomorphic to a Drinfeld module defined over a finite field, then the endomorphism ring can grow infinitely often. Pink proved that this occurs if and only if $r' = 1$ and that in this case $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$ is finite for all places $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F; more precisely, he proved that after replacing K by a finite extension we obtain $\Gamma_{\mathfrak{p}}^{\mathrm{geom}} = 1$ for all $\mathfrak{p} \neq \mathfrak{p}_0, \infty$. This effectively describes the image of the adelic representation in the case $r' = 1$.

On the other hand, Pink proved that if $\hat{\varphi}$ is not isomorphic to a Drinfeld module defined over a finite field, which is equivalent to saying $r' \geq 2$, then there exists a uniquely determined infinite subring $\hat{B}$ of $\hat{A}$ such that $\mathrm{End}_{\overline{K}}(\hat{\varphi}|\hat{B})$ over $\overline{K}$ is an order in a central simple algebra over the quotient field of $\hat{B}$ and that for every infinite subring $\hat{C} \subset \hat{A}$ we have $\mathrm{End}_{\overline{K}}(\hat{\varphi}|\hat{C}) \subset \mathrm{End}_{\overline{K}}(\hat{\varphi}|\hat{B})$. The adelic Galois representation associated to $\hat{\varphi}$ coincides with the adelic Galois representation associated to $\hat{\varphi}|\hat{B}$; thus, if $r' \geq 2$, then we can always reduce ourselves to the case of a Drinfeld module with the characteristics of $\hat{\varphi}|\hat{B}$, and studying the behavior of such Drinfeld modules describes the image of the adelic representation for arbitrary ones. This is what has inspired us to formulate our Main Theorem as follows:

Theorem 1.1.
Let φ be a Drinfeld A-module over a finitely generated field K of special characteristic $\mathfrak{p}_0$. Assume that $D := \mathrm{End}_K(\varphi)$ is an order in a central simple algebra over F of dimension d^2 and that for every infinite subring $B \subset A$ we have $\mathrm{End}_{\overline{K}}(\varphi|B) = D$. Let r be the positive integer such that the rank of φ is equal to rd and assume that $r \geq 2$. Then

$$\rho_{ad}(G_K^{\mathrm{geom}}) \cap \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$$

is open in both $\rho_{ad}(G_K^{\mathrm{geom}})$ *and* $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D\otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$.

Compare the formulation of this theorem with Theorem 6.1 of [**Pin06b**]: Under the assumptions of our theorem the latter says that for any finite set P of primes of A not containing $\mathfrak{p}_0$, the intersection $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_P)}(D\otimes_A A_P)\cap\Gamma_P^{\mathrm{geom}}$ is open in both groups. This was the result that helped us determine the outcome to aim for in the adelic case.

The reduction steps mentioned above and the general result for arbitrary Drinfeld modules in special characteristic that are not isomorphic to a Drinfeld module defined over a finite field will be explained at greater length in Chapter 9.

To complete this section, we describe a special case of Theorem 1.1. Assume that φ has minimal endomorphism ring which does not grow when restricting φ to infinite subrings of A. This is the simplest case that can occur and the one in which the result obtained mimics closely the one for Drinfeld modules in generic characteristic ([**PR09a**], Theorem 0.1):

COROLLARY 1.2.
Let φ *be a Drinfeld* A*-module of rank* $r\geq 2$ *over a finitely generated field* K *of special characteristic* $\mathfrak{p}_0$*. Assume that for every infinite subring* $B\subset A$ *we have* $\mathrm{End}_{\overline{K}}(\varphi|B)=A$*. Then*

$$\rho_{ad}(G_K^{\mathrm{geom}})\cap\mathrm{SL}_r(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$$

is open in both $\rho_{ad}(G_K^{\mathrm{geom}})$ *and* $\mathrm{SL}_r(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$.

1.3. Outline of the chapters

In Chapter 2 we present a few preparatory results that do not involve the theory of Drinfeld modules. To begin with we use the general theory of representations of linear algebraic groups to show that, if certain explicitly stated algebraic relations are satisfied on a connected semisimple algebraic group G, then G is isomorphic to SL_n for some $n\geq 1$. Next we prove an analogous result for finite subgroups of linear algebraic groups that, combined with a previous result by Larsen and Pink [**LP98**], allows us to establish certain criteria that help approximate finite subgroups of SL_n in non-zero characteristic by a subgroup of the form $\mathrm{SL}_n(k)$ or $\mathrm{SU}_n(k)$ for some finite field k. The exact formulation of the result thus obtained can be found in Theorem 2.19.

Chapter 3 is also devoted to preparatory results, this time on the side of the theory of Drinfeld modules. In the first two sections we list previously known results about Drinfeld modules in special characteristic, in some cases after reformulating them to fit our setting. In Section 3.3 we collect and explain a few important reduction steps that one can carry out before attacking the proof of

Theorem 1.1. Finally, we devote the last section of the chapter to the properties of Frobenius elements in the representations $\rho_\mathfrak{p}$ and $\overline{\rho_\mathfrak{p}}$ at a given prime $\mathfrak{p}$ of A.

The proof of the Main Theorem is carried out in Chapters 4 to 8. We assume throughout that φ satisfies the conditions of Theorem 1.1 and that the reduction steps introduced in Section 3.3 are in effect. In Chapters 4 to 7 we make the additional assumption that the field K has transcendence degree 1; this assumption will only be lifted in Chapter 8. We split the proof into chapters as follows:

In Chapter 4 we prove that the image of the residual representation contains $\mathrm{SL}_r(k_\mathfrak{p})$ for almost all primes $\mathfrak{p}$ of A. The key ingredients are the irreducibility of the residual representation [**PT06**], the Zariski density of $\Gamma_\mathfrak{p}$ [**Pin06a**], the characterization of $k_\mathfrak{p}$ in terms of the image of Frobenius elements and Theorem 2.19. In the last part we prove that the image of G_K in the product of two residual representations at distinct primes cannot be contained in the graph of an isomorphism between the factors.

In Chapter 5 we collect a few auxiliary results from group theory and cohomology that will be used in the subsequent chapters.

In Chapter 6 we prove that the image of $\rho_\mathfrak{p}$ contains $\mathrm{SL}_r(A_\mathfrak{p})$ for almost all primes $\mathfrak{p}$ of A. We accomplish this by proving a purely algebraic result first: if a closed subgroup H of $\mathrm{SL}_r(A_\mathfrak{p})$ maps surjectively onto $\mathrm{SL}_r(k_\mathfrak{p})$ and contains a non-scalar matrix of the form $1 + M_\mathfrak{p}$ with $M_\mathfrak{p} \in \mathfrak{gl}_r(\mathfrak{p}) \setminus \mathfrak{gl}_r(\mathfrak{p}^2)$, then H is equal to $\mathrm{SL}_r(A_\mathfrak{p})$. The group $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ satisfies the first condition for almost all $\mathfrak{p}$ by the results of Chapter 4; thus we are left with proving the existence of a corresponding non-scalar element $M_\mathfrak{p}$ for almost all $\mathfrak{p}$. This is achieved with the help of Frobenius elements.

In Chapter 7 we use the results of Chapter 6 and the openness of the image of Γ_P^{geom} at a finite set P of primes proved in [**Pin06b**] to establish the Main Theorem for fields of transcendence degree 1.

In Chapter 8 the field K can have arbitrary transcendence degree, but φ is still assumed to satisfy the assumptions of Theorem 1.1. We use a reduction argument similar to the one in [**PR09a**] in order to deduce the general case of Theorem 1.1 from the results of Chapter 7.

Chapter 9 is a natural completion of Section 1.2. It gives a precise description of the results that we can deduce from Theorem 1.1 for arbitrary Drinfeld modules of special characteristic that are not isomorphic to a Drinfeld module defined over a finite field.

CHAPTER 2

Linear algebraic groups and their finite subgroups

This chapter builds towards its main result, Theorem 2.19, which will play an important role in determining the image of the restricted residual representation $\overline{\rho_{\mathfrak{p}}}|_{G_K^{\mathrm{geom}}}$ at a given place $\mathfrak{p}$ of F.

2.1. Root system combinatorics

In this section we prove the following result: the only root systems where there is an orbit of the Weyl group that generates the ambient vector space while not satisfying a certain simple relation of linear dependence are of type A_n. Moreover, we show that if the dimension of the root system is different from 2, then the orbit in question is, up to a non-zero scalar multiple, the orbit of the first fundamental weight relative to a given base of the root system.

Then, assuming that a second simple linear dependence relation is not satisfied, we show that the general result also holds when the dimension of the root system equals 2.

THEOREM 2.1.
Let Φ be a non-trivial root system generating the Euclidean vector space V. Let $\mathcal{W}$ be the associated Weyl group and S a $\mathcal{W}$-orbit in V.

Assume the following conditions are satisfied:

(a) *V is generated by S as a vector space;*
(b) *There are no distinct elements $\lambda_1, \ldots, \lambda_4 \in S$ such that $\lambda_1+\lambda_2 = \lambda_3+\lambda_4$.*

Then either

(1) *there is an integer $n \geq 1$ and a constant $c \neq 0$ such that*
$$\Phi \cong \mathsf{A}_n = \{\pm(e_i - e_j) \mid 0 \leq i < j \leq n\} \subset V = \mathbb{R}^{n+1}/\operatorname{diag}(\mathbb{R})$$
and $S = \{ce_i \mid 0 \leq i \leq n\}$, or
(2) *$\Phi \cong \mathsf{A}_2$.*

Assuming a third condition similar to the second one from above, we get an even stronger result:

THEOREM 2.2.
Let $\Phi, V, \mathcal{W}$ and S be as defined in Theorem 2.1 and assume that in addition to Assumptions (a) and (b) of that theorem, the following condition also holds:

(c) *There are no distinct elements* $\lambda_1, \dots, \lambda_6 \in S$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_4 + \lambda_5 + \lambda_6$.

Then there is an integer $n \geq 1$ *and a constant* $c \neq 0$ *such that*

$$\Phi \cong \mathsf{A}_n = \{\pm(e_i - e_j) \mid 0 \leq i < j \leq n\} \subset V = \mathbb{R}^{n+1}/\operatorname{diag}(\mathbb{R})$$

and $S = \{ce_i \mid 0 \leq i \leq n\}$.

Proof of Theorem 2.1. In what follows we suppose that the assumptions of the theorem are satisfied. First we show that Φ is simple and only contains roots of the same length, thereby excluding the cases $\mathsf{B}_n, \mathsf{C}_n, \mathsf{F}_4$ and G_2. Next we impose some restrictions on the position of S in V relative to Φ in order to exclude the cases $\mathsf{D}_n, \mathsf{E}_6, \mathsf{E}_7$ and E_8. Finally we make use of the well-known structure theory of A_n to prove that, up to a non-zero scalar multiple, S is the $\mathcal{W}$-orbit of the first fundamental weight relative to the standard base of Φ.

Before we start, let us note that it follows from the assumptions of the theorem that S does not contain 0; indeed, under the action of the Weyl group the orbit of 0 is $\{0\}$, which cannot generate the non-trivial vector space V.

Lemma 2.3.
Let $\lambda \in S$ *and* α_1, α_2 *be two orthogonal roots in* Φ. *Then* $\lambda \perp \alpha_1$ *or* $\lambda \perp \alpha_2$.

Proof. Let s_{α_1} (respectively s_{α_2}) in $\mathcal{W}$ denote the reflexions corresponding to α_1 (respectively α_2). Then

$$\lambda + s_{\alpha_1}s_{\alpha_2}(\lambda) = s_{\alpha_1}(\lambda) + s_{\alpha_2}(\lambda)$$

and in order to avoid a contradiction to Assumption (b), we must have one of the following equalities:

$$s_{\alpha_1}(\lambda) = \lambda \quad \text{or} \quad s_{\alpha_2}(\lambda) = \lambda \quad \text{or} \quad \lambda = s_{\alpha_1}s_{\alpha_2}(\lambda).$$

The last equality yields $\lambda = s_{\alpha_1}(\lambda) = s_{\alpha_2}(\lambda)$ and it follows that in each case we have $\lambda \perp \alpha_1$ or $\lambda \perp \alpha_2$. □

Proposition 2.4.
The root system Φ *is simple.*

Proof. Let us assume that $\Phi = \Psi_1 + \Psi_2$ is decomposable and let $\lambda \in S$. Since Φ generates V there exists $\alpha \in \Phi$ such that α is not orthogonal to λ. Assume without loss of generality that $\alpha \in \Psi_2$. Then, by Lemma 2.3, the vector λ is orthogonal to all roots that are orthogonal to α, in particular $\lambda \perp \Psi_1$. Then $w(\lambda) \perp \Psi_1$ for all $w \in \mathcal{W}$ and therefore $S \perp \Psi_1$. However, this is a contradiction to Assumption (a). □

Proposition 2.5.
All roots in Φ have the same length and are therefore conjugate under the action of the Weyl group.

Proof. Let us assume that Φ contains roots of different lengths. By Proposition 2.4 the root system Φ is simple and hence contains two roots of different lengths that are not orthogonal; consequently these generate a sub-root system Ψ of type B_2 or G_2. Let $\{\alpha_1, \alpha_2\}$ be a basis of Ψ. Since S generates V, we can find an element $\lambda \in S$ that is not orthogonal to Ψ. We show that there are distinct elements $w_1, \ldots, w_4 \in \mathcal{W}(\Psi) \subset \mathcal{W}$ with $w_1(\lambda) + w_2(\lambda) = w_3(\lambda) + w_4(\lambda)$, thus obtaining a contradiction.

Since S is a $\mathcal{W}$-orbit, we can without loss of generality assume that the projection λ' of λ onto the plane generated by Ψ lies in the Weyl chamber corresponding to α_1 and α_2. Since $\mathcal{W}(\Psi)$ acts the same way on λ as on λ', it is enough to find relations for λ'. This effectively reduces the problem to the two-dimensional case.

Case B_2*:* Let us assume that α_1 is the longer root. We distinguish three cases according to whether λ' is on one of the boundaries of the Weyl chamber or in the interior. In each case we proceed similarly to find a quadruple $w_1, \ldots, w_4$ that yields the desired contradiction: we put $w_1 = \mathrm{Id}$ and for w_2 we choose an element of the Weyl group that sends λ' to $-\lambda'$. (If λ' lies on the exterior of the Weyl chamber, it is the scalar multiple of a root, thus we can choose the symmetry respective to the root in question; otherwise we can take the product of two symmetries relative to orthogonal roots.) By this choice, we get two distinct elements $w_1(\lambda') = \lambda'$ and $w_2(\lambda') = -\lambda'$ whose sum is zero; if we apply to this sum a symmetry relative to a root that is neither parallel nor orthogonal to λ', we obtain a pair of distinct elements $w_3(\lambda')$ and $w_4(\lambda')$ with sum zero that are also distinct from the first pair. Here are the exact computations for each case:

- If $\lambda' = c(\alpha_1 + \alpha_2), c \in \mathbb{R}_{>0}$, then
$$\lambda' + s_{\alpha_1+\alpha_2}(\lambda') = s_{\alpha_1}(\lambda') + s_{\alpha_1} s_{\alpha_1+\alpha_2}(\lambda');$$
- If $\lambda' = c(\alpha_1 + 2\alpha_2), c \in \mathbb{R}_{>0}$, then
$$\lambda' + s_{\alpha_1+2\alpha_2}(\lambda') = s_{\alpha_2}(\lambda') + s_{\alpha_2} s_{\alpha_1+2\alpha_2}(\lambda');$$
- If λ' is in the interior of the Weyl chamber, then
$$\lambda' + s_{\alpha_1} s_{\alpha_1+2\alpha_2}(\lambda') = s_{\alpha_1}(\lambda') + s_{\alpha_1+2\alpha_2}(\lambda').$$

Case G_2*:* Let us assume that α_1 is the shorter root. Again we distinguish three cases according to the same principle as in the case B_2 and apply the same method to find linear relations of the desired form:

- If $\lambda' = c(3\alpha_1 + 2\alpha_2), c \in \mathbb{R}_{>0}$, then
$$\lambda' + s_{3\alpha_1+2\alpha_2}(\lambda') = s_{2\alpha_1+\alpha_2}(\lambda') + s_{2\alpha_1+\alpha_2} s_{3\alpha_1+2\alpha_2}(\lambda');$$
- If $\lambda' = c(2\alpha_1 + \alpha_2), c \in \mathbb{R}_{>0}$, then
$$\lambda' + s_{2\alpha_1+\alpha_2}(\lambda') = s_{\alpha_1}(\lambda') + s_{\alpha_1} s_{2\alpha_1+\alpha_2}(\lambda');$$

- If λ' is in the interior of the Weyl chamber, then
$$\lambda' + s_{\alpha_2}s_{2\alpha_1+\alpha_2}(\lambda') = s_{\alpha_2}(\lambda') + s_{2\alpha_1+\alpha_2}(\lambda').$$

We found non-trivial relations contradicting Assumption (b) for B_2 and for G_2 as well. Consequently the root system cannot contain roots of different lengths. The well-known fact that the Weyl group acts transitively on every simple root system where all roots are of equal length completes the proof. □

Lemma 2.6.
Suppose $\dim(V) \geq 3$.

(1) *Let* $\lambda \in S$ *and* $\alpha \in \Phi$. *Then* $\lambda \in \langle\alpha\rangle^\perp \cup (\{\alpha\}^\perp \cap \Phi)^\perp$.
(2) *If* Φ *is of type* $\mathsf{D}_n\,(n \geq 4), \mathsf{E}_6, \mathsf{E}_7$ *or* E_8, *then* $(\{\alpha\}^\perp \cap \Phi)^\perp = \langle\alpha\rangle$ *for all* $\alpha \in \Phi$.
(3) *If* Φ *is of type* $\mathsf{A}_n\,(n \geq 3)$ *and* $\alpha = e_i - e_j$ *with* $i \neq j$, *then* $(\{\alpha\}^\perp \cap \Phi)^\perp = \langle e_i, e_j\rangle$.

Proof.

(1) If $\lambda \perp \alpha$, then $\lambda \in \langle\alpha\rangle^\perp$. Now suppose λ and α are not orthogonal. Since $\dim(V) \geq 3$, we can find $\beta \in \Phi$ such that $\alpha \perp \beta$. By Lemma 2.3 we know that λ is orthogonal to either α or β; therefore $\lambda \perp \beta$. It follows that $\lambda \perp \left\langle \{\alpha\}^\perp \cap \Phi \right\rangle$. Thus in each case we have
$$\lambda \in \langle\alpha\rangle^\perp \cup (\{\alpha\}^\perp \cap \Phi)^\perp.$$
(2) Since all roots are conjugate under the action of the Weyl group, it is enough to prove the assumption for an arbitrary element α of Φ. Clearly we have $\langle\alpha\rangle \subset (\{\alpha\}^\perp \cap \Phi)^\perp$. Henceforth we proceed case by case:

Case D_n*:* We choose the following construction of the root system:
$$\Phi = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq n\} \subset \mathbb{R}^n$$
and the corresponding simple roots
$$\alpha_i = e_i - e_{i+1} \text{ for } i = 1, \ldots, n-1 \quad \text{ and } \quad \alpha_n = e_{n-1} + e_n.$$
By explicit calculations for $\alpha = e_1 - e_2$ we then find
$$\langle \{\alpha\}^\perp \cap \Phi \rangle = \langle e_1 + e_2, \alpha_3, \ldots, \alpha_n\rangle.$$
This last subspace of $\mathbb{R}^n$ has dimension $n-1$. Combining this with the fact that $\langle\alpha\rangle \subset (\{\alpha\}^\perp \cap \Phi)^\perp$ yields $\langle\alpha\rangle = (\{\alpha\}^\perp \cap \Phi)^\perp$.

Case E_6*:* We choose the following construction of the root system:
$$\Phi = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq 5\}$$
$$\cup \left\{ \left(\tfrac{1}{2}\varepsilon_1, \ldots, \tfrac{1}{2}\varepsilon_5, \tfrac{\sqrt{3}}{2}\varepsilon_6\right) \mid \varepsilon_1, \ldots, \varepsilon_6 \in \{\pm 1\}, \prod_{i=1}^{6} \varepsilon_i = 1 \right\} \subset \mathbb{R}^6$$

and the corresponding simple roots

$$\begin{aligned}\alpha_i &= e_{i+1} - e_i \text{ for } i = 1, \ldots, 4, \\ \alpha_5 &= e_1 + e_2, \\ \alpha_6 &= (-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}).\end{aligned}$$

Again, by explicit calculations for $\alpha = e_1 - e_2$ we find

$$\langle \{\alpha\}^\perp \cap \Phi \rangle = \langle e_3 + e_4, \alpha_3, \ldots, \alpha_6 \rangle.$$

This last subspace of $\mathbb{R}^6$ has dimension 5. Therefore we can again conclude $\langle \alpha \rangle = (\{\alpha\}^\perp \cap \Phi)^\perp$.

Case E_7*:* We choose the following construction of the root system:

$$\Phi = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq 6\} \cup \{(0, \ldots, 0, \pm\sqrt{2})\}$$
$$\cup \left\{ \left(\tfrac{1}{2}\varepsilon_1, \ldots, \tfrac{1}{2}\varepsilon_6, \tfrac{1}{\sqrt{2}}\varepsilon_7\right) \mid \varepsilon_1, \ldots, \varepsilon_7 \in \{\pm 1\}, \prod_{i=1}^{6} \varepsilon_i = 1 \right\} \subset \mathbb{R}^7$$

and the corresponding simple roots

$$\begin{aligned}\alpha_i &= e_{i+1} - e_i \text{ for } i = 1, \ldots, 5, \\ \alpha_6 &= e_1 + e_2, \\ \alpha_7 &= (-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, -\tfrac{1}{\sqrt{2}}).\end{aligned}$$

Again, by explicit calculations for $\alpha = e_1 - e_2$ we find

$$\langle \{\alpha\}^\perp \cap \Phi \rangle = \langle e_3 + e_4, \alpha_3, \ldots, \alpha_7 \rangle.$$

This last subspace of $\mathbb{R}^7$ has dimension 6. Thus we can again conclude $\langle \alpha \rangle = (\{\alpha\}^\perp \cap \Phi)^\perp$.

Case E_8*:* We choose the following construction of the root system:

$$\Phi = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq 8\}$$
$$\cup \left\{ \left(\tfrac{1}{2}\varepsilon_1, \ldots, \tfrac{1}{2}\varepsilon_8\right) \mid \varepsilon_1, \ldots, \varepsilon_8 \in \{\pm 1\}, \prod_{i=1}^{8} \varepsilon_i = 1 \right\} \subset \mathbb{R}^7$$

and the corresponding simple roots

$$\begin{aligned}\alpha_i &= e_i - e_{i+1} \text{ for } i = 1, \ldots, 7, \\ \alpha_8 &= (-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).\end{aligned}$$

Again, by explicit calculations for $\alpha = e_1 - e_2$ we find

$$\langle \{\alpha\}^\perp \cap \Phi \rangle = \langle e_1 + e_2, \alpha_3, \ldots, \alpha_8 \rangle.$$

This last subspace of $\mathbb{R}^8$ has dimension 7. Therefore we can again conclude $\langle \alpha \rangle = (\{\alpha\}^\perp \cap \Phi)^\perp$.

(3) Just like in the second part of the proof, here it is also sufficient to prove the statement for an arbitrary element α of Φ, say $\alpha = e_0 - e_1$. For this choice of α we indeed have

$$(\{\alpha\}^\perp \cap \Phi)^\perp = \langle e_2 - e_3, \ldots, e_{n-1} - e_n \rangle^\perp = \langle e_0, e_1 \rangle. \qquad \square$$

PROPOSITION 2.7.
The root system Φ cannot be of type $\mathsf{D}_n\,(n \geq 4), \mathsf{E}_6, \mathsf{E}_7$ and E_8 and is therefore equal to A_n for some $n \geq 1$.

PROOF. Let Φ be of one of the types $\mathsf{D}_n\,(n \geq 4), \mathsf{E}_6, \mathsf{E}_7$ or E_8 and let $\mu \in S$. Combining the first two statements of the previous lemma, for all $\alpha \in \Phi$ we get

$$\mu \in \langle \alpha \rangle \cup \langle \alpha \rangle^\perp.$$

Since Φ generates V, we have $\bigcap_{\alpha\in\Phi} \langle \alpha \rangle^\perp = \{0\}$. Therefore we find an $\alpha \in \Phi$ with $\mu \in \langle \alpha \rangle$. Let $\beta \in \Phi$ be a root not orthogonal to α with $\beta \neq \pm\alpha$, in other words such that $\langle \beta \rangle \not\subset \langle \alpha \rangle \cup \langle \alpha \rangle^\perp$. Since $\mathcal{W}$ acts transitively on Φ, there exists $w \in \mathcal{W}$ with $w(\alpha) = \beta$. Then it follows from $\mu \in \langle \alpha \rangle$ that $\lambda := w(\mu)$ is contained in $\langle w(\alpha) \rangle = \langle \beta \rangle$ and therefore not contained in $\langle \alpha \rangle \cup \langle \alpha \rangle^\perp$. On the other hand, since S is stable under the operation of $\mathcal{W}$, we have $w(\mu) = \lambda \in S$. Applying Lemma 2.6 (1) to λ leads to a contradiction. Since we have already excluded root systems with different root lengths, the only remaining possibilities are the root systems of type A_n for some $n \geq 1$. $\square$

One part of Theorem 2.1 is now proven. It only remains to show that if $n \neq 2$, then up to a non-zero scalar multiple S is the $\mathcal{W}$-orbit of the first fundamental weight. This is the object of the following proposition.

PROPOSITION 2.8.
Let

$$\Delta = \{\alpha_i := e_{i-1} - e_i \mid 1 \leq i \leq n\}$$

denote the standard base of the root system A_n. If $n \neq 2$, then there is a constant $c \neq 0$ such that

$$S = \{ce_i \mid 0 \leq i < n\}.$$

PROOF. The claim is trivial for $n = 1$. Let us therefore suppose $n \geq 3$. Relative to Δ the fundamental weights are

$$\left\{ \lambda_i := \sum_{j=0}^{i-1} e_j \mid 1 \leq i \leq n \right\}.$$

Let $\mathfrak{C}(\Delta)$ denote the Weyl chamber relative to Δ. A vector $s = (s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1}/\operatorname{diag}(\mathbb{R})$ then lies in $\mathfrak{C}(\Delta)$ if and only if $s_0 \geq s_1 \cdots \geq s_n$.

Let us now fix $\lambda \in S$ that lies in $\mathfrak{C}(\Delta)$ and suppose there exist $1 \leq i < j \leq n$ such that $\lambda \not\perp \alpha_i$ and $\lambda \not\perp \alpha_j$. By Lemma 2.3 this is only possible if α_i and α_j

are not orthogonal, in other words if $j = i+1$. By Lemma 2.6 this means

$$\lambda \in \langle e_{i-1}, e_i \rangle \cap \langle e_{j-1}, e_j \rangle = \langle e_{i-1}, e_i \rangle \cap \langle e_i, e_{i+1} \rangle = \langle e_i \rangle,$$

hence $\lambda = ce_i$ with $c \in \mathbb{R}_{>0}$ and $0 < i < n$. However,

$$\mathfrak{C}(\Delta) \cap \{ce_i \mid c \in \mathbb{R}_{>0}, 0 < i < n\} = \emptyset,$$

a contradiction.

Therefore there exists a unique $1 \leq i \leq n$ such that $\lambda \not\perp \alpha_i$. Thus λ is a non-zero scalar multiple of the fundamental weight $\lambda_i = \sum_{j=0}^{i-1} e_j$. We now show that $i = 1$ or $i = n$.

Indeed, let us suppose $2 \leq i \leq n-1$. Then $\lambda \not\perp (e_0 - e_{n-1})$ and $\lambda \not\perp (e_1 - e_n)$. Given that $e_0 - e_{n-1}$ and $e_1 - e_n$ are orthogonal, we obtain a contradiction by Lemma 2.3. Hence we have $i = 1$ or $i = n$.

For $i = 1$ we find $\lambda = ce_0$ and $S = \{ce_i \mid 0 \leq i \leq n\}$. For $i = n$ we find $\lambda = c\sum_{j=0}^{n-1} e_i = -ce_n$, which yields the same result for S in this case also. $\square$

This in turn finishes the proof of Theorem 2.1. Now we turn to the proof of Theorem 2.2.

Proof of Theorem 2.2. Given Theorem 2.1, it only remains to show that if we add Assumption (c) to the original hypotheses, in the two-dimensional case we get $S = \{ce_i \mid 0 \leq i \leq 2\}$ for some $c \neq 0$.

Let $\Phi \cong \mathsf{A}_2$. Then $\mathcal{W}$ is the symmetric group on 3 elements and it acts on the vector space $V = \mathbb{R}^3/\operatorname{diag}(\mathbb{R})$ by permuting the coefficients. Let $\lambda_1 = (x, y, z)$ be an element of S. Since S is $\mathcal{W}$-stable, the conjugates of λ_1, namely

$$\lambda_2 = (y, z, x), \quad \lambda_3 = (z, x, y), \quad \lambda_4 = (x, z, y),$$

$$\lambda_5 = (y, x, z), \quad \lambda_6 = (z, y, x)$$

are all elements of S. Clearly we have

$$\lambda_1 + \lambda_2 + \lambda_3 = \lambda_4 + \lambda_5 + \lambda_6,$$

which leads to a contradiction unless two of the λ_i are equal. This happens if at least two of the coefficients x, y, z are equal. Having three equal coefficients would mean $\lambda_1 = 0$, which is impossible by Assumption (a), so exactly two of the coefficients are equal. Let us assume without loss of generality $x \geq y \geq z$. The two cases to consider are $z = y$ and $y = x$. In the first case we get $\lambda_1 = (x, y, y) = (x - y, 0, 0) = (x - y) \cdot e_0$ and, putting $c := x - y$, we find $S = \{ce_0, ce_1, ce_2\}$. The second case yields $\lambda_1 = (x, x, z) = (0, 0, z - x) = (z - x) \cdot e_2$ that, writing $c := z - x$, also gives $S = \{ce_0, ce_1, ce_2\}$. $\square$

2.2. Defining algebraic relations

Let R be a commutative ring, $n \geq 1$ an integer and $A \in \mathrm{GL}_n(R)$. Moreover, let

$$c_A(T) = T^n + \beta_1 T^{n-1} + \cdots + \beta_n \in R[T]$$

be the characteristic polynomial of A and let $c_A(T) = \prod_{i=1}^n (T - \alpha_i)$ be its decomposition into linear factors in $R'[T]$ for a suitable ring extension R' of R. We define

$$\begin{aligned} f(A) &= \prod_{\substack{i_1,\dots,i_4 \\ \text{distinct}}} (\alpha_{i_1}\alpha_{i_2} - \alpha_{i_3}\alpha_{i_4}), \\ g(A) &= \prod_{\substack{i_2,i_3,i_4 \\ \text{distinct}}} (\alpha_{i_2}^2 - \alpha_{i_3}\alpha_{i_4}), \\ h(A) &= \prod_{\substack{i_1,\dots,i_6 \\ \text{distinct}}} (\alpha_{i_1}\alpha_{i_2}\alpha_{i_3} - \alpha_{i_4}\alpha_{i_5}\alpha_{i_6}). \end{aligned}$$

The expressions f, g and h are symmetric in the roots of c_A; therefore they are polynomial expressions in $\beta_1, \dots, \beta_n$ with coefficients in $\mathbb{Z}$. This shows that the above constructions can be carried out over any commutative ring R. Thus they yield algebraic morphisms

$$f, g, h : \mathrm{GL}_n \to \mathbb{A}^1.$$

Lemma 2.9.
Let k be a field and let $f, g, h : \mathrm{GL}_{n,k} \to \mathbb{A}^1_k$ be defined as above. For an integer $c \geq 1$ let f_c, g_c, h_c denote the composition of the morphism $\gamma \mapsto \gamma^c$ with f, g and h, respectively. Then f_c, g_c, h_c do not vanish identically on $\mathrm{SL}_{n,k}$.

Proof. Let q be the characteristic of k, let $\overline{k}$ denote an algebraic closure of k and let p be a prime $\neq 2, q$. Consider in $\mathrm{SL}_n(\overline{k})$ the diagonal matrices of the form

$$A = \begin{pmatrix} \alpha & & & & & \\ & \alpha^p & & & & \\ & & \alpha^{p^2} & & & \\ & & & \ddots & & \\ & & & & \alpha^{p^{n-2}} & \\ & & & & & \alpha^{-\sum_{i=0}^{n-2} p^i} \end{pmatrix}.$$

Then

$$f_c(A) = \prod_{\substack{i_1,\dots,i_4 \leq n-2 \\ \text{distinct}}} (\alpha^{c(p^{i_1}+p^{i_2})} - \alpha^{c(p^{i_3}+p^{i_4})}) \prod_{\substack{i_1,i_2,i_3 \leq n-2 \\ \text{distinct}}} (\alpha^{c(p^{i_1}+p^{i_2})} - \alpha^{c(p^{i_3} - \sum_{j=0}^{n-2} p^j)}).$$

It is easy to check that for $i_1, \dots, i_4$ distinct and p prime, we have $c(p^{i_1} + p^{i_2}) \neq c(p^{i_3} + p^{i_4})$, so the first product does not vanish identically on $\mathrm{SL}_{n,k}$. In the second product one of the exponents is positive, the other negative; thus they cannot be equal and this product does not vanish identically on $\mathrm{SL}_{n,k}$ either.

Using the same diagonal matrices we can show that h_c and g_c do not vanish identically on matrices of the above form. For h_c we can take any prime exponent p different from the characteristic; for g_c we need the extra condition $p \neq 2$. $\square$

2.3. Linear algebraic groups

We now use the results obtained about root systems to find certain conditions under which a given linear algebraic group is equal to the special linear group.

THEOREM 2.10.
Let G be a connected semisimple linear algebraic group over an algebraically closed field L and V a finite dimensional, irreducible and faithful representation of G over L.
If f, g and h do not vanish identically on G, then $G = \mathrm{SL}_V$.

We start by proving the following lemma:

LEMMA 2.11.
Let G be a connected semisimple linear algebraic group over an algebraically closed field L. Let $\mathcal{W}$ denote its Weyl group, Φ the associated root system and E the Euclidean vector space generated by Φ. Let V be a finite dimensional, irreducible and faithful representation of G over L with highest weight λ. Then E is generated by $\mathcal{W} \cdot \lambda = \{w(\lambda) \mid w \in \mathcal{W}\}$.

PROOF. Let $G_1, \dots, G_m$ be simple connected linear algebraic groups defined over L with $G = G_1 \cdots G_m$. For all $1 \leq i \leq m$ let $\mathcal{W}_i$ denote the Weyl group of G_i and Φ_i the associated root system and, by abuse of notation, $\mathbb{R}\Phi_i$ the Euclidean vector space generated by Φ_i. Then

$$\mathcal{W} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_m \quad \text{and} \quad E = \mathbb{R}\Phi_1 \oplus \cdots \oplus \mathbb{R}\Phi_m.$$

Let $\lambda_i \in \mathbb{R}\Phi_i$ denote the highest weight of the representation restricted to G_i. For the global highest weight λ we have the decomposition $\lambda = \lambda_1 + \cdots + \lambda_m$. Since the representation V is faithful, it cannot be trivial on any of the components and thus for all $1 \leq i \leq m$ we have $\lambda_i \neq 0$. The factor $\mathcal{W}_i$ of the Weyl group $\mathcal{W}$ acts trivially on $\mathbb{R}\Phi_j$ for $i \neq j$ and irreducibly on $\mathbb{R}\Phi_i$. In particular, since λ_i is non-zero, we find that $\mathcal{W}_i \cdot \lambda_i$ generates $\mathbb{R}\Phi_i$.

Let $W_\lambda \subseteq E$ denote the subspace generated by $\mathcal{W} \cdot \lambda$. In order to prove that $W_\lambda = E$, it is now enough to prove that $\mathcal{W}_i \cdot \lambda_i \subseteq W_\lambda$ for all $1 \leq i \leq m$. Since W_λ is $\mathcal{W}$-invariant, it is enough to show that $\lambda_i \in W_\lambda$. By symmetry, it suffices to prove this for $i = 1$.

We proceed by induction. By the definition of W_λ we have $\lambda_1 + \cdots + \lambda_m = \lambda \in W_\lambda$. Now let $2 \leq k \leq m$ and assume that $\lambda_1 + \cdots + \lambda_k \in W_\lambda$. Then for all $\omega_k \in \mathcal{W}_k$ we find

$$\lambda_1 + \cdots + \lambda_{k-1} + \omega_k(\lambda_k) = \omega_k(\lambda_1 + \cdots + \lambda_k) \in W_\lambda$$

and hence also

$$\begin{aligned} \sum_{\omega_k \in \mathcal{W}_k} \omega_k(\lambda_1 + \cdots + \lambda_k) &= |\mathcal{W}_k|(\lambda_1 + \cdots + \lambda_{k-1}) + \sum_{\omega_k \in \mathcal{W}_k} \omega_k(\lambda_k) \\ &= |\mathcal{W}_k|(\lambda_1 + \cdots + \lambda_{k-1}) \in W_\lambda, \end{aligned}$$

where the last equality follows from the fact that, as a fixed point of the action of $\mathcal{W}_k$ on $\mathbb{R}\Phi_k$, the vector $\sum_{\omega_k \in \mathcal{W}_k} \omega_k(\lambda_k)$ is trivial. Hence $\lambda_1 + \cdots + \lambda_{k-1} \in W_\lambda$. The induction then yields $\lambda_1 \in W_\lambda$. □

Proof of Theorem 2.10. Let $\mathcal{W}$ denote the Weyl group of G and Φ the associated root system and E the Euclidean vector space generated by Φ. Since the representation V is irreducible, it is characterised by its unique highest weight λ. From Lemma 2.11 it follows that $S := \mathcal{W} \cdot \lambda$ generates the whole vector space E.

As the Weyl orbit of the highest weight, S consists only of weights. Since f is not identically zero on G, for four distinct weights $\lambda_1, \ldots, \lambda_4$ of V the relation $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$ cannot hold. Replacing f with h, we find that the analogous relation for sextuples of weights cannot hold either. Hence the assumptions of Theorem 2.2 are satisfied and we get $\Phi \cong \mathsf{A}_n$ for some positive integer n, as well as

$$S = \{ce_i \mid 0 \leq i \leq n\}, \quad c \neq 0.$$

Since S consists of weights, c is an integer. The intersection of S with the set of dominant weights consists of either ce_0 or ce_n, depending on the sign of c. Given that the highest weight of a representation is by definition dominant, we thus find that $\lambda = ce_0$ if $c > 0$, and $\lambda = ce_n$ otherwise. Since these two cases correspond to dual representations, which are interchanged by the outer automorphism of A_n, we can assume $c > 0$ and $\lambda = ce_0$.

Lemma 2.12.
Supose that $\mathrm{char}(L) = p > 0$. *Then* $0 < c \leq p - 1$.

Proof. Let us suppose that the projective representation induced by V is tensor-decomposable, i.e. that we find $r_1, r_2 > 1$ with $r_1 r_2 = \dim(V)$ such that in the projective representation G acts on $V = L^{r_1} \otimes L^{r_2}$ through

$$\mathrm{PGL}_{r_1,L} \times \mathrm{PGL}_{r_2,L} \to \mathrm{PGL}_V .$$

Let $g \in G$ and λ, λ' (resp. μ, μ') be two distinct eigenvalues of g on $\mathrm{PGL}_{r_1,L}$ (resp. $\mathrm{PGL}_{r_2,L}$). Then $\nu_1 := \lambda\mu$, $\nu_2 := \lambda'\mu'$, $\nu_3 := \lambda'\mu$, $\nu_4 := \lambda\mu'$ are four distinct eigenvalues of γ in the projective representation with $\nu_1\nu_2 = \nu_3\nu_4$. Since f does not vanish identically on the projection of G if it does not vanish on G itself, there is an element in G for which the above relation on the eigenvalues yields a contradiction. Consequently V is not projectively tensor-decomposable and therefore also not tensor-decomposable in the linear representation. It follows that G is almost simple and according to Steinberg's Tensor Product Theorem (cf. [**Hum06**], Theorem 2.7) we get $0 < c \leq p-1$. $\square$

We now show, independently of the characteristic of L, that c is, in fact, equal to 1. Let $\alpha := e_0 - e_1$ be the simple root corresponding to the fundamental weight e_0. Then there exists a homomorphism $\varphi_\alpha : \mathrm{SL}_2 \to G$ that sends the matrices of the form $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ to the root subgroup of G corresponding to α. Let $S_\alpha := \varphi_\alpha(\mathrm{SL}_2)$. The subspace $\bigoplus_{i\in\mathbb{Z}} V_{\lambda+i\alpha}$ is S_α-invariant and irreducible with highest weight λ by [**Jan03**], Part II, Proposition 2.11. By classical results in characteristic 0 and [**Pre87**], Theorem 1, in positive characteristic under the assumption $0 < c \leq p-1$, that is a consequence of Lemma 2.12, the associated representation of the Lie algebra $\mathfrak{l}_\alpha$ of S_α is irreducible with the same highest weight. According to [**Hum78**], Proposition 21.3, the set of weights of this representation is saturated. Namely, it consists of $\lambda, \lambda-\alpha, \dots, \lambda - r\alpha$, where $r := \langle \lambda, \alpha \rangle = c\langle e_0, e_0 - e_1\rangle = c$. More concretely,

$$
\begin{array}{l}
(c, 0, 0, \dots, 0), \\
(c-1, 1, 0, \dots, 0), \\
(c-2, 2, 0, \dots, 0), \\
\quad\vdots \\
(0, c, 0, \dots, 0)
\end{array}
$$

all appear in the representation of S_α, and hence in the representation V, as weights. However, due to the fact that f and g do not vanish identically on G, the equality

$$(c, 0, 0, \dots, 0) + (0, c, 0, \dots, 0) = (c-1, 1, 0, \dots, 0) + (1, c-1, 0, \dots, 0)$$

leads to a contradiction if $c \geq 2$.

By [**Che58**] it follows from $\Phi \cong \mathsf{A}_n$ that there is an epimorphism $\mathrm{SL}_{n+1} \twoheadrightarrow G$. The induced representation $\mathrm{SL}_{n+1} \twoheadrightarrow G \hookrightarrow \mathrm{GL}_V$ then also has highest weight e_0, which corresponds to the standard representation of SL_{n+1}. Consequently the image of the representation is $\mathrm{SL}_V = \mathrm{SL}_{n+1}$. Hence $G = \mathrm{SL}_V = \mathrm{SL}_{n+1}$. $\square$

2.4. Finite subgroups of linear algebraic groups

In this section we prove an analogue of the previous results about linear algebraic groups for their finite subgroups.

In the following let L denote an algebraically closed field of characteristic $p > 0$.

Definition 2.13.
Let G be a simply connected simple semisimple linear algebraic group over L and let $F : G \to G$ be a surjective endomorphism whose fixed point subgroup G^F is finite. Such a map F is called a *Frobenius map* and, in this setting, any non-abelian Jordan-Hölder constituent of G^F is called a *finite simple group of Lie type*.

The first result describes the general structure of finite simple groups of Lie type.

Proposition 2.14.
For almost all finite simple groups of Lie type Γ there exists a simply connected simple semisimple linear algebraic group G defined over L and a Frobenius map $F : G \to G$ such that

(1) $\Gamma = G^F/Z(G^F)$,
(2) G^F *is perfect, and*
(3) *the universal central covering group of Γ as an abstract group is G^F.*

Proof. By Definition 2.13 there exists a simply connected simple semisimple linear algebraic group Γ over L and a Frobenius map $F : G \to G$ such that Γ is a non-abelian Jordan-Hölder constituent of G^F. Then by [**GLS98**], Theorem 2.2.6 (f), the group G^F is generated by the elements whose order is a power of $p = \mathrm{char}(L)$. We can therefore apply [**GLS98**], Theorem 2.2.7, to G^F. The first part of this theorem says that with finitely many exceptions, $G^F/Z(G^F)$ is simple and therefore isomorphic to Γ; the second part says that, with the same exceptions as in the first part, the group G^F is perfect, proving (2). This also shows that Γ can only appear as the last non-trivial subgroup in any Jordan-Hölder decomposition of G^F; hence the above isomorphism between Γ and $G^F/Z(G^F)$ is an equality, which proves (1).

By [**GLS98**], Theorem 5.1.2, the simple and hence perfect group Γ has a universal central covering Γ^c which is unique up to isomorphism. The kernel $M(\Gamma)$ of the covering $\Gamma^c \to \Gamma$ is then called the *Schur multiplier* of Γ. Now assume that Γ satisfies (1) and (2). Then by [**GLS98**], Theorem 6.1.4, the Schur multiplier $M(\Gamma)$ is in almost all cases (with the exceptions listed in Table 6.1.3) isomorphic to $Z(G^F)$. By (1) the group G^F is a central extension of Γ; there exists therefore a uniquely determined homomorphism $\alpha : \Gamma^c \to G^F$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma^c \arrow[r, "\alpha"] \arrow[d, two heads] & G^F \arrow[dl, two heads] \\ \Gamma & \end{tikzcd}$$

From $M(\Gamma) \cong Z(G^F)$ it follows that α is injective and that Γ^c and G^F have the same cardinality. Consequently α is an isomorphism between G^F and the universal central covering of Γ. $\square$

Definition 2.15.
We call the finite simple groups of Lie type for which there exists a simply connected simple semisimple linear algebraic group G satisfying the conditions of Proposition 2.14 *regular*.

The next result concerns irreducible representations of finite simple groups of Lie type. It is a direct consequence of the stronger result stated in [**Hum06**], Theorems 2.11 and 20.2.

Proposition 2.16.
Let G be a simply connected simple algebraic group, $F : G \to G$ a Frobenius map and $\rho : G^F \to \mathrm{GL}(V)$ an irreducible representation of G^F on a finite dimensional L–vector space V. Then there is an irreducible representation $\rho_G : G \to \mathrm{GL}(V)$ such that ρ is the restriction of ρ_G to G^F.

Now we can finally state an analogue of Theorem 2.10.

Theorem 2.17.
Let V be an L–vector space of dimension $n \geq 2$ and $\Gamma \leqslant \mathrm{SL}(V)$ a subgroup that acts irreducibly on V. Assume that Γ is perfect and that $\Gamma/(\Gamma \cap (scalars))$ is a direct product of finite simple groups of Lie type that are regular in the sense of Definition 2.15.

If f, g and h do not vanish identically on Γ, then there is a finite subfield $k' \subset L$ and a model G' of SL_V over k' such that $\Gamma = G'(k')$.

Proof. By assumption there exist regular finite simple groups of Lie type $\Gamma_1, \ldots, \Gamma_m$ such that $\Gamma/(\Gamma \cap (\text{scalars})) = \Gamma_1 \times \cdots \times \Gamma_m$. Then by Proposition 2.14 there exist simply connected simple semisimple algebraic groups $G_1, \ldots, G_m$ and Frobenius maps $F_i : G_i \to G_i$ for all $1 \leq i \leq m$ such that $\Gamma_i = G_i^{F_i}/Z(G_i^{F_i})$ and $G_i^{F_i}$ is the universal central covering of Γ_i. Let us write $Z := \Gamma \cap (\text{scalars})$ and $\overline{\Gamma} := \Gamma/Z$.

Lemma 2.18.
There exists a surjective homomorphism $\rho : G_1^{F_1} \times \cdots \times G_m^{F_m} \to \Gamma$ such that the

following diagram is commutative:

$$\begin{tikzcd} G_1^{F_1} \times \cdots \times G_m^{F_m} \arrow[r, "\rho"] \arrow[d] & \Gamma \arrow[dl] \\ \Gamma_1 \times \cdots \times \Gamma_m & \end{tikzcd}$$

PROOF. For $1 \leq i \leq m$ let $\tilde{\Gamma}_i \subset \Gamma$ denote the preimage of Γ_i under the projection map. The short exact sequence

$$1 \to Z \to \tilde{\Gamma}_i \to \Gamma_i \to 1$$

then shows that $\tilde{\Gamma}_i$ is a central extension of Γ_i. Since $G_i^{F_i}$ is the universal central cover of Γ_i, there exists a homomorphism $\tilde{\alpha}_i : G_i^{F_i} \to \tilde{\Gamma}_i$ such that

$$\begin{tikzcd} G_i^{F_i} \arrow[r, "\tilde{\alpha}_i"] \arrow[d] & \tilde{\Gamma}_i \arrow[dl] \\ \Gamma_i & \end{tikzcd}$$

commutes. Let α_i denote the composition map $G_i^{F_i} \xrightarrow{\tilde{\alpha}_i} \tilde{\Gamma}_i \hookrightarrow \Gamma$. We define

$$\begin{array}{rcl} \rho : G_1^{F_1} \times \cdots \times G_m^{F_m} & \to & \Gamma, \\ (g_1, \ldots, g_m) & \mapsto & \alpha_1(g_1) \cdots \alpha_m(g_m). \end{array}$$

Let $1 \leq i < j \leq m$ and let $g_i \in \tilde{\Gamma}_i, g_j \in \tilde{\Gamma}_j$. Then $\overline{g_i g_j} = \overline{g_j g_i}$ in $\overline{\Gamma}$, so there exists $z \in Z$ such that $g_i g_j z = g_j g_i$, in other words such that $[g_i, g_j] = z$. This shows that the image of the commutator homomorphism

$$[\ ,\ ] : \tilde{\Gamma}_i \times \tilde{\Gamma}_j \longrightarrow \Gamma$$

is contained in the scalar subgroup Z. On the other hand, $[\ ,\ ]$ is bilinear, so it factors through a homomorphism $\tilde{\Gamma}_i/Z \times \tilde{\Gamma}_j/Z \cong \Gamma_i \times \Gamma_j \to Z$. Since Z is abelian, this map is trivial on the commutator subgroup of $\Gamma_i \times \Gamma_j$. As a direct product of non-abelian simple groups, $\Gamma_i \times \Gamma_j$ is perfect, so the map $[\ ,\ ]$ itself is trivial. It follows that $\tilde{\Gamma}_i$ and $\tilde{\Gamma}_i$ commute with each other, which in turn means that the homomorphism ρ is well-defined.

Let $g, g' \in \Gamma$. Then there exist $g_0, g_0' \in \mathrm{Im}(\rho)$ and $z, z' \in Z \subset Z(\Gamma)$ such that $g = g_0 z$ and $g' = g_0' z'$. Then $[g, g'] = [g_0 z, g_0' z'] = [g_0, g_0'] \in \mathrm{Im}(\rho)$ and hence $[\Gamma, \Gamma] \leqslant \mathrm{Im}(\rho)$. Since we have assumed Γ to be perfect, i.e., that $[\Gamma, \Gamma] = \Gamma$, we conclude that ρ is surjective. $\square$

Let $\rho : G_1^{F_1} \times \cdots \times G_m^{F_m} \to \Gamma$ be as in the above lemma. Since Γ acts irreducibly on V, the map ρ induces an irreducible representation of $G_1^{F_1} \times \cdots \times G_m^{F_m}$ on V. By [**Gor68**], Theorem 3.7.1, there exist non-trivial irreducible representations $\rho_i : G_i^{F_i} \to \mathrm{GL}(V_i)$, unique up to isomorphism, with $V \cong V_1 \otimes \cdots \otimes V_m$ and $\rho \cong \rho_1 \otimes \cdots \otimes \rho_m$. By Proposition 2.16 the representation ρ_i is the restriction of an irreducible representation of G_i on V_i that we again call ρ_i. Let us write $G := G_1 \times \cdots \times G_m$ and consider the representation

$$\rho_G := \rho_1 \otimes \cdots \otimes \rho_m : G \longrightarrow \mathrm{GL}(V).$$

As an exterior tensor product of irreducible representations, ρ_G is itself irreducible.

Since G is a connected semisimple algebraic group, $\rho_G(G) \subset \mathrm{GL}(V)$ is also connected semisimple. Let $\mathcal{W}$ be the associated Weyl group, Φ the root system of G and E the Euclidean vector space generated by Φ. Let λ denote the highest weight of the representation ρ_G. Then, by Lemma 2.11, the set $\mathcal{W} \cdot \lambda$ generates E.

Moreover, since f, g and h do not vanish identically on $\Gamma \subset \rho_G(G)$, they do not vanish identically on $\rho_G(G)$. We can therefore apply Theorem 2.10 to $\rho_G(G)$ in order to find that $\Phi \cong \mathsf{A}_{n-1}$ and $\rho_G(G) = \mathrm{SL}_V$. Since the representations ρ_i are all non-trivial and SL_V is simple, it follows that $m = 1$ and $G = G_1$. Since SL_V is simply connected, it even follows that the epimorphism $G \to \mathrm{SL}_V$ is an isomorphism. Write $F := F_1$. Then $\Gamma = G^F = \mathrm{SL}_V^F$ and, by standard classification results, as in [**Car87**], Proposition 4.5, the Frobenius map F is standard, i.e., there is a finite subfield $k' \subset L$ and a model G' of SL_V over k' such that $G^F = G'(k')$. $\square$

2.5. Subgroups acting irreducibly

The next theorem is the main result of this chapter.

THEOREM 2.19.
For every positive integer n there is an integer constant N such that for every algebraically closed field L of non-zero characteristic and every finite subgroup $\Gamma \leqslant \mathrm{SL}_n(L)$: if

(1) *every subgroup of Γ of index $\leq N$ acts irreducibly on L^n, and*
(2) *the map $\gamma \mapsto fgh(\gamma^N)$ does not vanish identically on Γ,*

then there is a finite subfield k' of L and a model G' of SL_n over k' such that $G'(k')$ is a normal subgroup of Γ of index $\leq N$.

REMARK.
We expect that the result of the theorem can be strengthened as follows: Let $\pi : \mathrm{SL}_n \to \mathrm{PGL}_n$ be the standard isogeny and G'^{ad} the image of G' under π. Then $\pi(G'(k')) \subset \pi(\Gamma) \subset G'^{\mathrm{ad}}(k')$.

Now we gather some results concerning the structure of Γ that we will use later on in the proof of the above theorem. Let us start by recalling a general result established by Larsen and Pink in [**LP98**], Theorem 0.2.

Proposition 2.20.
For every integer $n \geq 1$ there exists a constant c_n depending only on n such that any finite subgroup Γ of GL_n over any field k possesses normal subgroups $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1$ such that

(1) $[\Gamma : \Gamma_1] \leq c_n$.
(2) *Either $\Gamma_1 = \Gamma_2$, or $p := \mathrm{char}(k)$ is positive and Γ_1/Γ_2 is a direct product of finite simple groups of Lie type in characteristic p.*
(3) *Γ_2/Γ_3 is abelian of order not divisible by $\mathrm{char}(k)$.*
(4) *Either $\Gamma_3 = \{1\}$, or $p := \mathrm{char}(k)$ is positive and Γ_3 is a p-group.*

We deduce from it the following special case that arises in our setting.

Proposition 2.21.
For every integer $n \geq 1$ there exists a constant d_n such that for every algebraically closed field L of non-zero characteristic and every finite subgroup $\Gamma \subset \mathrm{GL}_n(L)$ whose subgroups of index $\leq n!$ act irreducibly on $V := L^n$, there exists a normal subgroup Γ' of Γ such that

(1) $[\Gamma : \Gamma'] \leq d_n$.
(2) *$\Gamma'/(\Gamma' \cap (scalars))$ is a direct product of finite simple groups of Lie type in characteristic p.*
(3) *If $\Gamma \subset \mathrm{SL}_n(L)$, then Γ' is perfect.*

Proof. Let c_n and $\Gamma_1, \Gamma_2, \Gamma_3$ be as in Proposition 2.20 and let $d_n := c_n \cdot n$. First we show that in our case Γ_3 is trivial. By definition, Γ_3 is a unipotent normal subgroup of Γ. Since Γ_3 is unipotent, we have $V^{\Gamma_3} \neq 0$ and since Γ_3 is normal in Γ, the subspace V^{Γ_3} of V is stabilized by Γ. On the other hand V is an irreducible representation of Γ and thus $V^{\Gamma_3} = V$. Consequently $\Gamma_3 = \{1\}$.

Now we show that Γ_2 is a scalar subgroup of Γ. Let us consider the representation of Γ_2 on V. Since Γ_2 is abelian of order not divisible by $\mathrm{char}(L)$, we get a decomposition into weight spaces

$$V = V_1 \oplus \cdots \oplus V_m.$$

By the normality of Γ_2 in Γ, the weight spaces are permuted by Γ. Let C be the centralizer of Γ_2 in Γ. Then C is the intersection of the stabilizers of $V_1, \ldots, V_m$ under the action of Γ. This yields an injection $\Gamma/C \hookrightarrow S_m$, where, for the purposes of this proof, S_m denotes the symmetric group on m elements and we find

$$[\Gamma : C] \leq |S_m| = m! \leq n!.$$

Hence the index of C in Γ is bounded by $n!$ and it follows from the assumption that C acts irreducibly on V. On the other hand C stabilizes V_1, so we get $V_1 = V$. Thus Γ_2 acts by scalar multiplication on V.

If $\Gamma \not\subset \mathrm{SL}_n(L)$, then we can take $\Gamma' = \Gamma_1$ and we have finished. Otherwise Γ_2 is a scalar subgroup of SL_n and thereby it has order at most n. Moreover, as Γ_1/Γ_2 is a product of simple groups, we find $(\Gamma_1/\Gamma_2)^{\mathrm{der}} = \Gamma_1/\Gamma_2$. Let Γ' denote in this case the derived group of Γ_1. Then the surjection

$$\Gamma' = \Gamma_1^{\mathrm{der}} \to (\Gamma_1/\Gamma_2)^{\mathrm{der}} = \Gamma_1/\Gamma_2$$

yields the estimate $[\Gamma_1 : \Gamma'] \leq |\Gamma_2| \leq n$. It follows that

$$[\Gamma : \Gamma'] = [\Gamma : \Gamma_1][\Gamma_1 : \Gamma'] \leq c_n \cdot n = d_n.$$

We can thus conclude that Γ' and d_n defined as above satisfy the conditions of the theorem. $\square$

Now we finally have all the ingredients together to prove Theorem 2.19.

Proof of Theorem 2.19. Suppose that Γ satisfies the conditions of the theorem and let d_n and Γ' be as in Proposition 2.21. Moreover, let e be the order of the largest finite simple group of Lie type that is not regular in the sense of Definition 2.15 and let $N = e \cdot \max\{n!, d_n!\}$. Then in particular we have $d_n \leq N$ and, with the above definition, Γ' is a normal subgroup of Γ of index $\leq N$. Hence Γ' acts irreducibly on $V := L^n$, and $\Gamma'/(\Gamma' \cap (\text{scalars}))$ is a product of finite simple groups of Lie type.

Let us suppose that one of the simple groups appearing in the decomposition of $\Gamma'/(\Gamma' \cap (\text{scalars}))$ is not regular. Let Γ_0 be its preimage in Γ' and Γ_c the preimage of its complement. Then, on the one hand, Γ' is a central product of Γ_0 and Γ_c and by [**Gor68**], Theorem 3.7.1, there exist irreducible representations V_0 of Γ_0 and V_c of Γ_c such that $V \cong V_0 \otimes V_c$ as a representation of Γ'. Since V is faithful and Γ_0 is non-abelian, V_0 has dimension > 1. Therefore $\dim V_c \neq \dim V$ and Γ_c does not act irreducibly on V. On the other hand

$$[\Gamma : \Gamma_c] = [\Gamma : \Gamma'][\Gamma' : \Gamma_c] \leq d_n \cdot e \leq N,$$

so by the first assumption of the theorem Γ_c acts irreducibly on V, a contradiction. Consequently all simple factors appearing in the decomposition of $\Gamma'/(\Gamma' \cap (\text{scalars}))$ are regular.

Moreover, since N is a multiple of $d_n!$, for all $\gamma \in \Gamma$ we find that $\gamma^N \in \Gamma'$. Consequently $\gamma \mapsto fgh(\gamma)$ does not vanish identically on Γ' by the second assumption of the theorem. In particular neither of the functions f, g and h vanishes identically on Γ'. Now we can apply Theorem 2.17 to Γ' and its representation on $V = L^n$: there is a finite field $k' \subset L$ and a model G' of SL_n over k' such that $G'(k') = \Gamma'$. $\square$

We close this chapter by establishing an auxiliary result that in some cases can give a more precise description of the field k' and of the algebraic group G' determined by Theorem 2.19.

Proposition 2.22.
Let k, k' be finite subfields of L and G' a model of SL_n over k' such that $G'(k')$ is a subgroup of $\mathrm{SL}_n(k)$. If k is a subfield of k', then $k = k'$ and $G'(k') = \mathrm{SL}_n(k)$.

PROOF. Let $q = |k|$ and $q' = |k'|$. From $k \subseteq k'$ follows $q \leq q'$. By [**Hum06**], Table 1.6.1, if G' is non-split, then

$$|G'(k')| = (q')^{n(n+1)/2} \prod_{i=1}^{n} ((q')^{i+1} - (-1)^{i+1}) > q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - 1) = |\operatorname{SL}_n(k)|,$$

a contradiction to $G'(k') \leqslant \operatorname{SL}_n(k)$, so this case cannot occur. If G' is split, then

$$|G'(k')| = (q')^{n(n+1)/2} \prod_{i=1}^{n} ((q')^{i+1} - 1) \geq q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1} - 1) = |\operatorname{SL}_n(k)|,$$

with equality if and only if $q = q'$. In that case $k = k'$; the second desired equality follows from $G'(k') \leqslant \operatorname{SL}_n(k)$. $\square$

CHAPTER 3

Preliminary results on Drinfeld modules

3.1. General results

In this section we list the most important known results about Drinfeld modules in special characteristic, using the notation introduced in Chapter 1.

Let φ be a Drinfeld A-module over a finitely generated field K, of special characteristic $\mathfrak{p}_0$.

PROPOSITION 3.1 (cf. [**Gos96**], Proposition 4.7.4 and Remark 4.7.5).
There exists a finite separable extension K' of K inside of $\overline{K}$ such that

$$\mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_{K'}(\varphi).$$

Let $D := \mathrm{End}_{\overline{K}}(\varphi)$, let Z denote the center of $D \otimes_A F$ and let us write

$$\dim_Z D \otimes_A F = d^2 \quad \text{and} \quad [Z/F] = e.$$

There exists an iteger $r \geq 1$ such that $\mathrm{rank}(\varphi) = rde$. Fix a maximal commutative subring $\hat{A}$ of D and let $\hat{\varphi} : \hat{A} \to \overline{K}\{\tau\}$ denote its tautological embedding. This is a Drinfeld $\hat{A}$-module of rank r, except that $\hat{A}$ is not necessarily a maximal order in its quotient field. Let $\tilde{A}$ denote the integral closure of $\hat{A}$ in its quotient field and $\tilde{F}$ denote the common quotient field of $\hat{A}$ and $\tilde{A}$. By [**Gos96**], Corollary 4.7.15, the ring $D \otimes_A F$ is a division algebra over F; thus its commutative subring $\hat{A} \otimes_A F$ is a field. It follows that $\hat{A} \otimes_A F = \tilde{F}$ and $\tilde{F}$ is a subfield of $D \otimes_A F$. By [**Hay79**], Proposition 3.2, there exists a Drinfeld module $\tilde{\varphi} : \tilde{A} \to \overline{K}\{\tau\}$ such that $\tilde{\varphi}|\hat{A}$ is isogenous to $\hat{\varphi}$ and the isogeny in question induces an isomorphism

$$\mathrm{End}_{\overline{K}}(\hat{\varphi}) \otimes_A \tilde{F} \cong \mathrm{End}_{\overline{K}}(\tilde{\varphi}) \otimes_A \tilde{F}.$$

On the other hand the definition of endomorphisms implies that

$$\mathrm{End}_{\overline{K}}(\hat{\varphi}) \otimes_A \tilde{F} \cong \mathrm{Cent}_{\mathrm{End}_{\overline{K}}(\varphi) \otimes_A F}(\hat{A}) = \tilde{F}$$

and thus $\mathrm{End}_{\overline{K}}(\tilde{\varphi}) = \tilde{A}$.

It was shown in [**Gos96**], Proposition 4.7.17, that $\mathrm{End}_{\overline{K}}(\varphi) \otimes_A F_\infty$ is a division algebra over F_∞; consequently its commutative subring $\tilde{F} \otimes_A F_\infty$ is a field, which shows that the place ∞ does not split in $\tilde{F}$. For later use, let us denote by $\tilde{\infty}$ the place of $\tilde{F}$ above ∞ and by $\mathfrak{P}_0$ the characteristic of $\tilde{\varphi}$. The latter is a place above the characteristic $\mathfrak{p}_0$ of φ.

Let K' be a finite extension of K as in Proposition 3.1. For a place $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F we have $G_{K'}$-equivariant isomorphisms

$$\bigoplus_{\mathfrak{P}|\mathfrak{p}} T_{\mathfrak{P}}(\tilde{\varphi}) \otimes_{\tilde{A}_{\mathfrak{P}}} \tilde{F}_{\mathfrak{P}} \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} T_{\mathfrak{P}}(\hat{\varphi}) \otimes_{\hat{A}_{\mathfrak{P}}} \tilde{F}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}; \tag{3.1}$$

hence the image of $G_{K'}$ in the representation on $T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}$ coincides with the one on $\bigoplus_{\mathfrak{P}|\mathfrak{p}} T_{\mathfrak{P}}(\tilde{\varphi}) \otimes_{\tilde{A}_{\mathfrak{P}}} \tilde{F}_{\mathfrak{P}}$.

Let us from now on assume that all endomorphisms of φ are already defined over K, i.e., that $D = \mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_K(\varphi)$.

Lemma 3.2.
Let $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ be a place of F and $\mathfrak{P}$ a place of $\tilde{F}$ above $\mathfrak{p}$. The representations $\rho_{\mathfrak{p}}$ and $\rho_{\mathfrak{P}}$ become isomorphic upon tensoring with $\tilde{F}_{\mathfrak{P}}$.

Proof. Since $\hat{A}$ is a maximal commutative subring of D and its quotient field is $\tilde{F}$, we have $D \otimes_A \tilde{F} \cong \mathrm{M}_{d\times d}(\tilde{F})$ and in turn

$$D \otimes_A \tilde{F}_{\mathfrak{P}} \cong \mathrm{M}_{d\times d}(\tilde{F}_{\mathfrak{P}}).$$

Thus, tensoring $\rho_{\mathfrak{p}}$ with $\tilde{F}_{\mathfrak{P}}$ yields a representation

$$\rho_{\mathfrak{p}} \otimes \tilde{F}_{\mathfrak{P}} : G_K \longrightarrow \mathrm{Cent}_{\mathrm{GL}_{rd}(\tilde{F}_{\mathfrak{P}})}(D \otimes_A \tilde{F}_{\mathfrak{P}}) \cong \mathrm{Cent}_{\mathrm{GL}_{rd}(\tilde{F}_{\mathfrak{P}})}(\mathrm{M}_{d\times d}(\tilde{F}_{\mathfrak{P}})) \cong \mathrm{GL}_r(\tilde{F}_{\mathfrak{P}}).$$

On the other hand, starting with $\rho_{\mathfrak{P}}$ we find

$$\rho_{\mathfrak{P}} \otimes \tilde{F}_{\mathfrak{P}} : G_K \longrightarrow \mathrm{GL}_r(\tilde{A}_{\mathfrak{P}}) \otimes \tilde{F}_{\mathfrak{P}} \cong \mathrm{GL}_r(\tilde{F}_{\mathfrak{P}}).$$

The isomorphism of the representations follows from the above G_K-equivariant isomorphism of rational Tate modules. □

Let $G_{\mathfrak{p}}$ denote the Zariski closure of $\Gamma_{\mathfrak{p}}$, which is an algebraic subgroup of the centralizer of $D \otimes_A F_{\mathfrak{p}}$ in the algebraic group $\underline{\mathrm{Aut}}_{F_{\mathfrak{p}}}(T_{\mathfrak{p}}(\varphi) \otimes F_{\mathfrak{p}}) \cong \mathrm{GL}_{rd,F_{\mathfrak{p}}}$.

Theorem 3.3.
For all places $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F we have $G_{\mathfrak{p}} = \mathrm{Cent}_{\mathrm{GL}_{rd,F_{\mathfrak{p}}}}(D \otimes_A F_{\mathfrak{p}})$, in other words $\Gamma_{\mathfrak{p}}$ is Zariski dense in $\mathrm{Cent}_{\mathrm{GL}_{rd,F_{\mathfrak{p}}}}(D \otimes_A F_{\mathfrak{p}})$.

Proof. Let $\mathfrak{P}$ be a place of $\tilde{F}$ above $\mathfrak{p}$. By [**Pin06a**], Theorem 1.1, the group $\Gamma_{\mathfrak{P}}$ is Zariski dense in $\mathrm{GL}_{r,\tilde{F}_{\mathfrak{P}}}$. On the other hand by Lemma 3.2 we have

$$G_{\mathfrak{p}} \times_{F_{\mathfrak{p}}} \tilde{F}_{\mathfrak{P}} \cong \mathrm{GL}_{r,\tilde{F}_{\mathfrak{P}}},$$

i.e., $G_{\mathfrak{p}} \subset \mathrm{Cent}_{\mathrm{GL}_{rd,F_{\mathfrak{p}}}}(D \otimes_A F_{\mathfrak{p}})$ is a model of $\mathrm{GL}_{r,\tilde{F}_{\mathfrak{P}}}$ over $F_{\mathfrak{p}}$. The desired equality follows. □

Combining the above theorem with [**PT06**], Lemma 3.8, yields

Corollary 3.4.
Let $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ be a place of F such that $\Gamma_{\mathfrak{p}}^{\mathrm{geom}} \subset \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_{\mathfrak{p}})}(D \otimes_A A_{\mathfrak{p}})$. Then $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$ is Zariski dense in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd,F_{\mathfrak{p}}}}(D \otimes_A F_{\mathfrak{p}})$.

3.2. Results building towards Theorem 1.1

Let us in this section assume that φ satisfies the conditions of Theorem 1.1. We make explicit the implications for this case of a few previously established results.

Theorem 3.5.
For every non-empty finite set P of places $\neq \mathfrak{p}_0, \infty$ of F, the subgroup

$$\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_P)}(D \otimes_A A_P) \cap \Gamma_P^{\mathrm{geom}}$$

is open in both $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_P)}(D \otimes_A A_P)$ and Γ_P^{geom}.

Proof. A careful reading of the proofs of Theorems 6.1 and 6.2 of [**Pin06b**] shows that, even though one of the original assumptions on φ required it not to be isomorphic to a Drinfeld module defined over a finite field, for the theorems to hold it is sufficient to have the analogous assumption for $\tilde{\varphi}$. By [**Pin06b**], Proposition 2.1, this is equivalent to $r = \mathrm{rank}(\tilde{\varphi}) \geq 2$. Since this is one of the assumptions of Theorem 1.1, we can apply [**Pin06b**], Theorems 6.1 and 6.2, to φ. Combining them shows that there exists a subfield E of F with $[F/E] < \infty$ and $B := E \cap A$ that is uniquely defined by either one of the following two properties:

(1) For every infinite subring $C \subset A$ we have $\mathrm{End}_{\overline{K}}(\varphi|C) \subset \mathrm{End}_{\overline{K}}(\varphi|B)$.
(2) For every non-empty finite set P of places $\neq \mathfrak{p}_0, \infty$ of F, let Q denote the set of places of E below those in P and let G_Q denote the centralizer of $\mathrm{End}_{\overline{K}}(\varphi|B) \otimes E_Q$ in $\underline{\mathrm{Aut}}_{E_Q}(T_Q(\varphi|B) \otimes E_Q)$. Then $G_Q^{\mathrm{der}}(B_Q) \cap \Gamma_Q^{\mathrm{geom}}$ is open in both $G_Q^{\mathrm{der}}(B_Q)$ and Γ_Q^{geom}.

Since F satisfies property (1) by the assumptions of Theorem 1.1 and E is uniquely determined, we have $E = F$. The theorem then follows from property (2). $\square$

The following result is a special case a theorem proved by Matthias Traulsen in his thesis ([**PT06**], Theorem B) for the case where K has transcendence degree 1 and later generalized by Egon Rütsche ([**PR09b**], Theorem 0.2) for fields of arbitrary transcendence degree.

Theorem 3.6.
For almost all primes $\mathfrak{p}$ of A the rings $D \otimes_A A_{\mathfrak{p}}$ and $A_{\mathfrak{p}}[\Gamma_{\mathfrak{p}}]$ are commutants of

each other in $\mathrm{End}_{A_\mathfrak{p}}(T_\mathfrak{p}(\varphi))$. *More precisely, for almost all primes* $\mathfrak{p}$ *we have*

$$D \otimes_A A_\mathfrak{p} \cong \mathrm{M}_{d\times d}(A_\mathfrak{p}) \quad \textit{and} \quad A_\mathfrak{p}[\Gamma_\mathfrak{p}] \cong \mathrm{M}_{r\times r}(A_\mathfrak{p}).$$

Let $\mathfrak{p}$ be a prime of A for which Theorem 3.6 holds. Then

$$\mathrm{Cent}_{\mathrm{GL}_r(A_\mathfrak{p})}(D \otimes_A A_\mathfrak{p}) \cong \mathrm{Cent}_{\mathrm{GL}_{rd}(A_\mathfrak{p})}(\mathrm{M}_{d\times d}(A_\mathfrak{p})) \cong \mathrm{GL}_r(A_\mathfrak{p})$$

and the Galois representation associated to φ at $\mathfrak{p}$ can simply be rewritten as

$$\rho_\mathfrak{p} : G_K \longrightarrow \mathrm{GL}_r(A_\mathfrak{p}).$$

The following result is then a direct corollary of the above theorem:

Corollary 3.7.
For all primes $\mathfrak{p}$ *of* A *for which Theorem 3.6 holds, the residual representation* $\overline{\rho_\mathfrak{p}} : G_K \to \mathrm{GL}_r(k_\mathfrak{p})$ *is absolutely irreducible.*

3.3. Reduction steps building towards Theorem 1.1

Let us again assume that φ satisfies the assumptions of Theorem 1.1. The Galois representations remain the same under replacing φ by an isomorphic Drinfeld module, thus doing so does not alter the outcome of the aforementioned theorem. Since it only attempts to describe the image of G_K^{geom} up to commensurability, the theorem is also invariant under replacing $K\overline{\kappa}$ by a finite extension, and thus under replacing K by a finite extension. Therefore we may make the following additional assumptions, all direct consequences of previously established well-known facts, on φ before tackling its proof:

(a) $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ is contained in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_\mathfrak{p})}(D \otimes_A A_\mathfrak{p})$ for every place $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F. Indeed, fix a maximal commutative subring $\hat{A}$ of D and let $\tilde{A}, \tilde{F}, \tilde{\varphi}, \mathfrak{P}_0$ and $\tilde{\infty}$ be as in Section 3.1. By [**Pin06b**], Proposition 2.3 we may assume that $\Gamma_\mathfrak{P}^{\mathrm{geom}} \subset \mathrm{SL}_r(\tilde{A}_\mathfrak{P})$ for all places $\mathfrak{P} \neq \mathfrak{P}_0, \tilde{\infty}$ of $\tilde{F}$. The desired result for $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ then follows from Lemma 3.2.

(b) φ has semistable reduction everywhere.

Let x be one of the finitely many places of K at which φ has bad reduction. The Tate uniformization of φ at x (cf. [**Dri74**], §7) is a pair (φ_x, Λ_x) where φ_x is a Drinfeld A-module over K_x of rank $r'd < rd$ with good reduction at x and Λ_x is, via φ_x, an A-lattice in K_x^{sep} of rank $rd - r'd$. Since D_x acts on Λ_x through a finite quotient, after replacing K by a finite extension we may also assume that

(c) For every place x of bad reduction, the decomposition group D_x acts trivially on Λ_x.

These assumptions will be in use from Chapter 4 until the end of Chapter 8.

3.4. Frobenius action

In this section we temporarily drop most assumptions of Theorem 1.1 on φ; initially we only assume that $\mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_K(\varphi)$.

Let x be a place of K where φ has good reduction. We let $\mathrm{Frob}_x \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ denote any element of the decomposition group above x that acts by $u \mapsto u^{|k_x|}$ on the residue field k_x^{sep}.

We start by adapting a few well-known results about Drinfeld modules with minimal endomorphism ring to the general case. Lemmas 3.8 and 3.9 are useful tools for carrying out these adaptations.

Lemma 3.8.
The Drinfeld $\tilde{A}$-module $\tilde{\varphi}$ has good reduction at a place x of K if and only if φ has good reduction at x.

Proof. The following good reduction criterion for Drinfeld modules in special characteristic is a special case of the criterion proved by Takahashi in [**Tak82**], Theorem 1, for arbitrary Drinfeld modules:

Reduction Criterion (in special characteristic). *Let φ be a Drinfeld A-module over a field K of characteristic $\mathfrak{p}_0 \neq 0$. Let x be a place of K and let $\mathfrak{p}$ be a prime ideal of A different from $\mathfrak{p}_0$. Then φ has good reduction at x if and only if the $\mathfrak{p}$-adic Tate module $T_{\mathfrak{p}}(\varphi)$ is unramified at v.*

Let $\mathfrak{p} \neq \mathfrak{p}_0$ be a place of A. By (3.1) we have a G_K-equivariant isomorphism

$$T_{\mathfrak{p}}(\varphi) \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{P}|\mathfrak{p}} T_{\mathfrak{P}}(\tilde{\varphi}) \otimes_{\tilde{A}_{\mathfrak{P}}} \tilde{F}_{\mathfrak{P}}.$$

Assume that φ has good reduction at x. Then, by the Reduction Criterion, the left hand side is unramified at x. Since the isomorphism is G_K-equivariant, the right hand side is also unramified at x; applying the Reduction Criterion to the Drinfeld module $\tilde{\varphi} : \tilde{A} \to K\{\tau\}$ in the other direction, we find that $\tilde{\varphi}$ has good reduction at x.

The converse follows similarly from applying the Reduction Criterion to the above isomorphism of rational Tate modules. □

The result of the last proposition will be used implicitly in every argument that involves passing from φ to $\tilde{\varphi}$ and then considering the set of places of good reduction for $\tilde{\varphi}$.

Lemma 3.9.
Let $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ be a place of F and $\mathfrak{P}$ a place of $\tilde{F}$ above $\mathfrak{p}$. Then the characteristic polynomial of $\rho_{\mathfrak{p}}(\mathrm{Frob}_x)$ is equal to the characteristic polynomial of $\rho_{\mathfrak{P}}(\mathrm{Frob}_x)$.

Proof. By Lemma 3.2 we have $\rho_{\mathfrak{p}} \otimes \tilde{F}_{\mathfrak{P}} \cong \rho_{\mathfrak{P}} \otimes \tilde{F}_{\mathfrak{P}}$. The result follows directly from the fact that tensoring with $\tilde{F}_{\mathfrak{P}}$ does not change the characteristic polynomial on either side of the isomorphism. $\square$

Proposition 3.10.
For every place $\mathfrak{p}$ of F different from the characteristic $\mathfrak{p}_0$ of φ and from ∞, the representation $\rho_{\mathfrak{p}}$ is unramified at x and the characteristic polynomial f_x of $\rho_{\mathfrak{p}}(\mathrm{Frob}_x)$ has coefficients in A and is independent of $\mathfrak{p}$.

Proof. Applying [**Gos96**], Theorem 4.12.12 (b), to the Drinfeld $\tilde{A}$-module $\tilde{\varphi}$, we find that for every place $\mathfrak{P}$ of $\tilde{F}$ different from the characteristic $\mathfrak{P}_0$ of $\tilde{\varphi}$ and from $\tilde{\infty}$, the representation $\rho_{\mathfrak{P}}$ is unramified at x and the characteristic polynomial g_x of $\rho_{\mathfrak{P}}(\mathrm{Frob}_x)$ has coefficients in $\tilde{A}$ and is independent of $\mathfrak{P}$.

Let $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ be a place of F and $\mathfrak{P}$ a place of $\tilde{F}$ above $\mathfrak{p}$. By Lemma 3.9 we have $f_x = g_x$ and thus g_x has coefficients in $\tilde{A} \cap A_{\mathfrak{p}} = A$. The other two properties are direct consequences of the isomorphism

$$\rho_{\mathfrak{p}} \otimes \tilde{F}_{\mathfrak{P}} \cong \rho_{\mathfrak{P}} \otimes \tilde{F}_{\mathfrak{P}}$$

that was established in Lemma 3.2. $\square$

Let $\alpha_1, \ldots, \alpha_r$ be the roots of f_x in an algebraic closure $\overline{F}$ of F, with repetitions if necessary. Consider any normalized valuation v of F and consider an extension $\overline{v}$ of v to $\overline{F}$. Let k_v denote the residue field at v.

Proposition 3.11.
There exists an integer d_0 independent of x with $1 \leq d_0 \leq d$ such that the following properties hold:

(1) *If v does not correspond to $\mathfrak{p}_0$ or ∞, then for all $1 \leq i \leq r$ we have*

$$\overline{v}(\alpha_i) = 0.$$

(2) *If v corresponds to ∞, then for all $1 \leq i \leq r$ we have*

$$\overline{v}(\alpha_i) = -\frac{1}{rd} \cdot \frac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]}.$$

(3) *If v corresponds to $\mathfrak{p}_0$, then there exists an integer $0 < s_x \leq r$ such that*

$$\overline{v}(\alpha_i) = \begin{cases} \frac{1}{s_x d_0} \cdot \frac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]} & \text{for precisely } s_x \text{ of the } \alpha_i, \text{ and} \\ 0 & \text{for the remaining } r - s_x \text{ of the } \alpha_i. \end{cases}$$

Proof. In this proof let v always denote a normalized valuation of $\tilde{F}$ and $\overline{v}$ an extension of v to $\overline{F}$.

By Lemma 3.9 the characteristic polynomial of Frob_x associated to φ is the same as the one associated to the Drinfeld $\tilde{A}$-module $\tilde{\varphi}$. Applying [**Dri77**], Proposition 2.1, to $\tilde{\varphi}$, we find

(1) If v does not correspond to $\mathfrak{P}_0$ or $\tilde{\infty}$, then for all $1 \leq i \leq r$ we have

$$\overline{v}(\alpha_i) = 0.$$

(2) If v corresponds to $\tilde{\infty}$, then for all $1 \leq i \leq r$ we have

$$\overline{v}(\alpha_i) = -\frac{1}{r} \cdot \frac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]}.$$

(3) If v corresponds to $\mathfrak{P}_0$, then there exists an integer $0 < s_x \leq r$ such that

$$\overline{v}(\alpha_i) = \begin{cases} \frac{1}{s_x} \cdot \frac{[k_x/\mathbb{F}_q]}{[k_v/\mathbb{F}_q]} & \text{for precisely } s_x \text{ of the } \alpha_i, \text{ and} \\ 0 & \text{for the remaining } r - s_x \text{ of the } \alpha_i. \end{cases}$$

Let us recall that $[\tilde{F}/F] = d$. The result then follows directly from passing from normalized valuations of $\tilde{F}$ to the corresponding normalized valuations of F. $\square$

Let Ad denote the adjoint representation of GL_r. Proposition 3.10 implies that the characteristic polynomial of $\mathrm{Ad}(\rho_\mathfrak{p}(\mathrm{Frob}_x))$ has coefficients in F and is independent of $\mathfrak{p}$. In the case where φ satisfies the assumptions of Theorem 1.1, these characteristic polynomials can be used to give a characterization of the field F.

Proposition 3.12 (cf. [**Pin06b**], Theorem 1.3).
Let φ be a Drinfeld A-module satisfying the assumptions of Theorem 1.1. Let X be an integral scheme of finite type over $\mathbb{F}_p$, whose function field K' is a finite extension of K, and over which φ has good reduction. Let Σ be any set of closed points $x \in X$ of Dirichlet density 1.

(1) *If $p \neq 2$ or $r \neq 2$, then the subfield F^{trad} generated by the traces of* $\mathrm{Ad}(\rho_\mathfrak{p}(\mathrm{Frob}_x))$ *for all $x \in X$ is equal to F.*
(2) *If $p = r = 2$, then either the subfield F^{trad} generated by the traces of* $\mathrm{Ad}(\rho_\mathfrak{p}(\mathrm{Frob}_x))$ *for all $x \in X$ or the subfield generated by their square roots is equal to F.*

Proof. Applying [**Pin06b**], Theorems 1.2 and 1.3, to $\tilde{\varphi} : \tilde{A} \to K\{\tau\}$ yields the analogous result for the subfield $F^{\mathrm{trad}}_{\tilde{\varphi}}$ of F generated by the traces of $\mathrm{Ad}(\rho_\mathfrak{P}(\mathrm{Frob}_x))$ for all $x \in X$. The proposition then follows from Lemma 3.9. $\square$

We deduce from this a result concerning the field generated by the traces of Frobenius elements in the residual adjoint representation.

Proposition 3.13.
Let φ be a Drinfeld A-module satisfying the assumptions of Theorem 1.1. There exists a finite set of primes S of A such that

(1) *for all primes $\mathfrak{p} \notin S$ the traces of* $\mathrm{Ad}(\rho_\mathfrak{p}(\mathrm{Frob}_x)) \mod \mathfrak{p}$ *for all places of good reduction x of K generate $k_\mathfrak{p}$, and*

(2) *for all distinct primes* $\mathfrak{p}_1, \mathfrak{p}_2 \notin S$ *the traces of* $\mathrm{Ad}(\rho_{\mathfrak{p}_1}(\mathrm{Frob}_x)) \mod \mathfrak{p}_1\mathfrak{p}_2$ *for all places of good reduction* x *of* K *generate* $k_{\mathfrak{p}_1} \times k_{\mathfrak{p}_2}$.

PROOF. Let Σ denote the set of places of K at which φ has good reduction and let B denote the ring of elements of F that are regular outside of $\mathfrak{p}_0$. For $x \in \Sigma$ let $\alpha_1, \ldots, \alpha_r \in \overline{F}$ denote the eigenvalues of Frob_x. By Proposition 3.11 they all have trivial valuation at places not above $\mathfrak{p}_0, \infty$ and constant valuation above ∞. The eigenvalues of $\mathrm{Ad}(\mathrm{Frob}_x)$, which are the ratios α_i/α_j, are thus units at all places not above $\mathfrak{p}_0$. Consequently

$$\mathrm{tr}(\mathrm{Ad}(\rho_{\mathfrak{p}}(\mathrm{Frob}_x))) = \sum_{i,j=1}^{r} \frac{\alpha_i}{\alpha_j} \in B.$$

Let B' denote the subring generated by the traces of $\mathrm{Ad}(\rho_{\mathfrak{p}}(\mathrm{Frob}_x))$ for all $x \in \Sigma$. We distinguish two cases according to Proposition 3.12.
First let us suppose that the field of fractions of B' is equal to F. Then B' has finite index in B. Let $\mathfrak{f} := \{b \in B' \mid bB \subseteq B'\}$. Then $\mathfrak{f}$ is a non-trivial ideal of B and the set S of primes of B containing ∞ and those dividing $\mathfrak{f}$ is finite. Moreover for all $\mathfrak{p} \notin S$ the intersection $\mathfrak{p} \cap B'$ is a prime of B' and $B'/\mathfrak{p} \cap B' \cong B/\mathfrak{p}$. On the other hand for all $\mathfrak{p} \notin S$ we have $B/\mathfrak{p} \cong k_{\mathfrak{p}}$. Hence for all $\mathfrak{p} \notin S$ the map $B' \to B/\mathfrak{p} \to k_{\mathfrak{p}}$ is surjective and the first assertion follows. For the second, we use the fact that for $\mathfrak{p}_1 \neq \mathfrak{p}_2$ outside of S, the ideal $\mathfrak{p}_1\mathfrak{p}_2 \cap B'$ can be decomposed into a product of prime ideals in B' as $(\mathfrak{p}_1 \cap B')(\mathfrak{p}_2 \cap B')$. Then, using the Chinese Remainder Theorem and the above observations, we find

$$\begin{aligned} B'/\mathfrak{p}_1\mathfrak{p}_2 \cap B' &= B'/\mathfrak{p}_1 \cap B' \times B'/\mathfrak{p}_1 \cap B' \\ &= B/\mathfrak{p}_1 \times B/\mathfrak{p}_2 \\ &= k_{\mathfrak{p}_1} \times k_{\mathfrak{p}_2}, \end{aligned}$$

proving the surjectivity of the map $B' \to B/\mathfrak{p}_1\mathfrak{p}_2 \to k_{\mathfrak{p}_1} \times k_{\mathfrak{p}_2}$. This is the second assertion.

By Proposition 3.12 the only remaining case is where $p = r = 2$ and the field of fractions of B' is equal to F^2. Let $\mathfrak{q}_0$ denote the place of F^2 below $\mathfrak{p}_0$ and B_{F^2} the ring of elements of F^2 that are regular outside of $\mathfrak{q}_0$. Then $[B_{F^2} : B'] < \infty$ and, as above, there exists a finite set S_{F^2} of places of F^2 such that for all $\mathfrak{q} \notin S_{F^2}$ the map $B' \to B_{F^2}/\mathfrak{q}$ is surjective. For a prime $\mathfrak{q} \notin S_{F^2}$ of F^2 let $\mathfrak{p}$ be a place of F above $\mathfrak{q}$. Since F/F^2 is purely inseparable, we have $B_{F^2}/\mathfrak{q} \cong k_{\mathfrak{p}}$. Let S be the set of primes of F lying above the places contained in S_{F^2}. For this choice of S, both assertions now follow analogously to the first case. $\square$

CHAPTER 4

The surjectivity of the residual representation

Let K denote a field of transcendence degree 1 and let $\varphi : A \to K\{\tau\}$ be a Drinfeld module satisfying the assumptions of Theorem 1.1. Proving that the residual representation is surjective at almost all places is the first step towards the proof of Theorem 1.1.

Throughout the chapter we assume that the reduction steps introduced in Section 3.3 are in effect.

4.1. Surjectivity at a given prime.

For all primes $\mathfrak{p}$ of A let $\Delta_{\mathfrak{p}}$ denote the image of G_K under $\overline{\rho_{\mathfrak{p}}}$ and $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$ the image of G_K^{geom}. Our aim is to prove the following result:

Theorem 4.1.
In the above situation, we have $\mathrm{SL}_r(k_{\mathfrak{p}}) = \Delta_{\mathfrak{p}}^{\mathrm{geom}}$ *for almost all primes* $\mathfrak{p}$ *of* A.

Outline of the proof. First we prove that for almost all primes $\mathfrak{p}$ of A the finite group $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$ satisfies the assumptions of Theorem 2.19. Next we prove that the field k' given by Theorem 2.19 is almost always equal to $k_{\mathfrak{p}}$. Using Proposition 2.22, we then deduce the desired equality.

Definition 4.2.
We denote by S_1 the finite set of primes of A for which Theorem 3.6 does not hold.

Since we are mainly interested in statements that hold for almost all primes of A, we can focus our attention on primes not contained in S_1. One particular advantage of this is that for $\mathfrak{p} \notin S_1$ the residual representation at $\mathfrak{p}$ can be simply written as

$$\overline{\rho_{\mathfrak{p}}} : G_K \longrightarrow \mathrm{GL}_r(k_{\mathfrak{p}}).$$

Proposition 4.3.
For every integer $c \geq 1$ *there exists a finite set of primes* $S(c) \supset S_1$ *of* A *such that for all* $\mathfrak{p} \notin S(c)$ *every subgroup of* $\Delta_{\mathfrak{p}}$ *of index* $\leq c$ *acts absolutely irreducibly.*

Proof. For any prime $\mathfrak{p}$ of A and any subgroup $H \leqslant \Delta_{\mathfrak{p}}$ let H' denote $\bigcap_{\gamma\in\Delta_{\mathfrak{p}}} \gamma H \gamma^{-1}$. This is a normal subgroup of $\Delta_{\mathfrak{p}}$; we define $\alpha_{\mathfrak{p},H}$ as the composite of the following homomorphisms

$$\alpha_{\mathfrak{p},H} : G_K \longrightarrow \Delta_{\mathfrak{p}} \twoheadrightarrow \Delta_{\mathfrak{p}}/H'.$$

Lemma 4.4.
For every integer $c \geq 1$ there exists a finite set of primes $S_0(c)$ of A such that for all $\mathfrak{p} \notin S_0(c)$ and all subgroups H of $\Delta_{\mathfrak{p}}$ of index $\leq c$ the homomorphism $\alpha_{\mathfrak{p},H}$ is unramified at all places of K.

Proof. For every place x of K at which φ has good reduction, the inertia group I_x acts trivially on $\varphi[\mathfrak{p}]$ and therefore the homomorphism $\alpha_{\mathfrak{p},H}$ is unramified at these places. Since there are only finitely many places x of K where φ has bad reduction, it is then enough to prove the lemma for one of them. By reduction step (b), φ has semistable reduction at x. Let (φ_x, Λ_x) be its Tate uniformization at x and let $\mathfrak{p}$ be any prime of A. Then there is an exact sequence

$$0 \longrightarrow \varphi_x[\mathfrak{p}] \longrightarrow \varphi[\mathfrak{p}] \longrightarrow \Lambda_x/\mathfrak{p}\Lambda_x \longrightarrow 0$$

of representations of the decomposition group D_x that is invariant under the action of $\mathrm{End}_{\overline{K}}(\varphi)$. By good reduction the inertia group I_x acts trivially on $\varphi_x[\mathfrak{p}]$ and by reduction step (c) it also acts trivially on $\Lambda_x/\mathfrak{p}\Lambda_x$. Therefore its image under $\overline{\rho_{\mathfrak{p}}}$ lies in a subgroup of the form

$$\left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array}\right) \cong \mathrm{Hom}_{\mathrm{End}_{\overline{K}}(\varphi)\otimes_A k_{\mathfrak{p}}}(\Lambda_x/\mathfrak{p}\Lambda_x, \varphi_x[\mathfrak{p}]).$$

Let H be a subgroup of $\Delta_{\mathfrak{p}}$ of index $\leq c$. Then $|\Delta_{\mathfrak{p}}/H'|$, and thereby every element of $\Delta_{\mathfrak{p}}/H'$, has order dividing $c!$. In particular we have $\alpha_{\mathfrak{p},H}(\mathrm{Frob}_x^{c!}) = 1$. It follows that the restriction of $\alpha_{\mathfrak{p},H}$ to I_x factors through the group of coinvariants

$$\mathrm{Hom}_{\mathrm{End}_{\overline{K}}(\varphi)\otimes_A k_{\mathfrak{p}}}(\Lambda_x/\mathfrak{p}\Lambda_x, \varphi_x[\mathfrak{p}])_{\mathrm{Frob}_x^{c!}}.$$

It suffices to prove that this group is trivial for almost all $\mathfrak{p}$. Since $\mathrm{Frob}_x^{c!}$ acts trivially on $\Lambda_x/\mathfrak{p}\Lambda_x$ by reduction step (c), it is enough to prove that the group of coinvariants $\varphi_x[\mathfrak{p}]_{\mathrm{Frob}_x^{c!}}$ vanishes.

Let f_x denote the characteristic polynomial of $\mathrm{Frob}_x^{c!}$ on the Tate module of φ_x at the prime $\mathfrak{p}$, which by Proposition 3.10 has coefficients in A and is independent of $\mathfrak{p}$. Moreover, by Proposition 3.11 every eigenvalue of Frob_x has valuation < 0 at ∞. It follows that 1 is not an eigenvalue of $\mathrm{Frob}_x^{c!}$ and so $f_x(1)$ is a non-zero element of A. For all $\mathfrak{p}$ not dividing $f_x(1)$, no eigenvalue of $\mathrm{Frob}_x^{c!}$ is congruent to 1 modulo a place lying above $\mathfrak{p}$; consequently for these $\mathfrak{p}$ we have $\varphi_x[\mathfrak{p}]_{\mathrm{Frob}_x^{c!}} = 0$. ☐

For every integer $c \geq 1$ let $S_0(c)$ denote the finite set of primes given by the above lemma and $S(c) := S_0(c) \cup S_1$. For every $\mathfrak{p} \notin S_0(c)$ and for every subgroup H of $\Delta_{\mathfrak{p}}$ of index $\leq c$ let $K^{(\mathfrak{p},H)}$ be the field fixed by the kernel of $\alpha_{\mathfrak{p},H}$.

By Lemma 4.4 it is unramified over K. Moreover, its degree $[K^{(\mathfrak{p},H)}/K] \leq c!$ is bounded independently of $\mathfrak{p}$ and H. By a function field analogue of the Hermite-Minkowski Theorem (cf. [**Gos96**], Theorem 8.23.5) about unramified extensions, there are only finitely many possibilities for $K^{(\mathfrak{p},H)}$. Therefore their compositum K' is a finite extension of K such that the restriction $\alpha_{\mathfrak{p},H}|_{K'} : G_{K'} \to \Delta_{\mathfrak{p}}/H'$ is trivial for all $\mathfrak{p} \notin S_0(c)$, in particular for all $\mathfrak{p} \notin S(c)$. For these primes $\mathfrak{p}$ we find that

$$\overline{\rho_{\mathfrak{p}}}(G_{K'}) \subset H' \subset H.$$

By the assumption on $S_1 \subset S(c)$ the Galois group $G_{K'}$ acts absolutely irreducibly under $\overline{\rho_{\mathfrak{p}}}$ for all $\mathfrak{p} \notin S(c)$. This yields the desired conclusion. □

Definition 4.5.
For $\mathfrak{p} \notin S(r!)$ let $\Delta_{\mathfrak{p},1}$ denote a fixed choice of normal subgroup of $\Delta_{\mathfrak{p}}$ satisfying the conditions of Proposition 2.21 and let $\Delta_{\mathfrak{p},2}$ denote $\Delta_{\mathfrak{p},1} \cap (\text{scalars})$.

Lemma 4.6.
For $\mathfrak{p} \notin S(r!)$ we have $\Delta_{\mathfrak{p},1} \subset \Delta_{\mathfrak{p},2}\Delta_{\mathfrak{p}}^{\text{geom}}$.

Proof. By construction, $\Delta_{\mathfrak{p}}^{\text{geom}}$ is a normal subgroup of $\Delta_{\mathfrak{p}}$ and their quotient is cyclic. Therefore

$$\Delta_{\mathfrak{p},1}\Delta_{\mathfrak{p}}^{\text{geom}}/\Delta_{\mathfrak{p}}^{\text{geom}} \cong \Delta_{\mathfrak{p},1}/(\Delta_{\mathfrak{p},1} \cap \Delta_{\mathfrak{p}}^{\text{geom}})$$

is also cyclic and the derived group $\Delta_{\mathfrak{p},1}^{\text{der}}$ is contained in $\Delta_{\mathfrak{p},1} \cap \Delta_{\mathfrak{p}}^{\text{geom}}$. On the other hand, $\Delta_{\mathfrak{p},1}/\Delta_{\mathfrak{p},2}$ is perfect, so there is a surjection $\Delta_{\mathfrak{p},1}^{\text{der}} \twoheadrightarrow \Delta_{\mathfrak{p},1}/\Delta_{\mathfrak{p},2}$. Combining these two statements we find $\Delta_{\mathfrak{p},1} = \Delta_{\mathfrak{p},2}(\Delta_{\mathfrak{p},1} \cap \Delta_{\mathfrak{p}}^{\text{geom}})$. This in turn yields the desired result. □

The following statement is an analogue of Proposition 4.3 for subgroups of $\Delta_{\mathfrak{p}}^{\text{geom}}$ of bounded index.

Proposition 4.7.
For every integer $c \geq 1$ there exists a finite set of primes $S \supset S_1$ of A such that for all $\mathfrak{p} \notin S$, every subgroup of $\Delta_{\mathfrak{p}}^{\text{geom}}$ of index $\leq c$ acts absolutely irreducibly.

Proof. Let c be fixed and let S be the union of the finite sets $S(r!)$ and $S(d_r c)$. For all $n \geq 1$ we have by definition $S(n) \supset S_1$; hence $S \supset S_1$. Let $\mathfrak{p}$ be a prime outside of S and H a subgroup of $\Delta_{\mathfrak{p}}^{\text{geom}}$ of index $\leq c$. Since $\Delta_{\mathfrak{p},2}$ is a scalar group, H acts absolutely irreducibly if and only if $\Delta_{\mathfrak{p},2}H$ does. Then, using Lemma 4.6 and the definition of $\Delta_{\mathfrak{p},1}$, we find

$$\begin{aligned}
[\Delta_{\mathfrak{p}} : \Delta_{\mathfrak{p},2}H] &= [\Delta_{\mathfrak{p}} : \Delta_{\mathfrak{p},2}\Delta_{\mathfrak{p}}^{\text{geom}}][\Delta_{\mathfrak{p},2}\Delta_{\mathfrak{p}}^{\text{geom}} : \Delta_{\mathfrak{p},2}H] \\
&\leq [\Delta_{\mathfrak{p}} : \Delta_{\mathfrak{p},1}][\Delta_{\mathfrak{p},2}\Delta_{\mathfrak{p}}^{\text{geom}} : \Delta_{\mathfrak{p},2}H] \\
&\leq d_r \cdot c.
\end{aligned}$$

Therefore $\Delta_{\mathfrak{p},2}H$ acts absolutely irreducibly by the choice of S. □

Proposition 4.8.
Let f, g and h be the algebraic morphisms defined in Section 2.2. For every integer $c \geq 1$ there exists a finite set of primes $S \supset S_1$ of A such that for all $\mathfrak{p} \notin S$, the map $\gamma \mapsto fgh(\gamma^c)$ does not vanish identically on $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$.

Proof. Let us suppose that there is an infinite set P of primes $\mathfrak{p}$ of A for which $\gamma \mapsto fgh(\gamma^c)$ vanishes identically on $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$. By Lemma 4.6, possibly after taking a finite number of primes out of P, we have $P \cap S_1 = \emptyset$ and for all $\mathfrak{p} \in P$ and all $\delta \in \Delta_{\mathfrak{p}}$ there exist $\alpha \in k_{\mathfrak{p}}^*$ and $\delta_g \in \Delta_{\mathfrak{p}}^{\mathrm{geom}}$ with $\delta^{d_r!} = \alpha\delta_g$. Consequently

$$\frac{\delta^{d_r!r}}{\det \delta^{d_r!}} = \frac{(\alpha\delta_g)^r}{\det(\alpha\delta_g)} = \delta_g^r \in \Delta_{\mathfrak{p}}^{\mathrm{geom}}.$$

Let x be a place of K at which φ has good reduction and let $\mathrm{Frob}_x \in G_K$ be an associated Frobenius element. Then there exists $\delta_x \in \Delta_{\mathfrak{p}}^{\mathrm{geom}}$ such that $\frac{\overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_x^{d_r!r})}{\det \overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_x)^{d_r!}} = \delta_x$. For $\mathfrak{p} \in P$ we have by assumption

$$fgh\left(\frac{\overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_x^{d_r!rc})}{\det \overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_x)^{d_r!c}}\right) = fgh(\delta_x^c) = 0$$

and therefore $fgh(\overline{\rho_{\mathfrak{p}}}(\mathrm{Frob}_x^{d_r!rc})) = 0$.

It is a consequence of Proposition 3.10 that the characteristic polynomial f_x of $\rho_{\mathfrak{p}}(\mathrm{Frob}_x^{d_r!rc})$ has coefficients in A and is independent of $\mathfrak{p}$. Since there are only finitely many possibilities to choose a bounded amount of eigenvalues of $\mathrm{Frob}_x^{d_r!rc}$, there is either a quadruple of distinct eigenvalues $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\mathrm{Frob}_x^{d_r!rc}$ in $\overline{F}$ such that $\alpha_1\alpha_2 - \alpha_3\alpha_4 \equiv 0$, or a triple of distinct eigenvalues $\alpha_1, \alpha_3, \alpha_4$ of $\mathrm{Frob}_x^{d_r!rc}$ in $\overline{F}$ such that $\alpha_1^2 - \alpha_3\alpha_4 \equiv 0$, or a sextuple of distinct eigenvalues $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ of $\mathrm{Frob}_x^{d_r!rc}$ in $\overline{F}$ such that $\alpha_1\alpha_2\alpha_3 - \alpha_4\alpha_5\alpha_6 \equiv 0$ modulo a prime lying above $\mathfrak{p}$ for infinitely many primes $\mathfrak{p}$ of P. Since the two other cases work analogously, let us suppose that it is the first case that occurs. Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 = 0$ and therefore

$$f_c : \mathrm{GL}_{r,F_{\mathfrak{p}}} \to \mathbb{A}^1_{F_{\mathfrak{p}}}, \quad \gamma \mapsto f(\gamma^{d_r!rc})$$

vanishes on $\rho_{\mathfrak{p}}(\mathrm{Frob}_x)$ for all places x of good reduction of K. Since these Frob_x form a dense set of conjugacy classes of G_K and the morphism f_c is conjugation-invariant, we obtain $f_c|_{\Gamma_{\mathfrak{p}}} = 0$ and in particular $f_c|_{\Gamma_{\mathfrak{p}}^{\mathrm{geom}}} = 0$. By Corollary 3.4 the image $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$ of the geometric Galois group is Zariski dense in $\mathrm{SL}_{r,F_{\mathfrak{p}}}$. Since f_c is an algebraic morphism, from $f_c|_{\Gamma_{\mathfrak{p}}^{\mathrm{geom}}} = 0$ it follows that f_c also vanishes on $\mathrm{SL}_{r,F_{\mathfrak{p}}}$. However, this is a contradiction by Lemma 2.9. ☐

Proposition 4.9.
For almost all primes $\mathfrak{p}$ of A there is a finite subfield $k'_{\mathfrak{p}}$ of $\overline{k_{\mathfrak{p}}}$ and a model G' of SL_r over $k'_{\mathfrak{p}}$ such that $G'(k'_{\mathfrak{p}})$ is a normal subgroup of $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$ of index bounded independently of $\mathfrak{p}$ that acts absolutely irreducibly.

PROOF. Let N_r be the integer depending on r provided by Theorem 2.19 and let $S_{N_r} \supset S_1$ be the finite set of primes of A for which, applied with the constant N_r, either Proposition 4.7 or Proposition 4.8 does not hold. By reduction step (a), for all primes $\mathfrak{p} \notin S_{N_r}$ the image $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$ of the geometric Galois group is contained in $\mathrm{SL}_r(k_{\mathfrak{p}})$ and thus we can apply Theorem 2.19 to $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$. By the definition of S_{N_r}, assumptions (1) and (2) of Theorem 2.19 are satisfied for all $\mathfrak{p} \notin S_{N_r}$. It follows that for every such prime $\mathfrak{p}$ there is a finite subfield $k'_{\mathfrak{p}}$ of $\overline{k_{\mathfrak{p}}}$ and a model G' of SL_r over $k'_{\mathfrak{p}}$ such that $G'(k'_{\mathfrak{p}})$ is a normal subgroup of $\Delta_{\mathfrak{p}}^{\mathrm{geom}}$ of index $\leq N_r$, thereby acting absolutely irreducibly. $\square$

PROPOSITION 4.10.
For almost all primes $\mathfrak{p}$ of A as in Proposition 4.9 we have $k_{\mathfrak{p}} \subseteq k'_{\mathfrak{p}}$.

PROOF. Let $S \supset S_1$ be the finite set of primes of A for which either Proposition 3.13 or Proposition 4.9 does not hold and let $\mathfrak{p}$ be a prime outside of S.

Let $\Delta_{\mathfrak{p},1}$ be the normal subgroup of $\Delta_{\mathfrak{p}}$ as in Definition 4.5. By the construction carried out in the proof of Proposition 2.19, on which Proposition 4.9 is based, we may assume that $G'(k'_{\mathfrak{p}}) = \Delta_{\mathfrak{p},1}^{\mathrm{der}}$. As the derived group of a normal subgroup, $G'(k'_{\mathfrak{p}})$ itself is normal in $\Delta_{\mathfrak{p}}$.

Let $\gamma \in \Delta_{\mathfrak{p}}$ and $\mathrm{int}_{G'}(\gamma) \in \mathrm{Aut}(G'(k'_{\mathfrak{p}}))$ denote the conjugation action of γ on $G'(k'_{\mathfrak{p}})$. This action, extended to G', is defined over $k'_{\mathfrak{p}}$; thus its derived map $\mathrm{Ad}_{G'}(\gamma) \in \mathrm{End}(\mathrm{Lie}\, G')$ has trace in $k'_{\mathfrak{p}}$. Tensoring $\mathrm{Lie}\, G'$ with $\overline{k'_{\mathfrak{p}}}$ and considering $\mathrm{Ad}_{G'}(\gamma)$ as an element of $\mathrm{End}(\mathrm{Lie}\, G' \otimes_{k'_{\mathfrak{p}}} \overline{k'_{\mathfrak{p}}})$ does not change the characteristic polynomial of $\mathrm{Ad}_{G'}(\gamma)$; therefore the trace of the latter still lies in $k'_{\mathfrak{p}}$.

On the other hand, since $\Delta_{\mathfrak{p}} \subset \mathrm{GL}_r(k_{\mathfrak{p}})$, we can also consider the conjugation action $\mathrm{int}_{\mathrm{SL}_r}(\gamma)$ of γ on $\mathrm{SL}_r(k_{\mathfrak{p}})$. As in the case of $\mathrm{int}_{G'}(\gamma)$ above, we conclude that the derived map $\mathrm{Ad}_{\mathrm{SL}_r}(\gamma) \in \mathrm{End}(\mathfrak{sl}_{r,k_{\mathfrak{p}}} \otimes_{k_{\mathfrak{p}}} \overline{k_{\mathfrak{p}}})$ has trace in $k_{\mathfrak{p}}$.

Since G' is a model of SL_r over $k'_{\mathfrak{p}}$, we have

$$\mathrm{Lie}\, G' \otimes_{k'_{\mathfrak{p}}} \overline{k'_{\mathfrak{p}}} = \mathfrak{sl}_{r,k_{\mathfrak{p}}} \otimes_{k_{\mathfrak{p}}} \overline{k_{\mathfrak{p}}}.$$

Moreover, given the inclusion $G'(k'_{\mathfrak{p}}) \leqslant \mathrm{SL}_r(k_{\mathfrak{p}})$, we have

$$\mathrm{int}_{G'}(\gamma) = \mathrm{int}_{\mathrm{SL}_r}(\gamma)|_{G'(k'_{\mathfrak{p}})}.$$

Together with the equality of Lie algebras this yields that $\mathrm{Ad}_{G'}(\gamma) = \mathrm{Ad}_{\mathrm{SL}_r}(\gamma)$ on $\mathfrak{sl}_{r,\overline{k_{\mathfrak{p}}}}$. Thus $\mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_r}(\gamma))$ lies in $k_{\mathfrak{p}} \cap k'_{\mathfrak{p}}$.

Since $\mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_r}(\gamma)) = \mathrm{tr}(\mathrm{Ad}(\gamma)) - 1$, it follows that $\mathrm{tr}(\mathrm{Ad}(\gamma))$ lies in $k_{\mathfrak{p}} \cap k'_{\mathfrak{p}}$. On the other hand, by Proposition 3.13, the field generated by $\{\mathrm{tr}(\mathrm{Ad}(\gamma))\}_{\gamma \in \Delta_{\mathfrak{p}}}$ is equal to $k_{\mathfrak{p}}$. Therefore $k_{\mathfrak{p}} \subseteq k'_{\mathfrak{p}}$. $\square$

The ingredients to finish the proof of the main theorem of this chapter are now all gathered together.

Proof of Theorem 4.1. By Propositions 4.9 and 4.10 for almost all primes $\mathfrak{p}$ of A there is a model G' of SL_r over an extension $k'_\mathfrak{p}$ of $k_\mathfrak{p}$ such that

$$G'(k'_\mathfrak{p}) \leqslant \Delta_\mathfrak{p}^{\mathrm{geom}} \leqslant \mathrm{SL}_r(k_\mathfrak{p}).$$

By Proposition 2.22 both inclusions are then equalities. $\square$

4.2. Surjectivity of products of residual representations.

Here we consider the image of the product representation $\overline{\rho_{\mathfrak{p}_1}} \times \overline{\rho_{\mathfrak{p}_2}}$ for pairs of distinct primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of A.

Proposition 4.11.
There exists a finite set S of primes of A such that for all pairs of distinct primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ not in S the image of $(\overline{\rho_{\mathfrak{p}_1}}\times\overline{\rho_{\mathfrak{p}_2}})(G_K)$ in $\mathrm{PGL}_r(k_{\mathfrak{p}_1})\times\mathrm{PGL}_r(k_{\mathfrak{p}_1})$ contains $\mathrm{PSL}_r(k_{\mathfrak{p}_1}) \times \mathrm{PSL}_r(k_{\mathfrak{p}_2})$.

Proof. Let $S \supset S_1$ be the finite set containing all primes $\mathfrak{p}$ of A for which the residual representation does not map surjectively onto $\mathrm{SL}_r(k_\mathfrak{p})$, all those with $|k_\mathfrak{p}| \leq 3$ and all those for which Proposition 3.13 is not satisfied. Let $\mathfrak{p}_1, \mathfrak{p}_2 \notin S$ be distinct primes and let Δ denote the image of G_K in $\mathrm{PGL}_r(k_{\mathfrak{p}_1}) \times \mathrm{PGL}_r(k_{\mathfrak{p}_2})$. Suppose that

$$\Delta^{\mathrm{der}} \neq \mathrm{PSL}_r(k_{\mathfrak{p}_1}) \times \mathrm{PSL}_r(k_{\mathfrak{p}_2}).$$

The assumptions on $\mathfrak{p}_1$ and $\mathfrak{p}_2$ ensure that Δ^{der} surjects to both factors. Moreover, as $|k_{\mathfrak{p}_1}|, |k_{\mathfrak{p}_2}| > 3$, these groups are simple. We can therefore apply [**Pin00**], Lemmas 9.4 and 9.5, that show the existence of a field isomorphism $\sigma : k_{\mathfrak{p}_2} \xrightarrow{\sim} k_{\mathfrak{p}_1}$ and of a corresponding isomorphism $\alpha : \sigma^* \mathrm{PGL}_{r,k_{\mathfrak{p}_1}} \longrightarrow \mathrm{PGL}_{r,k_{\mathfrak{p}_2}}$ such that $\Delta \subset \mathrm{Graph}(\alpha)$. Noting that the adjoint representation of GL_r factors through PGL_r, we thus have $\mathrm{Ad} \circ \varphi \cong \sigma^* \mathrm{Ad}$.

Calculating inside $A/\mathfrak{p}_2 \cong k_{\mathfrak{p}_2}$, for every place x of K where φ has good reduction we find

$$\sigma^{-1}(\mathrm{tr}\,\mathrm{Ad}(\rho_{\mathfrak{p}_1}(\mathrm{Frob}_x)) \mod \mathfrak{p}_1) = (\mathrm{tr}\,\mathrm{Ad}(\rho_{\mathfrak{p}_1}(\mathrm{Frob}_x)) \mod \mathfrak{p}_2).$$

This implies that the image modulo $\mathfrak{p}_1\mathfrak{p}_2$ of the ring generated by the traces of all such $\mathrm{Ad}(\rho_{\mathfrak{p}_1}(\mathrm{Frob}_x))$ is contained in $\mathrm{Graph}(\sigma^{-1}) \subsetneq k_{\mathfrak{p}_1} \times k_{\mathfrak{p}_2}$, in contradiction to Proposition 3.13 (b). $\square$

CHAPTER 5

Cohomological remarks and some group theory

We collect a few general results that are used in the next chapter. Let $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{pgl}_n$ and $\mathfrak{psl}_n$ denote the Lie algebras of $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ and PSL_n, respectively and let $\mathfrak{c}$ denote the center of $\mathfrak{gl}_n$.

PROPOSITION 5.1.
Let $n \geq 1$ and k be a finite field.

(1) *If $|k| > 3$, then*
$$H^1(\mathrm{SL}_n(k), \mathfrak{gl}_n(k)) = 0.$$

(2) *If $|k| > 9$, then*
$$H^1(\mathrm{SL}_n(k), \mathfrak{pgl}_n(k)) = 0.$$

(3) *Let H be a subgroup of $\mathrm{GL}_n(k)$ that contains $\mathrm{SL}_n(k)$. If $|k| > 9$, then*
$$H^1(H, \mathfrak{pgl}_n(k)) = 0.$$

PROOF. Part (1) was proved in [**TZ70**], Theorem 9. For (2), we show that the natural map
$$H^1(\mathrm{SL}_n(k), \mathfrak{gl}_n(k)) \longrightarrow H^1(\mathrm{SL}_n(k), \mathfrak{pgl}_n(k))$$
is an isomorphism. Indeed, let us consider the exact cohomology sequence
$$\begin{aligned} H^1(\mathrm{SL}_n(k), \mathfrak{c}(k)) &\longrightarrow H^1(\mathrm{SL}_n(k), \mathfrak{gl}_n(k)) \\ &\longrightarrow H^1(\mathrm{SL}_n(k), \mathfrak{pgl}_n(k)) \longrightarrow H^2(\mathrm{SL}_n(k), \mathfrak{c}(k)) \end{aligned}$$
associated to the short exact sequence
$$0 \longrightarrow \mathfrak{c}(k) \longrightarrow \mathfrak{gl}_n(k) \longrightarrow \mathfrak{pgl}_n(k) \longrightarrow 0$$
of $\mathrm{SL}_n(k)$-modules. Since $\mathrm{SL}_n(k)$ is perfect if $|k| > 3$ and $\mathfrak{c}(k)$ is abelian, the group $H^1(\mathrm{SL}_n(k), \mathfrak{c}(k)) \cong \mathrm{Hom}(\mathrm{SL}_n(k), \mathfrak{c}(k))$ is trivial. In a similar way, the group $H^2(\mathrm{SL}_n(k), \mathfrak{c}(k))$ of central extensions of $\mathrm{SL}_n(k)$ by $\mathfrak{c}(k)$ is trivial since, if $|k| > 9$, the group $\mathrm{SL}_n(k)$ has no central extensions by [**Ste81**], Theorem 1.1. Consequently the required map is indeed an isomorphism. Combined with Part (1) this yields (2).

Let $\mathrm{SL}_n(k) \leqslant H \leqslant \mathrm{GL}_n(k)$. Then $[H : \mathrm{SL}_n(k)]$ divides $[\mathrm{GL}_n(k) : \mathrm{SL}_n(k)] = |k| - 1$. In particular it is prime to the characteristic of k; therefore by [**CPS75**], Proposition 2.3 (g), the restriction map
$$H^1(H, \mathfrak{pgl}_n(k)) \longrightarrow H^1(\mathrm{SL}_n(k), \mathfrak{pgl}_n(k))$$

is injective. Part (3) then follows from (2). $\square$

The following proposition is an adaptation of [**PR09a**], Proposition 2.1.

PROPOSITION 5.2.
Let n be a natural number, k a finite field with at least 4 elements and H an additive subgroup of $\mathfrak{gl}_n(k)$. Assume that H is invariant under conjugation by $\mathrm{SL}_n(k)$. Then either H is contained in the group of scalar matrices or H contains $\mathfrak{sl}_n(k)$.

PROOF. Consider the restriction of the adjoint representation of $\mathrm{GL}_{n,k}$ to $\mathrm{SL}_{n,k}$, the weights of which are $e_i - e_j \in \mathbb{R}^n/\mathrm{diag}(\mathbb{R})$ for $i \neq j$ with multiplicity 1 and 0 with multiplicity n. The weight space W_0 of weight 0 is the group of diagonal matrices in $\mathfrak{gl}_n(k)$ and the weight space $W_{i,j}$ of weight $e_i - e_j$ is the group of matrices with all entries zero, except, possibly, in the position (i,j). Thus we can decompose $\mathfrak{gl}_n(k)$ as

$$\mathfrak{gl}_n(k) = W_0 \oplus \bigoplus_{i,j} W_{i,j}.$$

Since the multiplicative group k^* has at least 3 elements, any two distinct weights of the form $e_i - e_j$ remain distinct and different from 0 upon restricting the representation to $\mathrm{SL}_n(k)$. Therefore H can be decomposed as

$$H = (H \cap W_0) \oplus \bigoplus_{i,j} (H \cap W_{i,j}).$$

Each $W_{i,j}$ is a k-vector space of dimension 1 and the diagonal matrices $T(k)$ in $\mathrm{SL}_n(k)$ act on it through a homomorphism $T(k) \to k^*$.

If $n \geq 3$ then the above homomorphism is surjective. Hence $H \cap W_{i,j}$ is either 0 or equal to $W_{i,j}$.

If $n = 2$, then the homomorphism $T(k) \to k^*$ is not necessarily surjective. Let us suppose that there is a non-zero $h \in H \cap W_{i,j}$. Then at least $\{\alpha^2 h \mid \alpha \in k\}$ is a subset of $H \cap W_{i,j}$. Since $H \cap W_{i,j}$ is an additive group, $\{(\alpha^2 + \beta^2)h \mid \alpha \in k\}$ is also a subset thereof. As every element in the finite field k can be written as the sum of two squares, we then have $H \cap W_{i,j} = W_{i,j}$. Thus in this case also $H \cap W_{i,j}$ is either 0 or equal to $W_{i,j}$.

Consider the subgroup of $\mathrm{SL}_n(k)$ generated by the permutation matrices of positive signature and the products of a permutation matrix of negative signature with a scalar matrix of determinant -1. This subgroup permutes the weight spaces $W_{i,j}$ transitively. Since H is invariant under conjugation by $\mathrm{SL}_n(k)$, we find that either every $H \cap W_{i,j} = 0$ or every $H \cap W_{i,j} = W_{i,j}$. In other words, either H is contained in the group of diagonal matrices or contains the sum of all $W_{i,j}$, which is the group of matrices with diagonal 0.

If H is contained in the group of diagonal matrices, then take any element h of H and denote its diagonal entries by $h_1, \ldots, h_n$. Let $i \neq j$ and $u \in \mathrm{SL}_n(k)$ be the matrix with entry 1 on the diagonal and in the position (i,j) and 0 elsewhere.

Then uhu^{-1} has entry $h_i - h_j$ in the position (i, j). However this entry has to be 0 because $uhu^{-1} \in H$, and hence $h_i = h_j$. This can be done for any pair (i, j), which shows that H is contained in the group of scalar matrices.

If H contains the group of matrices with diagonal 0, we consider the trace form

$$\mathfrak{gl}_n(k) \times \mathfrak{gl}_n(k) \to k, \quad (A, B) \mapsto \mathrm{tr}(AB),$$

which is a perfect pairing invariant under $\mathrm{SL}_n(k)$. The orthogonal complement $H^\perp$ of H is again an $\mathrm{SL}_n(k)$-invariant subgroup, and since the inclusion for orthogonal complements is reversed, it is contained in the group of diagonal matrices. The arguments in the other case show that $H^\perp$ is contained in the group of scalar matrices. Taking orthogonal complements again, we deduce that H contains all of $\mathfrak{sl}_n(k)$, as desired. □

Corollary 5.3.
Let k be a finite field of characteristic 2 with at least 4 elements and H a non-zero additive subgroup of $\mathfrak{pgl}_2(k)$. Assume that H is invariant under conjugation by $\mathrm{SL}_2(k)$. Then H contains $\mathfrak{psl}_2(k)$.

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{c}(k) \longrightarrow \mathfrak{gl}_2(k) \xrightarrow{\mathrm{proj}} \mathfrak{pgl}_2(k) \longrightarrow 0.$$

Let us suppose that $\mathfrak{psl}_2(k) \not\subset H$. Then $(\mathfrak{psl}_2(k) \cap H) \lneq \mathfrak{psl}_2(k)$ and therefore $\mathrm{proj}^{-1}(\mathfrak{psl}_2(k) \cap H) \lneq \mathfrak{sl}_2(k)$. Given that $\mathrm{proj}^{-1}(\mathfrak{psl}_2(k) \cap H)$ is $\mathrm{SL}_2(k)$-invariant if $\mathfrak{psl}_2(k) \cap H$ is, by Proposition 5.2, the group $\mathrm{proj}^{-1}(\mathfrak{psl}_2(k) \cap H)$ is contained in the group of scalars $\mathfrak{c}(k)$. Hence $\mathrm{proj}(\mathrm{proj}^{-1}(\mathfrak{psl}_2(k) \cap H)) = \mathfrak{psl}_2(k) \cap H$ is trivial.

Let $\left(\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix}\right)$ be a non-zero element of H. From $h \notin \mathfrak{psl}_2(k)$ follows that x is non-zero. Since H is $\mathrm{SL}_2(k)$-invariant, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & z \\ y & 0 \end{pmatrix} \in H.$$

Since H is an additive group, it follows that

$$\begin{pmatrix} x & y \\ z & 0 \end{pmatrix} + \begin{pmatrix} x & z \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & y+z \\ y+z & 0 \end{pmatrix} \in H.$$

This is also an element of $\mathfrak{psl}_2(k)$, thus it must be zero. Consequently we have $y = z$. Now let $a \in k^*$. Then

$$\begin{pmatrix} a & a \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ y & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & a \\ 0 & a \end{pmatrix} = \begin{pmatrix} x & a^2x \\ a^{-2}y & 0 \end{pmatrix} \in H,$$

as well as

$$\begin{pmatrix} x & y \\ y & 0 \end{pmatrix} + \begin{pmatrix} x & a^2x \\ a^{-2}y & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2x + y \\ a^{-2}y + y & 0 \end{pmatrix} \in H.$$

Since this is an element of $\mathfrak{psl}_2(k)$, it must be zero. In particular we have

$$a^2x + y = 0.$$

Since $|k^*| > 1$ and $x \neq 0$ and this holds for all $a \in k^*$, we obtain a contradiction. Hence H must contain $\mathfrak{psl}_2(k)$. $\square$

CHAPTER 6

Second order and higher order approximation

In this chapter we return to the notation and the assumptions of Chapter 4. Most notably, we assume that the field K has transcendence degree 1, that the Drinfeld A-module φ satisfies the conditions of Theorem 1.1 and that the reduction steps introduced in Section 3.3 are in effect.

6.1. Congruence filtration.

Let π be a uniformizer of A at $\mathfrak{p}$. The congruence filtration of $\mathrm{GL}_r(A_\mathfrak{p})$ is defined by

$$\begin{array}{rcll} G_\mathfrak{p}^0 & := & \mathrm{GL}_r(A_\mathfrak{p}) & \text{and} \\ G_\mathfrak{p}^i & := & 1+\mathfrak{gl}_r(\mathfrak{p}^i) & \text{for all } i \geq 1. \end{array}$$

Its successive subquotients possess natural isomorphisms

$$\begin{array}{rll} v_0: & G_\mathfrak{p}^{[0]} := G_\mathfrak{p}^0/G_\mathfrak{p}^1 \xrightarrow{\sim} \mathrm{GL}_r(k_\mathfrak{p}) & \text{and} \\ v_i: & G_\mathfrak{p}^{[i]} := G_\mathfrak{p}^i/G_\mathfrak{p}^{i+1} \xrightarrow{\sim} \mathfrak{gl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}),\ [1+y] \mapsto [y] & \text{for all } i \geq 1. \end{array}$$

For any subgroup H of $\mathrm{GL}_r(A_\mathfrak{p})$ we define $H^i := H \cap G_\mathfrak{p}^i$ and $H^{[i]} := H^i/H^{i+1}$. Via v_i we identify the latter with a subgroup of $\mathrm{GL}_r(k_\mathfrak{p})$ or $\mathfrak{gl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1})$, respectively. In particular, let

$$G_\mathfrak{p}'^i := \mathrm{SL}_r(A_\mathfrak{p})^i \quad \text{for all} \quad i \geq 1.$$

Via v_i we get isomorphisms

$$\begin{array}{ll} G_\mathfrak{p}'^{[0]} := G_\mathfrak{p}'^0/G_\mathfrak{p}'^1 \xrightarrow{\sim} \mathrm{SL}_r(k_\mathfrak{p}) & \text{and} \\ G_\mathfrak{p}'^{[i]} := G_\mathfrak{p}'^i/G_\mathfrak{p}'^{i+1} \xrightarrow{\sim} \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}). & \end{array}$$

Similarly, for $\mathrm{PGL}_r(A_\mathfrak{p})$ we set

$$\begin{array}{rcll} PG_\mathfrak{p}^0 & := & \mathrm{PGL}_r(A_\mathfrak{p}) & \text{and} \\ PG_\mathfrak{p}^i & := & 1+\mathfrak{pgl}_r(\mathfrak{p}^i) & \text{for all } i \geq 1. \end{array}$$

For $i \geq 0$ the natural isomorphisms v_i for GL_r induce a series of natural isomorphisms

$$\begin{array}{rll} \overline{v_0}: & PG_\mathfrak{p}^{[0]} := PG_\mathfrak{p}^0/PG_\mathfrak{p}^1 \xrightarrow{\sim} \mathrm{PGL}_r(k_\mathfrak{p}) & \text{and} \\ \overline{v_i}: & PG_\mathfrak{p}^{[i]} := PG_\mathfrak{p}^i/PG_\mathfrak{p}^{i+1} \xrightarrow{\sim} \mathfrak{pgl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}). & \end{array}$$

For any subgroup H of $\mathrm{PGL}_r(A_\mathfrak{p})$ we define $H^i := H \cap PG_\mathfrak{p}^i$ and $H^{[i]} := H^i/H^{i+1}$. Via $\overline{v_i}$ we identify the latter with a subgroup of $\mathrm{PGL}_r(k_\mathfrak{p})$ or $\mathfrak{pgl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1})$, respectively.

For $i \geq 1$ the isogeny $P : \mathrm{SL}_r \to \mathrm{PGL}_r$ induces a natural commutative diagram

$$\begin{tikzcd}
G'^{i}_{\mathfrak{p}} \arrow[r] \arrow[d, "P"] & \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \arrow[d, "dP"] \\
PG^{i}_{\mathfrak{p}} \arrow[r] & \mathfrak{pgl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}).
\end{tikzcd}$$

6.2. Generalized commutator maps.

The commutator map of SL_r factors through a map

$$[\ ,\ ]^{\sim} : \mathrm{PGL}_r \times \mathrm{PGL}_r \longrightarrow \mathrm{SL}_r\,.$$

Its total derivative at the identity element defines a generalized Lie bracket

$$[\ ,\ ]^{\sim} : \mathfrak{pgl}_r \times \mathfrak{pgl}_r \longrightarrow \mathfrak{sl}_r,$$

denoted by the same symbol. Its composite with the map $dP : \mathfrak{sl}_r \to \mathfrak{pgl}_r$ is the usual Lie bracket $[\ ,\ ]$ on $\mathfrak{pgl}_r$, respectively on $\mathfrak{sl}_r$. We also denote the induced pairing $\mathfrak{pgl}_r \times \mathfrak{sl}_r \to \mathfrak{sl}_r$ by $[\ ,\ ]^{\sim}$. Proposition 1.2 of [**Pin00**] shows that the images of these pairings generate the following subspaces.

Proposition 6.1.
(a) *We have* $[\mathfrak{sl}_r, \mathfrak{sl}_r] = \mathfrak{sl}_r$ *unless* $r = 2$ *and we are in characteristic 2. In that case we have* $[\mathfrak{sl}_2, \mathfrak{sl}_2] \subset \mathfrak{c}$*, where* $\mathfrak{c}$ *denotes the center of* $\mathfrak{sl}_2$*.*
(b) *In all cases we have* $[\mathfrak{pgl}_r, \mathfrak{sl}_r]^{\sim} = \mathfrak{sl}_r$*.*

The generalized commutator maps

$$\mathrm{SL}_r \times \mathrm{SL}_r \longrightarrow \mathrm{PGL}_r \times \mathrm{SL}_r \longrightarrow \mathrm{PGL}_r \times \mathrm{PGL}_r \longrightarrow \mathrm{SL}_r$$

induce for any $i, j \geq 1$ a commutative diagram

$$\begin{tikzcd}
G'^{i}_{\mathfrak{p}} \times G'^{j}_{\mathfrak{p}} \arrow[r] \arrow[d] & G'^{[i]}_{\mathfrak{p}} \times G'^{[j]}_{\mathfrak{p}} \arrow[r, "\sim"] \arrow[d] & \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \times \mathfrak{sl}_r(\mathfrak{p}^j/\mathfrak{p}^{j+1}) \arrow[d] \arrow[ddd, bend left=60, "{[\,,]}"] \\
PG^{i}_{\mathfrak{p}} \times G'^{j}_{\mathfrak{p}} \arrow[r] \arrow[d] & PG^{[i]}_{\mathfrak{p}} \times G'^{[j]}_{\mathfrak{p}} \arrow[r, "\sim"] \arrow[d] & \mathfrak{pgl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \times \mathfrak{sl}_r(\mathfrak{p}^j/\mathfrak{p}^{j+1}) \arrow[d] \\
PG^{i}_{\mathfrak{p}} \times PG^{j}_{\mathfrak{p}} \arrow[r] \arrow[d, "{[\,,]^{\sim}}"] & PG^{[i]}_{\mathfrak{p}} \times PG^{[j]}_{\mathfrak{p}} \arrow[r, "\sim"] \arrow[d] & \mathfrak{pgl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \times \mathfrak{pgl}_r(\mathfrak{p}^j/\mathfrak{p}^{j+1}) \arrow[d, "{[\,,]^{\sim}}"] \\
G'^{i+j}_{\mathfrak{p}} \arrow[r] & G'^{[i+j]}_{\mathfrak{p}} \arrow[r, "\sim"] & \mathfrak{sl}_r(\mathfrak{p}^{i+j}/\mathfrak{p}^{i+j+1}),
\end{tikzcd}$$

involving, in the rightmost column, the generalized Lie brackets defined above.

6.3. Second order approximation.

For any subgroup H of $\mathrm{GL}_r(A_\mathfrak{p})$, let $\tilde{H}^{[1]}$ denote the image of $H^{[1]}$ in the quotient $\mathfrak{gl}_r(\mathfrak{p}/\mathfrak{p}^2)/\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)$ and $\tilde{H}$ denote the image of H/H^2 in the quotient $\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)/(1+\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2))$. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^{[1]} \arrow[r] \arrow[d] & H/H^2 \arrow[r] \arrow[d] & H^{[0]} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \tilde{H}^{[1]} \arrow[r] \arrow[d, hook] & \tilde{H} \arrow[r] \arrow[d, hook] & H^{[0]} \arrow[r] \arrow[d, hook] & 1 \\
0 \arrow[r] & \mathfrak{gl}_r(\mathfrak{p}/\mathfrak{p}^2)/\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[r] & \mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)/(1+\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)) \arrow[r] & \mathrm{GL}_r(k_\mathfrak{p}) \arrow[r] & 1.
\end{tikzcd}$$

Let us recall that $\Gamma_\mathfrak{p}$ and $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ denote the image of G_K, respectively G_K^{geom} under $\rho_\mathfrak{p}$, as well as $\Delta_\mathfrak{p}$ and $\Delta_\mathfrak{p}^{\mathrm{geom}}$ their images under $\overline{\rho_\mathfrak{p}}$. Let us also note that, with the notation introduced at the beginning of the chapter, we have $\Delta_\mathfrak{p} = \Gamma_\mathfrak{p}^{[0]}$ and $\Delta_\mathfrak{p}^{\mathrm{geom}} = \Gamma_\mathfrak{p}^{\mathrm{geom},[0]}$.

Let us also recall from Definition 4.2 that S_1 is the finite set of primes of A for which Theorem 3.6 does not hold. In this chapter we once more focus our attention on primes outside of S_1.

Proposition 6.2.
For almost all primes of $\mathfrak{p}$ of A we have $\tilde{\Gamma}_\mathfrak{p}^{[1]} \neq 0$; in other words, $\Gamma_\mathfrak{p}^{[1]}$ contains a non-scalar element.

Proof. Let $S \supset S_1$ be the set of primes $\mathfrak{q}$ with $|k_\mathfrak{q}| \leq 9$ and of all those for which the residual representation does not surject onto $\mathrm{SL}_r(k_\mathfrak{q})$. Let $\mathfrak{p}$ be a prime outside of S. There is a natural section

$$s_0 : \mathrm{GL}_r(k_\mathfrak{p}) \longrightarrow \mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)$$

using Teichmüller representatives. This in turn gives rise to a section

$$\overline{s}_0 : \mathrm{GL}_r(k_\mathfrak{p}) \longrightarrow \mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)/(1+\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)).$$

We denote the restriction of $\overline{s}_0$ to $\Delta_\mathfrak{p}$ again by $\overline{s}_0$. Suppose that $\tilde{\Gamma}_\mathfrak{p}^{[1]} = 0$. Since $\tilde{\Gamma}_\mathfrak{p}$ surjects onto $\Delta_\mathfrak{p}$, this then yields another section

$$\overline{s}_1 : \Delta_\mathfrak{p} \longrightarrow \mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)/(1+\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)).$$

By the assumptions on $\mathfrak{p}$ we have $|k_\mathfrak{p}| > 9$ and $\mathrm{SL}_r(k_\mathfrak{p}) \subseteq \Delta_\mathfrak{p}$. Proposition 5.1 (3) then shows that $H^1(\Delta_\mathfrak{p}, \mathfrak{gl}_r(\mathfrak{p}/\mathfrak{p}^2)/\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)) = 0$, from which we conclude that sections $\overline{s}_0$ and $\overline{s}_1$ are conjugate. We may therefore assume that they are equal. Then $\tilde{\Gamma}_\mathfrak{p} = \overline{s}_0(\Delta_\mathfrak{p}) \subseteq \overline{s}_0(\mathrm{GL}_r(k_\mathfrak{p}))$ in $\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)/(1+\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2))$. It follows that every element $\gamma \in \Gamma_\mathfrak{p}/\Gamma_\mathfrak{p}^2 \subseteq \mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}^2)$ can be written uniquely in the form

$\gamma = (1+\pi\lambda)\gamma_1$ with $\lambda \in k_{\mathfrak{p}}$ and $\gamma_1 \in \mathrm{GL}_r(k_{\mathfrak{p}})$. Since scalars act trivially in the adjoint representation, we have $\mathrm{Ad}\,\gamma = \mathrm{Ad}\,\gamma_1$. Thus

$$\{\mathrm{tr}(\mathrm{Ad}\,\gamma) \mid \gamma \in \Gamma_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}^2\} \subseteq k_{\mathfrak{p}}$$

and in turn

$$\{\mathrm{tr}(\mathrm{Ad}\,\gamma) \mid \gamma \in \Gamma_{\mathfrak{p}}\} \subseteq k_{\mathfrak{p}} \oplus \mathfrak{p}^2 \text{ in } A.$$

If $F^{\mathrm{trad}} = F$, then this is a contradiction to the fact that the traces of $\mathrm{Ad}(\rho_{\mathfrak{p}}(\mathrm{Frob}_x))$ for all places of good reduction x of K generate $A_{\mathfrak{p}} = k_{\mathfrak{p}} \oplus \mathfrak{p}$ and the proof is finished.

Suppose from now on that $F^{\mathrm{trad}} = F^2$, which can only occur if $p = r = 2$. Then we have to use further information related to the structure of the Drinfeld module φ in order to arrive to a contradiction. This will be achieved through a series of reduction steps contained in the following lemmas.

Lemma 6.3.
After replacing K by a finite extension, we can assume for all $\mathfrak{p} \notin S$ that

$$\Gamma_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}^2 \subset \mathrm{GL}_2(k_{\mathfrak{p}}) \subset \mathrm{GL}_2(A_{\mathfrak{p}}/\mathfrak{p}^2).$$

Proof. Let $\mathfrak{p} \notin S$. If $\gamma \in \Gamma_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}^2$, then we have already shown that there are uniquely determined $\lambda \in k_{\mathfrak{p}}$ and $\gamma_1 \in \mathrm{GL}_2(k_{\mathfrak{p}})$ such that $\gamma = (1+\pi\lambda)\gamma_1$.

Let us consider the composite map

$$\begin{array}{rcccl} \beta_{\mathfrak{p}} : G_K & \to & (1+\pi k_{\mathfrak{p}}) \times \mathrm{GL}_2(k_{\mathfrak{p}}) & \to & k_{\mathfrak{p}} \cong \mathbb{F}_2^{d_{\mathfrak{p}}}, \\ g & \mapsto & \rho_{\mathfrak{p},2}(g) = (1+\pi\lambda)\gamma_1 & \mapsto & \lambda, \end{array}$$

where $d_{\mathfrak{p}}$ denotes the dimension of $k_{\mathfrak{p}}$ as a vector space over the prime field $\mathbb{F}_2$. Being the composition of two homomorphisms, $\beta_{\mathfrak{p}}$ itself is a homomorphism. Moreover $\beta_{\mathfrak{p}}$ is unramified at all places of K. Indeed:

(1) If x is a place at which φ has good reduction, then $\rho_{\mathfrak{p},2}(I_x) = \{1\}$ and thus $\beta_{\mathfrak{p}}(I_x) = \{0\}$.
(2) Let x be one of the finitely many places of bad reduction of K and (ψ_x, Λ_x) the corresponding Tate uniformization. By reduction step (c) of Chapter 4 we have that $\rho_{\mathfrak{p},2}(I_x)$ lies in a subgroup of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}_r(A_{\mathfrak{p}}/\mathfrak{p}^2)$. The second map composing $\beta_{\mathfrak{p}}$ maps all matrices of this form to 0; therefore in this case we also have $\beta_{\mathfrak{p}}(I_x) = \{0\}$.

Let us consider the $d_{\mathfrak{p}}$ projection maps to the direct simple factors of $\mathbb{F}_2^{d_{\mathfrak{p}}}$. Since $\beta_{\mathfrak{p}}$ is unramified, these maps are again unramified. The kernels of all such maps, for all $\mathfrak{p} \notin S$, correspond to unramified extensions of K of degree ≤ 2. By the Hermite-Minkowski Theorem for function fields (cf. [**Gos96**], Theorem 8.23.5), there are only finitely many such extensions. Taking their compositum K', which is again a finite extension of K, and replacing K by K', we get the desired result. $\square$

Lemma 6.4.
After replacing K by a finite extension, we can assume for all primes $\mathfrak{p}$ of A that $\det(\rho_{\mathfrak{p},3}(G_K)) \subset (A_{\mathfrak{p}}/\mathfrak{p}^3)^*$ is contained in $k_{\mathfrak{p}}^*$.

Proof. By the definition of G_K^{geom} there exists $\sigma \in G_K$ such that $G_K = G_K^{\mathrm{geom}} \cdot \overline{\sigma^{\mathbb{Z}}}$, where $\overline{\sigma^{\mathbb{Z}}}$ denotes the closed subgroup of G_K that is topologically generated by σ. For all $\mathfrak{p}$, writing $A_{\mathfrak{p}}^* = k_{\mathfrak{p}}^*(1+\pi A_{\mathfrak{p}})$, we have

$$\det(\rho_{\mathfrak{p}}(\sigma^4)) \in k_{\mathfrak{p}}^*(1+\pi^3 A_{\mathfrak{p}}) \subset A_{\mathfrak{p}}^*.$$

It follows that the image of $G_K^{\mathrm{geom}} \cdot \overline{\sigma^{4\mathbb{Z}}}$ under $\det \circ \rho_{\mathfrak{p},3}$ is contained in $k_{\mathfrak{p}}^*$. The group $G_K^{\mathrm{geom}} \cdot \overline{\sigma^{4\mathbb{Z}}}$ is a subgroup of index 4 of G_K; replacing K by the corresponding extension of degree 4, we get the desired result. $\square$

Lemma 6.5.
After replacing K by a finite extension, we can assume for all $\mathfrak{p} \notin S$ that for all $g \in G_K$ we can write $\rho_{\mathfrak{p},3}(g)$ in the form $(1+\pi^2\gamma_2)\gamma_1$ with $\gamma_2 \in \mathfrak{sl}_2(k_{\mathfrak{p}})$ and $\gamma_1 \in \mathrm{GL}_2(k_{\mathfrak{p}})$.

Proof. Let us replace K by a finite extension for which both previous lemmas are satisfied. Let $\mathfrak{p} \notin S$ and $g \in G_K$. By Lemma 6.3 the element $\rho_{\mathfrak{p},3}(g)$ can be written in the form $(1+\pi^2\gamma_2)\gamma_1$ with $\gamma_2 \in \mathfrak{gl}_2(k_{\mathfrak{p}})$ and $\gamma_1 \in \mathrm{GL}_2(k_{\mathfrak{p}})$. Computing the determinant, we find

$$\det(\rho_{\mathfrak{p},3}(g)) = \det((1+\pi^2\gamma_2)\gamma_1) = (1+\pi^2\operatorname{tr}(\gamma_2))\det(\gamma_1).$$

By Lemma 6.4 this expression is contained in $k_{\mathfrak{p}}^*$. However, this is only possible if $\operatorname{tr}(\gamma_2) = 0$, in other words if $\gamma_2 \in \mathfrak{sl}_2(k_{\mathfrak{p}})$. $\square$

Lemma 6.6.
For the adjoint representation $\mathrm{Ad}_{\mathrm{SL}_2}$ of $\mathrm{GL}_{2,\mathbb{F}_2}$ on $\mathfrak{sl}_{2,\mathbb{F}_2}$, there is a short exact sequence of representations

$$0 \longrightarrow \mathbb{F}_2 \xrightarrow{\iota} \mathfrak{sl}_{2,\mathbb{F}_2} \longrightarrow (\mathrm{std}_2^{(2)} \otimes \det{}^{-1})_{\mathbb{F}_2} \longrightarrow 0, \tag{6.1}$$

where the representation on the left is the trivial representation of $\mathrm{GL}_{2,\mathbb{F}_2}$ and ι denotes the inclusion of scalars.

Proof. In order to alleviate the notation, we omit the subscript $\mathbb{F}_2$ for the length of this proof. The Lie algebra $\mathfrak{sl}_2$ is generated by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. The center $\mathfrak{c}$ of $\mathfrak{sl}_2$, generated by $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, is a subspace of $\mathfrak{sl}_2$ on which GL_2 acts trivially in the adjoint representation. Consequently $\mathrm{Ad}_{\mathrm{SL}_2}$ factors through $\mathfrak{sl}_2/\mathfrak{c}$; we denote the representation thus obtained by $\overline{\mathrm{Ad}_{\mathrm{SL}_2}}$. Let $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$ denote the images of $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ in $\mathfrak{sl}_2/\mathfrak{c}$, respectively and let $A = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in \mathrm{GL}_2$.

Then

$$\begin{aligned}
\overline{\mathrm{Ad}_{\mathrm{SL}_2}}(A)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \frac{1}{\det(A)}\begin{bmatrix} 0 & a^2 \\ c^2 & 0 \end{bmatrix}, \\
\overline{\mathrm{Ad}_{\mathrm{SL}_2}}(A)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \frac{1}{\det(A)}\begin{bmatrix} 0 & b^2 \\ d^2 & 0 \end{bmatrix}.
\end{aligned}$$

On the other hand, let $\mathrm{std}_2^{(2)} : \mathrm{GL}_2 \to \mathrm{GL}_2$ be given by $g \mapsto g^{(2)}$, where $g^{(2)}$ denotes the matrix obtained by raising the coefficients of g to the second power. Then

$$\begin{aligned}
(\mathrm{std}_2^{(2)} \otimes \det{}^{-1})(A)\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \frac{1}{\det(A)}\begin{pmatrix} a^2 \\ c^2 \end{pmatrix}, \\
(\mathrm{std}_2^{(2)} \otimes \det{}^{-1})(A)\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{1}{\det(A)}\begin{pmatrix} b^2 \\ d^2 \end{pmatrix}.
\end{aligned}$$

It follows that $\overline{\mathrm{Ad}_{\mathrm{SL}_2}}$ and $\mathrm{std}_2^{(2)} \otimes \det^{-1}$ are isomorphic representations of GL_2; this proves the desired result. $\square$

Tensoring (6.1) with $A_{\mathfrak{p}}/\mathfrak{p}^3$, we find the short exact sequence of representations

$$0 \longrightarrow A_{\mathfrak{p}}/\mathfrak{p}^3 \xrightarrow{\iota} \mathfrak{sl}_{2,A_{\mathfrak{p}}/\mathfrak{p}^3} \longrightarrow (\mathrm{std}_2^{(2)} \otimes \det{}^{-1})_{A_{\mathfrak{p}}/\mathfrak{p}^3} \longrightarrow 0,$$

where the representation on the left is the trivial representation of $\mathrm{GL}_{2,A_{\mathfrak{p}}/\mathfrak{p}^3}$.

For $\gamma \in \mathrm{GL}_2(A_{\mathfrak{p}}/\mathfrak{p}^3)$ this yields the equality of traces

$$\mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_2}(\gamma)) = 1 + \mathrm{tr}(\mathrm{std}_2^{(2)} \otimes \det{}^{-1})(\gamma) = 1 + \mathrm{tr}(\gamma^{(2)}) \cdot \det(\gamma)^{-1},$$

where $(\gamma)^{(2)}$ denotes the matrix obtained by raising the coefficients of γ to the second power.

Let $g \in G_K$ and $\rho_{\mathfrak{p},3}(g) = (1 + \pi^2\gamma_2)\gamma_1$ written in the form given by Lemma 6.5. Then

$$\begin{aligned}
\mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_2})(\rho_{\mathfrak{p},3}(g)) &= 1 + \mathrm{tr}((1+\pi^2\gamma_2)^{(2)}\gamma_1^{(2)}) \cdot \det(1+\pi^2\gamma_2)^{-1}\det(\gamma_1)^{-1} \\
&= 1 + \mathrm{tr}((1+\pi^2\gamma_2)^{(2)}\gamma_1^{(2)}) \cdot (1+\pi^2\,\mathrm{tr}(\gamma_2))^{-1}\det(\gamma_1)^{-1} \\
&= 1 + \mathrm{tr}(\gamma_1^{(2)})\det(\gamma_1)^{-1},
\end{aligned}$$

where, in order to obtain the last equality, we used that $(1+\pi^2\gamma_2)^{(2)} = 1+\pi^4\gamma_2^{(2)} = 1$ in $\mathrm{GL}_2(A_{\mathfrak{p}}/\mathfrak{p}^3)$, as well as the fact that $\mathrm{tr}(\gamma_2) = 0$.

Hence $\mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_2}(\gamma))$ is an element of $k_{\mathfrak{p}}$ for all $\gamma \in \Gamma_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}^3$. Since $\mathrm{tr}(\mathrm{Ad}\,\gamma) = \mathrm{tr}(\mathrm{Ad}_{\mathrm{SL}_2}(\gamma)) + 1$, it follows that

$$\{\mathrm{tr}(\mathrm{Ad}\,\gamma) \mid \gamma \in \Gamma_{\mathfrak{p}}/\Gamma_{\mathfrak{p}}^3\} \subseteq k_{\mathfrak{p}},$$

thereby giving

$$\{\mathrm{tr}(\mathrm{Ad}\,\gamma) \mid \gamma \in \Gamma_{\mathfrak{p}}\} \subseteq k_{\mathfrak{p}} \oplus \mathfrak{p}^3 \text{ in } A,$$

a contradiction to the fact that the traces of $\mathrm{Ad}(\rho_{\mathfrak{p}}(\mathrm{Frob}_x))$ for all places of good reduction x of K generate $A_{\mathfrak{p}}^{(2)} = \{a^2 \mid a \in A_{\mathfrak{p}}\} \not\subseteq k_{\mathfrak{p}} \oplus \mathfrak{p}^3$. This completes the proof of Proposition 6.2. $\square$

Corollary 6.7.
For almost all primes of $\mathfrak{p}$ *of* A *we have* $\tilde{\Gamma}_{\mathfrak{p}}^{\mathrm{geom},[1]} \neq 0$*; in other words,* $\Gamma_{\mathfrak{p}}^{\mathrm{geom},[1]}$ *contains a non-scalar element.*

Proof. Let S be the finite set of primes defined in the proof of Proposition 6.2 and let $\mathfrak{p} \notin S$. Since $\rho_{\mathfrak{p},2}(G_K^{\mathrm{geom}}) \lhd \rho_{\mathfrak{p},2}(G_K)$, the commutator group $[\Gamma_{\mathfrak{p}}^{[1]}, \rho_{\mathfrak{p},2}(G_K^{\mathrm{geom}})]$ is a subset of $\Gamma_{\mathfrak{p}}^{\mathrm{geom},[1]}$. Suppose that $\Gamma_{\mathfrak{p}}^{\mathrm{geom},[1]}$ only contains scalar elements. Then, after quotienting by $1 + \mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2)$, we find that

$$[\tilde{\Gamma}_{\mathfrak{p}}^{[1]}, \tilde{\Gamma}_{\mathfrak{p}}^{\mathrm{geom}}] \subseteq \tilde{\Gamma}_{\mathfrak{p}}^{\mathrm{geom},[1]} = \{0\},$$

which means that the commutator action of $\tilde{\Gamma}_{\mathfrak{p}}^{\mathrm{geom}}$ on $\tilde{\Gamma}_{\mathfrak{p}}^{[1]}$ is trivial. However, by the assumptions on S this action coincides with the action of $\mathrm{SL}_2(k_{\mathfrak{p}})$. Since the space of invariants $(\mathfrak{pgl}_{2,k_{\mathfrak{p}}})^{\mathrm{SL}_2(k_{\mathfrak{p}})}$ is trivial, this yields a contradiction. $\square$

6.4. Higher order approximation.

The following result is a straightforward consequence of the previous sections if $(p, r) \neq (2, 2)$. The proof in the remaining case is slightly more involved and will largely be based on [**Pin00**], Section 12.

Proposition 6.8.
Let H *be a closed subgroup of* $\mathrm{SL}_r(A_{\mathfrak{p}})$*. Assume that* $|k_{\mathfrak{p}}| \geq 4$*, that* $H^{[0]} = \mathrm{SL}_r(k_{\mathfrak{p}})$ *and* $H^{[1]} \subset \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2)$ *contains a non-scalar matrix. Then* $H = \mathrm{SL}_r(A_{\mathfrak{p}})$*.*

Proof. Since H is a closed subgroup of $\mathrm{SL}_r(A_{\mathfrak{p}})$, the claim is equivalent to $H^{[i]} = \mathrm{SL}_r(A_{\mathfrak{p}})^{[i]}$ for all $i \geq 0$. By assumption we have $H^{[0]} = \mathrm{SL}_r(k_{\mathfrak{p}})$.

Lemma 6.9.
We have $H^{[1]} = \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2)$*.*

Proof. Consider the conjugation action

$$H^{[0]} \times H^{[1]} \to H^{[1]}, \quad ([g], [h]) \mapsto [ghg^{-1}].$$

Under v_0 and v_1 this corresponds to the map

$$\mathrm{SL}_r(k_{\mathfrak{p}}) \times \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2) \to \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2), \quad (g, X) \mapsto gXg^{-1}.$$

Since $H^{[0]} = \mathrm{SL}_r(k_{\mathfrak{p}})$, it follows that $H^{[1]} \subset \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2)$ is closed under conjugation by $\mathrm{SL}_r(k_{\mathfrak{p}})$. Since it also contains a non-scalar matrix, by Proposition 5.2 it is equal to $\mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2)$. $\square$

Lemma 6.10.
If $(p, r) \neq (2, 2)$*, then we have* $H^{[i]} = \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1})$ *for all* $i \geq 1$*.*

Proof. Assume that $H^{[i]} = \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1})$ for some $i \geq 1$ and consider the commutator map

$$[\ ,\ ] : H^{[1]} \times H^{[i]} \to H^{[i+1]}.$$

Under v_1, v_i and v_{i+1} it corresponds to the Lie bracket

$$[\ ,\ ] : \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2) \times \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \to \mathfrak{sl}_r(\mathfrak{p}^{i+1}/\mathfrak{p}^{i+2}).$$

If $(p,r) \neq (2,2)$ then by Proposition 6.1 (a) the image of this latter map generates $\mathfrak{sl}_r(\mathfrak{p}^{i+1}/\mathfrak{p}^{i+2})$ as an additive group. Since $H^{[1]} = \mathfrak{sl}_r(\mathfrak{p}/\mathfrak{p}^2)$ and $H^{[i]} = \mathfrak{sl}_r(\mathfrak{p}^i/\mathfrak{p}^{i+1})$, we find $H^{[i+1]} = \mathfrak{sl}_r(\mathfrak{p}^{i+1}/\mathfrak{p}^{i+2})$. By induction the claim holds for all $i \geq 1$. $\square$

If $(p,r) \neq (2,2)$, this finishes the proof of the proposition. Let us assume from now on that $(p,r) = (2,2)$. In this case there is an exact sequence

$$0 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{sl}_2 \xrightarrow{dP} \mathfrak{pgl}_2 \longrightarrow \mathfrak{pgl}_2 / \mathfrak{psl}_2 \longrightarrow 0, \tag{6.2}$$

where $\mathfrak{c}$ and $\mathfrak{pgl}_2 / \mathfrak{psl}_2$ both have rank 1. The map

$$\mathrm{SL}_2(k_\mathfrak{p}) \cong G'^{[0]}_\mathfrak{p} \longrightarrow PG^{[0]}_\mathfrak{p} \cong \mathrm{PGL}_2(k_\mathfrak{p})$$

is an isomorphism, since the isogeny $P : \mathrm{SL}_2 \to \mathrm{PGL}_2$ is totally inseparable. It follows that $P^{-1}(PG^1_\mathfrak{p}) = G'^1_\mathfrak{p}$ inside $G'^0_\mathfrak{p}$. This yields the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P^{-1}(PG^2_\mathfrak{p}) \arrow[r] \arrow[d] & G'^1_\mathfrak{p} \arrow[r] \arrow[d, two heads] & PG^{[1]}_\mathfrak{p} \arrow[d, "\cong"] \\
0 \arrow[r] & 1 + \mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[r] \arrow[d, "\cong"] & G'^{[1]}_\mathfrak{p} \arrow[r] \arrow[d, "\cong"] & PG^{[1]}_\mathfrak{p} \arrow[d, "\cong"] \\
0 \arrow[r] & \mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[r] & \mathfrak{sl}_2(\mathfrak{p}/\mathfrak{p}^2) \arrow[r] & \mathfrak{pgl}_2(\mathfrak{p}/\mathfrak{p}^2).
\end{tikzcd} \tag{6.3}$$

By the Four Lemma we find that the composite vertical map on the left, henceforth denoted by μ, is surjective. Its kernel is $G'^2_\mathfrak{p}$. Consider the maps indicated by solid arrows in the diagram

$$\begin{tikzcd}
P^{-1}(PG^2_\mathfrak{p}) \arrow[r, "P"] \arrow[dd, "\mu"] & PG^2_\mathfrak{p} \arrow[d] \\
 & PG^{[2]}_\mathfrak{p} \arrow[d, "\cong"] \\
\mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[dr, dotted] & \mathfrak{pgl}_2(\mathfrak{p}^2/\mathfrak{p}^3) \arrow[d] \\
 & (\mathfrak{pgl}_2 / \mathfrak{psl}_2)(\mathfrak{p}^2/\mathfrak{p}^3).
\end{tikzcd} \tag{6.4}$$

By the exactness of the sequence (6.2) the composite morphism from the upper left corner to the lower right corner restricts to zero on $G'^2_{\mathfrak{p}}$. Hence it factors through a unique dotted arrow making the diagram commutative.

Lemma 6.11.
The dotted arrow in the diagram (6.4) *is an isomorphism.*

Proof. Let $\lambda : \mathbb{G}_m \to \mathrm{PGL}_2$ denote the cocharacter given by $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and let $\tilde\lambda : \mathbb{G}_m \to \mathrm{SL}_2$ denote the one given by $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. Then λ does not lift to a cocharacter of SL_2, but λ^2 lifts to $\tilde\lambda$. In other words we have a commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, "t\mapsto t^2"] \arrow[d, hook, "\tilde\lambda"] & \mathbb{G}_m \arrow[d, hook, "\lambda"] \\
\mathrm{SL}_2 \arrow[r, "P"] & \mathrm{PGL}_2 .
\end{tikzcd}$$

Let $\mathfrak{a}$ denote the Lie algebra of $\mathbb{G}_m$. Taking Lie algebras in the above diagram, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r, "\sim"] \arrow[d] & \mathfrak{a} \arrow[r, "0"] \arrow[d, hook, "d\tilde\lambda"] & \mathfrak{a} \arrow[r, "\sim"] \arrow[d, hook, "d\lambda"] & \mathfrak{a} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{c} \arrow[r] & \mathfrak{sl}_2 \arrow[r, "dP"] & \mathfrak{pgl}_2 \arrow[r] & \mathfrak{pgl}_2 / \mathfrak{psl}_2 \arrow[r] & 0.
\end{tikzcd} \tag{6.5}$$

The leftmost vertical map is an isomorphism for dimension reasons. The fact that λ is not congruent modulo 2 to a cocharacter coming from SL_2 implies that $\mathrm{Im}(d\lambda) \not\subset \mathrm{Im}(dP)$. Thus again for dimension reasons the rightmost vertical map is an isomorphism.

Taking $A_{\mathfrak{p}}$-valued points in the respective groups, we get a commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m(A_{\mathfrak{p}}) \arrow[rr] & & \mathbb{G}_m(A_{\mathfrak{p}}) & \\
\cup & & \cup & \\
1+\mathfrak{p} \arrow[rr, "t\mapsto t^2"] \arrow[dr, "\tilde\lambda"] \arrow[dd] & & 1+\mathfrak{p}^2 \arrow[dr, "\lambda"] \arrow[dd] & \\
& P^{-1}(PG^2_{\mathfrak{p}}) \arrow[rr, "P"] \arrow[dd] & & PG^2_{\mathfrak{p}} \arrow[dd] \\
\mathfrak{p}/\mathfrak{p}^2 \arrow[rr, dotted, "\mu"] \arrow[dr, "\cong"] & & \mathfrak{p}^2/\mathfrak{p}^3 \arrow[dr, "\cong"] & \\
& \mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[rr, dotted] & & (\mathfrak{pgl}_2 / \mathfrak{psl}_2)(\mathfrak{p}^2/\mathfrak{p}^3).
\end{tikzcd}$$

The lower oblique maps correspond to the leftmost and rightmost vertical isomorphisms of the diagram (6.5), respectively. The vertical maps in the back are defined by $1 + x \mapsto x$. Thus the dotted arrow in the back is given by

$$x \mapsto (1 + x)^2 - 1 = x^2 + 2x.$$

Since we are in characteristic 2, this is just the Frobenius map $x \mapsto x^2$, which clearly induces an isomorphism. Therefore the dotted arrow in front is an isomorphism, as desired. □

Lemma 6.12.
The composite map $P(H) \cap PG_{\mathfrak{p}}^2 \to (\mathfrak{pgl}_2 \,/\, \mathfrak{psl}_2)(\mathfrak{p}^2/\mathfrak{p}^3)$ is surjective.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H \cap G'^1_{\mathfrak{p}} \arrow[d] & \supset & H \cap P^{-1}(PG^2_{\mathfrak{p}}) \arrow[r] \arrow[dd, "\mu"] & P(H) \cap PG^2_{\mathfrak{p}} \arrow[d] \\
G'^{[1]}_{\mathfrak{p}} \arrow[d, "\cong"] & & & PG^{[2]}_{\mathfrak{p}} \arrow[d] \\
\mathfrak{sl}_2(\mathfrak{p}/\mathfrak{p}^2) & \supset & \mathfrak{c}(\mathfrak{p}/\mathfrak{p}^2) \arrow[r, dotted] & (\mathfrak{pgl}_2 \,/\, \mathfrak{psl}_2)(\mathfrak{p}^2/\mathfrak{p}^3)
\end{tikzcd}$$

deduced from diagrams (6.3) and (6.4). The leftmost composite vertical map is surjective by Lemma 6.9. By diagram (6.3) the left half is cartesian; hence the middle map is surjective. The dotted arrow is bijective by Lemma 6.11. Thus the rightmost composite vertical map is surjective. □

Lemma 6.13.
We have $H^{[2]} = \mathfrak{sl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$.

Proof. We proceed similarly to the proof of Lemma 6.9. Lemma 6.12 implies that the image of $P(H) \cap PG^2_{\mathfrak{p}}$ in $PG^{[2]}_{\mathfrak{p}} = \mathfrak{pgl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$ is non-zero. By Corollary 5.3 the image of $P(H) \cap PG^2_{\mathfrak{p}}$ in $PG^{[2]}_{\mathfrak{p}}$ thus contains $\mathfrak{psl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$. The generalized commutators

$$[P(H) \cap PG^2_{\mathfrak{p}}, H]^{\sim}$$

are contained in $H \cap G'^2_{\mathfrak{p}}$ and their images under the composite map

$$H \cap G'^2_{\mathfrak{p}} \longrightarrow G'^{[2]}_{\mathfrak{p}} \xrightarrow{\sim} \mathfrak{sl}_2(\mathfrak{p}^2/\mathfrak{p}^3) \twoheadrightarrow \mathfrak{psl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$$

contain all commutators of $\mathrm{SL}_2(k_{\mathfrak{p}})$ with $\mathfrak{psl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$. As $\mathrm{SL}_2(k_{\mathfrak{p}})$ acts non-trivially on this last group, the above composite map $H \cap G'^2_{\mathfrak{p}} \to \mathfrak{psl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$ must be non-zero. Thus the image of $H \cap G'^2_{\mathfrak{p}}$ in $G'^{[2]}_{\mathfrak{p}}$ is not contained in the scalars. By Proposition 5.2 the map $H \cap G'^2_{\mathfrak{p}} \to G'^{[2]}_{\mathfrak{p}}$ is surjective. □

Lemma 6.14.
We have $H^{[i]} = \mathfrak{sl}_2(\mathfrak{p}^i/\mathfrak{p}^{i+1})$ *for all* $i \geq 1$.

Proof. By the preceding lemmas and induction on i we may take $i \geq 2$, assume the assertion holds for all $i' \leq i$ and prove it for $i+1$. By Lemma 6.12 we may choose an element $\delta \in P(H) \cap PG_\mathfrak{p}^2$ whose image $X \in PG_\mathfrak{p}^{[2]} = \mathfrak{pgl}_2(\mathfrak{p}^2/\mathfrak{p}^3)$ projects to a non-zero element of $(\mathfrak{pgl}_2 \,/\, \mathfrak{psl}_2)(\mathfrak{p}^2/\mathfrak{p}^3)$. Let us consider the following commutative diagram, where the vertical arrow on the left hand side is surjective by the induction hypothesis.

$$\begin{tikzcd}
{[\delta, H \cap G_\mathfrak{p}^{\prime i-1}]^\sim} \arrow[r, phantom, "\subset"] \arrow[d] & H \cap G_\mathfrak{p}^{\prime i+1} \arrow[d] \arrow[ddd, bend left] \\
{[X, G_\mathfrak{p}^{\prime i-1}/G_\mathfrak{p}^{\prime i}]^\sim} \arrow[r, phantom, "\subset"] \arrow[dddr, phantom] \arrow[ddr] & G_\mathfrak{p}^{\prime [i+1]} \arrow[d, "\cong"] \\
& \mathfrak{sl}_2(\mathfrak{p}^i/\mathfrak{p}^{i+1}) \arrow[d] \\
& \mathfrak{psl}_2(\mathfrak{p}^i/\mathfrak{p}^{i+1}).
\end{tikzcd}$$

Proposition 6.1 implies that even though the Lie bracket pairing $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{psl}_2$ vanishes, we have $[\mathfrak{pgl}_2, \mathfrak{sl}_2]^\sim = \mathfrak{sl}_2$. Let us recall that $\mathfrak{pgl}_2 \,/\, \mathfrak{psl}_2$ has rank one. Hence for any $Y \in \mathfrak{pgl}_2$ that maps to a generator of $\mathfrak{pgl}_2 \,/\, \mathfrak{psl}_2$ we find that $[Y, \mathfrak{sl}_2]^\sim$ maps onto $\mathfrak{psl}_2$. It follows that the oblique map in the diagram is surjective.

Thus the composite vertical map on the right is surjective. By Proposition 5.2 the upper vertical map is then also surjective, as desired. $\square$

Lemmas 6.10 and 6.14 together imply Proposition 6.8. $\square$

Theorem 6.15.
For almost all primes $\mathfrak{p}$ *of* A*, we have* $\Gamma_\mathfrak{p}^{\mathrm{geom}} = \mathrm{SL}_r(A_\mathfrak{p})$.

Proof. Let S be the finite set containing the primes for which either Theorem 3.6, Theorem 4.1 or Proposition 6.7 does not hold and those with a residue field having at most 3 elements. For $\mathfrak{p} \notin S$, the field $k_\mathfrak{p}$ and the group $\Gamma_\mathfrak{p}^{\mathrm{geom}}$ satisfy all the assumptions of Proposition 6.8. Consequently $\Gamma_\mathfrak{p}^{\mathrm{geom}} = \mathrm{SL}_r(A_\mathfrak{p})$. $\square$

CHAPTER 7

Adelic openness for fields of transcendence degree 1

This chapter brings to a close the case where K has transcendence degree 1. We keep the notation and assumptions of the previous chapters.

PROPOSITION 7.1.
There exist a finite set S_0 of primes of A such that for every finite set of primes S containing S_0 we have

$$\Gamma_S^{\mathrm{geom}} = \Gamma_{S_0}^{\mathrm{geom}} \times \prod_{\mathfrak{p}\in S\setminus S_0} \mathrm{SL}_r(A_\mathfrak{p}).$$

PROOF. First, let us fix a finite set S_{00} of primes of A that is sufficiently large to ensure that for all primes outside S_{00} *all* previous propositions hold. Since $\Gamma_{S_{00}}^{\mathrm{geom}}$ is a closed subgroup of $\prod_{\mathfrak{p}\in S_{00}} \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_\mathfrak{p})}(D\otimes_A A_\mathfrak{p})$, it has only finitely many non-abelian finite simple quotients. Let $\Omega_1,\ldots,\Omega_n$ denote these simple quotients and let N be the maximum of their orders. Let S_0 be the union of S_{00} with the set of primes $\mathfrak{p}\notin S_{00}$ for which $|\,\mathrm{PSL}_r(k_\mathfrak{p})|\le N$. We will prove the proposition for this choice of S_0.

We proceed by induction on S. Consider any finite set of primes $S\supset S_0$ for which the desired equality is proved and any $\mathfrak{p}'\notin S$. To prove the equality for $S\cup\{\mathfrak{p}'\}$, we have to show

$$\Gamma_{S\cup\{\mathfrak{p}'\}}^{\mathrm{geom}} = \Gamma_S^{\mathrm{geom}} \times \mathrm{SL}_r(A_{\mathfrak{p}'}).$$

Identifying $\mathrm{SL}_r(A_{\mathfrak{p}'})$ with $\prod_{\mathfrak{p}\in S}\{1\}\times\mathrm{SL}_r(A_{\mathfrak{p}'})$, it suffices to show that

$$\Delta := \Gamma_{S\cup\{\mathfrak{p}'\}}^{\mathrm{geom}} \cap \mathrm{SL}_r(A_{\mathfrak{p}'})$$

is equal to $\mathrm{SL}_r(A_{\mathfrak{p}'})$.

LEMMA 7.2.
The image of Δ modulo $\mathfrak{p}'$ is equal to $\mathrm{SL}_r(k_{\mathfrak{p}'})$.

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma_{S\cup\{\mathfrak{p}'\}}^{\mathrm{geom}} \arrow[r] \arrow[d] & \prod_{i=1}^n \Omega_i \times \prod_{\mathfrak{p}\in S\setminus S_0} \mathrm{PSL}_r(k_\mathfrak{p}) \times \mathrm{PSL}_r(k_{\mathfrak{p}'}) \arrow[d] \\
\Gamma_S^{\mathrm{geom}} \arrow[r] & \prod_{i=1}^n \Omega_i \times \prod_{\mathfrak{p}\in S\setminus S_0} \mathrm{PSL}_r(k_\mathfrak{p}).
\end{tikzcd}$$

All factors on the right hand side are non-abelian finite simple groups. The inductive assumption implies that the lower homomorphism is surjective. By Theorem 4.1 the map $\Gamma^{\mathrm{geom}}_{S\cup\{\mathfrak{p}'\}} \to \mathrm{PSL}_r(k_{\mathfrak{p}'})$ is surjective. Hence if the upper homomorphism is not surjective, by Goursat's Lemma its image lies over the graph of an isomorphism between $\mathrm{PSL}_r(k_{\mathfrak{p}'})$ and another simple factor. Since $|\mathrm{PSL}_r(k_{\mathfrak{p}'})| > N \geq |\Omega_i|$ by construction, this factor must be $\mathrm{PSL}_r(k_{\mathfrak{p}'})$ for some $\mathfrak{p} \in S \setminus S_0$. This would however contradict Proposition 4.11. Therefore the upper homomorphism is surjective.

Given that the terms on the lower right hand side are all possible non-abelian finite simple quotients of Γ^{geom}_S, we deduce that the surjective homomorphism $\Gamma^{\mathrm{geom}}_{S\cup\{\mathfrak{p}'\}} \to \mathrm{PSL}_r(k_{\mathfrak{p}'})$ does not factor through Γ^{geom}_S. Thus its restriction to Δ is non-trivial. Since Δ is a normal subgroup of $\Gamma^{\mathrm{geom}}_{S\cup\{\mathfrak{p}'\}}$, its image is a normal subgroup of $\mathrm{PSL}_r(k_{\mathfrak{p}'})$. But this group is simple, and the image is non-trivial; hence the image is equal to $\mathrm{PSL}_r(k_{\mathfrak{p}'})$.

Let $\overline{\Delta}$ denote the image of Δ modulo $\mathfrak{p}'$. From the above it follows that $\overline{\Delta}$ is a subgroup of $\mathrm{SL}_r(k_{\mathfrak{p}'})$ that surjects onto $\mathrm{PSL}_r(k_{\mathfrak{p}'})$. Let Z denote the center of $\mathrm{SL}_r(k_{\mathfrak{p}'})$. Then

$$\overline{\Delta}Z/Z \cong \overline{\Delta}/\overline{\Delta} \cap Z = \mathrm{PSL}_r(k_{\mathfrak{p}'})$$

and hence $\overline{\Delta}Z = \mathrm{SL}_r(k_{\mathfrak{p}'})$. It follows that

$$\overline{\Delta} \supseteq [\overline{\Delta}, \overline{\Delta}] = [\overline{\Delta}Z, \overline{\Delta}Z] = [\mathrm{SL}_r(k_{\mathfrak{p}'}), \mathrm{SL}_r(k_{\mathfrak{p}'})].$$

Since we chose S_{00} to be large enough so that $|k_{\mathfrak{p}'}| \geq 4$, the group $\mathrm{SL}_r(k_{\mathfrak{p}'})$ is perfect and the last term of the inclusion sequence is equal to $\mathrm{SL}_r(k_{\mathfrak{p}'})$. This proves the desired equality. □

Lemma 7.3.
The group $\Delta^{[1]}$ contains a non-scalar element.

Proof. In order to alleviate the notation, let us denote $\mathrm{SL}_r(A_{\mathfrak{p}'})$ by G in the following proof. On the one hand, by Theorem 6.15 we have $\Gamma^{\mathrm{geom}}_{\mathfrak{p}'} = G$. On the other hand $\Delta \trianglelefteq \Gamma^{\mathrm{geom}}_{\mathfrak{p}'}$. Hence Δ is a normal subgroup of G and thereby $[G^{[1]}, \Delta/\Delta^2] \subseteq \Delta^{[1]}$. Suppose that $\Delta^{[1]}$ only contains scalar elements. Then, after quotienting by $1 + \mathfrak{c}(\mathfrak{p}'/\mathfrak{p}'^2)$, we find that

$$[\tilde{G}^{[1]}, \tilde{\Delta}] \subseteq \tilde{\Delta}^{[1]} = \{0\}.$$

This means that the commutator action of $\tilde{\Delta}$ on $\tilde{G}^{[1]}$ is trivial. However, by the previous lemma this action coincides with the action of $\mathrm{SL}_r(k_{\mathfrak{p}'})$, which is known to be non-trivial. We thus obtain a contradiction. □

The two previous lemmas show that all assumptions of Proposition 6.8 are satisfied for Δ. From this we conclude $\Delta = \mathrm{SL}_r(A_{\mathfrak{p}'})$, which completes the proof. □

We can now prove the following special case of Theorem 1.1.

Theorem 7.4.
The image of G_K^{geom} under ρ_{ad} is open in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$.

Proof. Taking the limit over all S containing S_0, Proposition 7.1 implies that the image of G_K^{geom} in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$ is equal to

$$\Gamma_{S_0}^{\mathrm{geom}} \times \prod_{p \notin S_0} \mathrm{SL}_r(A_{\mathfrak{p}}).$$

On the other hand, by Theorem 3.5 the subgroup

$$\Gamma_{S_0}^{\mathrm{geom}} \subset \prod_{p \in S_0} \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_{\mathfrak{p}})}(D \otimes_A A_{\mathfrak{p}})$$

is open; hence $\rho_{\mathrm{ad}}(G_K^{\mathrm{geom}})$ is open in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$, as stated. $\square$

CHAPTER 8

The case of arbitrary transcendence degree

In this chapter we let $\varphi : A \to K\{\tau\}$ denote a Drinfeld module satisfying the assumptions of Theorem 1.1, but the transcendence degree of K is now arbitrary. We prove the general case of Theorem 1.1 by reducing it to the case of a field of transcendence degree 1, using a specialization argument in the vein of [**PR09a**], Section 5.

We once more assume that the reduction steps introduced in Section 3.3 are in effect. In particular we assume that $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$ is contained in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(A_{\mathfrak{p}})}(D \otimes_A A_{\mathfrak{p}})$ for every place $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ of F.

8.1. Some group theory

Let $\mathfrak{p} \neq \mathfrak{p}_0$ be a prime of A and π a uniformizer at $\mathfrak{p}$. We use the same notation for congruence filtrations as we did in Chapter 6.

Proposition 8.1.
Let H be a closed subgroup of $\mathrm{SL}_r(A_{\mathfrak{p}})$. Assume that there exists an $n \geq 1$ such that $H^n/H^{3n} = G_{\mathfrak{p}}'^{n}/G_{\mathfrak{p}}'^{3n}$. Then we have

$$G_{\mathfrak{p}}'^{n} = H^n.$$

Proof. Since H is closed in $\mathrm{SL}_r(A_{\mathfrak{p}})$, it is enough to show that

$$H^l/H^{l+1} = G_{\mathfrak{p}}'^{l}/G_{\mathfrak{p}}'^{l+1} \tag{8.1}$$

for all $l \geq n$.

Lemma 8.2.
We have $H^i/H^j = G_{\mathfrak{p}}'^{i}/G_{\mathfrak{p}}'^{j}$ for all $n \leq i \leq j \leq 3n$.

Proof. Let $n \leq j \leq 3n$. We have the commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & H^j/H^{3n} \arrow[r] \arrow[d, hook] & H^n/H^{3n} \arrow[r] \arrow[d, "\cong"] & H^n/H^j \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & G_{\mathfrak{p}}'^{j}/G_{\mathfrak{p}}'^{3n} \arrow[r] & G_{\mathfrak{p}}'^{n}/G_{\mathfrak{p}}'^{3n} \arrow[r] & G_{\mathfrak{p}}'^{n}/G_{\mathfrak{p}}'^{j} \arrow[r] & 1.
\end{tikzcd}$$

The middle vertical map is an isomorphism by assumption and thus the rightmost vertical map is surjective; it is therefore an isomorphism. It follows that for all

$n \leq j \leq 3n$ we have
$$H^n/H^j = G'^n_{\mathfrak{p}}/G'^j_{\mathfrak{p}}.$$
Then, for $n \leq i \leq j \leq 3n$, we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & H^i/H^j \arrow[r] \arrow[d, hook] & H^n/H^j \arrow[r] \arrow[d, "\cong"] & H^n/H^i \arrow[r] \arrow[d, "\cong"] & 1 \\
1 \arrow[r] & G'^i_{\mathfrak{p}}/G'^j_{\mathfrak{p}} \arrow[r] & G'^n_{\mathfrak{p}}/G'^j_{\mathfrak{p}} \arrow[r] & G'^n_{\mathfrak{p}}/G'^i_{\mathfrak{p}} \arrow[r] & 1,
\end{tikzcd}$$

where the middle and right vertical maps are isomorphisms. By the Five Lemma the left vertical map is also an isomorphism. $\square$

From the above lemma we conclude that (8.1) holds for all $n \leq l \leq 3n-1$. From here on we proceed by induction. Let us assume there exists $m \geq 3n-1$ such that (8.1) holds for all for all $n \leq l \leq m$. We distinguish two cases:

Let us first assume that $(p, r) \neq (2, 2)$. We have the commutative diagram

$$\begin{tikzcd}
G'^{2n}_{\mathfrak{p}}/G'^{2n+1}_{\mathfrak{p}} \times G'^{m-2n+1}_{\mathfrak{p}}/G'^{m-2n+2}_{\mathfrak{p}} \arrow[r, "{[\,,\,]}"] \arrow[d, "\cong"] & G'^{m+1}_{\mathfrak{p}}/G'^{m+2}_{\mathfrak{p}} \arrow[d, "\cong"] \\
\mathfrak{sl}_r(\mathfrak{p}^{2n}/\mathfrak{p}^{2n+1}) \times \mathfrak{sl}_r(\mathfrak{p}^{m-2n+1}/\mathfrak{p}^{m-2n+2}) \arrow[r, "{[\,,\,]}"] & \mathfrak{sl}_r(\mathfrak{p}^{m+1}/\mathfrak{p}^{m+2}),
\end{tikzcd}$$

where the upper horizontal map denotes the commutator pairing and the lower horizontal map the Lie bracket pairing. By Proposition 6.1 (a) the set of commutators $[\mathfrak{sl}_r, \mathfrak{sl}_r]$ generates $\mathfrak{sl}_r$. Hence $[G'^{2n}_{\mathfrak{p}}/G'^{2n+1}_{\mathfrak{p}}, G'^{m-2n+1}_{\mathfrak{p}}/G'^{m-2n+2}_{\mathfrak{p}}]$ generates $G'^{m+1}_{\mathfrak{p}}/G'^{m+2}_{\mathfrak{p}}$. By assumption
$$[G'^{2n}_{\mathfrak{p}}/G'^{2n+1}_{\mathfrak{p}}, G'^{m-2n+1}_{\mathfrak{p}}/G'^{m-2n+2}_{\mathfrak{p}}] = [H^{2n}/H^{2n+1}, H^{m-2n+1}/H^{m-2n+2}]$$
and therefore
$$G'^{m+1}_{\mathfrak{p}}/G'^{m+2}_{\mathfrak{p}} \subset H^{m+1}/H^{m+2}.$$
The desired equality for $m+1$ follows.

Let us now assume $(p, r) = (2, 2)$.

Lemma 8.3.
We have
$$PG^{2n} \subset P(G'^n) \cdot PG^{2n+1}.$$

Proof. The sets
$$B := \left\{ 1 + \pi^{2n} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \,\middle|\, b \in A_{\mathfrak{p}} \right\}, \quad C := \left\{ 1 + \pi^{2n} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \,\middle|\, c \in A_{\mathfrak{p}} \right\}$$
are contained in $P(G'^n)$. Let a be an arbitrary element of $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}/\mathfrak{p}$ is a finite field of characteristic 2, the Frobenius map $A_{\mathfrak{p}}/\mathfrak{p} \to A_{\mathfrak{p}}/\mathfrak{p}, \alpha \mapsto \alpha^2$ is an

isomorphism. Thus the image of a in $A_\mathfrak{p}/\mathfrak{p}$ is a square and therefore there exist $x, y \in A_\mathfrak{p}$ such that $a = x^2 + \pi y + \pi^{2n+1}x^2 y$. Then

$$\begin{aligned} & P\left(\left(\begin{array}{cc} 1+\pi^n x & 0 \\ 0 & (1+\pi^n)^{-1} \end{array}\right)\right)\left(1+\pi^{2n+1}\left(\begin{array}{cc} y & 0 \\ 0 & 0 \end{array}\right)\right) \\ = & \left(1+\pi^{2n}\left(\begin{array}{cc} x^2 & 0 \\ 0 & 0 \end{array}\right)\right)\left(1+\pi^{2n+1}\left(\begin{array}{cc} y & 0 \\ 0 & 0 \end{array}\right)\right) \\ = & \left(1+\pi^{2n}\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\right) \end{aligned}$$

and hence

$$A := \left\{1+\pi^{2n}\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right) \,\middle|\, a \in A_\mathfrak{p}\right\} \subset P(G'^n) \cdot PG^{2n+1}.$$

Since A, B and C together generate $PG_\mathfrak{p}^{2n}$, the lemma follows. $\square$

By Lemma 8.2 we have $H^n \cdot G'^{2n+1} = G'^n$, from which it follows that

$$P(H^n) \cdot P(G'^{2n+1}) = P(G'^n)$$

and in turn $P(H) \cdot PG_\mathfrak{p}^{2n+1} \supset P(G'^n)$. Combined with Lemma 8.3, this yields

$$P(H) \cdot PG_\mathfrak{p}^{2n+1} \supset PG_\mathfrak{p}^{2n}$$

and thus

$$P(H)^{2n}/P(H)^{2n+1} = PG_\mathfrak{p}^{2n}/PG_\mathfrak{p}^{2n+1}.$$

We have the commutative diagram

$$\begin{array}{ccc} PG_\mathfrak{p}^{2n}/PG_\mathfrak{p}^{2n+1} \times G_\mathfrak{p}'^{m-2n+1}/G_\mathfrak{p}'^{m-2n+2} & \xrightarrow{[\,,\,]^\sim} & G_\mathfrak{p}'^{m+1}/G_\mathfrak{p}'^{m+2} \\ \downarrow \cong & & \downarrow \cong \\ \mathfrak{pgl}_2(\mathfrak{p}^{2n}/\mathfrak{p}^{2n+1}) \times \mathfrak{sl}_2(\mathfrak{p}^{m-2n+1}/\mathfrak{p}^{m-2n+2}) & \xrightarrow{[\,,\,]^\sim} & \mathfrak{sl}_2(\mathfrak{p}^{m+1}/\mathfrak{p}^{m+2}), \end{array}$$

where the upper horizontal map denotes the generalized commutator pairing and the lower horizontal map the generalized Lie bracket pairing. By Proposition 6.1 (b) the set of commutators $[\mathfrak{pgl}_2, \mathfrak{sl}_2]^\sim$ generates $\mathfrak{sl}_2$. Thus

$$[PG_\mathfrak{p}^{2n}/PG_\mathfrak{p}^{2n+1}, G_\mathfrak{p}'^{m-2n+1}/G_\mathfrak{p}^{m-2n+2}]^\sim$$

generates $G_\mathfrak{p}'^{m+1}/G_\mathfrak{p}'^{m+2}$. By assumption

$$\begin{aligned} &[PG_\mathfrak{p}^{2n}/PG_\mathfrak{p}^{2n+1}, G_\mathfrak{p}'^{m-2n+1}/G_\mathfrak{p}'^{m-2n+2}]^\sim \\ &= [P(H)^{2n}/P(H)^{2n+1}, H^{m-2n+1}/H^{m-2n+2}]^\sim \end{aligned}$$

and therefore

$$G_\mathfrak{p}'^{m+1}/G_\mathfrak{p}'^{m+2} \subset H^{m+1}/H^{m+2}.$$

The desired equality for $m+1$ follows also in this case. $\square$

8.2. Specialization with unchanging endomorphism ring

Let us choose an integral scheme X of finite type over $\mathbb{F}_p$ with function field K such that φ extends to a family of Drinfeld A-modules of rank rd over X. For any point $x \in X$, we obtain a Drinfeld A-module φ_x over the residue field k_x at x. The characteristic of φ_x is still $\mathfrak{p}_0$; hence for any prime $\mathfrak{p} \neq \mathfrak{p}_0$ of A, the specialization map induces an isomorphism of Tate modules

$$T_{\mathfrak{p}}(\varphi) \xrightarrow{\sim} T_{\mathfrak{p}}(\varphi_x). \tag{8.2}$$

Let $\overline{k_x}$ be a separable closure of k_x and $\overline{x} := \mathrm{Spec}(\overline{k_x})$ the associated geometric point of X over x. The morphisms $\mathrm{Spec}(K) \hookrightarrow X \hookleftarrow x$ induce homomorphisms of the étale fundamental groups

$$G_K \twoheadrightarrow \pi_1^{\mathrm{et}}(X, \overline{x}) \leftarrow \pi_1^{\mathrm{et}}(x, \overline{x}) = G_{k_x}.$$

The action of G_K on $T_{\mathfrak{p}}(\varphi)$ factors through $\pi_1^{\mathrm{et}}(X, \overline{x})$ and the specialization isomorphism (8.2) is equivariant under the above étale fundamental groups.

Let $X_{\overline{\kappa}} := X \times \overline{\kappa}$. In this case the morphisms $\mathrm{Spec}(K\overline{\kappa}) \hookrightarrow X_{\overline{\kappa}} \hookleftarrow x_{\overline{\kappa}}$ induce homomorphisms of the étale fundamental groups

$$G_K^{\mathrm{geom}} \twoheadrightarrow \pi_1^{\mathrm{et}}(X_{\overline{\kappa}}, \overline{x}_{\overline{\kappa}}) \leftarrow \pi_1^{\mathrm{et}}(x_{\overline{\kappa}}, \overline{x}_{\overline{\kappa}}) = G_{k_x}^{\mathrm{geom}}.$$

Similarly to the case of G_K above, the action of G_K^{geom} on $T_{\mathfrak{p}}(\varphi)$ factors through $\pi_1^{\mathrm{et}}(X_{\overline{\kappa}}, \overline{x}_{\overline{\kappa}})$ and the specialization isomorphism (8.2) is equivariant under the étale fundamental groups.

Proposition 8.4.
In the above situation, if φ satisfies the conditions of Theorem 1.1, then there exists a point $y \in X$ such that k_y has transcendence degree 1, we have

$$\mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_{\overline{k_y}}(\varphi_y)$$

and φ_y also satisfies the conditions of Theorem 1.1.

Proof. Let $\mathfrak{p} \neq \mathfrak{p}_0, \infty$ be a place of F for which Theorem 3.6 holds. Then $\Gamma_{\mathfrak{p}}$ is a closed subgroup of $\mathrm{GL}_r(A_{\mathfrak{p}})$. For any point $x \in X$ let $\Gamma_{x,\mathfrak{p}}$ denote the image of G_{k_x} in the representation on $T_{\mathfrak{p}}(\varphi_x)$. This is also a closed subgroup of $\mathrm{GL}_r(A_{\mathfrak{p}})$. Since $\mathfrak{p} \neq \mathfrak{p}_0$, the specialization isomorphism (8.2) turns $\Gamma_{x,\mathfrak{p}}$ into a subgroup of $\Gamma_{\mathfrak{p}}$ and $\Gamma_{x,\mathfrak{p}}^{\mathrm{geom}}$ into a subgroup of $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$.

Lemma 8.5.
There exists a point $y \in X$ such that k_y has transcendence degree 1 and $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom}}$ is open in $\mathrm{SL}_r(A_{\mathfrak{p}})$.

Proof. It follows from Theorem 3.5 that $\Gamma_{\mathfrak{p}}^{\mathrm{geom}}$ is an open subgroup of $\mathrm{SL}_r(A_{\mathfrak{p}})$. Hence there exists $n \geq 1$ such that $G_{\mathfrak{p}}'^{n} \subset \Gamma_{\mathfrak{p}}^{\mathrm{geom}}$. Let K' be a finite Galois extension of K such that

$$\mathrm{Gal}(K'\overline{\kappa}/K\overline{\kappa}) = \Gamma_{\mathfrak{p}}^{\mathrm{geom}}/G_{\mathfrak{p}}'^{3n}. \tag{8.3}$$

Since $(K' \cap K\overline{\kappa})\overline{\kappa} = K\overline{\kappa}$, replacing K by $K' \cap K\overline{\kappa}$ does not change property (8.3). Let $\pi_{\tilde{X}} : \tilde{X} \to X$ be the normalization of X in $K' \cap K\overline{\kappa}$. If there exists a point $\tilde{y} \in \tilde{X}$ such that $k_{\tilde{y}}$ has transcendence degree 1 and the image of $G_{k_{\tilde{y}}}$ is open in $\mathrm{SL}_r(A_{\mathfrak{p}})$, then $\pi_{\tilde{X}}(\tilde{y})$ satisfies the conditions of the lemma; consequently we can assume that K and K' have the same constant field κ. Let $\pi_X : X' \to X$ be the normalization of X in K'.

Lemma 8.6.
Assuming that K and K' have the same constant field, there exists an irreducible closed curve $Y \subset X$ such that

(1) $\pi_X^{-1}(Y)$ *is also irreducible, and*
(2) *the function fields of Y and $\pi_X^{-1}(Y)$ have the same constant field.*

This result is proved in [**Pin97**], Lemma 1.6, even though the second claim is not explicitly stated there.

Let $Y \subset X$ be as in Lemma 8.6 and let y denote the generic point of Y. Then k_y is the function field of Y and thus has transcendence degree 1.

Since K and K' have the same constant field, we have

$$\mathrm{Gal}(K'\overline{\kappa}/K\overline{\kappa}) = \mathrm{Gal}(K'/K).$$

The irreducibility of $\pi_X^{-1}(Y)$ implies that

$$\mathrm{Gal}(K'/K) = \mathrm{Gal}(k_{\pi_X^{-1}(y)}/k_y)$$

and since k_y and $k_{\pi_X^{-1}(y)}$ have the same constant field, we also have

$$\mathrm{Gal}(k_{\pi_X^{-1}(y)}/k_y) = \mathrm{Gal}(k_{\pi_X^{-1}(y)}\overline{\kappa}/k_y\overline{\kappa}).$$

Combining these equalities with (8.3), we find

$$\mathrm{Gal}(k_{\pi_X^{-1}(y)}\overline{\kappa}/k_y\overline{\kappa}) = \Gamma_{\mathfrak{p}}^{\mathrm{geom}}/G_{\mathfrak{p}}'^{3n}.$$

This in turn implies

$$\Gamma_{y,\mathfrak{p}}^{\mathrm{geom}} \cdot G_{\mathfrak{p}}'^{3n} = \Gamma_{\mathfrak{p}}^{\mathrm{geom}}.$$

It follows that $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom},n} \cdot G_{\mathfrak{p}}'^{3n} = G_{\mathfrak{p}}'^{n}$. In other words we have

$$\Gamma_{y,\mathfrak{p}}^{\mathrm{geom},n}/\Gamma_{y,\mathfrak{p}}^{\mathrm{geom},3n} = G_{\mathfrak{p}}'^{n}/G_{\mathfrak{p}}'^{3n}.$$

By Proposition 8.1 this yields $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom},n} = G_{\mathfrak{p}}'^{n}$. In particular $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom}}$ contains an open subgroup of $\mathrm{SL}_r(A_{\mathfrak{p}})$. Thus $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom}}$ is itself open in $\mathrm{SL}_r(A_{\mathfrak{p}})$. □

Let $y \in X$ be as in Lemma 8.5. By Proposition 3.1 there exists a finite separable extension k_y' of k_y such that all endomorphisms of φ_y are defined over k_y'. This extension corresponds to an open subgroup Γ_y of $\Gamma_{y,\mathfrak{p}}^{\mathrm{geom}}$, which by Lemma 8.5 is again open in $\mathrm{SL}_r(A_{\mathfrak{p}})$. The Tate conjecture for Drinfeld modules (see (1.1)) yields an inclusion

$$\mathrm{End}_{\overline{k_y}}(\varphi_y) = \mathrm{End}_{k_y'}(\varphi_y) \hookrightarrow \mathrm{End}_{A_{\mathfrak{p}}[\Gamma_y]}(T_{\mathfrak{p}}(\varphi_y)).$$

Since Γ_y is open in $\mathrm{SL}_r(A_\mathfrak{p})$, it is Zariski dense in SL_r. Hence

$$\mathrm{End}_{A_\mathfrak{p}[\Gamma_y]}(T_\mathfrak{p}(\varphi_y)) = \mathrm{End}_{A_\mathfrak{p}[\mathrm{SL}_r]}(T_\mathfrak{p}(\varphi_y)) \cong \mathrm{End}_{A_\mathfrak{p}[\mathrm{SL}_r]}(A_\mathfrak{p}^{rd}) \cong \mathrm{M}_{d\times d}(A_\mathfrak{p}).$$

Combined with $\mathrm{End}_{\overline{K}}(\varphi) \subset \mathrm{End}_{\overline{k_y}}(\varphi_y)$ and $\mathrm{End}_{\overline{K}}(\varphi) \otimes_A A_\mathfrak{p} \cong \mathrm{M}_{d\times d}(A_\mathfrak{p})$, this yields

$$\mathrm{End}_{\overline{K}}(\varphi) \otimes_A A_\mathfrak{p} = \mathrm{End}_{\overline{k_y}}(\varphi_y) \otimes_A A_\mathfrak{p};$$

hence $\mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_{\overline{k_y}}(\varphi_y)$.

Choose a maximal commutative subring $\hat{A}$ of $\mathrm{End}_{\overline{K}}(\varphi)$ as in Section 3.1. Since $\mathrm{End}_{\overline{K}}(\varphi) = \mathrm{End}_{\overline{k_y}}(\varphi_y)$, the ring $\hat{A}$ is a maximal commutative subring of $\mathrm{End}_{\overline{k_y}}(\varphi_y)$. Let $\hat{\varphi}_y : \hat{A} \to \overline{k_y}\{\tau\}$ denote its tautological embedding. Let $\tilde{A}$ be a normalization of $\hat{A}$ in their common quotient field $\tilde{F}$ and let $\tilde{\varphi}_y : \tilde{A} \to \overline{k_y}\{\tau\}$ be a Drinfeld module isogenous to $\hat{\varphi}$. Then $\mathrm{rank}(\tilde{\varphi}_y) = r$ and by the assumptions of Theorem 1.1 on φ we have $r \geq 2$. In order to apply Theorems 6.1 and 6.2 of [**Pin06b**] to φ_y in a straightforward way, we need the assumption that φ_y is not isomorphic to a Drinfeld module defined over a finite field. However, a careful reading of the proofs of those theorems shows that it is sufficient to have the analogous assumption for $\tilde{\varphi}_y$. By [**Pin06b**], Proposition 2.1, this is equivalent to $r = \mathrm{rank}(\tilde{\varphi}_y) \geq 2$. Thus we can apply [**Pin06b**], Theorems 6.1 and 6.2, to φ_y. Combining them shows that there exists a subfield E of F with $[F/E] < \infty$ and $B := E \cap A$ that is uniquely defined by either one of the following two properties:

(1) For every infinite subring $C \subset A$ we have $\mathrm{End}_{\overline{K}}(\varphi_y|C) \subset \mathrm{End}_{\overline{K}}(\varphi_y|B)$.

(2) For every non-empty finite set P of places $\neq \mathfrak{p}_0, \infty$ of F, let Q denote the set of places below those in P and let G_Q denote the centralizer of $\mathrm{End}_{\overline{K}}(\varphi_y|B) \otimes E_Q$ in $\underline{\mathrm{Aut}}_{E_Q}(T_Q(\varphi_y|B) \otimes E_Q)$. Then $G^{\mathrm{der}}_{y,Q}(B_Q) \cap \Gamma^{\mathrm{geom}}_{y,Q}$ is open in both $G^{\mathrm{der}}_{y,Q}(B_Q)$ and $\Gamma^{\mathrm{geom}}_{y,Q}$.

By Lemma 8.5 the group $\Gamma^{\mathrm{geom}}_{y,\mathfrak{p}}$ is open in $\mathrm{SL}_r(A_\mathfrak{p})$ for all places $\mathfrak{p}$ of F for which Theorem 3.6 holds; hence the field F satisfies property (2). Given that E is uniquely determined, we have $E = F$ and φ_y satisfies the remaining assumptions of Theorem 1.1 by property (1). $\square$

Proof of Theorem 1.1. If K has transcendence degree 1, then the result is Theorem 7.4. In the general case let us choose y as in Proposition 8.4. Then Theorem 7.4 shows that the image of the adelic representation associated to φ_y is open in $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$. By the specialization isomorphism (8.2) this image is a subgroup of $\rho_{\mathrm{ad}}(G_K^{\mathrm{geom}})$. Thus the latter is an open subgroup of $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{rd}(\mathbb{A}_F^{(\mathfrak{p}_0,\infty)})}(D \otimes_A \mathbb{A}_F^{(\mathfrak{p}_0,\infty)})$ as well. $\square$

CHAPTER 9

The general case

In this chapter we generalize the result of the Main Theorem. Let φ be a Drinfeld A-module over a finitely generated field K, of special characteristic $\mathfrak{p}_0$ and assume that φ is not isomorphic over $\overline{K}$ to a Drinfeld module defined over a finite field. Let Z denote the center of $\mathrm{End}_{\overline{K}}(\varphi) \otimes_A F$. The following result was proved in [**Pin06b**], Theorems 6.1 and 6.2:

Proposition 9.1.
In the above situation, there exists a unique subfield E of Z with the following properties:

(a) *The intersection $B := E \cap \mathrm{End}_{\overline{K}}(\varphi)$ is infinite with quotient field E, and $[Z/E]$ is finite.*
(b) *The tautological embedding $\psi : B \to \overline{K}\{\tau\}$ is a Drinfeld B-module (except that B is not necessarily a maximal order in E) whose endomorphism ring $\mathrm{End}_{\overline{K}}(\psi)$ is an order in a central simple algebra over E of dimension d'^2. Moreover, there exists an integer $r' \geq 2$ such that ψ is of rank $r'd'$.*
(c) *For any other infinite subring $C \subset \mathrm{End}_{\overline{K}}(\varphi)$ let $\chi : C \to \overline{K}\{\tau\}$ denote the tautological embedding. Then $\mathrm{End}_{\overline{K}}(\chi) \subset \mathrm{End}_{\overline{K}}(\psi)$.*

Let E, B and ψ be as in the above proposition and let $D := \mathrm{End}_{\overline{K}}(\psi)$. By Proposition 3.1 there exists a finite separable extension $K' \subset \overline{K}$ of K such that $D = \mathrm{End}_{K'}(\psi)$.

We now introduce a common ring extension of A and B, and the corresponding Drinfeld module. These will allow us to compare the Galois actions associated to φ and ψ, respectively.

Let C denote the center of $\mathrm{End}_{\overline{K}}(\varphi)$ and $\chi : C \to \overline{K}\{\tau\}$ the tautological embedding. It follows from property (c) of Proposition 9.1 that χ is defined over K'. Since A and B are contained in C, by the definitions of ψ and χ we have $\chi|A = \varphi$ and $\chi|B = \psi$.

The quotient field of C is Z. As explained in Section 3.1, the ring $\mathrm{End}_{\overline{K}}(\varphi) \otimes_A F_\infty$ is a division algebra over F_∞; thus ∞ does not split in Z. Let ∞_Z denote the unique place of Z above ∞. Among the places of Z above $\mathfrak{p}_0$ let $\mathfrak{P}_0$ denote the one that corresponds to the characteristic of χ. Let ∞_E denote the place of E below ∞_Z and $\mathfrak{q}_0$ the place of E below $\mathfrak{P}_0$.

Let P denote the set of places of F outside of $\mathfrak{p}_0$ and ∞, let R be the set of places of Z above those in P and let Q be the set of places of E below those in R.

Propositions 3.5 and 3.6 of [**Pin06b**] imply that the only place of Z above ∞_E is ∞_Z and the only place of Z above $\mathfrak{q}_0$ is $\mathfrak{P}_0$; this shows that $\mathfrak{q}_0$ and ∞_E are not contained in Q. Let $\mathbb{A}_F^P := \mathbb{A}_F^{(\mathfrak{p}_0,\infty)}$ denote the ring of adeles of F at places in P, let $\mathbb{A}_Z^R \subset \mathbb{A}_Z^{(\mathfrak{P}_0,\infty_Z)}$ be the ring of adeles of Z at places in R and $\mathbb{A}_E^Q \subset \mathbb{A}_E^{(\mathfrak{q}_0,\infty_E)}$ the ring of adeles of E at places in Q.

The following chart summarizes the notation that we have introduced.

$$\begin{tikzcd} & \boxed{\begin{array}{c} C, Z, \chi \\ \mathfrak{P}_0, \infty_Z \\ \text{the set } R \\ \mathbb{A}_Z^R \subset \mathbb{A}_Z^{(\mathfrak{P}_0,\infty_Z)} \end{array}} \arrow[dl, dash] \arrow[dr, dash] & \\ \boxed{\begin{array}{c} B, E, \psi \\ \mathfrak{q}_0, \infty_E \\ \text{the set } Q \\ \mathbb{A}_E^Q \subset \mathbb{A}_E^{(\mathfrak{q}_0,\infty_E)} \end{array}} & & \boxed{\begin{array}{c} A, F, \varphi \\ \mathfrak{p}_0, \infty \\ \text{the set } P \\ \mathbb{A}_F^P = \mathbb{A}_F^{(\mathfrak{p}_0,\infty)} \end{array}} \end{tikzcd}$$

From $\chi|A = \varphi$ and $\chi|B = \psi$ we get $G_{K'}$-equivariant homomorphisms

$$\prod_{\mathfrak{q}\in Q} T_{\mathfrak{q}}(\psi)\otimes_B E \cong \prod_{\mathfrak{P}|\mathfrak{q},\, \mathfrak{q}\in Q} T_{\mathfrak{P}}(\chi)\otimes_C Z \twoheadrightarrow \prod_{\mathfrak{P}\in R} T_{\mathfrak{P}}(\chi)\otimes_C Z \cong \prod_{\mathfrak{p}\in P} T_{\mathfrak{p}}(\varphi)\otimes_A F. \quad (9.1)$$

We claim that the composite homomorphism induces an action of the algebraic group $\mathrm{Cent}_{\mathrm{GL}_{r'd',\mathbb{A}_E^Q}}(D\otimes_B \mathbb{A}_E^Q)$ on $\prod_{\mathfrak{p}\in P} T_{\mathfrak{p}}(\varphi)\otimes_A F$. This is a consequence of the following lemma:

LEMMA 9.2.
For all places $\mathfrak{q} \neq \mathfrak{q}_0, \infty_E$ of E the decomposition map

$$T_{\mathfrak{q}}(\psi) \otimes_B E \cong \prod_{\mathfrak{P}|\mathfrak{q}} T_{\mathfrak{P}}(\chi) \otimes_C Z$$

is $\mathrm{Cent}_{\underline{\mathrm{Aut}}(T_{\mathfrak{q}}(\psi)\otimes_B E)}(D\otimes_B E_{\mathfrak{q}})$-invariant.

PROOF. Let $\mathfrak{q} \neq \mathfrak{q}_0, \infty_E$ be a place of E. By Proposition 9.1 (c) we have $C \subset \mathrm{End}_{\overline{K}}(\varphi) \subset D$. This yields a series of inclusions

$$\mathrm{End}_{E_{\mathfrak{q}}}(T_{\mathfrak{q}}(\psi)\otimes_B E) \supset D\otimes_B E_{\mathfrak{q}} \supset C\otimes_B E_{\mathfrak{q}} \cong Z\otimes_E E_{\mathfrak{q}} \cong \prod_{\mathfrak{P}|\mathfrak{q}} Z_{\mathfrak{P}}$$

which, to start with, shows that $\prod_{\mathfrak{P}|\mathfrak{q}} Z_{\mathfrak{P}}$ acts on $T_{\mathfrak{q}}(\psi)\otimes_B E$ and further implies that this action commutes with the action of $\mathrm{Cent}_{\underline{\mathrm{Aut}}(T_{\mathfrak{q}}(\psi)\otimes_B E)}(D\otimes_B E_{\mathfrak{q}})$. The lemma follows. □

The inclusions $B \subset C \subset \mathrm{End}_{\overline{K}}(\varphi)$ and the definitions of P, Q and R yield a series of inclusions

$$\mathbb{A}_E^Q \subset \mathbb{A}_Z^R = C \otimes_A \mathbb{A}_F^P \subset \mathrm{End}_{\overline{K}}(\varphi) \otimes \mathbb{A}_F^P,$$

which shows that $\mathbb{A}_E^Q$ acts naturally on $\prod_{\mathfrak{p}\in P} T_{\mathfrak{p}}(\varphi) \otimes_A F$.

It follows that the action of $\mathrm{Cent}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^Q)}(D \otimes_B \mathbb{A}_E^Q)$ on $\prod_{\mathfrak{p}\in P} T_{\mathfrak{p}}(\varphi) \otimes_A F$ is faithful and we can consider the intersection

$$\rho_{\mathrm{ad}}(G_K^{\mathrm{geom}}) \cap \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^Q)}(D \otimes_B \mathbb{A}_E^Q).$$

Theorem 9.3 (Adelic openness in special characteristic).
In the above situation, the intersection

$$\rho_{ad}(G_K^{\mathrm{geom}}) \cap \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^Q)}(D \otimes_B \mathbb{A}_E^Q)$$

is open in both $\rho_{ad}(G_K^{\mathrm{geom}})$ *and* $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^Q)}(D \otimes_B \mathbb{A}_E^Q)$.

Proof. Since K' is a finite extension of K, the Galois group $G_{K'}^{\mathrm{geom}}$ is open in G_K^{geom}; hence it is sufficient to prove the statement for K' instead of K. Given that the map in (9.1) is $G_{K'}$-equivariant, we can thus reduce the statement to the following lemma:

Lemma 9.4.
Let $\psi : B \to K'\{\tau\}$ be as above, let $S \supset \{\mathfrak{q}_0, \infty_E\}$ denote a finite set of places of E and let $\mathbb{A}_E^{(S)}$ be the ring of adeles of E at places outside of S. Let ρ_{ad} denote the adelic representation associated to ψ. Then the intersection

$$\rho_{ad}(G_{K'}^{\mathrm{geom}}) \cap \mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^{(S)})}(D \otimes_B \mathbb{A}_E^{(S)})$$

is open in both $\rho_{ad}(G_{K'}^{\mathrm{geom}})$ *and* $\mathrm{Cent}^{\mathrm{der}}_{\mathrm{GL}_{r'd'}(\mathbb{A}_E^{(S)})}(D \otimes_B \mathbb{A}_E^{(S)})$.

Proof. Given the projection $\mathbb{A}_E^{(\mathfrak{q}_0,\infty_E)} \twoheadrightarrow \mathbb{A}_E^{(S)}$, it is enough to prove the lemma for the case $S = \{\mathfrak{q}_0, \infty_E\}$.

By Proposition 9.1 the Drinfeld B-module ψ satisfies the conditions of Theorem 1.1, except that B is not necessarily a maximal order in E. Let $\tilde{B}$ denote the normalization of B in E. By [**Hay79**], Proposition 3.2, there exists a Drinfeld module $\tilde{\psi} : \tilde{B} \to K'\{\tau\}$ such that $\tilde{\psi}|B$ is isogenous to ψ. Let $\tilde{D} := \mathrm{End}_{\overline{K}}(\tilde{\psi})$. Since any isogeny induces an isomorphism of endomorphism rings up to finite index, we have $\tilde{D} \otimes_{\tilde{B}} E = D \otimes_B E$, and thus $\tilde{D}$ is an order in a central simple algebra over E of dimension d'^2 by Proposition 9.1 (b). By the same argument, Proposition 9.1 (c) implies that for every infinite subring $\tilde{C}$ of $\tilde{B}$ we have $\mathrm{End}_{\overline{K}}(\tilde{\psi}|\tilde{C}) \subset \tilde{D}$. Moreover, since any isogeny preserves the rank, we have $\mathrm{rank}(\tilde{\psi}) = \mathrm{rank}(\psi) = r'd'$ with, let us recall, $r' \geq 2$. Thus $\tilde{\psi}$ satisfies all conditions of Theorem 1.1.

On the other hand, the isogeny between ψ and $\tilde{\psi}$ induces a $G_{K'}$-equivariant isomorphism

$$\prod_{\mathfrak{q}\neq\mathfrak{q}_0,\infty_E} T_{\mathfrak{q}}(\psi)\otimes_{\tilde{B}} E \cong \prod_{\mathfrak{q}\neq\mathfrak{q}_0,\infty_E} T_{\mathfrak{q}}(\psi)\otimes_{B} E.$$

Applying Theorem 1.1 to $\tilde{\psi}$ and using the above isomorphism then yields the desired result for ψ. □

Using the set S of places of E outside Q, which is finite by the definition of the set Q, the above lemma allows us to complete the proof of the theorem. □

This theorem effectively settles the question of adelic openness for arbitrary Drinfeld modules in special characteristic.

Bibliography

[Car87] Roger W. Carter. *On the structure and complex representation theory of finite groups of Lie type*, volume 16 of *Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen [Lecture Notes in Mathematics at the University of Essen]*. Universität Essen Fachbereich Mathematik, Essen, 1987. Notes taken by Burkhard Külshammer.

[Che58] Claude Chevalley. Les groupes de type A_n. In *Séminaire C. Chevalley, 1956–1958. Classification des groupes de Lie algébriques*, volume 2 of *2 vols*, pages 20.1–20.13, 1958.

[CPS75] Edward Cline, Brian Parshall, and Leonard Scott. Cohomology of finite groups of Lie type. I. *Inst. Hautes Études Sci. Publ. Math.*, 45:169–191, 1975.

[DH87] Pierre Deligne and Dale Husemoller. Survey of Drinfel′d modules. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume 67 of *Contemp. Math.*, pages 25–91. Amer. Math. Soc., Providence, RI, 1987.

[Dri74] V. G. Drinfeld. Elliptic modules. *Mat. Sb. (N.S.)*, 94(136):594–627, 656, 1974.

[Dri77] V. G. Drinfeld. Elliptic modules. II. *Mat. Sb. (N.S.)*, 102(144)(2):182–194, 325, 1977.

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. Almost simple K-groups.

[Gor68] Daniel Gorenstein. *Finite groups*. Harper & Row Publishers, New York, 1968.

[Gos92] David Goss. L-series of t-motives and Drinfeld modules. In *The arithmetic of function fields (Columbus, OH, 1991)*, volume 2 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 313–402. de Gruyter, Berlin, 1992.

[Gos96] David Goss. *Basic structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1996.

[Hay79] David R. Hayes. Explicit class field theory in global function fields. In *Studies in algebra and number theory*, volume 6 of *Adv. in Math. Suppl. Stud.*, pages 173–217. Academic Press, New York, 1979.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.

[Hum06] James E. Humphreys. *Modular representations of finite groups of Lie type*, volume 326 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[Jan03] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.

[LP98] Michael Larsen and Richard Pink. Finite subgroups of algebraic groups. Preprint, http://www.math.ethz.ch/∼pink/ftp/LP5.pdf, 1998.

[Pin97] Richard Pink. The Mumford-Tate conjecture for Drinfeld-modules. *Publ. Res. Inst. Math. Sci.*, 33(3):393–425, 1997.

[Pin00] Richard Pink. Strong approximation for Zariski dense subgroups over arbitrary global fields. *Comment. Math. Helv.*, 75(4):608–643, 2000.

[Pin06a] Richard Pink. The Galois representations associated to a Drinfeld module in special characteristic. I. Zariski density. *J. Number Theory*, 116(2):324–347, 2006.

[Pin06b] Richard Pink. The Galois representations associated to a Drinfeld module in special characteristic. II. Openness. *J. Number Theory*, 116(2):348–372, 2006.

[PR09a] Richard Pink and Egon Rütsche. Adelic openness for Drinfeld modules in generic characteristic. *J. Number Theory*, 129(4):882–907, 2009.

[PR09b] Richard Pink and Egon Rütsche. Image of the group ring of the Galois representation associated to Drinfeld modules. *J. Number Theory*, 129(4):866–881, 2009.

[Pre87] A. A. Premet. Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic. *Mat. Sb. (N.S.)*, 133(175)(2):167–183, 271, 1987.

[PT06] Richard Pink and Matthias Traulsen. The Galois representations associated to a Drinfeld module in special characteristic. III. Image of the group ring. *J. Number Theory*, 116(2):373–395, 2006.

[Ste63] Robert Steinberg. Representations of algebraic groups. *Nagoya Math. J.*, 22:33–56, 1963.

[Ste81] Robert Steinberg. Generators, relations and coverings of algebraic groups. II. *J. Algebra*, 71(2):527–543, 1981.

[Tag95] Yuichiro Taguchi. The Tate conjecture for t-motives. *Proc. Amer. Math. Soc.*, 123(11):3285–3287, 1995.

[Tak82] Toyofumi Takahashi. Good reduction of elliptic modules. *J. Math. Soc. Japan*, 34(3):475–487, 1982.

[Tam94a] Akio Tamagawa. Generalization of Anderson's t-motives and Tate conjecture. *Sūrikaisekikenkyūsho Kōkyūroku*, 884:154–159, 1994. Moduli spaces, Galois representations and L-functions (Japanese) (Kyoto, 1993, 1994).

[Tam94b] Akio Tamagawa. The Tate conjecture for A-premotives. Preprint, 1994.

[Tam95] Akio Tamagawa. The Tate conjecture and the semisimplicity conjecture for t-modules. *Sūrikaisekikenkyūsho Kōkyūroku*, 925:89–94, 1995. Algebraic number theory and arithmetic geometry (Japanese) (Kyoto, 1994).

[TZ70] Olga Taussky and Hans Zassenhaus. On the 1-cohomology of the general and special linear groups. *Aequationes Math.*, 5:129–201, 1970.

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