Zip Data

Master's Thesis

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1 Introduction

Let G be a connected reductive algebraic group over an algebraically closed field, P and P' two parabolic subgroups of G with Levi factors L and L' respectively. Let $\varphi\colon L \to L'$ be an isogeny. Consider the algebraic group

$$Z := (\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L,$$

where $\ell \in L$ acts on $\mathcal{R}_u P$ by conjugation with ℓ and on $\mathcal{R}_u P'$ by conjugation with $\varphi(\ell)$. The group Z acts on G from the left by

$$(u, u', \ell) \cdot g = u'\varphi(\ell)g\ell^{-1}u^{-1}.$$

We call such a datum $(G, P, P', L, L', \varphi)$ an *algebraic zip datum.* In this paper we study the orbit structure of such an action.

Let W be the Weyl group of G with respect to a maximal torus T of G contained in L and pick a Borel subgroup B of P containing T. This defines a set of simple reflections S. There exists a subset I of S such that the Weyl group $W(L)$ of L is generated by the elements of I. On W there is a natural partial order, the Bruhat order, which we denote by $\leq$. In W there exists a natural set of representatives for $W/W(L)$, namely

$$W^I = \{w \in W \mid w \leq ws \text{ for all } s \in I\}.$$

To each $w \in W^I$ we associate a locally closed subset G^w of G such that:

Theorem 1.1 (see 5.19)**.**
$$G = \coprod_{w \in W^I} G^w.$$

Theorem 1.2 (see 6.12)**.** *The closure of G^w is given by*

$$\overline{G^w} = \coprod_{w'} G^{w'},$$

where w' ranges over the $w' \in W^I$ for which there exists $v \in W(L)$ such that $\varphi(v)w'v^{-1} \leq w$.

If P and P' are Borel subgroups of G and $L = L'$ is a common maximal torus our decomposition is the Bruhat decomposition of G into double cosets $P'wP$ for $w \in W$.

The orbits in each G^w correspond to the orbits of the action of a certain reductive group on itself by twisted conjugation (see Section 5). Of particular interest is the case that there are only finitely many orbits in G. We call algebraic zip data having this property *Frobenius zip data.* Using the Lang-Steinberg Theorem, we deduce a criterion for a zip datum to be Frobenius. It is satisfied in particular if the differential of φ vanishes, for example if φ is a Frobenius morphism (see Section 8).

Theorem 1.3. *For Frobenius zip data, the pieces G^w are the orbits of Z. In particular there is a bijection between the set of orbits in G and W^I.*

We also obtain a description of the stabilizers:

Theorem 1.4 (see Theorem 8.8)**.** *The stabilizer of an element of G under the action of a Frobenius zip datum is the semidirect product of a finite group of Lie type and a connected unipotent algebraic group.*

In Section 9 we determine which elements of W lie in the same orbits under the action of a Frobenius zip datum. In order to do this, we introduce the notion of an abstract zip datum. This is a datum (W, X, X', ψ), where W is an abstract group with subgroups X and X' and $\psi\colon X \to X'$ is a homomorphism. Each algebraic zip datum gives rise to an abstract zip datum $(W, W(L), W(L'), \psi)$, where ψ is induced by φ. For each abstract zip datum we define an equivalence relation on W, which is the equivalence relation defined by intersecting the orbits in G with W in case the abstract zip datum comes from a Frobenius zip datum. We give an inductive and an explicit characterization of this relation.

The main tool we use to deduce our results is the following: For each $w \in W$, the Bruhat cell $P'wP$ is Z-invariant, and we show that the orbits in $P'wP$ correspond to the orbits in L' under the action defined by another algebraic zip datum. This allows us to prove statements inductively, starting from the case $L' = L = G$. In this case, we simply have the group G acting on itself by conjugation twisted with φ.

In order to study the closure of a piece G^w, we show that G^w is the minimal Z-invariant subset of G containing the Bruhat cell BwB. Then we use the fact that the closure order between the Bruhat cells is the Bruhat order to deduce our result about the closure of G^w.

An important application is the classification of F-zips. Let k be a field of characteristic $p > 0$. An F-zip over k is a datum $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ consisting of a finite-dimensional vector space M over k, a descending filtration $C^\bullet$ of M, an ascending filtration $D_\bullet$ of M and Frobenius-linear isomorphisms $\varphi_\bullet$ between the graded pieces of these filtrations (see 10.2). This notion was introduced by Moonen and Wedhorn in [4]. There, they classify the F-zips over an algebraically closed field as follows:

The *type* τ of an F-zip is the function $\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ sending $i \in \mathbb{Z}$ to the dimension of the i-th graded piece of $C^\bullet$.

Theorem 1.5. *Let k be an algebraically closed field of characteristic $p > 0$. Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support $i_1 > \ldots > i_r$. Let $n_j = \tau(i_j)$ and $n = n_1 + \ldots + n_r$. Then there is a bijection*

$$\{\textit{isomorphism classes of } F\textit{-zips of type } \tau \textit{ over } k\} \longleftrightarrow (S_{n_1} \times \cdots \times S_{n_r})\backslash S_n$$

To prove this, they define a variety X_τ with an action of $G = GL_n$ such that the orbits on X_τ correspond to the isomorphism classes of F-zips of type τ and classify the G-orbits on X_τ.

In [4], Moonen and Wedhorn use F-zips to define stratifications on certain moduli spaces. For this application, it is also important to know the closure order between the orbits in X_τ. This order was determined by Wedhorn in [9].

In Section 10, we show that these results also follow from our theory of algebraic zip data. We show that for a certain Frobenius zip datum $(G, P, P', L, L', \varphi)$ there exists a morphism $G \to X_\tau$ which induces a bijection between the orbits of Z in G and the orbits of G in X_τ preserving the closure order. This L satisfies $W(L) = S_{n_1} \times \cdots \times S_{n_r}$ for $n_1, \ldots, n_r$ as in Theorem 1.5, hence Theorem 1.5 follows from our classification of the orbits in G. Furthermore, our result on the closure order of the orbits in G implies the result of Wedhorn. Although we work in a different setting than Moonen and Wedhorn, our proof of the closure order uses methods similar to those of Wedhorn in [9].

Theorem 1.4 yields a similar statement about the automorphism group of an F-zip (see 10.16). We also define certain universal constructions for F-zips, namely direct sums and tensor products, and show how they can be realized as morphisms of the algebraic zip data which classify the F-zips of a certain type.

2 Reductive Groups

Except for the beginning of Section 10, we shall use the language of varieties over a fixed algebraically closed field k. By an algebraic group we shall always mean a linear algebraic group over k.

For any algebraic group G, we denote by $\mathcal{R}_u G$ its unipotent radical. For any $w \in G$, we shall denote the conjugation map $G \to G, g \mapsto wgw^{-1}$ by $\mathrm{int}(w)$ or $g \mapsto {}^w g$. Let G be a connected reductive algebraic group.

Lemma 2.1 ([7], 8.4.6 (ii))**.** *Let P and Q be parabolic subgroups of G. Then $(P \cap Q)\mathcal{R}_u P$ is a parabolic subgroup of G with unipotent radical $(P \cap \mathcal{R}_u Q)\mathcal{R}_u P$.*

Lemma 2.2. *Let H be a Levi factor of a parabolic subgroup of G and let T be a maximal torus of H. If P is a parabolic subgroup of G also containing T, then $H \cap P$ is a parabolic subgroup of H with unipotent radical $H \cap \mathcal{R}_u P$. If $P = L \ltimes \mathcal{R}_u P$ is a Levi decomposition of P with $T \subset L$, then*

$$H \cap P = (H \cap L) \ltimes (H \cap \mathcal{R}_u P)$$

is a Levi decomposition of $H \cap P$.

Proof. This follows from [3], II.1.8. □

3 Algebraic Zip Data

An isogeny between two connected algebraic groups is a surjective homomorphism with finite kernel.

Definition 3.1. An *algebraic zip datum* is a tuple $(G, P, P', L, L', \varphi)$ consisting of a connected reductive algebraic group G, two parabolic subgroups P and P' of G, Levi components L and L' of P and P' and an isogeny $\varphi\colon L \to L'$.

For each algebraic zip datum $Z := (G, P, P', L, L', \varphi)$, we consider the algebraic group

$$(\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L,$$

where $\ell \in L$ acts on $\mathcal{R}_u P$ by conjugation and on $\mathcal{R}_u P'$ by conjugation with $\varphi(\ell)$. This group acts on G from the left by

$$(u', u, \ell) \colon g \mapsto u' \varphi(\ell) g \ell^{-1} u^{-1}.$$

We call this *the action of Z on G*.

Definition 3.2. A morphism between two algebraic zip data $(G, P, P', L, L', \varphi)$ and $(\tilde{G}, \tilde{P}, \tilde{P}', \tilde{L}, \tilde{L}', \tilde{\varphi})$ is a homomorphism $f \colon G \to \tilde{G}$ such that $f(P) \subset \tilde{P}$, $f(P') \subset \tilde{P}'$, $f(L) \subset \tilde{L}$, $f(L') \subset \tilde{L}'$ and the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & \tilde{L} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \tilde{\varphi}} \\ L' & \xrightarrow{f} & \tilde{L}' \end{array}$$

commutes.

The composition of two morphisms of algebraic zip data is the obvious one, and in this way we obtain the category of algebraic zip data.

A different choice of Levi component L of P would differ from the given one only by conjugation by an element of $\mathcal{R}_u P$, and, for any $u \in \mathcal{R}_u P$, the orbits of the action of the algebraic zip datum $(G, P, P', {}^u L, L', \varphi \circ \operatorname{int}(u^{-1}))$ are the same as the orbits of the action of $(G, P, P', L, L', \varphi)$. The same is true for a different choice for L', hence the orbit structure of the action of an algebraic zip datum only depends on the isogeny $P/\mathcal{R}_u P \to P'/\mathcal{R}_u P'$ induced by φ.

There exists a maximal torus of G contained in $P \cap P'$ and Levi components of P and P' containing this torus. Since we are only interested in the orbit structure, we can take L and L' to be these Levi components. So we will assume from now on that $L \cap L'$ contains a maximal torus of G.

We pick a maximal torus T of G contained in $L \cap L'$. Let N be the normalizer of T and $W = N/T$ the Weyl group of G with respect to T. We denote by Φ the root system of G with respect to T. For any $\alpha \in \Phi$, we denote by U_α the associated root subgroup of G. For all $w \in W$, we fix a representative $\dot{w}$ in N.

The algebraic group $P' \times P$ acts on G from the left by

$$(p', p) \colon x \mapsto p' x p^{-1}.$$

Lemma 3.3. *For any $n \in N$, the Bruhat cell $P'nP$ is a locally closed subvariety of G that is invariant under the action of Z.*

Proof. Since $P'nP$ is an orbit under the action of $P' \times P$, it is locally closed (see [7], Lemma 2.3.3). The second part of the claim follows directly from the definition of the action of Z. □

We first show how to relate the orbits in $P'nP$ to the orbits in the reductive group L' under the action of another algebraic zip datum.

Construction 3.4. For each $n \in N$, we construct a new zip datum Z_n as follows: Let

$$\begin{aligned} Q &:= L \cap {}^{n^{-1}}P', \\ Q' &:= L' \cap {}^{n}P, \\ M &:= L \cap {}^{n^{-1}}L', \\ M' &:= L' \cap {}^{n}L. \end{aligned}$$

By Lemma 2.2, Q and Q' are parabolic subgroups of the reductive groups L and L', and M and M' are Levi factors of Q and Q'. The group $\varphi(Q)$ is a parabolic subgroup of L' and $\varphi(M)$ is a Levi factor of $\varphi(Q)$. Hence, if we set $\tilde{\varphi} := \varphi \circ \mathrm{int}(n^{-1})\colon M' \to \varphi(M)$, we obtain an algebraic zip datum

$$Z_n := (L', Q', \varphi(Q), M', \varphi(M), \tilde{\varphi}).$$

If H and H' are two algebraic groups acting on varieties X and X' respectively we say that a morphism $f\colon X \to X'$ is equivariant with respect to a homomorphism $g\colon H \to H'$ if for all $x \in X$ and $h \in H$

$$f(h \cdot x) = g(h) \cdot f(x).$$

In this case, f induces a map from the orbits in X to the orbits in X'.

By direct calculation, the morphism

$$\begin{aligned} i_n\colon L' &\to P'nP \\ \ell' &\mapsto \ell' n \end{aligned}$$

is equivariant with respect to the homomorphism

$$\begin{aligned} (\mathcal{R}_u Q' \times \mathcal{R}_u \varphi(Q)) \rtimes M' &\to (\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L \\ (v, v', m') &\mapsto ({}^{n^{-1}}v, v', {}^{n^{-1}}m'). \end{aligned}$$

We will show that i_n induces a bijection between the orbits of the action Z_n on L' and the orbits of the action of Z on $P'nP$.

The stabilizer of n is

$$\begin{aligned} \mathrm{Stab}_{P' \times P}(n) &= \left\{ (p', p) \in P' \times P \mid p' n p^{-1} = n \right\} \\ &= \left\{ (p', n^{-1} p' n) \mid p' \in P' \cap {}^{n}P \right\}. \end{aligned} \tag{1}$$

This implies in particular

Lemma 3.5. *The dimension of* $P'nP$ *is* $\dim P + \dim P' - \dim(P' \cap {}^{n}P)$.

Let H_n be the image of $\mathrm{Stab}_{P' \times P}(n)$ under the projection $P' \times P \to L' \times L$.

Lemma 3.6. $H_n = \{(v'm', vn^{-1}m'n) \mid m' \in M', v' \in \mathcal{R}_u Q', v \in \mathcal{R}_u Q\}$.

Proof. By definition,

$$H_n = \{(\ell', \ell) \in L' \times L \mid \exists u' \in \mathcal{R}_u P', u \in \mathcal{R}_u P \colon n^{-1}\ell' u' n = \ell u\}.$$

Let $(\ell', \ell) \in H_n$. Choose $u' \in \mathcal{R}_u P'$ and $u \in \mathcal{R}_u P$ such that $n^{-1}\ell' u' n = \ell u$.

Let $\tilde{P}' = (P' \cap {}^n P)\mathcal{R}_u P'$ and $\tilde{P} = ({}^{n^{-1}}P' \cap P)\mathcal{R}_u P$. By Lemma 2.1 these are parabolic subgroups of G. From the Levi decomposition $P = L \ltimes \mathcal{R}_u P$ it follows that

$$L \cap \tilde{P} = L \cap {}^{n^{-1}}P' = Q$$

and analogously we find

$$L' \cap \tilde{P}' = Q'.$$

Since $\ell' u' \in P' \cap {}^n P$ this implies $\ell' \in Q'$, so we can write $\ell' = v'm'$ for uniquely determined $m' \in M'$ and $v' \in \mathcal{R}_u Q'$, and we have

$$\ell = (n^{-1}v'n)(n^{-1}m'n)(n^{-1}u'n)u^{-1}.$$

From $u' \in \mathcal{R}_u P' \cap (P' \cap {}^n P)$ we get $n^{-1}u'n \in {}^{n^{-1}}\mathcal{R}_u P' \cap P$. Lemma 2.2 implies $v' \in \mathcal{R}_u Q' = L' \cap \mathcal{R}_u({}^n P)$, so we get $n^{-1}v'n \in \mathcal{R}_u P$. Also,

$$n^{-1}m'n \in {}^{n^{-1}}M' = M \subset L.$$

This implies

$$\ell \in n^{-1}m'n({}^{n^{-1}}\mathcal{R}_u P' \cap P)\mathcal{R}_u P.$$

By Lemma 2.1, we have $({}^{n^{-1}}\mathcal{R}_u P' \cap P)\mathcal{R}_u P = \mathcal{R}_u \tilde{P}$. Hence

$$\ell \in n^{-1}m'n(\mathcal{R}_u \tilde{P} \cap L).$$

Finally, again by Lemma 2.2, we have $\mathcal{R}_u \tilde{P} \cap L = \mathcal{R}_u(\tilde{P} \cap L) = \mathcal{R}_u Q$, so we can write $\ell = vn^{-1}m'n$ for some $v \in \mathcal{R}_u Q$.

On the other hand, let $(\ell', \ell) = (v'm', vn^{-1}m'n)$ for some $m' \in M'$, $v' \in \mathcal{R}_u Q'$ and $v \in \mathcal{R}_u Q = L \cap {}^{n^{-1}}\mathcal{R}_u P'$. From

$$\begin{aligned} n^{-1}\ell'^{-1}n\ell = (n^{-1}m'^{-1}n)(n^{-1}v'^{-1}n)v(n^{-1}m'n) &\in ({}^{n^{-1}}L' \cap \mathcal{R}_u P)(L \cap {}^{n^{-1}}\mathcal{R}_u P') \\ &\subset {}^{n^{-1}}\mathcal{R}_u P' \cdot \mathcal{R}_u P \end{aligned}$$

it follows that there exist $u \in \mathcal{R}_u P$ and $u' \in \mathcal{R}_u P'$ such that $n^{-1}\ell' u' n = \ell u$. This shows $(\ell', \ell) \in H_n$. □

Lemma 3.7. *The morphism i_n induces a bijection between the orbits of the action of Z_n on L' and the orbits of the action of Z on $P'nP$.*

Proof. Any $g \in P'nP$ can be written as $u'\ell'n\ell u$ for certain $u \in \mathcal{R}_uP, u' \in \mathcal{R}_uP', \ell \in L$ and $\ell' \in L'$. Since such a g lies in the same orbit as $\varphi(\ell)\ell'n$, the map induced by i_n on the orbits is surjective.

Let $\ell', \ell'' \in L'$ such that $i_n(\ell')$ and $i_n(\ell'')$ lie in the same orbit. Then there exist $u \in \mathcal{R}_uP, u' \in \mathcal{R}_uP'$ and $\ell \in L$ such that

$$\ell'n = u'\varphi(\ell)\ell''n\ell^{-1}u^{-1}.$$

This implies

$$(\ell'^{-1}\varphi(\ell)\ell'', \ell) \in H_n.$$

By Lemma 3.6 the exist $m' \in M', v \in \mathcal{R}_uQ$ and $v' \in \mathcal{R}_uQ'$ such that

$$\begin{aligned} \ell'^{-1}\varphi(\ell)\ell'' &= v'm' \\ \ell &= vn^{-1}m'n. \end{aligned}$$

Together, this yields

$$\ell' = \varphi(v)\varphi({}^{n^{-1}}m')\ell''m'^{-1}v^{-1},$$

so ℓ' and ℓ'' lie in the same orbit under the action of Z_n. □

Since Lemma 3.7 relates the action of an algebraic zip datum on an algebraic group with the action of another zip datum on a group of smaller dimension, it will allow us to prove facts about such actions inductively. The base case of such an induction will always be the case where the above construction does not yield a smaller group, that is the case where $G = L'$.

The following two lemmas will be needed later.

Lemma 3.8. *The morphism*

$$\begin{aligned} \pi\colon P' \times P &\to P'nP \\ (p', p) &\mapsto p'np \end{aligned}$$

is separable. In particular the differential of π at any point is surjective.

Proof. The claim is equivalent to the fact that the multiplication map $P' \times {}^nP \to P' \cdot {}^nP$ is separable. To prove this, it is sufficient to show that

$$\mathrm{Lie}(P' \cap {}^nP) = \mathrm{Lie}(P') \cap \mathrm{Lie}({}^nP),$$

where Lie denotes the Lie algebra functor. Since P' and ${}^nP$ both contain T, both sides of the equation are the direct sum of $\mathrm{Lie}(T)$ and the $\mathrm{Lie}(U_\alpha)$ for all $\alpha \in \Phi$ such that $U_\alpha \subset P' \cap {}^nP$.

Since π is a $P' \times P$-equivariant morphism of homogenous $P' \times P$-spaces, the last point follows from [7], Theorem 4.3.7(ii). □

For any variety X and any point x on X, we denote the tangent space of X at x by T_xX. If Y and Z are non-singular subvarieties of a non-singular variety X, we say that Y and Z *intersect transversally in* X if for any $x \in Y \cap Z$, the space T_xX is spanned by the subspaces T_xY and T_xZ.

The varieties $L'n$, $P'nP$ and any orbit of Z in G are non-singular, because they are orbits of certain group actions (see [7], Theorem 4.3.7(i)).

Lemma 3.9. *Any orbit o of Z in $P'nP$ intersects $L'n$ transversally in $P'nP$.*

Proof. Let $x \in o \cap L'n$. We need to show that $T_x(P'nP)$ is the sum of $T_x(o)$ and $T_x(L'n)$. Write $x = \ell'n$ for some $\ell' \in L'$. In $P' \times P$ we consider $X = L' \times 1$ and

$$Y = \{(u'\varphi(\ell)\ell', u\ell) \mid \ell \in L, u \in \mathcal{R}_uP, u' \in \mathcal{R}_uP'\},$$

which map onto $L'n$ and o respectively under $\pi\colon P' \times P \to P'nP$. By Lemma 3.8, the differential of π at $(\ell', 1)$ is surjective, so it is sufficient to show that $T_{(\ell',1)}X$ and $T_{(\ell',1)}Y$ span $T_{(\ell',1)}(P' \times P) \cong T_{\ell'}P' \times T_1P$. Since $X = L'\ell' \times 1$ and $(\mathcal{R}_uP')\ell' \times 1 \subset Y$, the sum of the two vector spaces contains $T_{\ell'}P' \times 1$. Since this sum also contains $1 \times T_1(\mathcal{R}_uP)$, it suffices to show that the differential at $(\ell', 1)$ of the projection $Y \to L, (u'\varphi(\ell)\ell', u\ell) \mapsto \ell$ is surjective. But this follows from the existence of the section $L \to Y, \ell \mapsto (\varphi(\ell)\ell', \ell)$. □

4 Coset Representatives in Coxeter Groups

We collect some facts about Coxeter groups which we shall need in the sequel. These can be found in [2], sections 2.3 and 2.7.

Let (W, S) be a finite Coxeter group, i.e. W is a group with a set of generators $S = \{s_1, \dots, s_n\}$ such that W has a presentation

$$W =< s_1 \dots, s_n \mid (s_is_j)^{m_{ij}} = 1 >$$

for certain $m_{ij} \in \mathbb{N}$ such that $m_{ii} = 1$ for all $1 \le i \le n$ and $m_{ij} = m_{ji}$ for all $1 \le i \ne j \le n$. Let ℓ denote the length function on W, i.e. for $w \in W$ the length $\ell(w)$ of w is the smallest number r such that w is the product of r elements of S. An expression of w as the product of $\ell(w)$ elements of S is called reduced.

Let I be a subset of S. We denote by W_I the subgroup of W generated by I and by W^I (respectively ${}^IW$) the set of elements w of W which have minimal length in their coset wW_I (respectively W_Iw). Then every $w \in W$ can be written uniquely as $w = w^I \cdot w_I = \tilde{w}_I \cdot {}^Iw$ with $w_I, \tilde{w}_I \in W_I$, $w^I \in W^I$ and ${}^Iw \in {}^IW$, and $\ell(w) = \ell(w_I) + \ell(w^I) = \ell(\tilde{w}_I) + \ell({}^Iw)$ (see [2], Proposition 2.3.3). In particular, W^I and ${}^IW$ are systems of representatives for W/W_I and $W_I\backslash W$ respectively. The fact that $\ell(w) = \ell(w^{-1})$ for all $w \in W$ implies $W^I = ({}^IW)^{-1}$.

Furthermore, if J is a second subset of S, let ${}^JW^I$ be set of $w \in W$ which have minimal length in the double coset W_JwW_I. Then ${}^JW^I = {}^JW \cap W^I$ and ${}^JW^I$ is a system of representatives for $W_J\backslash W/W_I$ (see [2], Proposition 2.7.3).

Theorem 4.1 (Kilmoyer, [2], Theorem 2.7.4)**.** *If* $w \in {}^J W^I$*, then*

$$W_J \cap {}^w W_I = W_K,$$

where $K = J \cap {}^w I$*.*

Proposition 4.2 (Howlett, [2], Proposition 2.7.5)**.** *Let* ${}^J w^I \in {}^J W^I$ *and* $K = J \cap {}^w I \subset S$*. Then every element* w *of the double coset* $W_J\, {}^J w^I W_I$ *is uniquely expressible in the form* $w = w_J\, {}^J w^I w_I$ *where* $w_J \in W_J \cap W^K$ *and* $w_I \in W_I$*. Moreover, this decomposition satisfies*

$$\ell(w_J\, {}^J w^I w_I) = \ell(w_J) + \ell({}^J w^I) + \ell(w_I).$$

Lemma 4.3. *The set* W^I *is the set of elements* w *for which in the decomposition of Proposition 4.2 the factor* w_I *is* 1*.*

Proof. Let ${}^J w^I \in {}^J W^I$ and $K = J \cap {}^{{}^J w^I} I$. If $w_J \in W_J \cap W^K$ then for any $w_I \in W_I$ the decomposition $w_J\, {}^J w^I w_I$ is the unique decomposition given by Proposition 4.2. So we get

$$\ell(w_J\, {}^J w^I w_I) = \ell(w_J) + \ell({}^J w^I) + \ell(w_I) \geq \ell(w_J) + \ell({}^J w^I) = \ell(w_J\, {}^J w^I),$$

which shows that $w_J\, {}^J w^I \in W^I$.

Now let $w \in W^I$ and $w = w_J\, {}^J w^I w_I$ the decomposition from Proposition 4.2. Since by the preceding paragraph $w_J\, {}^J w^I \in W^I$ we must have $w_I = 1$. □

On W there exists a natural partial order, the Bruhat order, which we shall denote by $\leq$. It is characterized by the following property: For $x, w \in W$ we have $x \leq w$ if and only if for some (or, equivalently, any) reduced expression $w = s_{i_1} \cdots s_{i_n}$ as a product of simple reflections , one gets a reduced expression for x by removing certain s_{i_j} from this product. More information about the Bruhat order can be found in [1], Chapter 2.

Using this order, the set W^I can be described as

$$W^I = \{w \in W \mid w < ws \text{ for all } s \in I\}$$

(see [1], Definition 2.4.2 and Corollary 2.4.5).

Assume that in addition W is the Weyl group of a root system Φ, with S corresponding to a basis of Φ. Denote the set of positive roots with respect to the given basis by Φ^+ and the set of negative roots by Φ^-. For $I \subset S$, let Φ_I be the root system spanned by the basis elements corresponding to I, and let $\Phi_I^{\pm} := \Phi_I \cap \Phi^{\pm}$. Then

$$W^I = \{w \in W \mid w\Phi_I^+ \subset \Phi^+\} \tag{2}$$

(see [2], Proposition 2.3.3).

Also, for $w \in W$, the length of w is the cardinality of the set

$$\{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\} \tag{3}$$

(see [2], Proposition 2.2.7).

5 The Orbits of an Algebraic Zip Datum

Definition 5.1. A zip datum $Z = (G, P, P', L, L', \varphi)$ is *nice* with respect to a maximal torus $T \subset L \cap L'$ and a Borel subgroup B of G, if $\varphi(T) = T$,

$$T \subset B \subset P \cap P'$$

and

$$\varphi(L \cap B) = L' \cap B.$$

If there exist such T and B, we shall also just say that Z is nice.

Proposition 5.2. *Let $Z = (G, P, P', L, L', \varphi)$ be an algebraic zip datum. Then there exists a nice algebraic zip datum $\tilde{Z} = (G, \tilde{P}, \tilde{P}', \tilde{L}, \tilde{L}', \tilde{\varphi})$ and an isomorphism of varietes $\psi\colon G \to G$ which maps each orbit of Z in G bijectively onto an orbit of $\tilde{Z}$ in G.*

Such $\tilde{Z}$ and ψ can be obtained as follows: There exist a Borel subgroup B of G, $w \in W$ and $z \in L$ such that

$$B \subset P,$$
$${}^{\dot{w}}\varphi({}^{z}(L \cap B)) = {}^{\dot{w}}L' \cap B$$

and

$${}^{\dot{w}}\varphi({}^{z}T) = T.$$

For any such B,w and z the algebraic zip datum

$$\tilde{Z} = (G, P, {}^{\dot{w}}P', L, {}^{\dot{w}}L', \mathrm{int}(\dot{w}) \circ \varphi \circ \mathrm{int}(z))$$

is nice with respect to T and B and the morphism $\psi\colon G \to G, g \mapsto \dot{w}gz$ is equivariant with respect to the isomorphism

$$(\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L \to (\mathcal{R}_u P \times \mathcal{R}_u({}^{\dot{w}}P')) \rtimes L$$
$$(u, u', \ell) \mapsto ({}^{z^1}u, {}^{\dot{w}}u', {}^{z^{-1}}\ell).$$

Proof. Let T be a maximal torus of G contained in $L \cap L'$. There exists $w \in W$ such that P and ${}^{\dot{w}}P'$ both contain a Borel subgroup B of G which contains T. Since $\mathrm{int}(\dot{w}) \circ \varphi\colon L \to {}^{\dot{w}}L'$ is an isogeny and ${}^{\dot{w}}L' \cap B$ is a Borel subgroup of ${}^{\dot{w}}L'$, its preimage $B' := (\mathrm{int}(w) \circ \varphi)^{-1}({}^{\dot{w}}L' \cap B)$ is a Borel subgroup of L. Similarly, the subgroup $T' := (\mathrm{int}(w) \circ \varphi)^{-1}(T)$ is a maximal torus of L.

Since the action of L by inner automorphisms on pairs consisting of a Borel subgroup and a maximal torus contained in the Borel subgroup is transitive, there exists $z \in L$ such that ${}^{z}(L \cap B) = B'$ and ${}^{z}T = T'$, that is such that

$${}^{\dot{w}}\varphi({}^{z}(L \cap B)) = {}^{\dot{w}}L' \cap B$$

and

$$^{\dot{w}}\varphi(^{z}T) = T.$$

Hence the algebraic zip datum $\tilde{Z} := (G, P, {}^{\dot{w}}P', L, {}^{\dot{w}}L', \mathrm{int}(\dot{w}) \circ \varphi \circ \mathrm{int}(z))$ is nice with respect to B and T.

Let $\psi\colon G \to G$ be the isormorphism of varieties sending $g \in G$ to $\dot{w}gz$. For $(u, u', \ell) \in (\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L$ and $g \in G$

$$\psi(u'\varphi(\ell)g\ell^{-1}u^{-1}) = {}^{\dot{w}}u'(\mathrm{int}(\dot{w}) \circ \varphi \circ \mathrm{int}(z))({}^{z^{-1}}\ell)\psi(g)({}^{z^{-1}}\ell)({}^{z^{-1}}u^{-1}).$$

This shows that ψ is equivariant with respect to the isomorphism

$$\begin{aligned}(\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L &\to (\mathcal{R}_u P \times \mathcal{R}_u({}^{\dot{w}}P')) \rtimes L \\ (u, u'\ell) &\mapsto ({}^{z^1}u, {}^{\dot{w}}u', {}^{z^{-1}}\ell),\end{aligned}$$

which implies that ψ maps each orbit of Z in G bijectively to an orbit of $\tilde{Z}$ in G. □

This allows us to reduce to the case of a nice algebraic zip datum for many questions, so we will only consider such data in the following.

So let $Z = (G, P, P', L, L', \varphi)$ be nice with respect to T and B. The Borel subgroup B defines a set of simple reflections S of W. Let I and J be the subsets of S such that P and P' are the standard parabolics of type I and J respectively. Then $W(L) = W_I$ and $W(L') = W_J$. Since $\varphi(T) = T$, the isogeny φ induces an isomorphism $W_I \to W_J$ which we also denote by φ. Since $\varphi(L \cap B) = \varphi(L' \cap B)$, this isomorphism maps I bijectively to J, and so it is an isomorphism of Coxeter groups $(W_I, I) \to (W_J, J)$.

We denote by Φ be the root system of G with respect to T, and for any closed subgroup H of G which is normalized by T, we denote by Φ_H the root system of H. Also, we denote by Φ^+ the set of positive roots with respect to B, by Φ^- the set of negative roots, and we let $\Phi_H^{\pm} := \Phi_H \cap \Phi^{\pm}$.

Definition 5.3. For $w \in W$, let $K^w := J \cap {}^{w}I$.

Lemma 5.4. *Let ${}^{J}w^{I} \in {}^{J}W^{I}$. Let $Z_{{}^{J}\dot{w}^{I}} = (L', Q', \varphi(Q), M', M, \tilde{\varphi})$ be the algebraic zip datum from 3.4. Then:*

(a) The algebraic zip datum $Z_{{}^{J}\dot{w}^{I}}$ is nice with respect to T and $L' \cap B$.

(b) The type of Q' is $K^{{}^{J}w^{I}}$.

(c) The elements $w \in W_J\, {}^{J}w^{I} W_I \cap W^I$ are exactly the elements of the form $w = w_J\, {}^{J}w^{I}$ for some $w_J \in W_J$ having minimal length in $w_J W(M')$. For such w, w_J and ${}^{J}w^{I}$ we have $\ell(w) = \ell(w_J) + \ell({}^{J}w^{I})$.

Proof. First we show:

Claim. *(i)* $L \cap {}^{({}^{J}\dot{w}^{I})^{-1}}B = L \cap B$.

(ii) $L' \cap {}^{{}^J\dot{w}^I}B = L' \cap B$.

Proof. By assumption on ${}^Jw^I$ we have ${}^Jw^I\Phi_L^+ \subset \Phi^+$. This implies ${}^Jw^I\Phi_L^+ = {}^Jw^I\Phi_L \cap \Phi^+$ or equivalently $\Phi_L^+ = \Phi_L \cap ({}^Jw^I)^{-1}\Phi^+$. Since Φ_L^+ is the root system of $L \cap B$ and $\Phi_L \cap ({}^Jw^I)^{-1}\Phi^+$ is the root system of $L \cap {}^{({}^J\dot{w}^I)^{-1}}B$, we get (i). The second part can be shown similarly. □

Now (i) shows $L' \cap B = \varphi(L \cap B) = \varphi(L \cap {}^{({}^J\dot{w}^I)^{-1}}B) \subset \varphi(Q)$ and from (ii) we get $L' \cap B = L' \cap {}^{{}^J\dot{w}^I}B \subset Q'$.

Using (i) again we get

$$\tilde{\varphi}(M' \cap B) = \varphi({}^{({}^J\dot{w}^I)^{-1}}B \cap M) \subset \varphi(M) \cap B.$$

Because both $\tilde{\varphi}(M' \cap B)$ and $\varphi(M) \cap B$ are Borel subgroups of $\varphi(M)$, they must be equal. Since $\tilde{\varphi}(T) = T$ this shows (a).

Theorem 4.1 implies $W^{K^{{}^Jw^I}} = W(M')$, which shows (b). Then (c) is just a restatement of Lemma 4.3. □

Definition 5.5. Let $w \in W$. For any collection of subsets of I which are mapped into themselves under the homomorphism $\mathrm{int}(w^{-1}) \circ \varphi \colon W_I \to W$, the union of these sets is also mapped into itself under $\mathrm{int}(w^{-1}) \circ \varphi$. Hence there exists a unique maximal subset of I having this property, which we denote by I_w.

Lemma 5.6. *For $w \in W$, the map $\mathrm{int}(w^{-1}) \circ \varphi \colon I_w \to I_w$ is a bijection.*

Proof. Since $\mathrm{int}(w^{-1}) \colon W \to W$ and $\varphi \colon W_I \to W_J$ are bijective, the composite $\mathrm{int}(w^{-1}) \circ \varphi$ is injective. Since I_w is finite, this implies the claim. □

We give an inductive description of I_w for $w \in W^I$:

Lemma 5.7. *Let $w \in W^I$ and $w = w_J\,{}^Jw^I$ the decomposition from Lemma 5.4 and $Z_{{}^J\dot{w}^I} = (L', Q', \varphi(Q), M', M, \tilde{\varphi})$ be the algebraic zip datum from 3.4. Let $K_{w_j}^{{}^Jw^I}$ be the largest subset of $K^{{}^Jw^I}$ invariant under $\mathrm{int}(w_J^{-1}) \circ \tilde{\varphi}$. Then*

$$K_{w_j}^{{}^Jw^I} = {}^{{}^Jw^I}I_w.$$

Proof. The definition of $K_{w_J}^{{}^Jw^I}$ implies

$${}^{({}^Jw^I)^{-1}}[(\mathrm{int}(w_J)^{-1} \circ \varphi \circ \mathrm{int}({}^Jw^I)^{-1})(K_{w_j}^{{}^Jw^I})] \subset {}^{({}^Jw^I)^{-1}}[K_{w_j}^{{}^Jw^I}].$$

This shows that the subset ${}^{({}^Jw^I)^{-1}}K_{w_j}^{{}^Jw^I}$ of I is invariant under $\mathrm{int}(w^{-1}) \circ \varphi$, so that

$${}^{({}^Jw^I)^{-1}}K_{w_j}^{{}^Jw^I} \subset I_w.$$

Lemma 5.6 implies

$$ {}^{{}^J w^I} I_w = {}^{w_J^{-1}} \varphi(I_w) = {}^{w_J^{-1}} \tilde{\varphi}({}^{{}^J w^I} I_w), $$

hence ${}^{{}^J w^I} I_w$ is contained in $K^{{}^J w^I} = J \cap {}^{{}^J w^I} I$ and invariant under $\mathrm{int}(w_J^{-1}) \circ \tilde{\varphi}$. This shows ${}^{{}^J w^I} I_w \subset K_{w_j}^{{}^J w^I}$. □

For any set of simple reflections R, there exists a unique parabolic subgroup of type R of G containing B, which is called the *standard parabolic subgroup of type* R. This parарabolic subgroup has a unique Levi factor containing T, which is called the *standard Levi subgroup of type* R.

Definition 5.8. For $w \in W$, let L_w be the standard Levi subgroup of G of type I_w.

Lemma 5.9. *For* $w \in W$, *the morphism* $\mathrm{int}(\dot{w}^{-1}) \circ \varphi \colon L \to G$ *maps* L_w *into itself.*

Proof. Since the group L_w is generated by T and the U_α for $\alpha \in \Phi_{I_w}$, it is sufficient to show that these subgroups are mapped into L_w by $\mathrm{int}(\dot{w}^{-1}) \circ \varphi$. For T this is clear and for the U_α it follows from the definition of I_w. □

We give an inductive description of L_w for $w \in W^I$:

Lemma 5.10. *Let* $w \in W^I$ *and* $w = w_J\, {}^J w^I$ *the decomposition from Lemma 5.4 and* $Z_{{}^J \dot{w}^I} = (L', Q', \varphi(Q), M', M, \tilde{\varphi})$ *be the algebraic zip datum from 3.4. Let* M'_{w_J} *be the standard Levi subgroup of* Q' *of type* $K_{w_J}^{{}^J w^I}$. *Then*

$$ M'_{w_J} = {}^{{}^J \dot{w}^I} L_w. $$

Proof. This follows directly from Lemma 5.7. □

Remark 5.11. Using the inductive description of L_w in the preceding Lemma, one can show that L_w is the unique maximal subgroup of L which is mapped into itself by $\mathrm{int}(\dot{w}^{-1}) \circ \varphi \colon L \to G$. But we shall not need this.

For any $X \subset G$, we denote the union of the orbits of all elements of X by $\mathrm{o}(X)$.

Lemma 5.12. *If* X *is a constructible subset of* G, *then* $\mathrm{o}(X)$ *is constructible.*

Proof. This follows from the fact that $\mathrm{o}(X)$ is the image of the constructible subset $((\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L) \times X$ of $((\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L) \times G$ under the multplication morphism to G. □

Definition 5.13. For $w \in W^I$, let $G^{\dot{w}} := \mathrm{o}(\dot{w} L_w)$.

Lemma 5.14. *For* $w \in W^I$, *the* Z*-stable piece* $G^{\dot{w}}$ *does not depend on the choice of representative* $\dot{w}$.

Proof. By definition the group L_w contains T. Hence $\dot{w}L_w$ does not depend on the choice of $\dot{w}$, which implies the claim. $\square$

This justifies the following definition:

Definition 5.15. For $w \in W^I$, let $G^w = G^{\dot{w}}$.

Remark 5.16. A priori the G^w for $w \in W^I$ are only constructible subsets of G. However we shall see later (Corollary 6.15) that they are in fact locally closed in G.

Definition 5.17. For $w \in W^I$ let $j_{\dot{w}}$ be the continuous map

$$\begin{aligned} L_w &\to G^w \\ \ell &\mapsto \dot{w}\ell \end{aligned}$$

Lemma 5.18. *Let $w \in W^I$ and $w = w_J\,{}^J w^I$ the decomposition from Lemma 5.4. Let $Z_{{}^J\dot{w}^I} = (L', Q', \varphi(Q), M', M, \tilde{\varphi})$ be the algebraic zip datum from 3.4. Assume $\dot{w} = \dot{w}_J\,{}^J\dot{w}^I$.*

(i) We have $M'_{\dot{w}_J} = {}^{{}^J\dot{w}^I}L_{\dot{w}}$ and the diagram

$$\begin{array}{ccc} L_w & \xrightarrow{\;j_{\dot{w}}\;} & P'\dot{w}P \\ {\scriptstyle \mathrm{int}({}^J\dot{w}^I)^{-1}}\big\uparrow & & \big\uparrow{\scriptstyle i_{{}^J\dot{w}^I}} \\ M'_{\dot{w}_J} & \xrightarrow{\;j_{\dot{w}_J}\;} & L' \end{array}$$

commutes.

(ii) The morphism $i_{{}^J\dot{w}^I}$ maps $(L')^{w_J}$ into G^w and induces a bijection between the $Z_{{}^J\dot{w}^I}$-orbits in $(L')^{w_J}$ and the Z-orbits in G^w. In particular

$$(L')^{w_J}\,{}^J\dot{w}^I = G^w \cap L'\,{}^J\dot{w}^I.$$

Proof. The first part of (i) follows from Lemma 5.10, and the second statement in (i) can be directly verified. Then (ii) follows from (i), the definition of $(L')^{w_J}$ and G^w and Lemma 3.7. $\square$

Theorem 5.19. *(i) The G^w for $w \in W^I$ form a disjoint decomposition of G.*

(ii) For all $w \in W^I$, the continuous map $j_{\dot{w}}\colon L_w \to G^w$ induces a bijection between the orbits on L_w of the action of L_w on itself by

$$(\ell, g) \mapsto ({}^{\dot{w}^{-1}}\varphi(\ell))g\ell^{-1}$$

and the orbits of Z in G^w.

Proof. We prove everything by induction on $\dim G$. If $L' = G$, we have $W^I = \{1\}$ and $L = L_1 = L'$. Since in this case the action of Z is just the action of L on itself given in (ii), both claims are true.

Assume $\dim L' < \dim G$. For ${}^J w^I \in {}^J W^I$ it follows from Lemmas 3.7, 5.4 and 5.18 and the induction hypothesis applied to the nice algebraic zip datum $Z_{{}^J\dot{w}^I}$ that $P' \, {}^J\dot{w}^I P$ is the disjoint union of the G^w for the $w \in W^I$ of the form $w = w_J \, {}^J w^I$ with $w_J \in W_J$. Since $G = \coprod_{{}^J\dot{w}^I \in {}^J W^I} P' \, {}^J\dot{w}^I P$ this proves (i).

The induction step for (ii) follows from Lemma 5.18. □

6 Closure

In this section, we will show that for $w \in W^I$, the closure of G^w is the union of other Z-stable pieces $G^{w'}$ for certain $w' \in W^I$.

We shall need the following lemma (see [8, Lemma 7.3]):

Lemma 6.1. *Let G be a connected linear algebraic group and φ a surjective endomorphism of G which leaves a Borel subgroup B of G invariant. Then the morphism $G \times B \to G, (x, b) \mapsto \varphi(x)bx^{-1}$ is surjective.*

Proposition 6.2. *For $w \in W^I$, we have $G^w = \mathrm{o}(B\dot{w}B)$.*

Proof. We proceed by induction on $\dim G$. In the base case we have $G = L'$, and the claim follows from Lemma 6.1.

Assume $\dim L' < \dim G$ and let $b, b' \in B$. We can decompose b and b' as $b = \ell u$ and $b' = u'\ell'$ with $\ell \in B \cap L$, $\ell' \in B \cap L'$, $u \in \mathcal{R}_u P$ and $u' \in \mathcal{R}_u P'$. Since $\mathrm{o}(b'\dot{w}b) = \mathrm{o}(\varphi(\ell)\ell'\dot{w})$ with $\varphi(\ell)\ell' \in L' \cap B$ we get

$$\mathrm{o}(B\dot{w}B) = \mathrm{o}((B \cap L')\dot{w}).$$

Hence it will be sufficient to prove that $G^w = \mathrm{o}((B \cap L')\dot{w})$.

Let $w = w_J \, {}^J w^I \in {}^J W^I$ be the decomposition given by Lemma 5.4. The morphism $i_{{}^J\dot{w}^I} : L' \to P'\dot{w}P$ maps $(L' \cap B)\dot{w}_J$ onto $(L' \cap B)\dot{w}$. Since by Lemma 5.4 the zip datum $Z_{{}^J\dot{w}^I}$ is nice with respect to T and $B \cap L'$, the claim now follows from the induction hypothesis applied to $Z_{{}^J w^I}$ and Lemma 3.7. □

Corollary 6.3. *If P is a Borel subgroup of G, then $G^w = BwB$ for all $w \in W$.*

The group $B \times B$ acts on G by $(b', b) \cdot g = b'gb^{-1}$. The set $\{\dot{w} \mid w \in W\}$ is a set of representatives for this action, and for $w, w' \in W$

$$w \leq w' \text{ if and only if } B\dot{w}B \subset \overline{B\dot{w}'B}.$$

Proposition 6.2 will allow us to use the closure order of this action to determine the closure order of the action of Z. For this, we shall need the following lemma (see [9, 5.3]):

Lemma 6.4. *Let H be any algebraic group acting on a variety Z and let $P \subset H$ be an algebraic subgroup such that H/P is proper. Then for any P-invariant subvariety $Y \subset Z$ we have*

$$H \cdot \overline{Y} = \overline{H \cdot Y}.$$

Lemma 6.5. *For $w \in W^I$, we have*

$$\overline{G^w} = \bigcup_{\substack{x \in W \\ x \leq w}} \mathrm{o}(B\dot{x}).$$

Proof. We let L act on G by $\ell * g = \varphi(\ell) g \ell^{-1}$. Since $B\dot{w}B = \mathcal{R}_u P'(B\dot{w}B)\mathcal{R}_u P$, it follows from Proposition 6.2 that

$$G^w = L * B\dot{w}B.$$

This together with Lemma 6.4, applied to $H = L$, $P = L \cap B$, $Z = G$ and $Y = B\dot{w}B$ yields

$$\overline{G^w} = L * \overline{B\dot{w}B} = \bigcup_{x \leq w} L * B\dot{x}B.$$

Because again for each such x we have $\mathcal{R}_u P'(B\dot{x}B)\mathcal{R}_u P = B\dot{x}B$, this implies the lemma. □

Lemma 6.6. *For all $x, z \in W$ and $b \in B$ there exists $v \in W$ such that $v \leq z$ and*

$$\dot{z} b \dot{x} \in B\dot{v}\dot{x}B.$$

Proof. We prove the statement by induction on $\ell(z)$. If $z = 1$, we may take $v = 1$. For the induction step write $z = sz'$ for some simple reflection s such that $\ell(z') = \ell(z) - 1$. By the induction hypothesis there exists $v' \leq z'$ such that $\dot{z}' b \dot{x} \in B\dot{v}'\dot{x}B$. Hence $\dot{z} b \dot{w} \in \dot{s}B\dot{v}'\dot{x}B \subset B\dot{s}\dot{v}'\dot{x}B \cup B\dot{v}'\dot{x}B$, so either $v = sv'$ or $v = v'$ will have the required property. □

Lemma 6.7. *Let $w \in W^I$, $b, b' \in B$ and $x \in W$ such that $\mathrm{o}(b\dot{w}) = \mathrm{o}(b'\dot{x})$. Then there exists $y \in W_I$ such that $\varphi(y) w y^{-1} \leq x$.*

Proof. We proceed by induction on $\dim G$. In the base case we have $G = L$, so w must be 1 and we may take $y = 1$.

So assume that $\dim L' < \dim G$. We also may and do assume that $b, b' \in L \cap B$. Let $x = x_J\, {}^J x^I x_I$ be the decomposition of x given by Proposition 4.2, so we have $x_J \in W_J$, $x_I \in W_I$ and ${}^J x^I \in {}^J W^I$. It follows from Lemma 4.3 that $x^I := x_J\, {}^J x^I \in W^I$.

By Lemma 6.6, there exists $v \in W$ such that $v \leq \varphi(x_I)$ and $\varphi(\dot{x}_I) b' \dot{w}^I \in B\dot{v}\dot{x}^I B$. Hence there exists $b'' \in L' \cap B$ such that $\varphi(\dot{x}_I) b' \dot{w}^I$ lies in the same orbit as $b'' \dot{v}\dot{x}^I$. Altogether we get

$$\mathrm{o}(b\dot{w}) = \mathrm{o}(b'\dot{x}^I \dot{x}_I) = \mathrm{o}(\varphi(\dot{x}_I) b' \dot{x}^I) = \mathrm{o}(b'' \dot{v}\dot{x}^I) = \mathrm{o}(b'' \dot{v}\dot{x}_J\, {}^J\dot{x}^I).$$

Let $Z_{{}^J\dot{x}^I} = (L', Q', \varphi(Q), M', \varphi(M), \tilde{\varphi})$. By Lemma 5.4 we can decompose $w = w_J\, {}^J w^I$ with $w_J \in W_J$ minimal in $w_J W(M')$ and ${}^J w^I \in {}^J W^I$. From $\mathrm{o}(b\dot{w}) = \mathrm{o}(b'\dot{x})$ we get

$$P_J\, {}^J\dot{w}^I P_I = P_J b \dot{w} P_I = P_J b' \dot{x} P_I = P_J\, {}^J\dot{x}^I P_I,$$

which implies ${}^{J}x^{I} = {}^{J}w^{I}$.

Now Lemma 5.4 implies that $b\dot{w}_J$ and $b''\dot{v}\dot{x}_J$ lie in the same orbit under the action of $Z_{{}^{J}\dot{x}^{I}}$ on L'. Hence, by the induction hypothesis, there exists a $y' \in W(M')$ such that

$$\tilde{\varphi}(y')w_J y'^{-1} \leq vx_J.$$

Since both sides lie in W_J and since ${}^{J}x^{I} \in {}^{J}W$, this implies

$$z := \tilde{\varphi}(y')w_J y'^{-1}\, {}^{J}x^{I} \leq vx_J\, {}^{J}x^{I} = vx^{I}.$$

Since $\tilde{\varphi} = \varphi \circ \operatorname{int}(({}^{J}\dot{w}^{I})^{-1})$, if we let $\tilde{y} = ({}^{J}w^{I})^{-1}y'\, {}^{J}w^{I} \in W_I$ we can write z as $\varphi(\tilde{y})w\tilde{y}^{-1}$.

Because of the Bruhat relation $z \leq vx^{I}$ we can write $z = v'x'$ for certain $v', x' \in W$ with $v' \leq v$ and $x' \leq x^{I}$. Since $\varphi(I) = J$, we get $\varphi^{-1}(v') \leq \varphi^{-1}(v) \leq x_I$, where φ^{-1} is the inverse of the isomorphism $\varphi\colon W_I \to W_J$ induced by φ. Since $x^{I} \in W^{I}$, we get

$$v'^{-1}z\varphi^{-1}(v') = x'\varphi^{-1}(v') \leq x^{I}\varphi^{-1}(v) \leq x^{I}x_I = x.$$

So $y := \varphi^{-1}(v')^{-1}\tilde{y}$ has the required property. □

For $w \in W$, the set $T\dot{w}$ is independent of the choice of representative $\dot{w}$. This justifies the following definition:

Definition 6.8. For $w \in W^{I}$ let $\tilde{G}^{w} := \mathrm{o}(T\dot{w}) \subset G^{w}$.

We shall show that $\tilde{G}^{w}$ is dense in G^{w}. The crucial case is the following lemma. The proof we give is a slight modification of the proof of Lemma 6.1 given in [8].

Lemma 6.9. *Let G be a reductive algebraic group and φ a surjective endomorphism of G leaving invariant a Borel subgroup B of G and a maximal torus T of B. Then the morphism $\alpha\colon G \times T \to G, (g,t) \mapsto \varphi(g)tg^{-1}$ has dense image.*

Proof. Equivalently we may show that for some $t_0 \in T$, the image of the morphism $\tilde{\alpha}\colon G \times T \to G, (g,t) \mapsto t_0\varphi(g)t_0^{-1}t^{-1}g^{-1}$ is dense. It will be enough to show that the differential of $\tilde{\alpha}$ at 1 is surjective. This differential is the linear map

$$\begin{aligned} L(G) \times L(T) &\to L(G) \\ (X,Y) &\mapsto T + L(\operatorname{int}(t_0) \circ \varphi)(X) - X. \end{aligned}$$

This linear map has image

$$\operatorname{Lie}(T) + (L(\operatorname{int}(t_0) \circ \varphi) - 1)\operatorname{Lie}(G).$$

Let $\varphi_{t_0} = \operatorname{int}(t_0) \circ \varphi$. Let B^{-} be the Borel subgroup opposite to B with respect to T. Since $\varphi(B) = B$, the differential of φ_{t_0} at 1 preserves $L(\mathcal{R}_u B)$ and $L(\mathcal{R}_u B^{-})$. If we find a t_0 such that $L(\varphi_{t_0})$ has no fixed points on $L(\mathcal{R}_u B)$ and $L(\mathcal{R}_u B^{-})$ we will be done.

For each $\alpha \in \Phi$, let x_α be a basis vector of $L(U_\alpha)$. The isogeny φ induces a bijection $\tilde{\varphi}\colon \Phi \to \Phi$ such that $\varphi(U_\alpha) = U_{\varphi(\alpha)}$. For each $\alpha \in \Phi$ there exists a $c(\alpha) \in k$ such that $L(\varphi)(x_\alpha) = c(\alpha)x_{\tilde{\varphi}(\alpha)}$. This implies $L(\varphi_{t_0})(x_\alpha) = \alpha(t_0)c(\alpha)x_{\tilde{\varphi}(\alpha)}$. Since φ_{t_0} fixes $\mathcal{R}_u B$ and $\mathcal{R}_u B^-$, its differential permutes Φ^+ and Φ^-. Hence $L(\varphi_{t_0})$ can only have a fixed point in $L(\mathcal{R}_u B)$ or $L(\mathcal{R}_u B^-)$ if there exists a cycle $(\alpha_1, \cdots, \alpha_n)$ of the permutation $\tilde{\varphi}$ in Φ^+ or Φ^- such that

$$\prod_{i=1}^{n} \alpha_i(t_0)c(\alpha_i) = 1.$$

This shows that for t_0 in some non-empty open subset of T, the differential $L(\varphi_{t_0})$ has no fixed points on $L(\mathcal{R}_u B)$ and $L(\mathcal{R}_u B^-)$. □

Lemma 6.10. *For each $w \in W^I$, the set $\tilde{G}^w$ is dense in G^w.*

Proof. By Theorem 5.19, the continuous map $j_{\dot{w}}\colon L_{\dot{w}} \to G^w$ gives a bijection between the orbits in $L_{\dot{w}}$ under the action of $L'_{\dot{w}}$ on itself by twisted conjugation and the orbits of Z in G^w. By Lemma 6.9, the orbit of T is dense in $L_{\dot{w}}$. Since $j_{\dot{w}}$ is continous we get

$$L'_{\dot{w}}\dot{w} = j_{\dot{w}}(\overline{\mathrm{o}(T)}) \subset \overline{j_{\dot{w}}(\mathrm{o}(T))} \subset \overline{\tilde{G}^w}.$$

Since $\tilde{G}^w$ is Z-invariant, this implies $\overline{\tilde{G}^w} = G^w$. □

Definition 6.11. For w and w' in W^I we let $w \preccurlyeq w'$ if and only if there exists $y \in W_I$ such that $\varphi(y)wy^{-1} \leq w'$.

Theorem 6.12. *For $w \in W^I$*

$$\overline{G^w} = \coprod_{\substack{w' \in W^I \\ w' \preccurlyeq w}} G^{w'}.$$

Proof. First, consider $w' \in W^I$ such that $G^{w'}$ intersects $\overline{G^w}$. Then by Proposition 6.2 and Lemma 6.5 there exist $b, b' \in B$ and $x \in W$ such that $x \leq w$ and $\mathrm{o}(bw') = \mathrm{o}(b'x)$. Hence Lemma 6.7 implies $w' \preccurlyeq w$ and this shows " $\subset$ ".

For " $\supset$ " let $w' \in W^I$ with $w' \preccurlyeq w$. By definition there exists $y \in W^I$ such that $\varphi(y)w'y^{-1} \leq w$. Since by Lemma 6.10 the orbit of $T\dot{w}'$ is dense in $G^{\dot{w}'}$, in order to show $G^{w'} \subset \overline{G^w}$ it is sufficient to show $T\dot{w}' \subset \overline{G^w}$. For $t \in T$

$$\mathrm{o}(t\dot{w}') = \mathrm{o}(\varphi(\dot{y})t\dot{w}'\dot{y}^{-1}) = \mathrm{o}((\varphi(\dot{y})t\varphi(\dot{y}^{-1}))(\varphi(\dot{y})\dot{w}'\dot{y}^{-1})).$$

Hence, since $\varphi(\dot{y})t\varphi(\dot{y}^{-1}) \in T$ and $\varphi(y)w'y^{-1} \leq w$, Lemma 6.5 shows $t\dot{w}' \in \overline{G^w}$. □

Remark 6.13. This theorem was motivated by a similar result of Wedhorn in a different setting in [9]. Also, the proof we give here was inspired by Wedhorn's proof in [9].

Corollary 6.14. *The relation $\preccurlyeq$ is a partial order on W^I.*

This was previously proved by Wedhorn directly for arbitrary isomorphisms of Coxeter groups $\varphi\colon (W_I, I) \xrightarrow{\simeq} (W_J, J)$ in [9, section 4].

Corollary 6.15. *For $w \in W^I$, the set G^w is a locally closed subset of G.*

Proof. It follows from Theorem 6.12 that the boundary of G^w in G is

$$\coprod_{w'} G^{w'},$$

where the disjoint union ranges over all $w' \in W^I$ such that $w \neq w'$ and $w' \preccurlyeq w$. It follows from Theorem 6.12 that the boundary contains the closure of each such $G^{w'}$, hence it is closed. □

In particular, each G^w now has the structure of a variety.

Lemma 6.16. *For $w \in {}^J W^I$,*

$$\ell(w) = \dim(L' \cap {}^{\dot{w}}P) + \dim(\mathcal{R}_u P') - \dim(P' \cap {}^{\dot{w}}P).$$

Proof. Denote the right hand side by m. Let $\tilde{P} = (P' \cap {}^{\dot{w}}P)\mathcal{R}_u P'$. By Lemma 2.1 this is a parabolic subgroup of G. We can write $\tilde{P}$ as $(L' \cap {}^{\dot{w}}P)\mathcal{R}_u P'$ and the decomposition $P' = L' \ltimes \mathcal{R}_u P'$ implies

$$\tilde{P} = (L' \cap {}^{\dot{w}}P)\mathcal{R}_u P' = (L' \cap {}^{\dot{w}}P) \ltimes \mathcal{R}_u P'.$$

From this we find that the product morphism

$$(P' \cap {}^{\dot{w}}P) \times \bigtimes_{\alpha} U_\alpha \to \tilde{P}$$

is a bijection, where the product ranges over the set

$$\{\alpha \in \Phi^+ \setminus \Phi_{L'} \mid w^{-1}\alpha \notin \Phi_P\}$$

taken in any fixed order. Hence m is the cardinality of this set. Since $w \in {}^J W$, by (2)

$$w^{-1}\Phi^+_{L'} \subset \Phi^+ \subset \Phi_P,$$

hence the above set can be written as

$$\{\alpha \in \Phi^+ \mid w^{-1}\alpha \notin \Phi_P\}.$$

Since $w \in W^I$, by (2) we have $w\Phi^-_L \subset \Phi^-$, so m is the cardinality of the set

$$\{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}.$$

By (3), this is $\ell(w)$, so we are done. □

Theorem 6.17. *For $w \in W^I$, the dimension of G^w is $\dim P + \ell(w)$.*

Proof. We proceed by induction on $\dim G$. If $G = L$ we have $w = 1$ and $G^w = G = P$, so the claim is true.

Assume $\dim L < \dim G$. Let $w = w_J\, {}^J w^I$ be the decomposition from Lemma 5.4. The induction hypothesis applied to the algebraic zip datum $Z_{{}^J\dot{w}^I}$ yields

$$\dim(L')^{w_J} = \dim(L' \cap {}^{{}^J\dot{w}^I}P) + \ell(w_J),$$

where $(L')^{w_J}$ is the $Z_{{}^J\dot{w}^I}$-stable piece of L' associated to w_J. By Lemma 5.18

$$(L')^{w_J}\, {}^J\dot{w}^I = G^w \cap L'\, {}^J\dot{w}^I.$$

The set of nonsingular points of G^w is Z-invariant, hence there exists a nonsingular point x of G^w in $(L')^{w_J}\, {}^J\dot{w}^I$. Since $\mathrm{o}(x) \subset G^w$, Lemma 3.9 implies $T_x(P'\, {}^J\dot{w}^I P) = T_x(G^w) + T_x(L'\, {}^J\dot{w}^I)$. Hence x is a nonsingular point of $(L')^{w_J}\, {}^J\dot{w}^I$ and

$$\dim T_x((L')^{w_J}\, {}^J\dot{w}^I) = \dim(G^w) + \dim(L'\, {}^J\dot{w}^I) - \dim(P'\, {}^J\dot{w}^I P).$$

Because $(L')^{w_J}$ is irreducible, this is also the dimension of $(L')^{w_J}$. By Lemma 3.5

$$\dim(P'\, {}^J\dot{w}^I P) = \dim P' + \dim P - \dim(P' \cap {}^{{}^J\dot{w}^I}P).$$

Altogether we get

$$\dim(G^w) = \dim(L' \cap {}^{{}^J\dot{w}^I}P) + \ell(w_J) + \dim P + \dim(\mathcal{R}_u P') - \dim(P' \cap {}^{{}^J\dot{w}^I}P).$$

Since $\ell(w) = \ell(w_J) + \ell({}^J w^I)$, the claim now follows from Lemma 6.16. □

7 Stabilizers

In this section we consider the stabilizers of the action of an algebraic zip datum. For an element $g \in G$ we denote the stabilizer of g by $\mathrm{Stab}_Z(g)$.

First we give an inductive description of the stabilizer of an element of G.

Lemma 7.1. *Let Z be any algebraic zip datum. Let $n \in N$ and $\ell' \in L'$. Let $Z_n = (L', Q', \varphi(Q), M', \varphi(M), \tilde{\varphi})$ as in 3.4. Then for*

$$(u, u', \ell) \in (\mathcal{R}_u P \times \mathcal{R}_u P') \rtimes L$$

the following are equivalent:

(i) $(u, u', \ell) \in \mathrm{Stab}_Z(\ell' n)$.

(ii) *There exist $m' \in M'$, $v \in \mathcal{R}_u Q$, $v' \in \mathcal{R}_u Q'$ and $\tilde{u} \in \mathcal{R}_u P \cap {}^n\mathcal{R}_u P'$ such that*

$$\begin{aligned}(v', \varphi(v), m') &\in \mathrm{Stab}_{Z_n}(\ell') \\ \ell &= v({}^{n^{-1}}m') \\ u &= ({}^{n^{-1}}v')\tilde{u} \\ u' &= {}^{\ell' v' n}(\tilde{u}v).\end{aligned}$$

Proof. Let $(u, u', \ell) \in \mathrm{Stab}_Z(\ell')$. Then

$$u'\varphi(\ell)\ell' n \ell^{-1} u^{-1} = \ell' n. \tag{4}$$

Hence $(\ell'^{-1}u'\varphi(\ell)\ell', u\ell) \in \mathrm{Stab}_{P' \times P}(n)$, where we let $P' \times P$ act on G as in Section 3. This implies $(\ell'^{-1}\varphi(\ell)\ell', \ell) \in H_n$, so Lemma 3.6 shows that there exist $m' \in M'$, $v \in \mathcal{R}_u Q$ and $v' \in \mathcal{R}_u Q'$ such that

$$\ell = v \, {}^{n^{-1}}m' \tag{5}$$

and

$$\ell'^{-1}\varphi(\ell)\ell' = v'm'. \tag{6}$$

Plugging (5) into (6) yields

$$\varphi(v)\varphi({}^{n^{-1}}m')\ell' m'^{-1} v'^{-1} = \ell',$$

so we get $(v', \varphi(v), m') \in \mathrm{Stab}_{Z_n}(\ell')$.

Plugging (5) and (6) into (4) yields

$$u'(\ell' v' m' \ell'^{-1})\ell' n (n^{-1} m'^{-1} n) v^{-1} u^{-1} = \ell' n,$$

which simplifies to

$$u' = \ell'({}^{n}(uv)v'^{-1})\ell'^{-1}. \tag{7}$$

We can decompose u as $u = u_1 u_2$ with $u_1 \in \mathcal{R}_u P \cap {}^{n^{-1}}L'$ and $u_2 \in \mathcal{R}_u P \cap {}^{n^{-1}}\mathcal{R}_u P'$. Since $v \in \mathcal{R}_u Q = L \cap {}^{n^{-1}}\mathcal{R}_u P'$ and $v' \in \mathcal{R}_u Q' = L' \cap {}^{n}\mathcal{R}_u P'$, decomposing the right side of (7) into it components in L' and $\mathcal{R}_u P'$ yields

$$u_1 = {}^{n^{-1}}v'$$

and

$$u' = {}^{\ell' v' n} u_2.$$

Hence, if we let $\tilde{u} = u_2$, we have shown that (i) implies (ii). That (ii) also implies (i) can be directly verified. □

Now let Z be an algebraic zip datum which is nice with respect to T and B.

Definition 7.2. For $w \in W^I$, let P_w be the standard parabolic subgroup of G of type I_w.

By definition L_w is a Levi factor of P_w. For $p \in P_w$, if $p = u\ell$ with $u \in \mathcal{R}_u P_w$ and $\ell \in L_w$ is its Levi decomposition, we call ℓ the *component of* p *in* L_w.

Definition 7.3. For $w \in W^I$, let L_w^f be the set of fixed points of the endomorphism $\mathrm{int}(\dot{w}^{-1}) \circ \varphi$ of L_w. This is a closed subgroup of L_w.

Lemma 7.4. *Let* $w \in W^I$ *and* $(u, u', \ell) \in \mathrm{Stab}_Z(\dot{w})$. *Then* $\ell \in L \cap P_w$ *and the component of* ℓ *in* L_w *lies in* L_w^f.

Proof. We proceed by induction on $\dim G$. If $G = L'$, we have $w = 1$ and $L = G = P_1$, so $\ell \in L \cap P_w$. Since ${}^{\dot w^{-1}}\varphi(\ell) = \ell$ is equivalent to $\varphi(\ell)\dot w \ell^{-1} = \dot w$ we have $\mathrm{Stab}_Z(1) = L_1^f$. Hence in this case the claim is true.

Assume $\dim L' < \dim G$. Decompose $w = w_J\, {}^J w^I$ as in Lemma 5.4. We may and do choose the representative $\dot w_J$ so that $\dot w = \dot w_J\, {}^J\dot w^I$. Let $(\ell, u, u') \in \mathrm{Stab}_Z(\dot w)$. Let $Z_{{}^J\dot w^I} = (L', Q', \varphi(Q), M', \varphi(M), \tilde\varphi)$. Applying Lemma 7.1 to $n = {}^J\dot w^I$ and $\ell' = \dot w_J$ shows that there exist $m' \in M'$, $v \in \mathcal{R}_u Q$, $v' \in \mathcal{R}_u Q'$ and $\tilde u \in \mathcal{R}_u P \cap {}^{{}^J\dot w^I}\mathcal{R}_u P'$ such that

$$
\begin{aligned}
(v', \varphi(v), m') &\in \mathrm{Stab}_{Z_{{}^J\dot w^I}}(\dot w_J)\\
\ell &= v({}^J\dot w^I)^{-1} m'\, {}^J\dot w^I\\
u &= ({}^J\dot w^I)^{-1} v'\, {}^J\dot w^I \tilde u\\
u' &= {}^{\dot w_J v'\, {}^J\dot w^I}(\tilde u v).
\end{aligned}
$$

By Lemma 5.4 $W_{K_{{}^J w^I}} = W(M')$. Let $K^{{}^J w^I}_{w_J}$ be the largest subset of $K^{{}^J w^I}$ invariant under $\mathrm{int}(w_J^{-1}) \circ \tilde\varphi$ and let Q'_{w_J} be the standard parabolic subgroup of L' of type $K^{{}^J w^I}_{w_J}$. Since $Z_{{}^J\dot w^I}$ is nice with respect to T and $L' \cap B$, the induction hypothesis applied to $Z_{{}^J\dot w^I}$ shows that m' can be written as $m' = \tilde v \tilde m$ with $\tilde m \in M'_{w_J}$ such that ${}^{\dot w_J^{-1}}\varphi({}^{({}^J\dot w^I)^{-1}}\tilde m) = \tilde m$ and $\tilde v \in M' \cap \mathcal{R}_u Q'_{w_J}$. By Lemma 5.10 we have $M'_{w_J} = {}^{{}^J\dot w^I}L_w$. Hence $\tilde\ell = {}^{({}^J\dot w^I)^{-1}}\tilde m$ lies in L_w and since ${}^{\dot w^{-1}}\varphi(\tilde\ell) = \tilde\ell$ we find $\tilde m \in L_w^f$. From Lemma 5.7 we get $K^{{}^J w^I}_{w_J} = {}^{{}^J w^I} I_w$, which implies ${}^{({}^J\dot w^I)^{-1}}\mathcal{R}_u Q'_{\dot w_J} \subset \mathcal{R}_u P_{\dot w}$. Hence the identity

$$\ell = v^{({}^J\dot w^I)^{-1}}\tilde v \tilde\ell$$

shows that $\ell \in P_w$ and that $\tilde\ell \in L_w^f$ is its component in L_w. □

The preceding lemma yields a morphism $\pi\colon \mathrm{Stab}_Z(\dot w) \to L_w^f$ which sends $(u, u', \ell) \in \mathrm{Stab}_Z(\dot w)$ to the component of ℓ in L_w. We denote the kernel of π by K_w. Then we get:

Theorem 7.5. *There is a split short exact sequence*

$$1 \longrightarrow K_w \hookrightarrow \mathrm{Stab}_Z(\dot w) \xrightarrow{\pi} L_w^f \longrightarrow 1\ ,$$

where K_w is a connected unipotent group.

Proof. The morphism $L_w^f \to \mathrm{Stab}_Z(\dot w)\colon \ell \mapsto (0, 0, \ell)$ is section of π, which shows that π is onto and that the sequence is split. It remains to show that K_w is unipotent and connected.

Because of Lemma 7.4 we have a homomorphism

$$
\begin{aligned}
K_w &\to \mathcal{R}_u P_w\\
(u, u', \ell) &\mapsto u\ell,
\end{aligned}
$$

which is injective. This shows that K_w is unipotent.

Let $(u, u', \ell) \in \mathrm{Stab}_Z(\dot{w})$. Then $\ell \in L \cap \mathcal{R}_u P_w \subset L \cap \mathcal{R}_u B$, hence $\varphi(\ell) \in \mathcal{R}_u B$ and, since $w \in W^I$, also ${}^{\dot{w}}\ell^{-1} \in \mathcal{R}_u B$. This shows that ${}^{\dot{w}}u = u'\varphi(\ell)^{\dot{w}}\ell^{-1} \in \mathcal{R}_u B$. Hence we have a morphism

$$\begin{aligned} \mathrm{pr}_2 \colon K_w &\to \mathcal{R}_u P \cap {}^{\dot{w}^{-1}}B \\ (u, u', \ell) &\mapsto u. \end{aligned}$$

To show that K_w is connected it will be sufficient to prove that pr_2 is bijective. We prove this by induction on $\dim G$. If $G = L'$ both K_w and $\mathcal{R}_u P$ are trivial, so the claim is true.

Assume $\dim L' < \dim G$. We decompose $w = w_J\, {}^J w^I$ as in Lemma 5.4. Let $(u, u', \ell), (u, \hat{u}', \hat{\ell}) \in \mathrm{Stab}_Z(\dot{w})$. Let $Z_{{}^J\dot{w}^I} = (L', Q', \varphi(Q), M', \varphi(M), \tilde{\varphi})$. Applying Lemma 7.1 to $n = {}^J\dot{w}^I$ and $\ell' = \dot{w}_J$ shows that there exist $m', \hat{m}' \in M'$, $v, \hat{v} \in \mathcal{R}_u Q$, $v', \hat{v}' \in \mathcal{R}_u Q'$ and $\tilde{u}, \hat{\tilde{u}} \in \mathcal{R}_u P \cap {}^{{}^J\dot{w}^I}\mathcal{R}_u P'$ such that

$$\begin{aligned} (v', \varphi(v), m') &\in \mathrm{Stab}_{Z_{{}^J\dot{w}^I}}(\dot{w}_J) \\ \ell &= v({}^J\dot{w}^I)^{-1} m'\, {}^J\dot{w}^I \\ u &= ({}^J\dot{w}^I)^{-1} v'\, {}^J\dot{w}^I \tilde{u} \\ u' &= {}^{\dot{w}_J v'\, {}^J\dot{w}^I}(\tilde{u}v) \end{aligned}$$

and

$$\begin{aligned} (\hat{v}', \varphi(\hat{v}), \hat{m}') &\in \mathrm{Stab}_{Z_{{}^J\dot{w}^I}}(\dot{w}_J) \\ \hat{\ell} &= \hat{v}({}^J\dot{w}^I)^{-1} \hat{m}'\, {}^J\dot{w}^I \\ u &= ({}^J\dot{w}^I)^{-1} \hat{v}'\, {}^J\dot{w}^I \hat{\tilde{u}} \\ \hat{u}' &= {}^{\dot{w}_J \hat{v}'\, {}^J\dot{w}^I}(\hat{\tilde{u}}\hat{v}). \end{aligned}$$

Since $v', \hat{v}' \in L'$ and ${}^{{}^J\dot{w}^I}\tilde{u}, {}^{{}^J\dot{w}^I}\hat{\tilde{u}} \in \mathcal{R}_u P'$ the identity

$$({}^J\dot{w}^I)^{-1} v'\, {}^J\dot{w}^I \tilde{u} = ({}^J\dot{w}^I)^{-1} \hat{v}'\, {}^J\dot{w}^I \hat{\tilde{u}}$$

implies $v' = \hat{v}'$ and $\tilde{u} = \hat{\tilde{u}}$. Hence the induction hypothesis applied to the nice algebraic zip datum $Z_{{}^J\dot{w}^I}$ shows that $m' = \hat{m}'$ and $\varphi(v) = \varphi(\hat{v})$. Since the kernel of the isogeny φ must be contained in T, we also get $v = \hat{v}$. Hence we have $\ell = \hat{\ell}$. This also implies $u' = \hat{u}'$, which shows that pr_2 is injective.

Now let $u \in \mathcal{R}_u P \cap {}^{\dot{w}^{-1}}B$. Then we have ${}^{{}^J\dot{w}^I}u \in P'$, so we can decompose $u = u_1 \tilde{u}$ for certain $u_1 \in \mathcal{R}_u P \cap {}^{({}^J\dot{w}^I)^{-1}}L'$ and $\tilde{u} \in \mathcal{R}_u P \cap {}^{({}^J\dot{w}^I)^{-1}}\mathcal{R}_u P'$. Let $v' = {}^{{}^J\dot{w}^I}u_1 \in L' \cap {}^{{}^J\dot{w}^I}\mathcal{R}_u P = \mathcal{R}_u Q'$ so that $u = {}^{({}^J\dot{w}^I)^{-1}}v'\tilde{u}$. By the induction hypothesis there exist $m' \in M'$ and $v \in \mathcal{R}_u Q$ such that

$$(v', \varphi(v), m') \in \mathrm{Stab}_{Z_{{}^J\dot{w}^I}}(\dot{w}_J).$$

Hence, if we let $\ell = v({}^J\dot{w}^I)^{-1} m'\, {}^J\dot{w}^I$ and $u' = {}^{\dot{w}_J v'\, {}^J\dot{w}^I}(\tilde{u}v)$, by Lemma 7.1 we have $(u, u', \ell) \in \mathrm{Stab}_Z(\dot{w})$. After multpliying ℓ by the inverse of its component in L_w we may assume that $(u, u', \ell) \in K_w$. This shows that pr_2 is surjective. $\square$

Remark 7.6. If one considers the scheme-theoretic stabilizer and not just the associated variety, the group K_w may be nonreduced.

8 Frobenius Zip Data

Let Z be nice with respect to T and B. In Theorem 5.19 we showed that we have a decomposition of G into Z-stable pieces G^w such that the orbits in each piece G^w correspond to the orbits in the reductive group L_w under the action of that group on itself by twisted conjugation. The Lang-Steinberg Theorem gives a good criterion for such an action to be transitive:

Theorem 8.1 ([8], Theorem 10.1)**.** *Let G be a connected linear group and φ a surjective endomorphism of G with a finite number of fixed points. Then the morphism*

$$\begin{aligned} G &\to G \\ g &\mapsto \varphi(g)g^{-1} \end{aligned}$$

is surjective.

Definition 8.2. A *Frobenius zip datum* is a nice algebraic zip datum such that for every $w \in W^I$, the endomorphism $\mathrm{int}(\dot{w}^{-1}) \circ \varphi$ of L_w has finitely many fixed points.

Example 8.3. If the differential of φ at 1 vanishes, then Z is a Frobenius zip datum.

Proof. If the differential of φ vanishes, then so does the differential of

$$\mathrm{int}(\dot{w}^{-1}) \circ \varphi \colon L_w \to L_w$$

for all $w \in W^I$. Hence it suffices to show that an endomorphism φ of an algebraic group H with vanishing differential has only finitely many fixed points. Let H^f be set set of fixed points of φ, this is a closed subgroup of H. The restriction of φ to H^f is the identity, but by assumption its differential is zero. This is only possible if H^f has dimension zero, i.e. if H^f is finite. □

Proposition 8.4. *For a nice algebraic zip datum Z the following are equivalent:*

(i) The algebraic zip datum Z is a Frobenius zip datum.

(ii) There are only finitely many orbits under the action of Z on G.

(iii) Each Z-stable piece G^w for $w \in W^I$ is a single orbit.

Proof. *(i)* $\implies$ *(iii)* follows from Theorems 5.19 and 8.1. *(iii)* $\implies$ *(ii)* is trivial.

Assume *(ii)* holds. Let $w \in W^I$. Since are finitely many orbits in G^w, there exists a dense orbit in L_w, where L_w acts on itself by conjugation twisted with

$\mathrm{int}(\dot{w}^{-1}) \circ \varphi$. Let ℓ be an element of the dense orbit. Then the stabilizer of ℓ must be finite, which is equivalent to saying that $\mathrm{int}(\ell'^{-1}) \circ \mathrm{int}(\dot{w}^{-1}) \circ \varphi$ has finitely many fixed points. Hence the Lang-Steinberg Theorem implies that the orbit of ℓ is all of L_w. But then the stabilizer of $1 \in L_w$ must also be finite, which shows that Z is Frobenius. □

Let Z be a Frobenius zip datum. Since then the Z-stable pieces G^w are just the orbits of w, Theorems 5.19, 6.12, 6.17 and 7.5 can be rephrased as follows:

Theorem 8.5. *The set $\{\dot{w} \mid w \in W^I\}$ is a set of representatives for the action of Z on G.*

Theorem 8.6. *For $w, w' \in W^I$ the following are equivalent:*

(i) $\overline{\mathrm{o}(w')} \subset \mathrm{o}(w)$.

(ii) $w' \preccurlyeq w$.

Theorem 8.7. *For $w \in W^I$ the dimension of* $\mathrm{o}(w)$ *is* $\dim P + \ell(w)$.

Theorem 8.8. *For $w \in W^I$, the stabilizer of $\dot{w}$ is the semidirect product of L_w^f and a connected unipotent algebraic group. In particular, the group of connected components of* $\mathrm{Stab}_Z(\dot{w})$ *is isomorphic to L_w^f.*

Furthermore, Lemma 5.14 implies

Lemma 8.9. *For $w \in W$, the orbit of $\dot{w}$ does not depend on the choice of representative $\dot{w} \in N$.*

Let $W = W(G)$, $X = W(L)$ and $X' = W(L')$ and $\psi\colon X \to X'$ the homomorphism induced by φ. By Lemma 8.9 the orbits in G induce a well-defined equivalence relation on W which, by Lemma 3.7 and Theorem 8.1, can be characterized as follows:

Two elements w and w' of W are equivalent if and only if: Either $X = W$, or there exist $x \in X$ and $x' \in X'$ such that $w' = x'wx$ and $\psi(x)x' \sim 1$ under the equivalence relation on X' obtained analogously from the algebraic zip datum $Z_{\dot{w}}$ as in Lemma 3.7. In the following section, we shall consider such equivalence relations in a more general context.

9 Abstract Zip Data

Definition 9.1. An abstract zip datum is a tuple (W, X, X', ψ), where W is a group with subgroups X and X' and $\psi : X \to X'$ is a group homomorphism.

Fix such an abstract zip datum A.

Definition 9.2. Let $w \in W$. If two subgroups of X are left invariant by the homomorphism $\mathrm{int}(w^{-1}) \circ \psi\colon X \to W$ so is the subgroup they generate. Hence there exists a unique maximal subgroup with this property, which we denote by X_w.

Definition 9.3. To each abstract zip datum we associate a relation on W as follows: For w and w' in W we let $w \sim w'$ if and only if there exist $x \in X$ and $u \in X_w$ such that $w' = \psi(x)wux^{-1}$.

Lemma 9.4. *This is an equivalence relation.*

Proof. Reflexivity is clear. To prove symmetry, let $w' = \psi(x)wux^{-1}$ as above. From $\mathrm{int}(w'^{-1}) \circ \psi = \mathrm{int}(xu^{-1}w^{-1}) \circ \psi \circ \mathrm{int}(x^{-1})$ we get $X_{w'} = {}^{x}X_w$. Hence the identity

$$w = \psi(x^{-1})w'(xu^{-1}x^{-1})x$$

shows $w' \sim w$.

Now let $w \sim w'$ and $w' \sim w''$, that is $w' = \psi(x)wux^{-1}$ and $w'' = \psi(\tilde{x})w'\tilde{u}\tilde{x}^{-1}$ for some $x, \tilde{x} \in X$, $u \in X_w$ and $\tilde{u} \in X_{w'}$. Since again $X_{w'} = {}^{x}X_w$, we get $w \sim w''$ from

$$w'' = \psi(\tilde{x}x)wu(x^{-1}\tilde{u}x)(\tilde{x}x)^{-1}.$$

□

Construction 9.5. Analogously to 3.4, we define a new abstract zip datum A_w for each $w \in W$ as

$$A_w := (X', X' \cap {}^{w}X, \psi({}^{w^{-1}}X' \cap X), \psi \circ \mathrm{int}(w^{-1})).$$

Theorem 9.6. *The equivalence relation is uniquely characterized by the following property:*

Two elements w and w' of W are equivalent if and only if:

Either $W = X$ or there exist $x \in X$ and $x' \in X'$ such that $w' = x'wx$ and $\psi(x)x' \in X'$ is equivalent to 1 under the equivalence relation on X' defined by A_w.

Proof. Since this property allows to determine wheter any two elements w, w' of W are equivalent, it characterizes the equivalence relation uniquely if it holds. It remains to show that the equivalence relation has this property.

If $W = X$, we have $X_w = W$ for all $w \in W$, so all elements of W are equivalent and the claim is true. So we may assume that X is not equal to W. Let $w' \sim w$, that is $w' = \psi(\tilde{x})wu\tilde{x}^{-1}$ for some $\tilde{x} \in X$ and $u \in X_w$. Let $x = u\tilde{x}^{-1} \in X$ and $x' = \psi(x) \in X'$. To show $\psi(x)x' = \psi(u) \sim 1$ under the equivalence relation defined by A_w, it is enough to show that $\psi(X_w)$ is contained in $(X' \cap {}^{w}X)_1$. But this follows from the fact that $\psi(X_w)$ is a subgroup of $X' \cap {}^{w}X$ which is mapped to itself by $\psi \circ \mathrm{int}(w^{-1})$. This proves the "only if" part of the claim.

For the other direction, let $w' = x'wx$ as above. Since $\psi(x)x' \sim 1$ in X' we can write $\psi(x)x' = \psi({}^{w^{-1}}y)uy^{-1}$ for some $y \in X' \cap {}^{w}X$ and $u \in (X' \cap {}^{w}X)_1$. Since the subgroup ${}^{w^{-1}}(X' \cap {}^{w}X)_1$ of X is mapped to itself by $\mathrm{int}(w^{-1}) \circ \psi$, it lies in X_w. Hence the identity

$$w' = x'wx = \psi(x^{-1 w^{-1}}y)w^{w^{-1}}u(x^{-1 w^{-1}}y)^{-1}$$

shows $w' \sim w$. □

Proposition 9.7. *Assume that X is finite. Each equivalence class in W has cardinality $|X|$. In particular, there are $[W : X]$ equivalence classes.*

Proof. Let $H_0 = X$ and define inductively $H_i = \{x \in X \mid {}^{w^{-1}}\psi(x) \in H_{i-1}\}$ for $i \geq 1$. Then $X_w = \cap_{i \geq 0} H_i$.

Pick $w \in W$ and let o be the equivalence class of w. Then we have a surjective map

$$\begin{aligned} \Psi : X \times X_w &\to o \\ (x, u) &\mapsto \psi(x) w u x^{-1}. \end{aligned}$$

Let (x, u) and $(\tilde{x}, \tilde{u}) \in X \times X_w$ have the same image z under ψ, that is

$$z = \psi(x) w u x^{-1} = \psi(\tilde{x}) w \tilde{u} \tilde{x}^{-1}.$$

If we let $y = x^{-1}\tilde{x}$, we get

$${}^{w^{-1}}\psi(y) = u y \tilde{u}^{-1}.$$

From this it follows inductively that $y \in H_i$ for all $i \geq 0$, hence $y \in X_w$. This implies

$$\Psi^{-1}(z) = \{(xy, {}^{w^{-1}}\psi(y^{-1}) u y) \mid y \in X_w\}.$$

Hence each fiber of Ψ has cardinality $|X_w|$, so the image of Ψ must have cardinality $|X|$. $\square$

Definition 9.8. An abstract zip datum (W, X, X', ψ) is *of Coxeter type* if W is a finite Coxeter group with set of simple reflections S such that $X = W_I$ and $X' = W_J$ for certain $I, J \subset S$ and $\psi \colon (W_I, I) \to (W_J, J)$ is an isomorphism of Coxeter groups.

Example 9.9. Any algebraic zip datum $Z = (G, P, P', L, L', \varphi)$ for which there exists a maximal torus T of G which is left invariant by φ gives rise to an abstract zip datum $(W, W(L), W(L'), \varphi)$. If Z is nice with respect to T and a Borel subgroup B of G, this abstract zip datum is of Coxeter type.

In case an abstract zip datum (W, X, X', ψ) arises from a nice algebraic zip datum, Theorem 8.5 implies that W^I is a system of representatives for the equivalence relation on W. In fact this holds for any abstract zip datum of Coxeter type:

Proposition 9.10. *Let (W, X, X', ψ) be of Coxeter type. Then the set W^I is a set of representatives for the equivalence relation on W.*

Proof. We prove this by induction on $|W|$. The base case is the case $W = X'$. In this case $W^I = \{1\}$ and there is exactly one equivalence class in W, so the claim is true.

Assume $|X'| < |W|$. Since $|W^I| = [W : W_I]$ it follows from Proposition 9.7 that it is enough to show that if $w \in W^I$ and $\hat{w} \in W^I$ are equivalent they are equal. Decompose $w = w_J\, {}^J w^I$ and $\hat{w} = \hat{w}_J\, {}^J\hat{w}^I$ as in Lemma 4.3. Since the equivalence class of any $w \in W$ is contained in $X'wX$, we must have ${}^J w^I = {}^J\hat{w}^I$. Hence, by Theorem 9.6, the elements w_J and $\hat{w}_J$ are equivalent under the equivalence relation defined on X' by $A_{\,{}^J w^I}$.

Let $K = J \cap {}^{{}^J w^I} I$ and $\tilde{K} = {}^{({}^J w^I)^{-1}} K$. Theorem 4.1 implies $X' \cap {}^{{}^J w^I} X = W_K$ and ${}^{({}^J w^I)^{-1}} X' \cap X = W_{\tilde{K}}$. Since ψ is an isomorphism of Coxeter groups, this also implies $\psi({}^{({}^J w^I)^{-1}} X' \cap X) = W_{\psi(\tilde{K})}$. Hence $A_{\,{}^J w^I}$ is of Coxeter type. Since $w_J, \hat{w}_J \in W^K$, the induction hypothesis applied to $Z_{\,{}^J w^I}$ implies $w_J = \hat{w}_J$, so we get $w = \hat{w}$. □

10 F-Zips

We define the notion of an F-zip as in [4]. Let p be a prime number and q a power of p. Let S be a scheme over $\mathbb{F}_q$. We denote by $F_S \colon S \to S$ the morphism which is the identity on the underlying topological space and the homomorphism $x \mapsto x^q$ on the structure sheaf. For an $\mathcal{O}_S$-module M, we set $M^{(q)} = F_S^* M$.

Let S be a scheme and M a locally free $\mathcal{O}_S$-module sheaf of finite rank.

Definition 10.1. A *descending filtration* $C^\bullet$ of M is a sequence of $(C^i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_S$-submodules of M such that C^i is locally on S a direct summand of C^{i-1} and such that $C^i = M$ for $i \ll 0$ and $C^i = 0$ for $i \gg 0$. We set $\mathrm{gr}^i_C = C^i/C^{i+1}$. We have an analogous definition of an *ascending filtration* $D_\bullet$ with associated graded modules $\mathrm{gr}^D_i = D_i/D_{i-1}$.

Let $C^\bullet$ be a descending filtration of M. If there exists a function $\tau \colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ such that $\tau(i) = \dim_{k(s)}(\mathrm{gr}^i_C \otimes k(s))$ for all $s \in S$ we say that $C^\bullet$ *is of type* τ. There is a similar definition for an ascending filtration.

Definition 10.2. Let S be a scheme over $\mathbb{F}_q$. An *F-zip over S* is a tuple $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$, where

(i) M is a locally free $\mathcal{O}_S$-module of finite rank,

(ii) $C^\bullet$ is a descending filtration of M,

(iii) $D_\bullet$ is an ascending filtration of M,

(iv) $\varphi_\bullet$ is a family of $\mathcal{O}_S$-linear isomorphisms

$$\varphi_i \colon (\mathrm{gr}^i_C)^{(q)} \xrightarrow{\simeq} \mathrm{gr}^D_i$$

for $i \in \mathbb{Z}$.

An F-zip *is of type* τ if $C^\bullet$ is of type τ.

Definition 10.3. An *isomorphism* between two F-zips $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ and $(\tilde{M}, \tilde{C}^\bullet, \tilde{D}_\bullet, \tilde{\varphi}_\bullet)$ is an isomorphism between the $\mathcal{O}_S$-modules M and $\tilde{M}$ which is compatible with the filtrations and the φ_i and $\tilde{\varphi}_i$.

Let S and S' be two schemes over $\mathbb{F}_q$ and $f\colon S \to S'$ a morphism over $\mathbb{F}_q$. For an F-zip $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ on S', we get an F-zip $f^*\underline{M}$ on S by $f^*\underline{M} = (f^*M, f^*C^\bullet, f^*D_\bullet, f^*\varphi_\bullet)$.

Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support and let $n = \sum_{i\in\mathbb{Z}} \tau(i)$. Let V be an n-dimensional vector space over $\mathbb{F}_q$. For a scheme S over $\mathbb{F}_q$, let $V_S = V \otimes \mathcal{O}_S$. Let X_τ^V be the scheme over $\mathbb{F}_q$ whose S-valued points are F-zips $(V_S, C^\bullet, D_\bullet, \varphi_\bullet)$ of type τ and such that for two schemes S and S' over $\mathbb{F}_q$ and for any $f\colon S \to S'$ the induced map $X_\tau(S') \to X_\tau(S)$ is f^*.

The group $G = \mathrm{GL}(V)$ acts on X_τ^V as follows: For a scheme S over $\mathbb{F}_q$, the action of $G(S)$ on S-valued points is given by

$$g \cdot (V_S, C^\bullet, D_\bullet, \varphi_\bullet) = (V_S, gC^\bullet, gD_\bullet, \psi_\bullet),$$

where ψ_i is the composition

$$\begin{aligned}(g(C^i)/g(C^{i+1}))^{(q)} &\overset{(g^{(q)})^{-1}}{\overset{\cong}{\to}} (C^i/C^{i+1})^{(q)} \\ &\overset{\varphi_i}{\overset{\cong}{\to}} D_i/D_{i-1} \overset{g}{\overset{\cong}{\to}} g(D_i)/g(D_{i-1}).\end{aligned}$$

The F-zips corresponding to two points of X_τ^V are isomorphic if and only if they are conjugate under the G-action. Any F-zip of type τ is Zariski-locally isomorphic to the F-zip corresponding to a point of X_τ^V. Hence in order to classify F-zips, we need to classify the orbits in X_τ^V.

Let $V = \oplus_{i\in\mathbb{Z}} V_i$ be a decomposition into subspaces V_i of V such that $\dim V_i = \tau(i)$. Such a decomposition defines a descending filtration $C_V^\bullet$ of type τ on V by $C_V^j = \oplus_{k\geq j} V_k$, and an ascending filtration $D_\bullet^V$ of type τ by $D_j^V = \oplus_{k\leq j} V_k$. For a scheme S over $\mathbb{F}_q$ let $C_{V,S}^\bullet = C_V^\bullet \otimes \mathcal{O}_S$ and $D_\bullet^{V,S} = D_\bullet^V \otimes \mathcal{O}_S$. Let $P^+ \subset G$ and $P^- \subset G$ be the stabilizers of $C_V^\bullet$ and $D_\bullet^V$ respectively. They are parabolic subgroups of G, whose intersection L, which is the stabilizer of the grading $(V_i)_{i\in\mathbb{Z}}$, is a common Levi factor. The submodules $V_i \otimes \mathcal{O}_S$ map isomorphically onto the graded pieces $\mathrm{gr}^i_{C_{V,S}}$ and $\mathrm{gr}_i^{D^{V,S}}$ under the projections. This gives rise to a bijective correspondence between elements of $L(S)$ and families of isomorphisms $(\mathrm{gr}^i_{C_{V,S}})^{(q)} \overset{\cong}{\to} \mathrm{gr}_i^{D_{V,S}}$.

Let T be a maximal torus of L and B a parabolic subgroup of G contained in P^+. This defines a set of simple reflections. Let I and J be the types of P^+ and P^- respectively.

Lemma 10.4. *Let S be scheme over $\mathbb{F}_q$ and $C^\bullet$ any descending filtration of type τ on V_S. Then the stabilizer $\mathrm{Stab}_G(C^\bullet)$ of $C^\bullet$ is a parabolic subgroup of G_S of type I.*

Proof. It suffices to show that this is true locally on S. Hence we may assume that $S = \mathrm{Spec}(A)$ for some $\mathbb{F}_q$-algebra A. By localizing further, we may assume that each C^i is a free summand of V_S. Then there exists a basis $(v_1, \dots, v_n)$ of $V \otimes A$ such that for all $i \in \mathbb{Z}$ the module C^i has basis $(v_1, \dots, v_{d_i})$ for suitable $d_i \in \mathbb{Z}$. Similarly there exists a basis $(w_1, \dots, w_n)$ of $V \otimes A$ such that

$(w_1, \ldots, w_{d_i})$ is a basis of $C^i_{V,S}$ for all $i \in \mathbb{Z}$. Then the linear map $V \otimes A \to V \otimes A$ sending v_i to w_i defines an element $g \in G(S)$ such that $gC^\bullet = C^\bullet_{V,S}$. Hence $\mathrm{Stab}_G(C^\bullet)$ and $\mathrm{Stab}_G(C^\bullet_{V,S}) = {}^g\mathrm{Stab}_G(C^\bullet)$ have the same type. $\square$

We call I the *parabolic type associated to* τ. Let w_0 be the unique element of W of maximal length and let $J = {}^{w_0}I$. Since P^- is opposite to P^+ it has type J. Hence by an argument similar to Lemma 10.4, the set J is the type of $\mathrm{Stab}_G(D_\bullet)$ for any ascending filtration $D_\bullet$ of type τ. For any set of simple reflections K we denote by $\mathcal{P}_K$ the variety classifying the parabolic subgroups of G of type K.

By the arguments in the preceding paragraph, there is a G-equivariant morphism

$$\begin{aligned} \pi\colon X^V_\tau &\to \mathcal{P}_I \times \mathcal{P}_J \\ (V_S, C^\bullet, D_\bullet, \varphi_\bullet) &\mapsto (\mathrm{Stab}_G(C^\bullet), \mathrm{Stab}_G(D_\bullet)). \end{aligned}$$

Let $\tilde{X}^V_\tau$ be the fiber over P^+ of the G-equivariant morphism $X^V_\tau \to \mathcal{P}_I$ obtained by composing π with the projection $\mathcal{P}_I \times \mathcal{P}_J \to \mathcal{P}_I$. This is the closed subscheme of X^V_τ whose S-valued points are the F-zips $(V_S, C^\bullet, D_\bullet, \varphi_\bullet)$ with $C^\bullet = C^\bullet_{V,S}$. The group P^+ stabilizes $\tilde{X}^V_\tau$ and we shall see that determining the orbits of G on X^V_τ is the same as determining the orbits of P^+ on $\tilde{X}^V_\tau$.

For a scheme S over $\mathbb{F}_q$ and $g \in G(S)$, let

$$\underline{M}_g := (V_S, C^\bullet_{V,S}, gD^{V,S}_\bullet, \varphi^g_\bullet) \in X^V_\tau(S)$$

be the F-zip defined by

$$\varphi^g_\bullet\colon (\mathrm{gr}^\bullet_{C_{V,S}})^{(q)} \overset{\simeq}{\to} \mathrm{gr}^{D^{V,S}}_\bullet \overset{g}{\to} \mathrm{gr}^{gD^{V,S}}_\bullet,$$

where the first maps form the family of isomorphisms $(\mathrm{gr}^\bullet_{C_{V,S}})^{(q)} \overset{\simeq}{\to} \mathrm{gr}^{D_{V,S}}_\bullet$ corresponding to $1 \in L(S)$. This defines a morphism $f\colon G \to \tilde{X}^V_\tau$.

Let k be an algebraic closure of $\mathbb{F}_q$. From now on we return to the language of varieties, so we consider G as variety over k and we replace X^V_τ and $\tilde{X}^V_\tau$ by the varieties over k corresponding to the associated reduced schemes. We also consider f to be a morphism of these varieties.

For any algebraic group G, any subgroup H of G and any H-variety X we denote by $G \times^H X$ the quotient of $G \times X$ by the left action of H defined by $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. It exists for example if the variety X is quasi-projective (see [5], Section 3.2). The action of G on $G \times X$ by multplication on the left on the first factor induces a left action of G on $G \times^H X$.

Lemma 10.5. *Let G, H and X be as above and assume that $G \times^H X$ exists. Then the morphism $q\colon X \to G \times^H X$ which sends $x \in X$ to the class of $(x, 1)$ induces a bijection between the orbits of H in X and the orbits of G in $G \times^H X$ which preserves the closure order.*

Proof. The morphism q is the composite of the inclusion

$$\begin{aligned} i\colon X &\to G \times X \\ x &\mapsto (1, x) \end{aligned}$$

and the projection $\mathrm{pr}\colon G \times X \to G \times^H X$. We let $G \times H$ act on $G \times X$ from the left by

$$(g, h)\colon (g', x) \mapsto (gg'h^{-1}, h \cdot x).$$

Then the $G \times H$-orbits in $G \times X$ are of the subvarieties $G \times o$ for all H-orbits o in X. Hence i induces a bijection between the orbits of H in X and the orbits of $G \times H$ in $G \times X$ preserving the closure order. Furthermore, it follows from the definition of $G \times^H X$ and the properties of the quotient morphism pr that pr induces a bijection between the orbits of $G \times H$ in $G \times X$ and the orbits of G in $G \times^H X$ preserving the closure order. Altogether this proves the claim about q. □

Lemma 10.6 (see [6], Lemma 3.7.4)**.** *Let G be an algebraic group and H a subgroup. Let X be a variety with a left action of G. Let $\Phi\colon X \to G/H$ be a G-equivariant morphism from X to the homogenous space G/H, and let $E \subset X$ be the fiber $\Phi^{-1}(H)$. Then E is stabilized by H, and the map $G \times^H E \to X$ sending the equivalence class of (g, e) to $g \cdot e$ defines an isomorphism of G-varieties.*

Applying this to $H = P^+$ and the morphism $X^V_\tau \to \mathcal{P}_I$ from above shows:

Proposition 10.7. *$X^V_\tau = G \times^{P^+} \tilde{X}^V_\tau$.*

In particular, the inclusion $\tilde{X}^V_\tau \to X^V_\tau$ induces a bijection between the orbits of P^+ in $\tilde{X}^V_\tau$ and the orbits of G in X^V_τ which preserves the closure order.

Lemma 10.8. *The morphism f is surjective. The fibers of f are the left cosets of $\mathcal{R}_u P^-$.*

Proof. Let $(V_k, C^\bullet_{V,k}, D_\bullet, \varphi_\bullet) \in \tilde{X}^V_\tau$. There exists $g \in G$ such that $D_\bullet = gD^{V,k}_\bullet$. By composing the inverse of the family of isomorphisms $(\mathrm{gr}^\bullet_{C_{V,k}})^{(q)} \overset{\simeq}{\to} \mathrm{gr}^{D_{V,k}}_\bullet$ corresponding to $1 \in L$ with $\varphi_\bullet$ we obtain a family of isomorphisms

$$\tilde{\varphi}_\bullet\colon \mathrm{gr}^{D_{V,k}}_\bullet \to \mathrm{gr}^{gD_{V,k}}_\bullet$$

After multiplying g with a suitable element of L, we may assume that this is the same as the family of isomorphisms

$$\mathrm{gr}^{D_{V,k}}_\bullet \overset{g}{\to} \mathrm{gr}^{gD_{V,k}}_\bullet$$

used to define $\varphi^g_\bullet$. Then we have $\underline{M}_g = (V_k, C^\bullet_{V,k}, D_\bullet, \varphi_\bullet)$ which shows surjectivity.

Now let $g, g' \in G$ such that $\underline{M}_g = \underline{M}_{g'}$. Then $gD^{V,k}_\bullet = g'D^{V,k}_\bullet$ implies that there exists $\lambda \in P^-$ such that $g' = g\lambda$. Furthermore, since $\varphi^g_\bullet = \varphi^{g'}_\bullet$, the family of automorphisms of $\mathrm{gr}^{D^{V,k}}_\bullet$ induced by λ must be trivial. This implies $\lambda \in \mathcal{R}_u P^-$.

On the other hand, if $\lambda \in \mathcal{R}_u P^-$, then reading the preceding paragraph in reverse shows that $\underline{M}_g = \underline{M}_{g\lambda}$ for all $g \in G$. □

Let P^+ act on G as follows: For $p \in P^+$, let $p = u\ell$ with $\ell \in L$ and $u \in \mathcal{R}_u P^+$ be its Levi decomposition. Then for $g \in G$

$$p \cdot g = u\ell g(\ell^{(q)})^{-1}.$$

Lemma 10.9. *The morphism f is P^+-equivariant.*

Proof. Let $p \in P^+$ and $p = u\ell$ its Levi decomposition. Let $g \in G$. Let

$$p \cdot \underline{M}_g = p \cdot (V_k, C^\bullet_{V,k}, gD^{V,k}_\bullet, \varphi^g_\bullet) = (V_k, C^\bullet_{V,k}, pgD^{V,k}_\bullet, \psi_\bullet).$$

Since $L \subset \mathrm{Stab}(D^{V,k}_\bullet)$, we have $pgD^{V,k}_\bullet = u\ell g(\ell^{(q)})^{-1} D^{V,k}_\bullet$. By definition, the family $\psi_\bullet$ is the composite

$$(\mathrm{gr}^\bullet_{C_{V,k}})^{(q)} \overset{(p^{-1})^{(q)}}{\to} (\mathrm{gr}^\bullet_{C_{V,k}})^{(q)} \overset{\simeq}{\to} \mathrm{gr}^{D^{V,k}}_\bullet \\ \overset{g}{\to} \mathrm{gr}^{gD^{V,k}}_\bullet \overset{p}{\to} \mathrm{gr}^{pgD^{V,k}}_\bullet .$$

Since the map induced by p on the graded pieces are the same as the maps induced by ℓ, we get $\psi_\bullet = \varphi^{pg(\ell^{(q)})^{-1}}_\bullet$. This shows $p \cdot \underline{M}_g = \underline{M}_{p\cdot g}$. □

Consider the algebraic group $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$, where $\ell \in L$ acts on $\mathcal{R}_u P^+$ by conjugation and on $\mathcal{R}_u P^-$ by conjugation with $\ell^{(q)}$. We have an action of $\mathcal{R}_u P^-$ on G by multiplication on the right and an action of P^+ on G. These fit together to a left action of $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$ on G which is given by

$$(u, u', \ell) \cdot g = u\ell g(\ell^{(q)})^{-1} u'^{-1}.$$

Proposition 10.10. *The morphism f induces a bijection between the orbits of $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$ on G and the orbits of P^+ on $\tilde{X}^V_\tau$. This bijection preserves the closure order.*

Proof. By Lemma 10.9, the morphism f induces a map between the orbits in G and the orbits in X'_τ. Lemma 10.8 implies that this map is surjective. Let $g, g' \in G$ such that $f(g)$ and $f(g')$ lie in the same orbit under P^+. Then there exists $p \in P^+$ such that $\underline{M}_g = p \cdot \underline{M}_{g'} = \underline{M}_{p\cdot g'}$. Then, by Lemma 10.8 there exists $u' \in \mathcal{R}_u P^-$ such that $(p \cdot g')u'^{-1} = g$. Let $p = u\ell$ be the Levi decomposition of p. Then $u\ell g'(\ell^{(q)})^{-1} u'^{-1} = g$. This shows that the map induced by f is injective.

Lemma 10.8 implies that f factors through a morphism $\bar{f} \colon G/\mathcal{R}_u P^- \to \tilde{X}^V_\tau$ which must be bijective and hence a homeomorphism. Hence $\tilde{X}^V_\tau$ carries the quotient topology induced by f. This implies that the bijection preserves the closure order. □

Together, Propositions 10.7 and 10.10 show that the composite

$$G \overset{f}{\twoheadrightarrow} \tilde{X}^V_\tau \hookrightarrow X^V_\tau$$

induces a bijection between the orbits of the action of $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$ on G and the orbits of G on X_τ^V which preserves the closure order.

Let $Z^{(V_i)}$ be the algebraic zip datum $(G, P^+, P^-, L, L, \varphi)$, where $\varphi\colon \ell \mapsto \ell^{(q)}$ is the Frobenius homomorphism. The isomorphism of varieties $G \to G, g \mapsto g^{-1}$ is equivariant with respect to the action of $Z^{(V_i)}$ on the domain and the action of $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$ defined above on the codomain. This, together with Propositions 10.7 and 10.10 shows

Theorem 10.11. *The morphism $\psi^{(V_i)}$ sending $g \in G$ to $\underline{M}_{g^{-1}} \in X_\tau^V$ induces a bijective correspondence between the orbits of $Z^{(V_i)}$ on G and the orbits of G on X_τ^V which preserves the closure order.*

Corollary 10.12. *(i) Every F-zip over k is isomorphic to $\underline{M}_g$ for some $g \in G$.*

(ii) For $g, g' \in G$, the F-zips $\underline{M}_g$ and $\underline{M}_{g'}$ are isomorphic if and only if there exist $u^+ \in \mathcal{R}_u P^+$, $u^- \in \mathcal{R}_u P^-$ and $\ell \in L$ such that $g' = u^+ \ell g (\ell^{-1})^{(q)} u^-$.

Theorem 10.11 allows us to apply our results about the orbits of the action of an algebraic zip datum to the classification of F-zips. Let $V = \mathbb{F}_q^n$ and $(e_1, \ldots, e_n)$ the standard basis of V. For $i \in \mathbb{Z}$ let $d_i = \sum_{j \geq i} \tau(i)$ and V_i the span of $(e_{d_{i-1}+1}, \ldots, e_{d_i})$. Then $\dim V_i = \tau(i)$ and V is the direct sum of the V_i. We get an algebraic zip datum $Z := Z^{(V_i)} = (G, P^+, P^-, L, L, \varphi)$. In order to apply our results to classify the orbits of the action of Z, we use Proposition 5.2 to find a nice algebraic zip datum $\tilde{Z}$ having isomorphic orbit structure.

Let B be the group of upper triangular matrices and T be the group of diagonal matrices in $G = GL_n$. Then B is a Borel subgroup of P^+. Let $i_1 > \ldots > i_r$ be the support of τ and let $n_j = \tau(i_j)$ for $1 \leq j \leq r$. The Weyl group W can be identified with S_n such that $W(L)$ corresponds to $S_{n_1} \times \cdots S_{n_r}$. For $w \in W$, let $\dot{w} \in \mathrm{GL}_n(k)$ be the representative of w having only entries 0 and 1. Let $w_{0,I}$ be the longest element in $W_I = W(L)$. Let $w = w_0$ and $z = w_{0,I}$. Then

$$^{w}\varphi(^{z}(L \cap B)) = L \cap B$$

and

$$^{w}\varphi(^{z}T) = T,$$

so by Proposition 5.2 the algebraic zip datum

$$\tilde{Z} := (G, P^+, {}^{w_0}P^-, L, {}^{w_0}L, \mathrm{int}(w_0 w_{0,I}) \circ \varphi)$$

is nice with respect to T and B and the morphism $\psi\colon G \to G, g \mapsto w_0 g w_{0,I}$ maps the orbits of Z bijectively to the orbits of $\tilde{Z}$. Since φ is a Frobenius morphism, the algebraic zip datum $\tilde{Z}$ is Frobenius.

Let $x = w_0 w_{0,I}$.

Theorem 10.13. *(i) The set*

$$\{\underline{M}_{\dot{w}_{0,I}\dot{w}\dot{w}_0} \mid w \in {}^I W\}$$

is a set of representatives for the action of G on X_τ.

(ii) For $w, w' \in {}^I W$, the orbit of $\underline{M}_{\dot{w}_{0,I}\dot{w}'\dot{w}_0}$ is contained in the closure of the orbit of $\underline{M}_{\dot{w}_{0,I}\dot{w}\dot{w}_0}$ if and only if there exists $u \in W_I$ such that

$$uw'x\varphi(u^{-1})x^{-1} \leq w.$$

Proof. By Theorem 8.5, the set

$$\{\dot{w} \mid w \in W^I\}$$

is a set of representatives for the action of $\tilde{Z}$ on G. Both w_0 and $w_{0,I}$ are idempotent. Hence applying ψ shows that the set

$$\{\dot{w}_0\dot{w}\dot{w}_{0,I} \mid w \in W^I\}$$

is a set of representatives for the action of Z. Hence by Theorem 10.11 the set

$$\{\underline{M}_{\dot{w}_{0,I}\dot{w}^{-1}\dot{w}_0} \mid w \in W^I\}$$

is a set of representatives for the action of G on X_τ^V. Now (i) follows from

$$(W^I)^{-1} = {}^I W.$$

Since ψ is an isomorphism of varieties, it preserves the closure order. Hence, by Theorem 10.11, for $w, w' \in {}^I W$, the orbit of $\underline{M}_{\dot{w}_{0,I}\dot{w}'\dot{w}_0}$ is contained in the closure of the orbit of $\underline{M}_{\dot{w}_{0,I}\dot{w}\dot{w}_0}$ if and only if the orbit of $\dot{w}'^{-1}$ under $\tilde{Z}$ is contained in the closure of the orbit of w^{-1} under $\tilde{Z}$. By Theorem 8.6 this is the case if and only if there exists $u \in W_I$ such that

$$x\varphi(u)x^{-1}w'^{-1}u^{-1} \leq w^{-1}.$$

Since the Bruhat order satisfies

$$y \leq y' \text{ if and only if } y^{-1} \leq y'^{-1}$$

for all y and y' in W, this proves (ii). □

Remark 10.14. (i) was proven by Moonen and Wedhorn in [4] and (ii) was proven by Wedhorn in [9].

The automorphism group of a zip datum $\underline{M} \in X_\tau^V$ is its stabilizer in G. Hence we can get a description of this group using Theorem 8.8. For this, we need the following lemma.

Lemma 10.15. *For $g \in G$, the homomorphism*

$$\gamma \colon (\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L \to G$$
$$(u, u', \ell) \mapsto u\ell$$

restricts to an isomorphism

$$\mathrm{Stab}_{(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L}(g) \to \mathrm{Stab}_G(\underline{M}_g).$$

Proof. That γ restricts to a morphism

$$\gamma_g \colon \mathrm{Stab}_{(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L}(g) \to \mathrm{Stab}_G(\underline{M}_g)$$

follows from Lemma 10.9 and the definition of the action of $(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L$. If $(u, u', \ell) \in \mathrm{Stab}_{(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L}(g)$ and $(u, \tilde{u}', \ell) \in \mathrm{Stab}_{(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L}(g)$, then $u' = \tilde{u}'$. This implies the injectivity of γ_g.

Let $p \in \mathrm{Stab}_G(\underline{M}_g)$. Since $pC^\bullet_{V,k} = C^\bullet_{V,k}$, the element p must be in P^+. Since $\underline{M}_{p \cdot g} = p \cdot \underline{M}_g = \underline{M}_g$, Lemma 10.8 shows that there exists $u' \in \mathcal{R}_u P^-$ such that $(p \cdot g)u'^{-1} = g$. If $p = u\ell$ is the Levi decomposition of p, this shows

$$(u, u', \ell) \in \mathrm{Stab}_{(\mathcal{R}_u P^+ \times \mathcal{R}_u P^-) \rtimes L}(g).$$

Hence γ_g is surjective. ☐

Theorem 10.16. *Let $w \in {}^I W$. Let N_w be the unique maximal subgroup of L invariant under $\varphi_w := \mathrm{int}(\dot{w}\dot{w}_0\dot{w}_{0,I}) \circ \varphi$ and let N_w^f be the group of fixed points of φ_w, which is finite. Then the automorphism group of $\underline{M}_{\dot{w}_{0,I}\dot{w}\dot{w}_0}$ is the semidirect product of N_w^f and a connected unipotent group.*

Proof. Since $w \in {}^I W$, its inverse w^{-1} is in W^I and by definition $N_w = L_{w^{-1}}$ and $N_w^f = L_{w^{-1}}^f$. It follows from Proposition 5.2 that there is an isomorphism $\mathrm{Stab}_Z(\dot{w}_0\dot{w}^{-1}\dot{w}_{0,I}) \to \mathrm{Stab}_{\tilde{Z}}(\dot{w}^{-1})$. This, together with the preceding Lemma shows that the automorphism group of $\underline{M}_{\dot{w}_{0,I}\dot{w}\dot{w}_0}$ is isomorphic to $\mathrm{Stab}_{\tilde{Z}}(\dot{w}^{-1})$. Hence the claim follows from Theorem 8.8. ☐

Remark 10.17. That a result like Theorem 10.16 should hold was conjectured by Wedhorn in a conversation with the author.

Now we describe certain universal constructions for F-zips, and how these universal constructions can be realized as morphisms of the algebraic zip data which classify the F-zips of a certain type.

Definition 10.18. Let S be a scheme over $\mathbb{F}_q$ and $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ and $\underline{\tilde{M}} = (\tilde{M}, \tilde{C}^\bullet, \tilde{D}^\bullet, \tilde{\varphi}_\bullet)$ two F-zips over S. Then we get filtrations $(C \oplus \tilde{C})^\bullet$ and $(D \oplus \tilde{D})_\bullet$ on $M \oplus \tilde{M}$ by $(C \oplus \tilde{C})^i = C^i \oplus \tilde{C}^i$ and $(D \oplus \tilde{D})_i = D_i \oplus \tilde{D}_i$ for $i \in \mathbb{Z}$. There are natural isomorphisms $\mathrm{gr}^\bullet_{C \oplus \tilde{C}} \cong \mathrm{gr}^\bullet_C \oplus \mathrm{gr}^\bullet_{\tilde{C}}$ and $\mathrm{gr}^{D \oplus \tilde{D}}_\bullet \cong \mathrm{gr}^D_\bullet \oplus \mathrm{gr}^{\tilde{D}}_\bullet$. This allows to define an F-zip $\underline{M} \oplus \underline{\tilde{M}} = (M \oplus \tilde{M}, (C \oplus \tilde{C})^\bullet, (D \oplus \tilde{D})_\bullet, (\varphi \oplus \tilde{\varphi})_\bullet)$, which we call the *direct sum of $\underline{M}$ and $\underline{\tilde{M}}$*.

Definition 10.19. For two algebraic zip data $Z = (G, P, P', L, L', \varphi)$ and $\tilde{Z} = (\tilde{G}, \tilde{P}, \tilde{P}', \tilde{L}, \tilde{L}', \tilde{\varphi})$ we define their direct product to be the algebraic zip datum

$$Z \times \tilde{Z} := (G \times \tilde{G}, P \times \tilde{P}, P' \times \tilde{P}', L \times \tilde{L}, L' \times \tilde{L}', \varphi \times \tilde{\varphi}).$$

Now let τ and $\tilde{\tau}$ be two function $\mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with finite support and let $n = |\tau|$ and $\tilde{n} = |\tilde{\tau}|$. Let V and $\tilde{V}$ be two vector spaces of dimension n and $\tilde{n}$ respectively and let $V = \oplus_{i \in \mathbb{Z}} V_i$ and $\tilde{V} = \oplus_{i \in \mathbb{Z}} \tilde{V}_i$ be decompositions of V and $\tilde{V}$ such that $\dim V_i = \tau(i)$ and $\dim \tilde{V}_i = \tilde{\tau}(i)$ for $i \in \mathbb{Z}$. Then as above we get algebraic zip data $Z^{(V_i)}$ and $Z^{(\tilde{V}_i)}$ and morphisms $\psi^{(V_i)} \colon \mathrm{GL}(V) \to X_\tau^V$ and $\psi^{(\tilde{V}_i)} \colon \mathrm{GL}(\tilde{V}) \to X_{\tilde{\tau}}^{\tilde{V}}$ inducing bijections on the sets of orbits under the respective actions.

Since for two filtrations $C^\bullet$ and $\tilde{C}^\bullet$ of V_k of type τ and $\tilde{\tau}$ respectively, the type of $(C \oplus \tilde{C})^\bullet$ is $\tau + \tilde{\tau}$, the formation of direct sums gives a morphism $\oplus \colon X_\tau^V \times X_{\tilde{\tau}}^{\tilde{V}} \to X_{\tau+\tilde{\tau}}^{V \oplus \tilde{V}}$.

The decompositions of V and $\tilde{V}$ choosen above induce a decomposition of $V \oplus \tilde{V}$ as $V \oplus \tilde{V} = \oplus_{i \in \mathbb{Z}} (V_i \oplus \tilde{V}_i)$, from which we get an algebraic zip datum $Z^{(V_i \oplus \tilde{V}_i)}$ and a morphism $\psi^{(V_i \oplus \tilde{V}_i)} \colon \mathrm{GL}(V \oplus \tilde{V}) \to X_{\tau+\tilde{\tau}}$ which induces a bijection between the orbits of $Z^{(V_i \oplus \tilde{V}_i)}$ on $\mathrm{GL}(V \oplus \tilde{V})$ and the orbits of $\mathrm{GL}(V \oplus \tilde{V})$ on $X_{\tau+\tilde{\tau}}^{V \oplus \tilde{V}}$.

There is also the morphism $\oplus \colon \mathrm{GL}(V) \times \mathrm{GL}(\tilde{V}) \to \mathrm{GL}(V \oplus \tilde{V})$, which sends $(g, \tilde{g})$ to the automorphism $g \oplus \tilde{g}$ of $V \oplus \tilde{V}$ which maps $(v, \tilde{v})$ to $(gv, \tilde{g}\tilde{v})$. From $C_V^\bullet \oplus C_{\tilde{V}}^\bullet = C_{V \oplus \tilde{V}}^\bullet$ and $D_\bullet^V \oplus D_\bullet^{\tilde{V}} = D_\bullet^{V \oplus \tilde{V}}$ it follows that $\oplus$ is a morphism of zip data $Z^{(V_i)} \times Z^{(\tilde{V}_i)} \to Z^{(V_i \oplus \tilde{V}_i)}$ and that for $g \in \mathrm{GL}(V)$ and $\tilde{g} \in \mathrm{GL}(\tilde{V})$

$$\underline{M}_g \oplus \underline{M}_{\tilde{g}} = \underline{M}_{g \oplus \tilde{g}},$$

that is the diagram

$$\begin{array}{ccc} \mathrm{GL}(V) \times \mathrm{GL}(\tilde{G}) & \xrightarrow{\oplus} & \mathrm{GL}(V \oplus \tilde{V}) \\ \downarrow{\scriptstyle \psi^{(V_i)} \times \psi^{(\tilde{V}_i)}} & & \downarrow{\scriptstyle \psi^{(V_i \oplus \tilde{V}_i)}} \\ X_\tau^V \times X_{\tilde{\tau}}^{\tilde{V}} & \xrightarrow{\oplus} & X_{\tau+\tilde{\tau}}^{V \oplus \tilde{V}} \end{array}$$

commutes.

Definition 10.20. Let S be a scheme over $\mathbb{F}_q$ and $\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ and $\underline{\tilde{M}} = (\tilde{M}, \tilde{C}^\bullet, \tilde{D}^\bullet, \tilde{\varphi}_\bullet)$ two F-zips over S. The tensor product of $C^\bullet$ and $\tilde{C}^\bullet$ is defined to be the descending filtration on $M \otimes \tilde{M}$ given by

$$(C \otimes \tilde{C})^i = \sum_{n+n'=i} C^n \otimes \tilde{C}^{n'}.$$

The ascending filtration $(D \otimes \tilde{D})_\bullet$ is defined similarly. There are natural isomorphisms

$$\mathrm{gr}^i_{C \otimes \tilde{C}} \xrightarrow{\cong} \oplus_{n+n'=i} \mathrm{gr}^n_C \otimes \mathrm{gr}^{n'}_{\tilde{C}},$$

and similarly for $(D \otimes \tilde{D})_\bullet$. Hence if we let

$$(\varphi \otimes \tilde{\varphi})_i = \oplus_{n+n'=i} \varphi_n \otimes \tilde{\varphi}_{n'} \colon (\mathrm{gr}^i_{C \otimes \tilde{C}})^{(q)} \xrightarrow{\simeq} \mathrm{gr}_i^{D \otimes \tilde{D}},$$

we get an F-zip

$$\underline{M} \otimes \underline{\tilde{M}} := (M \otimes \tilde{M}, (C \otimes \tilde{C})^\bullet, (D \otimes \tilde{D})_\bullet, (\varphi \otimes \tilde{\varphi})_\bullet),$$

which we call the *tensor product of* $\underline{M}$ *and* $\underline{\tilde{M}}$.

Let $V = \oplus_{i \in \mathbb{Z}} V_i$ and $\tilde{V} = \oplus_{i \in \mathbb{Z}} \tilde{V}_i$ as above. Then we get a grading on $V \otimes \tilde{V}$ by $V \otimes \tilde{V} = \oplus_{i \in \mathbb{Z}} W_i$ with

$$W_i := \oplus_{n+n'=i} V_n \otimes \tilde{V}_{n'}.$$

Let $\tau \otimes \tilde{\tau} \colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}, i \mapsto \dim W_i$. Then analogously to the construction for direct sums above we get a morphism

$$\otimes \colon X^V_\tau \times X^{\tilde{V}}_{\tilde{\tau}} \to X^{V \otimes \tilde{V}}_{\tau \otimes \tilde{\tau}}$$

which sends two F-zips to their tensor product and a morphism of zip data

$$\otimes \colon Z^{(V_i)} \times Z^{(\tilde{V}_i)} \to Z^{(W_i)}$$

such that the diagram

$$\begin{array}{ccc} \mathrm{GL}(V) \times \mathrm{GL}(\tilde{G}) & \xrightarrow{\otimes} & \mathrm{GL}(V \otimes \tilde{V}) \\ \downarrow{\scriptstyle \psi^{(V_i)} \times \psi^{(\tilde{V}_i)}} & & \downarrow{\scriptstyle \psi^{(W_i)}} \\ X^V_\tau \times X^{\tilde{V}}_{\tilde{\tau}} & \xrightarrow{\otimes} & X^{V \otimes \tilde{V}}_{\tau + \tilde{\tau}} \end{array}$$

commutes.

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