Diss. ETH No. 19512

André-Oort Conjecture for Drinfeld Moduli Spaces

A dissertation submitted to
ETH ZURICH

for the degree of
Doctor of Sciences

presented by

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2011

Acknowledgements

I have profited a lot from Prof. Richard Pink's guidance. He suggested the topic of this project and took a lot of time to introduce me into the subject. In course of my PhD studies, he has always been a source for helpful ideas and suggestions which ensured the success of this thesis. I also benefited from his insistence on precise formulation of mathematical statements. For all this support during the past years I would like to express my deepest gratitude to him.

I want to thank Prof. Florian Breuer, Prof. Bruno Klingler and Prof. Gisbert Wüstholz for kindly accepting to be co-examiner and reading my work.

I am very grateful to Prof. Gebhard Böckle, Prof. Florian Breuer, Prof. Evelina Viada and Prof. Gisbert Wüstholz for giving me the opportunity to present my work and inviting me to international conferences.

A special thanks also goes to Philipp Habegger, Fritz Hörmann, Jonas Jermann and Anna von Pippich for their support, much helpful advice and interesting discussions during my doctoral studies.

Finally, I thank the Swiss National Science Foundation for the financial support of this PhD project.

Contents

Summary

In this thesis we consider the analogue of the André-Oort conjecture for Drinfeld modular varieties. This analogue was formulated by Breuer and says that every irreducible component of the Zariski closure of a set of special points in a Drinfeld modular variety is a special subvariety. Breuer proved it in the case where the given special points all lie in a curve and in the case where all special points have a certain behaviour at a fixed set of primes.

We extend the results of Breuer by proving the analogue for arbitrary sets of special points with separable reflex field over the base field. In particular, our result shows the correctness of the full analogue for Drinfeld modular varieties of rank coprime to the characteristic of the base field.

The proof of our result is an adaptation of the methods of Klingler and Yafaev in the classical case and consists of several steps of arithmetic and geometric nature:

- We show that, in any infinite family of Drinfeld modular subvarieties X of a Drinfeld modular variety, the degree of X is unbounded, where the degree of subvarieties is defined via the Satake compactification of a Drinfeld modular variety. We prove this using an explicit classification of Drinfeld modular subvarieties.

- We prove a geometric criterion for a Hodge-generic subvariety Z of a Drinfeld modular variety S to be equal to S. It says that Z is equal to S if Z is contained in a suitable Hecke translate of itself.

- We show the existence of primes satisfying certain technical conditions which are needed to construct a Hecke correspondence satisfying the assumptions in the above geometric criterion. This step uses an effective version of Čebotarev's theorem over function fields which relies on the correctness of the generalized Riemann conjecture over function fields.

- We finish the proof by induction using the above results.

Zusammenfassung

In der vorliegenden Arbeit betrachten wir das Analogon der André-Oort Vermutung für Drinfeld-Modulvarietäten. Dieses Analogon wurde von Breuer formuliert und besagt, dass jede irreduzible Komponente des Zariski-Abschlusses einer Menge von speziellen Punkten in einer Drinfeld-Modulvarietät eine spezielle Untervarietät ist. Breuer bewies es, falls die gegebenen speziellen Punkte alle in einer Kurve liegen, und im Fall, dass alle speziellen Punkte ein bestimmtes Verhalten an einer festen Menge von Stellen haben.

Wir erweitern die Resultate von Breuer, indem wir das Analogon für beliebige Mengen von speziellen Punkten mit separablem Reflexkörper über dem Grundkörper beweisen. Insbesondere zeigt unser Resultat die Richtigkeit des vollen Analogons für Drinfeld-Modulvarietäten, deren Rang teilerfremd zur Charakteristik des Grundkörpers ist.

Der Beweis unseres Resultats ist eine Anpassung der Methoden von Klingler und Yafaev im klassischen Fall und besteht aus mehreren Schritten von arithmetischer und geometrischer Natur:

- Wir zeigen, dass in jeder unendlichen Familie von Drinfeld-Moduluntervarietäten X einer Drinfeld-Modulvarietät der Grad von X unbeschränkt ist, wobei der Grad von Untervarietäten mit Hilfe der Satake-Kompaktifizierung einer Drinfeld-Modulvarietät definiert ist. Wir beweisen dies mit einer expiziten Klassifikation von Drinfeld-Untermodulvarietäten.

- Für eine Hodge-generische Untervarietät Z einer Drinfeld-Modulvarietät S beweisen wir ein geometrisches Kriterium, wann Z gleich S ist. Es besagt, dass Z gleich S ist, wenn Z in einem geeigneten Hecke-Translat von sich selber enthalten ist.

- Wir zeigen die Existenz von Stellen, die gewisse technische Bedingungen erfüllen. Diese verwenden wir um Hecke-Korrespondenzen zu konstruieren, die die Voraussetzungen des obigen geometrischen Kriteriums erfüllen. Dieser Schritt benutzt eine effektive Version des Satzes von Čebotarev über Funktionenkörpern, die auf der Richtigkeit der verallgemeinerten Riemann-Vermutung über Funktionenkörpern beruht.

- Wir beenden den Beweis mit Induktion mit Hilfe obiger Resultate.

Introduction

The André-Oort conjecture

The André-Oort conjecture asserts that every irreducible component of the Zariski closure of a set of special points in a Shimura variety is a special subvariety. This remarkable statement arose from research of André and Oort about the distribution of CM points in moduli spaces of abelian varieties in the late 1980's and the 1990's.

In the late 1990's, Edixhoven proved the conjecture for Hilbert modular surfaces and products of modular curves assuming the generalized Riemann hypothesis (GRH) in [11] and [12]. Both proofs exploit the Galois action on special points and use geometric properties of Hecke correspondences. In the special case of a product of two modular curves, André [2] gave a proof without assuming GRH.

These methods were extended in [13] by Edixhoven and Yafaev to prove the conjecture for curves in general Shimura varieties containing infinitely many special points all lying in the same Hecke orbit. Subsequently, Yafaev [37] also proved the conjecture for general curves assuming GRH.

Recently, Klingler, Ullmo and Yafaev have announced a proof of the full André-Oort conjecture assuming GRH, see [24] and [36]. Their methods use a combination of the methods of Edixhoven and Yafaev and equidistribution results of Clozel and Ullmo [7] established by ergodic theoretic methods.

For a more detailed exposition of results concerning the André-Oort conjecture for Shimura varieties, we refer to the survey article of Noot [29].

Drinfeld modular varieties

Drinfeld modular varieties are a natural analogue of Shimura varieties in the function field case. They are moduli spaces for Drinfeld A-modules over a global function field F of a given rank r with some level structure, where A is the ring

of elements of F that are regular outside of a fixed place ∞.

As for Shimura varieties, there is an analytic description of a Drinfeld modular variety S as a double quotient. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of the completion F_∞ of F and let $\mathbb{A}_F^f$ be the adeles of F outside of ∞. There is a natural rigid-analytic isomorphism

$$S(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}), \tag{0.0.1}$$

where $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is an open compact subgroup, called *level*, and Ω_F^r denotes Drinfeld's upper half-space obtained by removing all F_∞-rational hyperplanes from $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. In this situation, the Drinfeld modular variety S is of dimension $r-1$ and S is denoted by $S_{F,\mathcal{K}}^r$.

Also along the same lines as for Shimura varieties, one can define *Hecke correspondences* on Drinfeld modular varieties. These are finite algebraic correspondences defined over the base field F.

Special subvarieties and Drinfeld modular subvarieties

One can define special subvarieties of a Drinfeld modular variety $S = S_{F,\mathcal{K}}^r$ parametrising Drinfeld A-modules of rank r in analogy to the case of Shimura varieties. For each finite extension F' of F of degree r/r' with only one place above ∞ and integral closure A' of A in F', the restriction of Drinfeld A'-modules to A gives a morphism from the moduli space of Drinfeld A'-modules of rank r' (with a certain level structure) to S defined over F'. These morphisms are analogues of morphisms induced by a Shimura subdatum. A *special subvariety* V is defined to be a geometric irreducible component of a Hecke translate of the image of such a morphism. A *special point* is a special subvariety of dimension 0.

In fact, we can interpret each special subvariety as a geometric irreducible component of a *Drinfeld modular subvariety*. A Drinfeld modular subvariety X is the image of the composition of an above morphism defined by the restriction of Drinfeld A'-modules to A with a morphism given by a Hecke correspondence. Such a composition, called *inclusion morphism*, is associated to an extension F'/F of the above type and an $\mathbb{A}_F^f$-linear isomorphism $b : (\mathbb{A}_F^f)^r \xrightarrow{\cong} (\mathbb{A}_{F'}^f)^{r'}$ encoding the involved Hecke correspondence. We say that F' is the *reflex field* of X and its geometric irreducible components.

A Drinfeld modular subvariety X with reflex field F' is irreducible over F'. So if a special subvariety V is a geometric irreducible component of X, the union of the Galois conjugates of V over F' is equal to X.

André-Oort Conjecture for Drinfeld modular varieties

The following analogue of the André-Oort conjecture was formulated by Breuer in [5]:

Conjecture 1. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and Σ a set of special points in S. Then each irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of Σ is a special subvariety of S.*

Breuer [5] proved this analogue in two cases. Firstly, when the Zariski closure of Σ is a curve, and secondly when all special points in Σ have a certain behaviour at a fixed set of primes. Before, he proved an analogue of the André-Oort conjecture for products of modular curves in odd characteristic in [4]. These proofs use an adaptation of the methods of Edixhoven and Yafaev in [13], [12] and [37]. The results are unconditional because GRH holds over function fields.

In this thesis, we extend the arguments of Breuer using an adaptation of the methods of Klingler and Yafaev in [24]. Our main result is the following theorem:

Theorem 2. *Conjecture 1 is true if the reflex fields of all special points in Σ are separable over F.*

Since the reflex field of a special point in a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ is of degree r over F, special points with inseparable reflex field over F can only occur if r is divisible by $p = \mathrm{char}(F)$. So Theorem 2 implies

Theorem 3. *Conjecture 1 is true if r is not a multiple of $p = \mathrm{char}(F)$.*

Compactification of Drinfeld modular varieties and degree of subvarieties

In [30], Pink constructs the *Satake compactification* $\overline{S}^r_{F,\mathcal{K}}$ of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$. It is a normal projective variety over F which contains $S^r_{F,\mathcal{K}}$ as an open subvariety.

If $\mathcal{K}$ is sufficiently small in a certain sense, there is a natural ample invertible sheaf on $\overline{S}^r_{F,\mathcal{K}}$. We assume this condition on $\mathcal{K}$ in the following because the proof of Theorem 2 can be easily reduced to this case. So we can define the *degree* of a subvariety of $S^r_{F,\mathcal{K}}$ as the degree of its Zariski closure in $\overline{S}^r_{F,\mathcal{K}}$ with respect to this ample invertible sheaf. The degree of a subvariety can be seen as a measure for the "complexity" of the subvariety.

The construction of the natural ample invertible sheaf on the compactification is compatible with inclusion morphisms and the morphisms appearing in Hecke correspondences. Therefore we can control the degree of Drinfeld modular subvarieties and Hecke translates of subvarieties.

Reductions in the proof of Theorem 2

We need to show that a geometrically irreducible subvariety Z of S containing a Zariski dense subset of special points with separable reflex field over F is a special subvariety. By the separability assumption, Z is defined over a finite separable extension of F and the union of its Galois conjugates over F is defined and irreducible over F. Since the union of the Galois conjugates over F of a special point is a finite union of Drinfeld modular subvarieties of dimension 0, we are reduced to showing the following statement for $d = 0$:

Theorem 4. *Let Σ be a set of Drinfeld modular subvarieties of S of dimension d with separable reflex field over F whose union is Zariski dense in a subvariety $Z \subset S$ which is defined and irreducible over F. Then Z is a finite union of Drinfeld modular subvarieties of S.*

By descending induction on d, this follows from the following crucial statement:

Theorem 5. *Let Σ and Z be as in Theorem 4 with $d < \dim Z$. Then, for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.*

In [24], Klingler and Yafaev perform the same induction, however they work with special subvarieties instead of Drinfeld modular subvarieties.

Sketch of the proof of Theorem 5

For the proof of Theorem 5 we assume that Z is *Hodge-generic*. This means that no geometric irreducible component of Z is contained in a proper Drinfeld modular subvariety of S. We can make this assumption because otherwise we can replace S by a smaller Drinfeld modular variety.

Degree of Drinfeld modular subvarieties

For a Drinfeld modular subvariety X which is the image of $S^{r'}_{F',\mathcal{K}'}$ under an inclusion morphism, we show that the product $D(X)$ (called the predegree of X)

of the index of $\mathcal{K}'$ in a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ with the class number of F' is up to a constant a lower bound for the degree of X.

We give a classification of the Drinfeld modular subvarieties of S and then use it to show that the predegree $D(X)$ is unbounded as X ranges through an infinite set of Drinfeld modular subvarieties. This gives the following result:

Theorem 6. *If Σ is an infinite set of Drinfeld modular subvarieties of S, then $\deg X$ is unbounded as X varies over Σ.*

Note that, for a special subvariety V which is a geometric irreducible component of a Drinfeld modular subvariety X, the union of the Galois conjugates of V over its reflex field is equal to X. Therefore, $\deg X$ measures both the degree of V and the number of Galois conjugates of V. So our unboundedness statement tells us that it is not possible that in an infinite family of special subvarieties V, the degrees and the number of Galois conjugates of V are both bounded. Since we can exclude this case, we only need an adaptation of the Galois-theoretic and geometric methods in [24] and do not need equidistribution results as in [7].

Geometric criterion

Recall that we assume that Z is Hodge-generic and irreducible over F. These two assumptions allow us to deduce a geometric criterion for Z being equal to S. It is a key ingredient of our proof of Theorem 5 and says that Z is equal to the whole of S provided that Z is contained in a suitable Hecke translate of itself. A similar geometric criterion appears in the proof of Klingler and Yafaev in the classical case.

Theorem 7. *Suppose that $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$ and assume that $Z \subset T_{g_\mathfrak{p}} Z$ for some $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$. If, for all $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$, the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $k_1 \cdot g_\mathfrak{p} \cdot k_2$ is unbounded, then $Z = S$.*

The proof of this theorem is based on two results:

(i) Zariski density

We define the $T_{h_\mathfrak{p}} + T_{h_\mathfrak{p}^{-1}}$-orbit of a geometric point $x \in S(\mathbb{C}_\infty)$ to be the smallest subset of $S(\mathbb{C}_\infty)$ containing x which is invariant under $T_{h_\mathfrak{p}}$ and $T_{h_\mathfrak{p}^{-1}}$. Using the rigid-analytic structure of $S(\mathbb{C}_\infty)$ given by (0.0.1), we show that the $T_{h_\mathfrak{p}} + T_{h_\mathfrak{p}^{-1}}$-orbit of an arbitrary point $x \in S(\mathbb{C}_\infty)$ is Zariski dense in the geometric irreducible component of S containing x provided that $h_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ is chosen such that the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $h_\mathfrak{p}$ is unbounded.

(ii) A result of Pink [31, Theorem 0.1] on the Galois representations associated to Drinfeld modules implies that the image of the arithmetic étale fundamental group of a geometric irreducible component of Z is open in $\mathrm{GL}_r(F_\mathfrak{p})$, see Theorem 4 in [6]. Here we need our assumption that Z is Hodge-generic.

We deduce from (ii) and the assumption $Z \subset T_{g_\mathfrak{p}} Z$ that, in some finite cover of S, some $T_{h_\mathfrak{p}} + T_{h_\mathfrak{p}^{-1}}$-orbit is contained in an irreducible component of the preimage of Z, where $h_\mathfrak{p} = (k_1 g_\mathfrak{p} k_2)^n$ for suitable $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$ and $n \geq 1$. With the unboundedness assumption in the theorem and (i) we then conclude that Z is equal to the whole of S.

Induction

Our final step of the proof of Theorem 5 consists of an induction which uses a Hecke correspondence with specific properties. Precisely, we prove the following statement by induction:

Theorem 8. *Let X be a Drinfeld modular subvariety of S associated to F'/F and $b : (\mathbb{A}_F^f)^r \xrightarrow{\cong} (\mathbb{A}_{F'}^f)^{r'}$ and assume that X is contained in a Hodge-generic subvariety $Z \subset S$ which is irreducible over F.*

Suppose that $T_{g_\mathfrak{p}}$ is a Hecke correspondence localized at a prime $\mathfrak{p}$ with the following properties:

(i) *$g_\mathfrak{p}$ is defined by some $g'_{\mathfrak{p}'} \in \mathrm{GL}_{r'}(F'_{\mathfrak{p}'})$ where $\mathfrak{p}'$ is a prime of F' lying over $\mathfrak{p}$, i.e., $g_\mathfrak{p} = b^{-1} \circ g'_{\mathfrak{p}'} \circ b$.*

(ii) *$g_\mathfrak{p}$ satisfies the unboundedness condition in Theorem 7, i.e., $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$ and, for all $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$, the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $k_1 \cdot g_\mathfrak{p} \cdot k_2$ is unbounded,*

(iii) *If $\iota : S' \to S$ is an inclusion morphism with $X \subset \iota(S')$, then the Hecke correspondence T' on S' defined by $g'_{\mathfrak{p}'}$ satisfies (ii) and $\deg T' = \deg T_{g_\mathfrak{p}}$,*

(iv) *$\deg X > \deg(T_{g_\mathfrak{p}})^{2^s-1} \cdot (\deg Z)^{2^s}$ for $s := \dim Z - \dim X$.*

Then there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.

We perform an induction over $s := \dim Z - \dim X$. Property (i) implies $X \subset T_{g_\mathfrak{p}} X$, in particular we therefore have

$$X \subset Z \cap T_{g_\mathfrak{p}} Z.$$

The lower bound (iv) for $\deg X$ now says that X cannot be a union of geometric irreducible components of $Z \cap T_{g_\mathfrak{p}} Z$. Therefore we find an irreducible component Z' over F of $Z \cap T_{g_\mathfrak{p}} Z$ with $X \subset Z'$ and $\dim Z' > \dim X$. There are two cases:

If $Z' = Z$, we have $Z \subset T_{g_\mathfrak{p}} Z$ and conclude by Theorem 7 that $Z = S$, so the conclusion of Theorem 5 is true with $X' = S$.

If $Z' \subsetneq Z$, then $\dim Z' < \dim Z$ because Z is irreducible over F. We replace Z by Z' and apply the induction hypothesis. In this step, it is possible that Z' is not Hodge-generic any more. In this case, we replace S by a smaller Drinfeld modular variety S' and show that (i)-(iv) are still valid in S' using our property (iii).

Choice of a suitable Hecke correspondence

To finish the proof of Theorem 5, by Theorem 8 we need to show that, for almost all $X \in \Sigma$, there is a Hecke correspondence $T_{g_\mathfrak{p}}$ localized at a prime $\mathfrak{p}$ with the properties (i)-(iv). To construct such a $T_{g_\mathfrak{p}}$ for a $X \in \Sigma$, we need the prime $\mathfrak{p}$ to satisfy specific conditions under which we call the prime *good* for X:

Definition 9. *Let X be a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ associated to F'/F and $b : (\mathbb{A}_F^f)^r \xrightarrow{\cong} (\mathbb{A}_{F'}^f)^{r'}$. A prime $\mathfrak{p}$ of F is called* good *for $X \subset S^r_{F,\mathcal{K}}$ if there is an $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that the following holds for the $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} := s_\mathfrak{p} \cdot A_\mathfrak{p}^r$:*

(a) $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ *where* $\mathcal{K}_\mathfrak{p} = s_\mathfrak{p} \mathcal{K}(\mathfrak{p}) s_\mathfrak{p}^{-1}$ *for the principal congruence subgroup* $\mathcal{K}(\mathfrak{p})$ *of* $\mathrm{GL}_r(A_\mathfrak{p})$,

(b) $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ *is an* $A' \otimes_A A_\mathfrak{p}$*-submodule of* $(A' \otimes_A A_\mathfrak{p})^{r'}$,

(c) there exists a prime $\mathfrak{p}'$ *of* F' *above* $\mathfrak{p}$ *with local degree* 1 *over* F.

Theorem 10. *If $\mathfrak{p}$ is a good prime for a Drinfeld modular subvariety $X \subset S^r_{F,\mathcal{K}}$, then there is a Hecke correspondence $T_{g_\mathfrak{p}}$ localized at $\mathfrak{p}$ satisfying (i)-(iii) from Theorem 8 with*

$$\deg T_{g_\mathfrak{p}} = |k(\mathfrak{p})|^{r-1},$$

where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$.

We show this theorem by defining

$$g_\mathfrak{p} := s_\mathfrak{p} \mathrm{diag}(\pi_\mathfrak{p}^{-1}, 1, \ldots, 1) s_\mathfrak{p}^{-1}$$

for a uniformizer $\pi_\mathfrak{p}$ at $\mathfrak{p}$. In the proof, it is crucial that $\mathcal{K}_\mathfrak{p}$ is *not* a maximal compact subgroup of $\mathrm{GL}_r(F_\mathfrak{p})$, which is guaranteed by condition (a) in the definition of good prime. Otherwise we are not able to satisfy the unboundedness condition (ii).

However, (a) is a very strict condition on the prime $\mathfrak{p}$: For a fixed level $\mathcal{K}$ it can only be satisfied at most at a finite set of primes because $\mathcal{K}$ is maximal compact at almost all primes. Since (b) and (c) are both satisfied only for an infinite set of primes of density smaller than one, for a fixed level $\mathcal{K}$, in general we cannot find a prime $\mathfrak{p}$ satisfying (a)-(c). We get rid of this problem by starting with a prime $\mathfrak{p}$ for which there is an $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that

(a') $\mathcal{K} = s_\mathfrak{p}\mathrm{GL}_r(A_\mathfrak{p})s_\mathfrak{p}^{-1} \times \mathcal{K}^{(\mathfrak{p})}$

and also (b) and (c) are satisfied. We can find such a prime because (a') is satisfied for some $s_\mathfrak{p}$ for all but finitely many primes. With an effective version of Čebotarev's theorem which relies on the correctness of GRH for function fields we can even show that such a prime satisfying an upper bound for its degree exists provided that $\deg X$ is large enough:

Theorem 11. *For all $N > 0$, there is a $d_N > 0$ such that, for all Drinfeld modular subvarieties X of $S^r_{F,\mathcal{K}}$ with $\deg X \geq d_N$, there is a prime $\mathfrak{p}$ of F and a $s_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ such that the following holds where $\Lambda_\mathfrak{p} := s_\mathfrak{p} \cdot A^r_\mathfrak{p}$ and X is associated to F'/F and $b : (\mathbb{A}^f_F)^r \xrightarrow{\cong} (\mathbb{A}^f_{F'})^{r'}$:*

(a') $\mathcal{K} = s_\mathfrak{p}\mathrm{GL}_r(A_\mathfrak{p})s_\mathfrak{p}^{-1} \times \mathcal{K}^{(\mathfrak{p})}$,

(b) $b_\mathfrak{p}(\Lambda_\mathfrak{p})$ *is an* $A' \otimes_A A_\mathfrak{p}$*-submodule of* $(A' \otimes_A A_\mathfrak{p})^{r'}$,

(c) *there exists a prime* $\mathfrak{p}'$ *of* F' *above* $\mathfrak{p}$ *with local degree* 1 *over* F,

(d) $|k(\mathfrak{p})|^N < \deg X$.

We conclude the proof of Theorem 5 as follows. We choose a certain $N > 0$ and, for all $X \in \Sigma$ with $\deg X \geq d_N$, we choose a prime $\mathfrak{p}$ satisfying the properties in Theorem 11. Since $\deg X$ is unbounded as X ranges over Σ by Theorem 6, this works for almost all $X \in \Sigma$. We then make $\mathcal{K}_\mathfrak{p}$ smaller by passing to a finite cover of S. More precisely, we consider the Drinfeld modular variety $\tilde{S} := S^r_{F,\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} = s_\mathfrak{p}\mathcal{K}(\mathfrak{p})s_\mathfrak{p}^{-1} \times \mathcal{K}^{(\mathfrak{p})}$ which is a finite cover of $S = S^r_{F,\mathcal{K}}$. The conditions (a)-(c) from Definition 9 are now satisfied for some Drinfeld modular subvariety $\tilde{X}$ of $\tilde{S}$ lying over X, i.e., $\mathfrak{p}$ is a good prime for $\tilde{X} \subset \tilde{S}$.

By Theorem 10, we then find a Hecke correspondence $T_{g_\mathfrak{p}}$ on $\tilde{S}$ localized at $\mathfrak{p}$ satisfying (i)-(iii) from Theorem 8 for $\tilde{X} \subset \tilde{S}$. Furthermore, with property (d) of $\mathfrak{p}$ we can show that (iv) is also satisfied for $\tilde{X} \subset \tilde{S}$ and some irreducible

component $\tilde{Z}$ over F of the preimage of Z in $\tilde{S}$ provided that $\deg X$ is large enough and we have chosen $N > 0$ suitably before.

Since $\deg X$ is unbounded as X ranges over Σ by Theorem 6, with Theorem 8 we therefore get a Drinfeld modular subvariety $\tilde{X}'$ of $\tilde{S}$ with $\tilde{X} \subsetneq \tilde{X}' \subsetneq \tilde{Z}$ for almost all $X \in \Sigma$. The image $X' \subset S$ of $\tilde{X}'$ under the covering map $\tilde{S} \to S$ then satisfies the conclusion of Theorem 5.

Difficulties in the inseparable case

Unfortunately, the above methods do not work in the inseparable case, i.e., if Σ in Theorem 4 contains Drinfeld modular subvarieties of S with inseparable reflex field. This is caused by the fact that every prime ramifies in an inseparable field extension. Therefore, for a Drinfeld modular subvariety with inseparable reflex field, there is no prime for which condition (c) in Definition 9 is satisfied. So we cannot apply Theorem 10 to find a Hecke correspondence satisfying (i)-(iii) from Theorem 8.

Also other approaches to find such Hecke correspondences fail. For example, if X is a Drinfeld modular subvariety of dimension 0 with purely inseparable reflex field F'/F and $\mathfrak{p}$ any prime of F, then a Hecke correspondence $T_{g_\mathfrak{p}}$ localized at $\mathfrak{p}$ satisfying (i) of Theorem 8 does not satisfy the unboundedness condition (ii) in Theorem 8: Indeed, in this case there is exactly one prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with ramification index r and, if $\pi_{\mathfrak{p}'} \in F'_{\mathfrak{p}'}$ is a uniformizer, then $1, \pi_{\mathfrak{p}'}, \ldots, \pi_{\mathfrak{p}'}^{r-1}$ is an $F_\mathfrak{p}$-basis of $F'_{\mathfrak{p}'}$. Therefore, if $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$ is defined by $g'_{\mathfrak{p}'} = \pi_{\mathfrak{p}'}^k \in \mathrm{GL}_1(F'_{\mathfrak{p}'})$ as in (i) of Theorem 8, then $g_\mathfrak{p}$ is a conjugate of the matrix

$$\begin{pmatrix} & & & \pi_\mathfrak{p} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}^k \in \mathrm{GL}_r(F_\mathfrak{p})$$

for $\pi_\mathfrak{p} := \pi_{\mathfrak{p}'}^r$. Its r-th power is a scalar matrix, hence the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $g_\mathfrak{p}$ is bounded and we cannot apply our geometric criterion (Theorem 7) for the Hecke correspondence $T_{g_\mathfrak{p}}$.

Outline of the thesis

In Chapter 0, we introduce some notation and conventions and discuss a few algebro-geometric preliminaries.

In Chapter 1, we define *Drinfeld modular varieties* for arbitrary level $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ as quotients of fine moduli schemes of Drinfeld modules.

In Chapter 2, we first define *projection morphisms* and *Hecke correspondences* on Drinfeld modular varieties. Then we define *inclusion morphisms* of Drinfeld modular varieties which allow us to define *Drinfeld modular subvarieties* and *special subvarieties* of a Drinfeld modular variety S. Subsequently, we discuss a criterion under which two Drinfeld modular subvarieties are contained in each other and give a classification of the Drinfeld modular subvarieties of S. Finally, we show that the absolute Galois group naturally acts on the set of Drinfeld modular subvarieties of S and describe the Galois action on the irreducible components of S.

In Chapter 3, we define the *degree* of subvarieties of a Drinfeld modular variety using the Satake compactification constructed in [30] and give estimates for the degree of Hecke translates of subvarieties. We then show that, in any infinite family of Drinfeld modular subvarieties X of S, the degree of X is unbounded (Theorem 6). Here we need our classification of Drinfeld modular subvarieties from the previous chapter.

The next two chapters are devoted to the proof of our geometric criterion for being a Drinfeld modular subvariety (Theorem 7). Chapter 4 deals with Zariski density of $T_g + T_{g^{-1}}$-orbits and in Chapter 5 we give the proof of the actual criterion.

In Chapter 6, we first define *good primes* for Drinfeld modular subvarieties. We then explain, for a fixed Drinfeld modular subvariety, how we can find a suitable Hecke correspondence at a good prime as in Theorem 10. The last section of Chapter 6 is devoted to find a good prime for a given Drinfeld modular subvariety of large enough degree in some finite cover of S (Theorem 11).

In Chapter 7, we finally conclude the proof of Theorem 5 by proving Theorem 8 and applying the results of the previous chapters. Here we also explain why Theorem 5 implies our main result (Theorem 2).

Chapter 0

Preliminaries

0.1 Notation and conventions

The following notation and conventions will be used throughout this thesis:

$|M|$ denotes the cardinality of a set M.

$\mathbb{F}_q$ denotes a fixed finite field with q elements.

For an $\mathbb{F}_q$-algebra R, we denote by $R\{\tau\}$ the ring of non-commutative polynomials in the variable τ with coefficients in R and the commutator rule $\tau\lambda = \lambda^q\tau$ for $\lambda \in R$.

F always denotes a global function field of characteristic p with field of constants $\mathbb{F}_q$ and ∞ a fixed place of F.

For a pair (F, ∞), we use the following notation:

A	ring of elements of F regular outside ∞
F_∞	completion of F at ∞
$\mathbb{C}_\infty$	completion of an algebraic closure of F_∞
F^{sep}	separable closure of F inside $\mathbb{C}_\infty$
$\mathbb{A}_F^f$	ring of finite adeles of F (i.e., adeles outside ∞)
$\hat{A}$	profinite completion of A
$\mathrm{Cl}(F)$	class group of A
$g(F)$	genus of F
Ω_F^r	Drinfeld's upper half-space of dimension $r-1$ over F

A place $\mathfrak{p} \neq \infty$ of F is said to be a *prime* of F. We identify a prime $\mathfrak{p}$ of F with a prime ideal of A and denote its residue field by $k(\mathfrak{p})$. The completion of F at $\mathfrak{p}$ is denoted by $F_\mathfrak{p}$, its discrete valuation ring by $A_\mathfrak{p}$ and the ring of finite

adeles of F outside $\mathfrak{p}$ by $\mathbb{A}_F^{f,\mathfrak{p}}$.

We often consider a finite extension F' of F with exactly one place ∞' over ∞. We use the analogous notations for such extensions, e.g., A', $F'_{\infty'}$. Note that the above used algebraic closure of F_∞ is also an algebraic closure of $F'_{\infty'}$. Therefore, we can and do assume $\mathbb{C}_\infty = \mathbb{C}_{\infty'}$.

For a place $\mathfrak{p}$ of F, we set $F'_\mathfrak{p} := F' \otimes F_\mathfrak{p}$ and $A'_\mathfrak{p} := A' \otimes A_\mathfrak{p}$. Since F is a global field, there are canonical isomorphisms

$$\begin{aligned} F'_\mathfrak{p} &\xrightarrow{\sim} \prod_{\mathfrak{p}'|\mathfrak{p}} F'_{\mathfrak{p}'} \\ A'_\mathfrak{p} &\xrightarrow{\sim} \prod_{\mathfrak{p}'|\mathfrak{p}} A'_{\mathfrak{p}'} \end{aligned}$$

of $F_\mathfrak{p}$- resp. $A_\mathfrak{p}$-algebras. We identify $F'_\mathfrak{p}$ with $\prod_{\mathfrak{p}'|\mathfrak{p}} F'_{\mathfrak{p}'}$ and $A'_\mathfrak{p}$ with $\prod_{\mathfrak{p}'|\mathfrak{p}} A'_{\mathfrak{p}'}$ via these isomorphisms. Furthermore, for a place $\mathfrak{p}'$ over $\mathfrak{p}$, we denote by $(F'_\mathfrak{p})^{(\mathfrak{p}')}$ resp. $(A'_\mathfrak{p})^{(\mathfrak{p}')}$ the product of all $F'_\mathfrak{q}$ resp. $A'_\mathfrak{q}$ with $\mathfrak{q}|\mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}'$.

For $r, r' \geq 1$ with $r = r' \cdot [F'/F]$, we often consider an isomorphism of F-vector spaces $\varphi : F^r \xrightarrow{\sim} F'^{r'}$. In this situation, extending scalars to $F_\mathfrak{p}$ for a place $\mathfrak{p}$ of F and to $\mathbb{A}_F^f$, we get isomorphisms

$$\begin{aligned} F_\mathfrak{p}^r &\xrightarrow{\varphi} F_\mathfrak{p}'^{r'} \\ (\mathbb{A}_F^f)^r &\xrightarrow{\varphi} (\mathbb{A}_{F'}^f)^{r'} \end{aligned},$$

which we also denote by φ (by a slight abuse of notation).

For a second finite extension F'' of F with exactly one place ∞'' above ∞, we use the analogous conventions and notations.

For the formulation of algebro-geometric results, we use the following conventions:

- For a scheme X over a field K and a field extension L of K, we write X_L for its base extension $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$.
- Unless otherwise stated, *variety* means a reduced separated scheme of finite type over $\mathbb{C}_\infty$ and *subvariety* means a reduced closed subscheme of a variety.
- Since $\mathbb{C}_\infty$ is algebraically closed, we can and do identify the set $X(\mathbb{C}_\infty)$ of $\mathbb{C}_\infty$-valued points of a variety X with the set of its closed points.
- For a subfield $K \subset \mathbb{C}_\infty$, a variety X together with a scheme X_0 of finite type over K and an isomorphism of schemes $\alpha_X : X_{0,\mathbb{C}_\infty} \xrightarrow{\sim} X$ is called

a *variety over* K. We often write X in place of (X, X_0, α_X) and identify $X_{0,\mathbb{C}_\infty}$ with X via α_X if this leads to no confusion. Note that in this case X is also a variety over K' for any intermediate field K' of $K \subset \mathbb{C}_\infty$ because of $(X_{0,K'})_{\mathbb{C}_\infty} \cong X_{0,\mathbb{C}_\infty}$.

- Let X' and X be two varieties over K. A morphism $X' \to X$ of varieties over $\mathbb{C}_\infty$ which is the base extension to $\mathbb{C}_\infty$ of a morphism $X'_0 \to X_0$ of schemes over K is called a *morphism of varieties over* K.

- For a variety $X = X_{0,\mathbb{C}_\infty}$ over K, a subvariety X' together with a closed subscheme X'_0 of X_0 and an isomorphism $X'_{0,\mathbb{C}_\infty} \xrightarrow{\sim} X'$ such that $X'_{0,\mathbb{C}_\infty} \xrightarrow{\sim} X' \hookrightarrow X = X_{0,\mathbb{C}_\infty}$ is the base extension to $\mathbb{C}_\infty$ of $X'_0 \hookrightarrow X_0$ is said to be *defined over* K. Again, we often only write X' and identify $X'_{0,\mathbb{C}_\infty}$ and X' via the given isomorphism.

- For a variety $X = X_{0,\mathbb{C}_\infty}$ over K and a subfield $K' \subset \mathbb{C}_\infty$ containing K, we denote by $X(K')$ the set of K'-valued points of X_0, i.e.,

$$X(K') := X_0(K') = \mathrm{Mor}_K(\mathrm{Spec}(K'), X_0).$$

  Note that $X(K')$ is naturally a subset of the set of closed points of X via the natural inclusions and identifications

$$X(K') := X_0(K') \subset X_0(\mathbb{C}_\infty) = X(\mathbb{C}_\infty) = \{\text{closed points of } X\},$$

  in fact it is equal to the set of closed points of X defined over K', see, e.g., p. 26 of [3].

- A variety $X = X_{0,\mathbb{C}_\infty}$ over K is called K-irreducible if X_0 is an irreducible scheme over K.

- The *degree* of a finite surjective morphism $X \to Y$ of irreducible varieties (over $\mathbb{C}_\infty$) is defined to be the degree of the extension of the function fields $\mathbb{C}_\infty(X)/\mathbb{C}_\infty(Y)$. We say that a finite surjective morphism $f : X \to Y$ of (not necessarily irreducible) varieties is of degree d if for each irreducible component Z of Y

$$\sum_{\text{irr. components } X_i \text{ of } f^{-1}(Z)} \deg(f|X_i : X_i \to Z) = d.$$

  For a surjective finite morphism $f : X \to Y$ of varieties of degree d, this definition implies the equality

$$f_*([X]) = d \cdot [Y]$$

of cycles on Y (see, e.g., Section 1.4 of [15] for the definition of the push-forward of cycles).

If f is in addition flat, then $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module by Proposition III.9.2 (e) in [22]. By localization at the generic points of the irreducible components of Y, we see that $f_*\mathcal{O}_X$ is locally free of rank $d = \deg f$.

Remark: We could also formulate our results in the language of classical algebraic geometry. However, it turns out to be more convenient to use the language of schemes instead.

For a subfield $K \subset \mathbb{C}_\infty$ we denote by K^{sep} the separable and by $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. Since the field extension $\overline{K}/K^{\mathrm{sep}}$ is purely inseparable, each K-automorphism of K^{sep} has a unique continuation to a K-automorphism of $\overline{K}$. Therefore, we can and do identify the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ with the automorphism group $\mathrm{Aut}_K(\overline{K})$.

0.2 Galois action on subvarieties

Proposition 0.2.1. *Let $X = X_{0,\mathbb{C}_\infty}$ be a variety over $K \subset \mathbb{C}_\infty$. Then there is a natural action of the absolute Galois group G_K on the set of subvarieties of X which are defined over $\overline{K}$. Such a subvariety is already defined over K if and only if it is defined over K^{sep} and G_K-stable.*

Proof. First note that G_K acts on $X_{0,\overline{K}} = X_0 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ by

$$\begin{array}{ccc} G_K & \to & \mathrm{Aut}_{\mathrm{Schemes}}(X_{0,\overline{K}}) \\ \sigma & \mapsto & \mathrm{id} \times \mathrm{Spec}(\sigma^{-1}) \end{array} .$$

By our conventions in Section 0.1, a subvariety of X defined over $\overline{K}$ is of the form $X' = X'_{0,\mathbb{C}_\infty}$ for a closed subscheme X'_0 of $X_{0,\overline{K}}$. Using the above action, we set

$$\sigma(X') := \sigma(X'_0)_{\mathbb{C}_\infty}.$$

This is a subvariety of X defined over $\overline{K}$ because it is the base extension to $\mathbb{C}_\infty$ of the closed subscheme $\sigma(X'_0)$ of $X_{0,\overline{K}}$. Hence, we have a natural action of G_K on the set of subvarieties of X which are defined over $\overline{K}$.

The last statement follows from Theorem AG. 14.4 in [3]. □

Proposition 0.2.2. *Let $X = X_{0,\mathbb{C}_\infty}$ be a variety over $K \subset \mathbb{C}_\infty$. Then the irreducible components of X are defined over K^{sep} and the action of G_K from*

Proposition 0.2.1 restricts to a G_K-action on the set of irreducible components of X. The latter action is transitive if and only if X is K-irreducible.

Proof. Let $X_{0,K^{\mathrm{sep}}} = Z_1 \cup \cdots \cup Z_n$ be the decomposition of $X_{0,K^{\mathrm{sep}}}$ into irreducible components. Then, $Z_1, \ldots, Z_n$ stay irreducible after base extension to $\mathbb{C}_\infty$ (Exercise II.3.15 in [22]). Hence, $Z_{1,\mathbb{C}_\infty}, \ldots, Z_{n,\mathbb{C}_\infty}$ are the irreducible components of $X_{0,\mathbb{C}_\infty} = X$ and these are defined over K^{sep} because $Z_1, \ldots, Z_n$ are closed subschemes of $X_{0,K^{\mathrm{sep}}}$.

Since the irreducible components of X are therefore exactly the maximal irreducible subvarieties of X defined over K^{sep}, the action of G_K from Proposition 0.2.1 restricts to an action of G_K on the set of irreducible components of X over $\mathbb{C}_\infty$.

Note that, by Proposition 0.2.1, the union of all the irreducible components of X over $\mathbb{C}_\infty$ lying in one orbit of the latter action is defined over K because it is G_K-stable. Hence, if this action is not transitive, X can be written as a finite union of at least two proper subvarieties defined over K. In particular, the action is transitive if X is K-irreducible.

Conversely, assume by contradiction that this action is transitive and that $X = X_1 \cup X_2$ where X_1 and X_2 are proper subvarieties of X defined over K. Then there are irreducible components X_1' of X_1 and X_2' of X_2 over $\mathbb{C}_\infty$ such that $X_1' \subsetneq X_2$ and $X_2' \subsetneq X_1$. By the transitivity assumption, there is a $\sigma \in G_K$ with $\sigma(X_1') = X_2'$. This gives the contradiction

$$X_2' = \sigma(X_1') \subset \sigma(X_1) = X_1,$$

hence X is K-irreducible if the considered action is transitive. □

0.3 Quotient varieties

Proposition 0.3.1. *Let X be an affine variety (over $\mathbb{C}_\infty$) and G a finite subgroup of the group of automorphisms of X. Then the topological quotient X/G has the structure of an affine variety such that the canonical projection $\pi : X \to X/G$ is a morphism of algebraic varieties. Together with π it satisfies the following universal property: For each G-invariant morphism $f : X \to Y$ of affine varieties, there is a unique morphism $h : X/G \to Y$ with $f = h \circ \pi$.*

Note that $(X/G, \pi)$ is determined up to unique isomorphism by the universal property. We call X/G the *quotient variety* of X under the action of G.

Proof. We refer to the construction in Section III.12 in [34]. If $X = \mathrm{Spec}(B)$, then we have $X/G = \mathrm{Spec}(B^G)$, where B^G denotes the subring of the elements of B fixed by all automorphisms in G, and π is defined by the inclusion map $B^G \hookrightarrow B$. □

Proposition 0.3.2. *The quotient variety X/G and the canonical projection $\pi : X \to X/G$ have the following properties:*

(i) *The morphism π is finite of degree $|G|$.*

(ii) *If X and all the automorphisms in G are defined over a subfield $K \subset \mathbb{C}_\infty$, then X/G and π are also defined over K.*

(iii) *If X is a normal variety, then X/G is also normal.*

(iv) *If G acts freely on the closed points of X, then π is an étale morphism.*

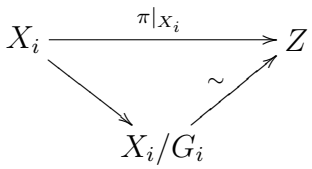
Proof. For (i), assume that $X = \mathrm{Spec}(B)$ and $X/G = \mathrm{Spec}(B^G)$. For $b \in B$, the monic polynomial

$$f_b(X) = \prod_{g \in G} (X - g(b))$$

with root b has coefficients in B^G. Therefore, B is integral over B^G. Since B is a finitely generated $\mathbb{C}_\infty$-algebra, it is also a finitely generated B^G-algebra and by integrality over B^G therefore a finitely generated B^G-module. Hence, π is a finite morphism.

If X is irreducible, the ring B, and therefore also B^G, is an integral domain. The function field $\mathbb{C}_\infty(X)$ of X is the quotient field of B and the function field $\mathbb{C}_\infty(X/G)$ of X/G the quotient field of B^G. The latter is equal to the subfield of invariants of $\mathbb{C}_\infty(X)$ under the unique extension of the action of G on B to $\mathbb{C}_\infty(X)$. Therefore, $\mathbb{C}_\infty(X)/\mathbb{C}_\infty(X/G)$ is a Galois extension of degree $|G|$ and π of degree $|G|$.

If X is reducible and Z is an irreducible component of X/G, the action of G on the irreducible components of $\pi^{-1}(Z)$ is transitive. This follows because X/G is the topological quotient of X under the action of G. Assume that $X_1, \ldots, X_k$ are the irreducible components of $\pi^{-1}(Z)$. The stabilizer G_i of such an irreducible component X_i is a index k subgroup of G. For each i, one can check that Z together with $\pi|_{X_i}$ satisfies the universal property of the quotient variety of X_i under G_i. Therefore, there is an isomorphism $X_i/G_i \cong Z$ such that the diagram



commutes. Since X_i is irreducible, it follows by the above discussion that $\pi|_{X_i} : X_i \to Z$ has degree $|G_i| = |G|/k$ and

$$\sum_{i=1}^{k} \deg(\pi|_{X_i} : X_i \to Z) = |G|.$$

Therefore, by our definition, π is of degree $|G|$.

Statements (ii) and (iii) follow from Remark 2) resp. Corollary c) in Section III.12 in [34], and (iv) follows from Section II.7 of [27]. □

If a group G acts on an affine variety X via

$$\rho : G \longrightarrow \mathrm{Aut}(X)$$

such that $\rho(G) \subset \mathrm{Aut}(X)$ is finite, we denote by X/G the quotient variety $X/\rho(G)$. By Proposition 0.3.2, the canonical projection $X \to X/G$ is finite of degree $|\rho(G)|$.

Chapter 1

Drinfeld modular varieties

1.1 Analytic description and modular interpretation

We consider the following datum:

- A global function field F together with a fixed place ∞,
- a positive integer r, called *rank*, and
- a compact open subgroup $\mathcal{K}$ of $\mathrm{GL}_r(\mathbb{A}_F^f)$, called *level*.

We define *Drinfeld's upper half-space* over F of dimension $r-1$ by

$$\Omega_F^r := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{F_\infty\text{-rational hyperplanes}\}.$$

Proposition 1.1.1. *The points of Drinfeld's upper half-space Ω_F^r are in bijective correspondence with the set of injective F_∞-linear maps $F_\infty^r \hookrightarrow \mathbb{C}_\infty$ up to multiplication by a constant in $\mathbb{C}_\infty^*$ via the assignment*

$$[\omega_1 : \cdots : \omega_r] \longmapsto [(a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r].$$

Proof. We have the canonical bijection

$$\begin{array}{rcl} \mathbb{C}_\infty^r & \longrightarrow & \{F_\infty\text{-linear maps } F_\infty^r \to \mathbb{C}_\infty\} \\ (\omega_1, \ldots, \omega_r) & \longmapsto & (a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r \end{array}.$$

The F_∞-linear map $(a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r$ is injective if and only if $\omega_1, \ldots, \omega_r$ are F_∞-linearly independent, i.e., if and only if $(\omega_1, \ldots, \omega_r)$ does not

lie in a F_∞-rational hyperplane through 0. Hence, factoring out the action of $\mathbb{C}_\infty^*$ on both sides, we get the desired bijection of Drinfeld's upper half-space with the set of injective F_∞-linear maps $F_\infty^r \hookrightarrow \mathbb{C}_\infty$ up to multiplication by a constant in $\mathbb{C}_\infty^*$. □

In the following, we use the identification given by Proposition 1.1.1 and denote the element of Ω_F^r associated to an injective F_∞-linear map $\omega : F_\infty^r \hookrightarrow \mathbb{C}_\infty$ by $\overline{\omega}$.

Using this notation, one sees that $\mathrm{GL}_r(F)$ acts on Ω_F^r from the left by

$$T \cdot \overline{\omega} := \overline{\omega \circ T^{-1}} \tag{1.1.1}$$

for $T \in \mathrm{GL}_r(F)$ considered as automorphism of F_∞^r.

Remark: This action can also be described regarding Ω_F^r as a subset of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. A short calculation shows that, for $\omega = [\omega_1 : \cdots : \omega_r] \in \Omega_F^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ and $T \in \mathrm{GL}_r(F)$ with $T^{-1} = (s_{ij})$, we have

$$T \cdot \omega = [s_{11}\omega_1 + \cdots + s_{r1}\omega_r : \cdots : s_{1r}\omega_1 + \cdots + s_{rr}\omega_r]. \tag{1.1.2}$$

In other words, the action of a $T \in \mathrm{GL}_r(F)$ on $\Omega_F^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ is the restriction to Ω_F^r of the natural action of $(T^{-1})^T \in \mathrm{GL}_r(\mathbb{C}_\infty)$ on $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$.

Theorem 1.1.2. *There is a normal affine variety $S_{F,\mathcal{K}}^r$ of dimension $r-1$ defined over F together with an isomorphism*

$$S_{F,\mathcal{K}}^r(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}) \tag{1.1.3}$$

of rigid-analytic spaces, where $\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ is viewed as a discrete set.

In the proof, we define a variety $S_{F,\mathcal{K}}^r$ over F together with a rigid-analytic isomorphism of the form (1.1.3) up to isomorphism over F. This variety is called the *Drinfeld modular variety* associated to the datum $(F, r, \mathcal{K})$. We will identify its $\mathbb{C}_\infty$-valued points with double cosets in $\mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K})$ via the rigid-analytic isomorphism given in the proof.

Proof. The proof consists of several steps:

(i) We use Drinfeld's construction of Drinfeld moduli schemes in [10] to define $S_{F,\mathcal{K}}^r$ and a rigid-analytic isomorphism of the form (1.1.3) for $\mathcal{K} = \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$ a principal congruence subgroup modulo a proper ideal I of A.

(ii) For $g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A})$ and proper ideals I, J of A with $J\hat{A}^r \subset gI\hat{A}^r$, we define morphisms

$$\pi_g : S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(I)},$$

which are defined over F and satisfy the compatibility relation

$$\pi_g \circ \pi_{g'} = \pi_{gg'}.$$

In particular, these morphisms define an action of $\mathrm{GL}_r(\hat{A})$ on $S^r_{F,\mathcal{K}(I)}$.

(iii) We use this action to extend the definition in (i) to all compact open subgroups $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$.

(iv) We extend the definition in (ii) to get morphisms

$$\pi_g : S^r_{F,\mathcal{K}'} \longrightarrow S^r_{F,\mathcal{K}}$$

for arbitrary $\mathcal{K}$, $\mathcal{K}' \subset \mathrm{GL}_r(\hat{A})$ and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$.

(v) We define $S^r_{F,\mathcal{K}}$ and a rigid-analytic isomorphism β of the form (1.1.3) for arbitrary levels $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$. We use the morphisms π_g from (iv) to show the well-definedness of $(S^r_{F,\mathcal{K}}, \beta)$ up to isomorphism over F.

Step (i): Recall that a *Drinfeld A-module of rank r* over an F-scheme S is a line bundle $\mathcal{L}$ over S together with a ring homomorphism φ from A to the ring $\mathrm{End}_{\mathbb{F}_q}(\mathcal{L})$ of $\mathbb{F}_q$-linear endomorphisms of $\mathcal{L}$ (as a group scheme over S) such that, over any trivializing affine open subset $\mathrm{Spec}(B) \subset S$, the homomorphism φ is given by

$$\varphi : \begin{array}{ccc} A & \longrightarrow & \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,\mathrm{Spec}(B)}) = B\{\tau\} \\ a & \longmapsto & \varphi_a = \sum_{i=0}^{m(a)} b_i(a)\tau^i \end{array}$$

where τ denotes the q-power Frobenius and, for all $a \in A$,

(a) $q^{m(a)} = |A/(a)|^r$,

(b) $b_{m(a)}(a) \in B^*$,

(c) $b_0(a) = \gamma(a)$ where γ is the ring homomorphism $F \to B$ corresponding to the morphism of affine schemes $\mathrm{Spec}(B) \hookrightarrow S \to \mathrm{Spec}(F)$.

For a proper ideal I of A, an *I-level structure* on a Drinfeld module $\mathcal{L}/S$ of rank r is an A-linear isomorphism of group schemes over S

$$\alpha : \underline{(I^{-1}/A)^r} \longrightarrow \mathcal{L}_I := \bigcap_{a \in I} \ker(\mathcal{L} \xrightarrow{a} \mathcal{L}),$$

where $\underline{(I^{-1}/A)^r}$ denotes the constant group scheme over S with fibers $(I^{-1}/A)^r$.

Remark: In general, one can also define Drinfeld A-modules together with level structures over A-schemes instead of F-schemes. In this case, one uses a different definition of I-level structure to deal smoothly with the fibers over $\mathfrak{p} \in \mathrm{Spec}(A)$ dividing I, see, e.g., Section I.6 in [9].

By Section 5 of [10], the functor

$$\begin{array}{rccl} & F\text{-schemes} & \longrightarrow & \text{Sets} \\ \mathcal{F}^r_{F,I}: & S & \longmapsto & \{\text{Isomorphism classes of Drinfeld } A\text{-modules} \\ & & & \text{of rank } r \text{ over } S \text{ with } I\text{-level structure}\} \end{array}$$

is representable by a nonsingular affine scheme of finite type over F of dimension $r-1$. Note that, in [10], it is actually shown that the corresponding functor from the category of schemes over Spec A to the category of sets is representable if I is contained in two distinct maximal ideals of A. The argument in the proof shows that it is enough that I is a proper ideal of A if we work with schemes over Spec F.

By our conventions in Section 0.1, the base extension to $\mathbb{C}_\infty$ of the above representing scheme is a non-singular variety of dimension $r-1$ defined over F. We denote it by $S^r_{F,\mathcal{K}(I)}$, where $\mathcal{K}(I)$ denotes the principal congruence subgroup modulo I. By Proposition 6.6 in [10], there is a natural isomorphism

$$S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F) \setminus (\Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)/\mathcal{K}(I)) \tag{1.1.4}$$

of rigid-analytic spaces. Under this isomorphism, the equivalence class of an element $(\overline{\omega}, h) \in \Omega^r_F \times \mathrm{GL}_r(\mathbb{A}^f_F)$ is mapped to the $\mathbb{C}_\infty$-valued point of $S^r_{F,\mathcal{K}(I)}$ corresponding to the Drinfeld module over $\mathbb{C}_\infty$ associated to the lattice

$$\Lambda := \omega(F^r \cap h\hat{A}^r)$$

with I-level structure given by the composition of the isomorphisms

$$(I^{-1}/A)^r \xrightarrow{h} I^{-1} \cdot (F^r \cap h\hat{A}^r)/(F^r \cap h\hat{A}^r) \xrightarrow{\omega} I^{-1} \cdot \Lambda/\Lambda,$$

where the first isomorphism is given by the multiplication by h on $(\mathbb{A}^f_F)^r$ via the natural identifications

$$\begin{array}{rcl} (I^{-1}/A)^r & \cong & I^{-1}\hat{A}^r/\hat{A}^r \\ I^{-1} \cdot (F^r \cap h\hat{A}^r)/(F^r \cap h\hat{A}^r) & \cong & I^{-1} \cdot h\hat{A}^r/h\hat{A}^r \end{array}$$

by the inclusion maps. For a detailed survey of this modular interpretation, we refer to the explanations in Section II.5 in [9].

Step (ii): Let I, J be proper ideals of A and $g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A}^r)$ such that $J\hat{A}^r \subset gI\hat{A}^r$. For such a datum, we construct a morphism of functors

$$\mathcal{F}^r_{F,J} \longrightarrow \mathcal{F}^r_{F,I}.$$

The given $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with matrix entries in $\hat{A}$ induces a surjective endomorphism of $(\mathbb{A}_F^f)^r/\hat{A}^r$ with kernel $g^{-1}\hat{A}^r/\hat{A}^r$. Since there is a natural isomorphism $(F/A)^r \cong (\mathbb{A}_F^f/\hat{A})^r$ induced by the inclusion maps, we therefore get a surjective homomorphism of A-modules

$$(F/A)^r \xrightarrow{g} (F/A)^r.$$

The kernel $U := \ker g$ of this homomorphism is contained in $(J^{-1}/A)^r$ because we have $g^{-1}\hat{A}^r \subset J^{-1}I\hat{A}^r \subset J^{-1}\hat{A}^r$ by our assumption $J\hat{A}^r \subset gI\hat{A}^r$.

For any Drinfeld module $\mathcal{L}$ over an F-scheme S with J-level structure $\alpha : \underline{(J^{-1}/A)^r} \xrightarrow{\sim} \mathcal{L}_J$, the image of $\underline{U} \subset \underline{(J^{-1}/A)^r}$ under α is a finite A-invariant subgroup scheme of $\mathcal{L}$ over S. Hence, the quotient $\mathcal{L}' := \mathcal{L}/\alpha(\underline{U})$ is also a Drinfeld A-module over S and contains the finite subgroup scheme $\mathcal{L}_J/\alpha(\underline{U})$. Since $g(J^{-1}/A)^r \cong (J^{-1}/A)^r/U$, there is a unique A-linear isomorphism α' of group schemes over S such that the diagram

$$\begin{array}{ccc} \underline{(J^{-1}/A)^r} & \xrightarrow[\alpha]{\sim} & \mathcal{L}_J \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\ \underline{g(J^{-1}/A)^r} & \xrightarrow[\alpha']{\sim} & \mathcal{L}_J/\alpha(\underline{U}) \end{array}$$

commutes, where $\pi : \mathcal{L}_J \to \mathcal{L}_J/\alpha(\underline{U})$ is the canonical projection. By the assumption $J\hat{A}^r \subset gI\hat{A}^r$, we have $(I^{-1}/A)^r \subset g(J^{-1}/A)^r$. Restricting the isomorphism α' to the I-torsion gives therefore an I-level structure

$$\underline{(I^{-1}/A)^r} \xrightarrow{\sim} \mathcal{L}'_I$$

of $\mathcal{L}'$.

The assignment $(\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha'|_{\underline{(I^{-1}/A)^r}})$ induces a morphism of functors $\mathcal{F}^r_{F,J} \to \mathcal{F}^r_{F,I}$ and therefore a morphism $\pi_g : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}(I)}$ defined over F.

We get the following description of π_g on $\mathbb{C}_\infty$-valued points identified with double quotients by the isomorphisms (1.1.4):

$$[(\omega, h)] \longmapsto [(\omega, hg^{-1})]. \tag{1.1.5}$$

Indeed, consider the Drinfeld module φ over $\mathbb{C}_\infty$ with J-level structure α corresponding to the double coset $[(\overline{\omega}, h)]$ in $\mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}(J))$. It is associated to the lattice

$$\Lambda := \omega(F^r \cap h\hat{A}^r)$$

and α is given by the composition

$$(J^{-1}/A)^r \xrightarrow{h} J^{-1}\cdot(F^r\cap h\hat{A}^r)/(F^r\cap h\hat{A}^r)\xrightarrow{\omega} J^{-1}\cdot\Lambda/\Lambda.$$

The image of $U:=\ker((F/A)^r\xrightarrow{g}(F/A)^r)\cong g^{-1}\hat{A}^r/\hat{A}^r$ under this level structure is equal to

$$\omega(F^r\cap hg^{-1}\hat{A}^r)/\Lambda,$$

therefore the image of (φ,α) under the above morphism of functors is the Drinfeld module associated to the lattice

$$\Lambda':=\omega(F^r\cap hg^{-1}\hat{A}^r)$$

with level structure given by the composition

$$(I^{-1}/A)^r\xrightarrow{hg^{-1}} I^{-1}\cdot(F^r\cap hg^{-1}\hat{A}^r)/(F^r\cap hg^{-1}\hat{A}^r)\xrightarrow{\omega} I^{-1}\cdot\Lambda'/\Lambda'.$$

This shows the above description of π_g on $\mathbb{C}_\infty$-valued points.

This description implies that we have the relation

$$\pi_g\circ\pi_{g'}=\pi_{gg'}$$

for two such morphisms

$$\begin{aligned}\pi_g : S^r_{F,\mathcal{K}(I')} &\longrightarrow S^r_{F,\mathcal{K}(I)},\\ \pi_{g'} : S^r_{F,\mathcal{K}(I'')} &\longrightarrow S^r_{F,\mathcal{K}(I')}.\end{aligned}$$

In particular, we have an action of $\mathrm{GL}_r(\hat{A})$ on $S^r_{F,\mathcal{K}(I)}$ by morphisms defined over F and hence also on isomorphism classes of Drinfeld A-modules with I-level structure.

Step (iii): Using the action of $\mathrm{GL}_r(\hat{A})$ on $S^r_{F,\mathcal{K}(I)}$ by the morphisms π_g, we define, for a compact open subgroup $\mathcal{K}\subset\mathrm{GL}_r(\hat{A})$,

$$S^r_{F,\mathcal{K}}:=S^r_{F,\mathcal{K}(I)}/\mathcal{K},$$

where $\mathcal{K}(I)$ is a principal congruence subgroup contained in $\mathcal{K}$. Since $\mathcal{K}(I)$ acts trivially on $S^r_{F,\mathcal{K}(I)}$, this quotient can be viewed as a quotient under the action of the finite group $\mathcal{K}/\mathcal{K}(I)$ by morphisms defined over F. By Proposition 0.3.1, the quotient is an affine variety defined over F of dimension $r-1=\dim S^r_{F,\mathcal{K}(I)}$. By Proposition 0.3.1, it is a normal variety because the non-singular variety $S^r_{F,\mathcal{K}(I)}$ is normal. By the description (1.1.5) of the above action on $\mathbb{C}_\infty$-valued points, the rigid-analytic isomorphism (1.1.4) induces one of the form

$$\beta_I : (S^r_{F,\mathcal{K}(I)}/\mathcal{K})(\mathbb{C}_\infty)\cong\mathrm{GL}_r(F)\setminus(\Omega^r_F\times\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}). \tag{1.1.6}$$

It remains to show that, up to F-isomorphism, $(S^r_{F,\mathcal{K}(I)}/\mathcal{K},\, \beta_I)$ is independent of the choice of I. For this note that, for two ideals I, J with $I \subset J$, the functors

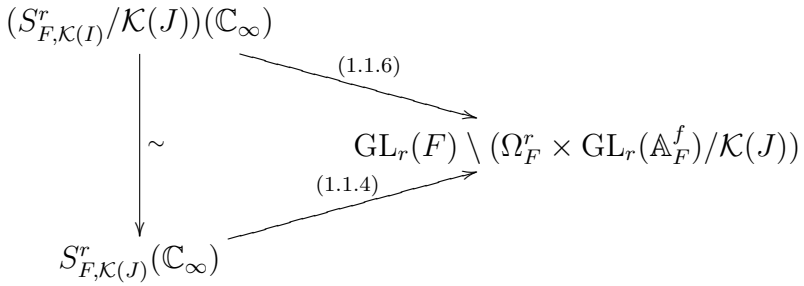
$$\begin{aligned} S &\longmapsto \mathcal{F}^r_{F,I}(S)/\mathcal{K}(J), \\ S &\longmapsto \mathcal{F}^r_{F,J}(S) \end{aligned}$$

are isomorphic, where the quotient is taken with respect to the action of $\mathrm{GL}_r(\hat{A})$ on $\mathcal{F}^r_{F,I}(S)$. The isomorphism is given by restricting I-level structures to $(J^{-1}/A)^r$.

Therefore, we have a natural isomorphism

$$S^r_{F,\mathcal{K}(I)}/\mathcal{K}(J) \cong S^r_{F,\mathcal{K}(J)}$$

defined over F, which is compatible with the isomorphisms (1.1.6) and (1.1.4), i.e., the diagram



commutes.

So for two ideals J, I with $\mathcal{K}(I) \subset \mathcal{K}$ and $\mathcal{K}(J) \subset \mathcal{K}$ we have

$$S^r_{F,\mathcal{K}(I)}/\mathcal{K} \cong S^r_{F,\mathcal{K}(I\cap J)}/\mathcal{K} \cong S^r_{F,\mathcal{K}(J)}/\mathcal{K},$$

and these isomorphisms are compatible with the isomorphisms (1.1.6). Therefore, we can well-define $S^r_{F,\mathcal{K}}$ up to isomorphism over F by $S^r_{F,\mathcal{K}(I)}$ together with the rigid-analytic isomorphism β_I.

Step (iv): Let $g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A})$ and $\mathcal{K}, \mathcal{K}' \subset \mathrm{GL}_r(\hat{A})$ with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ be given. Choose proper ideals I and J of A such that $\mathcal{K}(I) \subset \mathcal{K}$, $\mathcal{K}(J) \subset \mathcal{K}'$ and $J\hat{A}^r \subset gI\hat{A}^r$. Then, by Step (iii),

$$\begin{aligned} S^r_{F,\mathcal{K}'} &:= S^r_{F,\mathcal{K}(J)}/\mathcal{K}', \\ S^r_{F,\mathcal{K}} &:= S^r_{F,\mathcal{K}(I)}/\mathcal{K} \end{aligned}$$

and, by Step (ii), there is a morphism

$$\pi_g : S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(I)}.$$

Since $g\mathcal{K}'g^{-1} \subset \mathcal{K}$, for each $k' \in \mathcal{K}'$, there is a $k \in \mathcal{K}$ such that $gk' = kg$ and

$$\pi_g \circ \pi_{k'} = \pi_k \circ \pi_g$$

as morphisms $S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(I)}$. So the composition of π_g with the canonical projection $S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}}$ is $\mathcal{K}'$-invariant and induces therefore a morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ such that the diagram

$$\begin{array}{ccc} S^r_{F,\mathcal{K}(J)} & \xrightarrow{\pi_g} & S^r_{F,\mathcal{K}(I)} \\ \downarrow & & \downarrow \\ S^r_{F,\mathcal{K}'} & \xrightarrow{\pi_g} & S^r_{F,\mathcal{K}} \end{array}$$

commutes, where the vertical maps are the canonical projections. By (1.1.5), using the identifications $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ and $S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$ with double quotients given by (1.1.6), this morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is given by

$$[(\omega, h)] \longmapsto [(\omega, hg^{-1})] \tag{1.1.7}$$

on $\mathbb{C}_\infty$-valued points. Therefore, we have defined π_g independently of the choice of I and J if all matrix entries of g lie in $\hat{A}$.

If $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ is arbitrary, there is a $\lambda \in A \setminus \{0\}$ such that $\lambda \cdot g \in \mathrm{GL}_r(\mathbb{A}_F^f) \cap \mathrm{Mat}_r(\hat{A})$. We then define $\pi_g := \pi_{\lambda \cdot g}$. This morphism is independent of the choice of λ because we have

$$[(\omega, h(\lambda g)^{-1})] = [(\omega, hg^{-1})]$$

in $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ for all $\lambda \in A \setminus \{0\}$ and $[(\omega, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$. In particular, π_g is still described by (1.1.7) on $\mathbb{C}_\infty$-valued points.

The latter implies the relation

$$\pi_g \circ \pi_{g'} = \pi_{gg'} \tag{1.1.8}$$

for two such morphisms $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ and $\pi_{g'} : S^r_{F,\mathcal{K}''} \to S^r_{F,\mathcal{K}'}$.

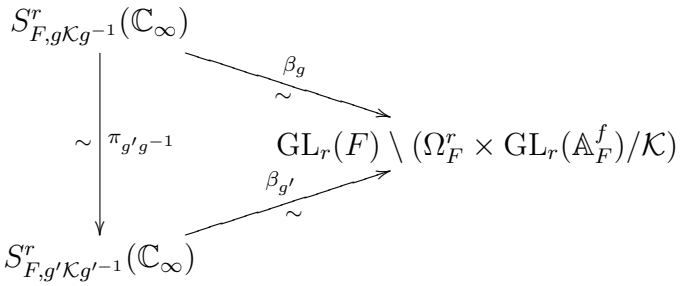
Step (v): For an arbitrary compact open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$, we choose a $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ such that $g\mathcal{K}g^{-1} \subset \mathrm{GL}_r(\hat{A})$. The composition of the rigid-analytic isomorphism (1.1.6)

$$S^r_{F,g\mathcal{K}g^{-1}}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/g\mathcal{K}g^{-1})$$

and $[(\overline{\omega}, h)] \mapsto [(\overline{\omega}, hg)]$ gives a rigid-analytic isomorphism

$$\beta_g : S^r_{F,g\mathcal{K}g^{-1}}(\mathbb{C}_\infty) \cong \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}).$$

For another $g' \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with $g'\mathcal{K}g'^{-1} \subset \mathrm{GL}_r(\hat{A})$, the diagram



commutes. By the relation (1.1.8), the vertical arrow $\pi_{g'g^{-1}}$ is an isomorphism over F with inverse $\pi_{gg'^{-1}}$.

Therefore, we can well-define $S^r_{F,\mathcal{K}}$ up to F-isomorphism as $S^r_{F,g\mathcal{K}g^{-1}}$ together with the rigid-analytic isomorphism β_g. Since we have seen in Step (iii) that $S^r_{F,g\mathcal{K}g^{-1}}$ is a normal affine variety of dimension $r-1$ defined over F, the same holds for $S^r_{F,\mathcal{K}}$. □

Proposition 1.1.3. *Let C be a set of representatives in $\mathrm{GL}_r(\mathbb{A}_F^f)$ for $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$, and set $\Gamma_g := g\mathcal{K}g^{-1} \cap \mathrm{GL}_r(F)$ for $g \in C$. Then the map*

$$\begin{aligned} \coprod_{g\in C} \Gamma_g \setminus \Omega_F^r &\longrightarrow \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}) \\ [\overline{\omega}]_g &\longmapsto [(\overline{\omega},\, g)] \end{aligned}$$

is a rigid analytic isomorphism which maps for each $g \in C$ the quotient space $\Gamma_g \setminus \Omega_F^r$ to the $\mathbb{C}_\infty$-valued points of an irreducible component Y_g of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$.

This theorem implies that the irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ are disjoint and that C is in bijective correspondence with the set of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ where $g \in C$ corresponds to the irreducible component Y_g with $Y_g(\mathbb{C}_\infty) \cong \Gamma_g \setminus \Omega_F^r$ via the isomorphism given in the theorem.

Proof. The definition of the map is independent of the choice of the representative $\overline{\omega} \in \Omega_F^r$ because for $gkg^{-1} \in \Gamma_g$ with $k \in \mathcal{K}$ we have

$$[((gkg^{-1}) \cdot \overline{\omega}, g)] = [(\overline{\omega}, (gkg^{-1})^{-1}g)] = [(\overline{\omega}, gk^{-1})] = [(\overline{\omega}, g)].$$

Let $[(\overline{\omega}, h)]$ be an arbitrary element of $\mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K})$ and $g \in C$ the representative of the double coset of h in $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$. Then there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that $g = T \cdot h \cdot k$. Hence, the element $[T \cdot \overline{\omega}]_g$ of $\Gamma_g \setminus \Omega_F^r$ is mapped to $[(T \cdot \overline{\omega},\, T \cdot h \cdot k)] = [(\overline{\omega},\, h)]$. This shows the surjectivity.

To show the injectivity consider $\overline{\omega_1}, \overline{\omega_2} \in \Omega_F^r$ and $g_1,\ g_2 \in C$ with $[(\overline{\omega_1}, g_1)] = [(\overline{\omega_2}, g_2)]$. In this case, there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that $T \cdot \overline{\omega_1} = \overline{\omega_2}$ and $T \cdot g_1 \cdot k = g_2$. The latter equation implies that g_1 and g_2 lie in the same double coset in $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$, hence $g_1 = g_2 =: g \in C$. Furthermore we conclude that $T = g \cdot k^{-1} \cdot g^{-1} \in g\mathcal{K}g^{-1} \cap \mathrm{GL}_r(F) = \Gamma_g$, which shows that $[\overline{\omega_1}]_g = [\overline{\omega_2}]_g$ in $\Gamma_g \setminus \Omega_F^r$.

Hence, the considered map is bijective. Since $\mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ is viewed as a discrete set, the map is also an isomorphism of rigid analytic spaces. Therefore, it only remains to show that the quotient spaces $\Gamma_g \setminus \Omega_F^r,\ g \in C$, are irreducible as rigid-analytic spaces because the irreducible components of the rigid analytification of $S_{F,\mathcal{K}}^r$ coincide with the rigid analytification of the irreducible components of $S_{F,\mathcal{K}}^r$ (see, e.g., Theorem 2.3.1 in [8]). Since $S_{F,\mathcal{K}}^r$ is a normal variety and therefore its rigid analytification a normal rigid analytic space, this is equivalent to the connectedness of the quotient spaces $\Gamma_g \setminus \Omega_F^r$. The latter follows because Ω_F^r is a connected rigid-analytic space by Theorem 2.4 in [25]. □

Definition 1.1.4. *For a $\mathbb{C}_\infty$-valued point $p = [(\overline{\omega}, h)] \in S(\mathbb{C}_\infty)$ of a Drinfeld modular variety $S = S_{F,\mathcal{K}}^r$ with $h \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and $\overline{\omega} \in \Omega_F^r$ associated to $\omega : F_\infty^r \hookrightarrow \mathbb{C}_\infty$, the elements of*

$$\mathrm{End}(p) := \{u \in \mathbb{C}_\infty : u \cdot \omega(F^r) \subset \omega(F^r)\}$$

are called endomorphisms *of p.*

Note that $\mathrm{End}(p)$ is well-defined because the homothety class of $\omega(F^r) \subset \mathbb{C}_\infty$ does not depend on the chosen representatives ω and h.

Remark: If $\mathcal{K} = \mathcal{K}(I)$ and $p \in S_{F,\mathcal{K}(I)}^r(\mathbb{C}_\infty)$ is corresponding to the Drinfeld module φ over $\mathbb{C}_\infty$ associated to the lattice $\Lambda \subset \mathbb{C}_\infty$, then $\omega(F^r) = F \cdot \Lambda$, and therefore

$$\mathrm{End}(p) = F \cdot \mathrm{End}(\varphi)$$

for the endomorphism ring $\mathrm{End}(\varphi) \subset \mathbb{C}_\infty$ of φ.

Lemma 1.1.5. *The set* $\mathrm{End}(p)$ *of endomorphisms of p is a field extension of F contained in $\mathbb{C}_\infty$ of finite degree dividing r.*

Proof. Since $V := \omega(F^r) \subset \mathbb{C}_\infty$ is an F-subvectorspace of $\mathbb{C}_\infty$, the subset $\mathrm{End}(p) \subset \mathbb{C}_\infty$ contains F and is closed under addition and multiplication. Furthermore, multiplicative inverses of elements $0 \neq u \in \mathrm{End}(p)$ also lie in $\mathrm{End}(p)$ because dimension reasons imply that $x \mapsto u \cdot x$ is an automorphism of V with inverse $x \mapsto u^{-1} \cdot x$. Therefore $\mathrm{End}(p)$ is a field extension of F contained in $\mathbb{C}_\infty$.

Now take an element $0 \neq \xi \in V$. We have $x \cdot \xi \in V$ for all $x \in \mathrm{End}(p)$, hence $F \subset \mathrm{End}(p) \subset \xi^{-1} \cdot V$. This implies

$$r = \dim_F(\xi^{-1} \cdot V) = \dim_{\mathrm{End}(p)}(\xi^{-1} \cdot V) \cdot [\mathrm{End}(p)/F].$$

Therefore the field extension $\mathrm{End}(p)/F$ is finite with degree dividing r. □

Lemma 1.1.6. *Each irreducible component X of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ contains a point $p \in X(\mathbb{C}_\infty)$ with $\mathrm{End}(p) = F$.*

Proof. Choose $\overline{\omega} \in \Omega^r_F$ such that $\omega(F^r) = F \oplus F \cdot \xi_2 \oplus \cdots \oplus F \cdot \xi_r$ with $\xi_2, \ldots, \xi_r \in \mathbb{C}_\infty$ algebraically independent over F. This is possible because $\mathbb{C}_\infty$ as uncountable field is of infinite transcendence degree over the countable field F.

Now choose $h \in \mathrm{GL}_r(\mathbb{A}_F^f)$ such that $p := [(\overline{\omega}, h)] \in X(\mathbb{C}_\infty)$ (use the description of the irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ given in Proposition 1.1.3). Since $1 \in \omega(F^r)$, we have on the one hand $\mathrm{End}(p) \subseteq \omega(F^r)$. On the other hand, all elements of $\mathrm{End}(p)$ are algebraic over F because the extension $\mathrm{End}(p)/F$ is finite. But by the choice of $\xi_2, \ldots, \xi_r$, every element of $\omega(F^r)$ which is algebraic over F lies in F. Hence, $\mathrm{End}(p) = F$. □

1.2 Rank one case

In the case $r = 1$ the variety $S^r_{F,\mathcal{K}}$ is zero-dimensional and defined over F for any compact open subgroup $\mathcal{K} \subset \mathrm{GL}_1(\mathbb{A}_F^f) = (\mathbb{A}_F^f)^*$. Hence, $S^1_{F,\mathcal{K}}$ consists only of finitely many closed points and it can be set-theoretically identified with $S^1_{F,\mathcal{K}}(\mathbb{C}_\infty)$. By Proposition 0.2.2, the closed points are all defined over F^{sep} and the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $S^1_{F,\mathcal{K}}$.

Drinfeld's upper half-space Ω^1_F just consists of one point. Therefore, we have

$$S^1_{F,\mathcal{K}} = F^* \setminus (\mathbb{A}_F^f)^* \,/\, \mathcal{K}$$

as a set. Since $(\mathbb{A}_F^f)^*$ is abelian, this set can be identified with the abelian group $(\mathbb{A}_F^f)^* \,/\, (F^* \cdot \mathcal{K})$.

Since $F^* \cdot \mathcal{K}$ is a closed subgroup of finite index of $(\mathbb{A}_F^f)^*$, by class field theory, there is a finite abelian extension H/F totally split at ∞ such that the Artin map

$$\psi_{H/F} : (\mathbb{A}_F^f)^* \longrightarrow \mathrm{Gal}(H/F)$$

induces an isomorphism $(\mathbb{A}_F^f)^* \,/\, (F^* \cdot \mathcal{K}) \cong \mathrm{Gal}(H/F)$. In particular we have

$$|S^1_{F,\mathcal{K}}| = [H : F].$$

Theorem 1.2.1. *If $\psi_{H/F}(g) = \sigma|_H$ for a $g \in (\mathbb{A}_F^f)^*$ and a $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$, then the action of σ on $S^1_{F,\mathcal{K}} = F^* \setminus (\mathbb{A}_F^f)^* / \mathcal{K}$ is given by multiplication with g^{-1}.*

Proof. This follows from Theorem 1 in Section 8 of Drinfeld's article [10]. Note that in this article the action of an element $g \in (\mathbb{A}_F^f)^*$ on $S^1_{F,\mathcal{K}} = F^* \setminus (\mathbb{A}_F^f)^* / \mathcal{K}$ is given by the morphism π_g, which is given by multiplication with g^{-1}. □

Corollary 1.2.2. *The absolute Galois group* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ *acts transitively on* $S^1_{F,\mathcal{K}}$. □

Chapter 2

Morphisms and Drinfeld modular subvarieties

2.1 Projection morphisms and Hecke correspondences

Let $S^r_{F,\mathcal{K}}$ be a fixed Drinfeld modular variety. For each $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and all compact open subgroups $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ of $\mathrm{GL}_r(\mathbb{A}_F^f)$, we have a well-defined map

$$\begin{array}{rcl} S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) & \to & S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \\ [(\overline{\omega}, h)] & \mapsto & [(\overline{\omega}, hg^{-1})]. \end{array} \tag{2.1.1}$$

Theorem 2.1.1. *This map is induced by a unique finite morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ defined over F of degree $[g^{-1}\mathcal{K}g : \mathcal{K}' \cdot (\mathcal{K} \cap F^*)]$.*

Proof. In the case that $\mathcal{K}$ and $\mathcal{K}'$ are contained in $\mathrm{GL}_r(\hat{A})$, we already showed the existence of a morphism π_g which is described by (2.1.1) on $\mathbb{C}_\infty$-valued points in the proof of Theorem 1.1.2. If $\mathcal{K}$ and $\mathcal{K}'$ are arbitrary with $\mathcal{K}' \subset g^{-1}\mathcal{K}g$, there is an $s \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with

$$s\mathcal{K}'s^{-1} \subset sg^{-1}\mathcal{K}gs^{-1} \subset \mathrm{GL}_r(\hat{A}).$$

By our definition in the proof of Theorem 1.1.2, we have $S^r_{F,\mathcal{K}'} = S^r_{F,s\mathcal{K}'s^{-1}}$, where under the identifications of $\mathbb{C}_\infty$-valued points introduced in the proof of Theorem 1.1.2

$$[(\omega, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) \longleftrightarrow [(\omega, hs^{-1})] \in S^r_{F,s\mathcal{K}'s^{-1}}(\mathbb{C}_\infty).$$

Similarly, we have $S^r_{F,\mathcal{K}} = S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}$ with

$$[(\omega, h)] \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \longleftrightarrow [(\omega, hgs^{-1})] \in S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}(\mathbb{C}_\infty).$$

Using these identifications, we can define the morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ as $\pi_1 : S^r_{F,s\mathcal{K}'s^{-1}} \to S^r_{F,sg^{-1}\mathcal{K}gs^{-1}}(\mathbb{C}_\infty)$. Since the latter morphism π_1 is given by $[(\omega, h)] \mapsto [(\omega, h)]$ on $\mathbb{C}_\infty$-valued points, by the above identifications π_g is indeed described by (2.1.1) on $\mathbb{C}_\infty$-valued points. So we have shown the existence of the morphism π_g defined over F. It is uniquely determined by (2.1.1) because $\mathbb{C}_\infty$ is algebraically closed.

To show finiteness of the morphism π_g and the statement about its degree, we use Proposition 0.3.1 and 0.3.2. By the above definition of a general morphism π_g, it is enough to show these statements for morphisms of the form $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ with $\mathcal{K}' \subset \mathcal{K} \subset \mathrm{GL}_r(\hat{A})$.

We first assume that $\mathcal{K}' = \mathcal{K}(I)$ is a principal congruence subgroup. Then π_1 is the canonical projection

$$S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}(I)}/\mathcal{K}$$

by the construction in the proof of Theorem 1.1.2. By the discussion at the end of Section 0.3, in this case π_1 is finite of degree $|\rho(\mathcal{K})| = [\mathcal{K} : \ker(\rho)]$ for

$$\rho : \begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathrm{Aut}(S^r_{F,\mathcal{K}(I)}) \\ g & \longmapsto & \pi_g \end{array}.$$

So we have to show that $\ker(\rho) = \mathcal{K}(I) \cdot (\mathcal{K} \cap F^*)$. For $g = k\lambda$ with $k \in \mathcal{K}(I)$ and $\lambda \in \mathcal{K} \cap F^*$, we have

$$\pi_g([(\overline{\omega}, h)]) = [(\overline{\omega}, h\lambda^{-1}k^{-1})] = [(\overline{\omega \circ \lambda^{-1}}, h)] = [(\overline{\omega}, h)]$$

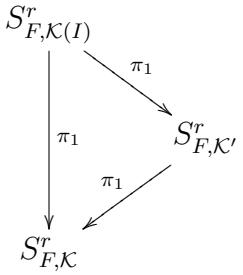
for all $[(\overline{\omega}, h)] \in S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty)$, hence $\mathcal{K}(I) \cdot (\mathcal{K} \cap F^*) \subset \ker(\rho)$.

Conversely, assume that $g \in \ker(\rho)$. By Lemma 1.1.6, there is a geometric point $p = [(\overline{\omega}, h)] \in S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty)$ with $\mathrm{End}(p) = F$. For this point, we have $[(\overline{\omega}, h)] = [(\overline{\omega}, hg^{-1})]$, hence there are $c \in \mathbb{C}^*_\infty$, $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}(I)$ with

$$\begin{aligned} \omega &= c \cdot \omega \circ T^{-1}, \\ h &= Thg^{-1}k. \end{aligned}$$

The first equality implies that $\omega(F^r) = c \cdot \omega(F^r)$, hence $c \in F^*$ because of $\mathrm{End}(p) = F$. The matrix $T \in \mathrm{GL}_r(F)$ is therefore equal to the scalar matrix $c \in F^*$. By the second equality, we conclude that $g = kT$ and $T \in \mathcal{K}$ because g and k both lie in $\mathcal{K}$. Hence, $g \in \mathcal{K}(I) \cdot (\mathcal{K} \cap F^*)$ and indeed $\ker(\rho) = \mathcal{K}(I) \cdot (\mathcal{K} \cap F^*)$. This concludes the proof for the case $\mathcal{K}' = \mathcal{K}(I)$.

For general subgroups $\mathcal{K}' \subset \mathcal{K} \subset \mathrm{GL}_r(\hat{A})$, choose a proper ideal I of A with $\mathcal{K}(I) \subset \mathcal{K}'$. Then we have the following commutative diagram of projection maps:



We have already shown that the morphisms $\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}}$ are finite. Therefore, $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is finite of degree

$$\frac{\deg(\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}})}{\deg(\pi_1 : S^r_{F,\mathcal{K}(I)} \to S^r_{F,\mathcal{K}'})} = \frac{[\mathcal{K} : \mathcal{K}(I) \cdot (\mathcal{K} \cap F^*)]}{[\mathcal{K}' : \mathcal{K}(I) \cdot (\mathcal{K}' \cap F^*)]}.$$

This is equal to

$$\frac{[\mathcal{K} : \mathcal{K}']}{[\mathcal{K}(I) \cdot (\mathcal{K} \cap F^*) : \mathcal{K}(I) \cdot (\mathcal{K}' \cap F^*)]} = \frac{[\mathcal{K} : \mathcal{K}']}{[\mathcal{K} \cap F^* : \mathcal{K}' \cap F^*]} = [\mathcal{K} : \mathcal{K}' \cdot (\mathcal{K} \cap F^*)],$$

as claimed. □

In the following, we call the morphisms π_g *projection morphisms* of Drinfeld modular varieties. In the case $g = 1$ we also call them *canonical projections* of Drinfeld modular varieties. For two elements $g, g' \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and two subgroups $\mathcal{K}' \subset g^{-1}\mathcal{K}g$, $\mathcal{K}'' \subset g'^{-1}\mathcal{K}'g'$, by the description on $\mathbb{C}_\infty$-valued points, we have

$$\pi_{gg'} = \pi_g \circ \pi_{g'}. \tag{2.1.2}$$

Definition 2.1.2. *A compact open subgroup $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is called* amply small *if there is a proper ideal I of A and a $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ such that $g\mathcal{K}g^{-1}$ is contained in the principal congruence subgroup $\mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$.*

Proposition 2.1.3. *Let $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ be amply small, $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and $\mathcal{K}' \subset g^{-1}\mathcal{K}g$. Then the morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is étale. Furthermore, if $\mathcal{K}'$ is a normal subgroup of $g^{-1}\mathcal{K}g$, it is an étale Galois cover over F with group $g^{-1}\mathcal{K}g/\mathcal{K}'$ where the automorphism of the cover corresponding to a coset $[x] \in g^{-1}\mathcal{K}g/\mathcal{K}'$ is given by $\pi_x : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}'}$.*

Proof. We first reduce ourselves to the case $g = 1$ and $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$ for some proper ideal I of A. For $\mathcal{K}, \mathcal{K}'$ and g arbitrary with $\mathcal{K}$ amply small, let

$h \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and I a proper ideal of A such that $h^{-1}\mathcal{K}h \subset \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$. By the relation (2.1.2), we have the commutative diagram

$$\begin{array}{ccc} S^r_{F,h^{-1}g\mathcal{K}'g^{-1}h} & \xrightarrow[\sim]{\pi_{g^{-1}h}} & S^r_{F,\mathcal{K}'} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_g} \\ S^r_{F,h^{-1}\mathcal{K}h} & \xrightarrow[\sim]{\pi_h} & S^r_{F,\mathcal{K}} \end{array}$$

where the horizontal morphisms are isomorphisms with $(\pi_h)^{-1} = \pi_{h^{-1}}$ and $(\pi_{g^{-1}h})^{-1} = \pi_{h^{-1}g}$. Furthermore, $\mathcal{K}'$ is a normal subgroup of $g^{-1}\mathcal{K}g$ if and only if $h^{-1}g\mathcal{K}'g^{-1}h$ is a normal subgroup of $h^{-1}\mathcal{K}h$. In this case there is an isomorphism $\alpha : g^{-1}\mathcal{K}g/\mathcal{K}' \cong h^{-1}\mathcal{K}h/h^{-1}g\mathcal{K}'g^{-1}h$ given by conjugation by $h^{-1}g$ such that, under the isomorphism $\pi_{g^{-1}h} : S^r_{F,h^{-1}g\mathcal{K}'g^{-1}h} \to S^r_{F,\mathcal{K}'}$ in the above diagram, for $x \in g^{-1}\mathcal{K}g/\mathcal{K}'$, the automorphism π_x of $S^r_{F,\mathcal{K}'}$ passes into the automorphism $\pi_{\alpha(x)}$ of $S^r_{F,h^{-1}g\mathcal{K}'g^{-1}h}$.

Hence, we can assume w.l.o.g. that $g = 1$ and $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{K}(I) \subset \mathrm{GL}_r(\hat{A})$.

Case (i): Let $\mathcal{K}'$ be a principal congruence subgroup $\mathcal{K}(J)$ modulo a proper ideal J of A, i.e., $\mathcal{K}' = \mathcal{K}(J) \lhd \mathcal{K} \subset \mathcal{K}(I)$.

Then, by our definition in the proof of Theorem 1.1.2, $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ is the canonical projection

$$S^r_{F,\mathcal{K}(J)} \longrightarrow S^r_{F,\mathcal{K}(J)}/\mathcal{K}.$$

We show that $\mathcal{K}/\mathcal{K}(J)$ acts freely on the closed points of $S^r_{F,\mathcal{K}(J)}$. By Proposition 0.3.2, this implies that this projection is an étale morphism. By the modular interpretation of $S^r_{F,\mathcal{K}(J)}$ in the proof of Theorem 1.1.2, it is enough to show that the action of $\mathcal{K}/\mathcal{K}(J)$ on isomorphism classes of Drinfeld A-modules over $\mathbb{C}_\infty$ together with J-level structure is free.

Indeed, assume that a coset $[k] \in \mathcal{K}/\mathcal{K}(J)$ stabilizes the isomorphism class of the Drinfeld module φ over $\mathbb{C}_\infty$ associated to a lattice $\Lambda \subset \mathbb{C}_\infty$ together with J-level structure $\alpha : (J^{-1}/A)^r \xrightarrow{\sim} J^{-1} \cdot \Lambda/\Lambda$. By our definition of the action of $\mathrm{GL}_r(\hat{A})$ on Drinfeld modules with J-level structure in the proof of Theorem 1.1.2, the image of (φ, α) under k is $(\varphi, \alpha \circ k^{-1})$. So there is an automorphism of φ given by a $c \in \mathbb{C}_\infty^*$ with $c \cdot \Lambda = \Lambda$ such that the diagram

$$\begin{array}{ccc} (J^{-1}/A)^r & \xrightarrow[\sim]{\alpha} & J^{-1} \cdot \Lambda/\Lambda \\ \downarrow{\scriptstyle k^{-1}} & & \downarrow{\scriptstyle c} \\ (J^{-1}/A)^r & \xrightarrow[\sim]{\alpha} & J^{-1} \cdot \Lambda/\Lambda \end{array}$$

commutes. Since c is an automorphism of φ, it is an element of the group of units of the endomorphism ring $\mathrm{End}(\varphi)$. By Theorem 4.9 (2) in [9] the latter

is an order in a finite extension F' of F in which there is exactly one place ∞' above ∞. In particular, $\mathrm{End}(\varphi)$ is contained in the integral closure A' of A in F' which is equal to all elements of F' regular outside ∞'. This implies $c \in A'^*$, hence c is an element of the field of constants of F'.

Now assume $c \neq 1$. Then there is an $n \geq 1$ such that $(c-1)^{q^n-1} = 1$ because $c-1$ is a nonzero element of the finite field of constants of F' containing $\mathbb{F}_q$. Therefore, $c-1$ is an element of the group of units of $\mathrm{End}(\varphi)$, i.e., $(c-1)\Lambda = \Lambda$. However, by the assumption $\mathcal{K} \subset \mathcal{K}(I)$, the restriction of k^{-1} to $(I^{-1}/A)^r$ is the identity, hence by the commutativity of the diagram c is the identity on $I^{-1} \cdot \Lambda/\Lambda$. Therefore, for all $x \in I^{-1} \cdot \Lambda$, we have $(c-1) \cdot x = cx - x \in \Lambda$. This contradicts $(c-1) \cdot \Lambda = \Lambda$ because $I^{-1} \cdot \Lambda \subsetneq \Lambda$.

Hence, we have $c = 1$ and, by the commutative diagram above, $k^{-1} : (J^{-1}/A)^r \to (J^{-1}/A)^r$ is the identity, i.e., $k \in \mathcal{K}(J)$ and $[k] = 1 \in \mathcal{K}/\mathcal{K}(J)$.

So we have shown that $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}$ is an étale cover. The group $\mathcal{K}/\mathcal{K}(J)$ injects into the automorphism group over F of this cover via $[k] \mapsto \pi_k$. Furthermore, by Corollary 3.13 in [26], an element of the automorphism group over F of this cover is uniquely determined by the image of one geometric point. Since the cover is of degree $[\mathcal{K} : \mathcal{K}(J)]$ by Theorem 2.1.1 (note that $\mathcal{K} \cap F^* = \{1\}$ because of $\mathcal{K} \subset \mathcal{K}(I)$), there are only $[\mathcal{K} : \mathcal{K}(J)]$ possibilities for the image of one geometric point. Hence, the automorphism group over F of the cover must be equal to $\mathcal{K}/\mathcal{K}(J)$ and the automorphism corresponding to a coset $[k] \in \mathcal{K}/\mathcal{K}(J)$ is given by π_k. The cover is Galois with group $\mathcal{K}/\mathcal{K}(J)$ (over F) because this group acts simply transitively on the geometric fibers.

Case (ii): Let $\mathcal{K}'$ be an arbitrary normal subgroup of $\mathcal{K}$, i.e., $\mathcal{K}' \lhd \mathcal{K} \subset \mathcal{K}(I)$.

Choose a proper ideal J of A such that $\mathcal{K}(J) \subset \mathcal{K}'$ and note that the diagram

$$\begin{array}{ccc} S^r_{F,\mathcal{K}(J)} & & \\ \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_1} & \\ S^r_{F,\mathcal{K}} = S^r_{F,\mathcal{K}(J)}/\mathcal{K} & \swarrow_{\pi_1} & S^r_{F,\mathcal{K}'} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}' \end{array} \tag{2.1.3}$$

commutes. Since $\mathcal{K}'$ is normal in $\mathcal{K}$, the action of $\mathcal{K}$ on $S^r_{F,\mathcal{K}(J)}$ induces an action of $\mathcal{K}/\mathcal{K}'$ on the quotient $S^r_{F,\mathcal{K}'} = S^r_{F,\mathcal{K}(J)}/\mathcal{K}'$. By the commutativity of the diagram, the variety $S^r_{F,\mathcal{K}}$ is the quotient of $S^r_{F,\mathcal{K}'}$ under this action. Furthermore, this action is free on the closed points of $S^r_{F,\mathcal{K}'}$ because $\mathcal{K}/\mathcal{K}(J)$ acts freely on the closed points of $S^r_{F,\mathcal{K}(J)}$. Therefore, we conclude by the same arguments

as above that $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is an étale cover with group $\mathcal{K}/\mathcal{K}'$ where the automorphism of the cover corresponding to a coset $[k] \in \mathcal{K}/\mathcal{K}'$ is given by π_k.

Case (iii): Let $\mathcal{K}'$ be an arbitrary subgroup of $\mathcal{K}$, i.e., $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{K}(I)$.

As in case (ii) above, choose a proper ideal J of A such that $\mathcal{K}(J) \subset \mathcal{K}'$. The diagram above then also commutes and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ are surjective étale morphisms by case (i). Furthermore, $S^r_{F,\mathcal{K}(J)}$ is a non-singular variety as explained in step (i) of the proof of Theorem 1.1.2.

Proposition 17.3.3.1 in EGA IV [19] says that if $X \to Y$ is a flat, surjective morphism of schemes and X is regular, then Y is also regular. Therefore, $S^r_{F,\mathcal{K}}$ and $S^r_{F,\mathcal{K}'}$ are both non-singular varieties.

By Proposition 10.4 in [22], a morphism $f : X \to Y$ of non-singular varieties of the same dimension over an algebraically closed field is étale if and only if, for every closed point $x \in X$, the induced map $T_x \to T_{f(x)}$ on Zariski tangent spaces is an isomorphism. We can apply this criterion because $S^r_{F,\mathcal{K}(J)}$, $S^r_{F,\mathcal{K}}$ and $S^r_{F,\mathcal{K}'}$ are all non-singular. Since the morphisms $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}'}$ and $\pi_1 : S^r_{F,\mathcal{K}(J)} \to S^r_{F,\mathcal{K}}$ are étale, the commutativity of the above diagram therefore implies that $\pi_1 : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is étale. □

Corollary 2.1.4. *If $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is amply small, then the Drinfeld modular variety $S^r_{F,\mathcal{K}}$ is non-singular.*

Proof. See case (iii) of the above proof of Proposition 2.1.3. □

Definition 2.1.5 (Hecke correspondence)**.** *For $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and $\mathcal{K}_g := \mathcal{K} \cap g^{-1}\mathcal{K}g$ the diagram*

$$\begin{array}{ccccc} & & S^r_{F,\mathcal{K}_g} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_g} & \\ S^r_{F,\mathcal{K}} & & & & S^r_{F,\mathcal{K}} \end{array}$$

is called the Hecke correspondence *T_g associated to g. For subvarieties $Z \subset S^r_{F,\mathcal{K}}$ we define*

$$T_g(Z) := \pi_g(\pi_1^{-1}(Z)).$$

Note that $T_g(Z)$ is a subvariety of $S^r_{F,\mathcal{K}}$ for any subvariety $Z \subset S^r_{F,\mathcal{K}}$ because π_g is finite and hence proper. By Theorem 2.1.1, the degree of the morphism π_1 equals

$$\deg(\pi_1) = [\mathcal{K} : (\mathcal{K} \cap g^{-1}\mathcal{K}g) \cdot (\mathcal{K} \cap F^*)] = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g].$$

It is called the *degree* $\deg T_g$ of the Hecke correspondence T_g.

2.2 Inclusions and Drinfeld modular subvarieties

Let $S = S^r_{F,\mathcal{K}}$ be a given Drinfeld modular variety. We consider the following datum:

- A finite extension $F' \subset \mathbb{C}_\infty$ of F of degree r/r' for some integer $r' \geq 1$ with only one place ∞' lying over ∞, and
- an $\mathbb{A}_F^f$-linear isomorphism $b : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$.

Furthermore, we choose an isomorphism

$$\varphi : F^r \xrightarrow{\sim} F'^{r'}$$

of vector spaces over F. By scalar extension to F_∞ and $\mathbb{A}_F^f$ it induces isomorphisms

$$\begin{array}{ccc} F_\infty^r & \xrightarrow{\varphi} & F'^{r'}_{\infty'} \\ (\mathbb{A}_F^f)^r & \xrightarrow{\varphi} & (\mathbb{A}_{F'}^f)^{r'} \end{array},$$

which we also denote by φ. We now define a morphism from the lower-rank Drinfeld modular variety $S' = S^{r'}_{F',\mathcal{K}'}$ with $\mathcal{K}' := (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ into S:

Theorem 2.2.1. *There is a finite morphism $\iota^{F'}_{F,b} : S' \to S$ defined over F' which on $\mathbb{C}_\infty$-valued points is given by the injective map*

$$\begin{array}{ccc} S'(\mathbb{C}_\infty) & \longrightarrow & S(\mathbb{C}_\infty) \\ [(\overline{\omega'}, h')] & \longmapsto & [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)], \end{array} \tag{2.2.1}$$

where $\overline{\omega'} \in \Omega^{r'}_{F'}$ and $h' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$. The morphism $\iota^{F'}_{F,b}$ is independent of the choice of $\varphi : F^r \xrightarrow{\sim} F'^{r'}$.

Proof. We denote by A' the integral closure of A in F'. Since there is only one place ∞' of F' above ∞, this is the ring of elements of F' regular away from ∞'.

Case (i): We first consider the case where $b(\widehat{A}^r) = \widehat{A'}^{r'}$ and $\mathcal{K} = \mathcal{K}(I)$ is a principal congruence subgroup modulo a proper ideal I of A. In this case, $\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ is the principal congruence subgroup modulo $I' := IA'$. Indeed $(b\mathcal{K}(I)b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^{r'})$ exactly consists of the elements of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^{r'})$ which stabilize $b(\widehat{A}^r)$ and induce the identity on the quotient $b(\widehat{A}^r)/I \cdot b(\widehat{A}^r)$. These are exactly the elements of the principal congruence subgroup $\mathcal{K}(I') \subset \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ because of $b(\widehat{A}^r) = \widehat{A'}^{r'}$.

In this situation, b induces an A-linear isomorphism $(I^{-1}/A)^r \xrightarrow{\sim} (I'^{-1}/A')^{r'}$, which we again denote by b. Therefore, for a Drinfeld A'-module $(\mathcal{L},\, \varphi)$ of rank r' over an F'-scheme with I'-level structure

$$\alpha : \underline{(I'^{-1}/A')^{r'}} \xrightarrow{\sim} \mathcal{L}_{I'},$$

the restriction $(\mathcal{L},\, \varphi|_A)$ to $A \subset A'$ is a Drinfeld A-module of rank $r = r' \cdot [F'/F]$ over S and the composition

$$\underline{(I^{-1}/A)^r} \xrightarrow{b} \underline{(I'^{-1}/A')^{r'}} \xrightarrow{\alpha} \mathcal{L}_I$$

is an I-level structure on $(\mathcal{L},\, \varphi|_A)$ (note that the I-torsion subgroup scheme $\mathcal{L}_I$ of $\mathcal{L}$ coincides with the I'-torsion subgroup scheme $\mathcal{L}_{I'}$ because I generates I' as an ideal of A'). The assignment

$$(\mathcal{L},\, \varphi,\, \alpha) \longmapsto (\mathcal{L},\, \varphi|_A, \alpha \circ b) \tag{2.2.2}$$

defines a morphism of functors from $\mathcal{F}^{r'}_{F',I'}$ to the restriction of $\mathcal{F}^r_{F,I}$ to the subcategory of F'-schemes (see step (i) of the proof of Theorem 1.1.2 for the definition of these functors). Therefore, we have a morphism

$$\iota^{F'}_{F,\,b} : S^{r'}_{F',\mathcal{K}(I')} \longrightarrow S^r_{F,\mathcal{K}(I)}$$

defined over F'. By Lemma 3.1 and Proposition 3.2 in [5], it is a proper morphism which is injective on $\mathbb{C}_\infty$-valued points. Since $S^{r'}_{F',\mathcal{K}(I')}$ and $S^r_{F,\mathcal{K}(I)}$ are both affine schemes of finite type over $\mathbb{C}_\infty$, the morphism $\iota^{F'}_{F,\,b}$ is therefore a proper morphism of finite presentation with finite fibers. This implies that $\iota^{F'}_{F,\,b}$ is finite by Theorem 8.11.1 of EGA IV [20].

We now check that $\iota^{F'}_{F,\,b}$ is given by (2.2.1) on $\mathbb{C}_\infty$-valued points. By our discussion in the proof of Theorem 1.1.2, a $\mathbb{C}_\infty$-valued point $p = [(\overline{\omega'}, h')] \in S'(\mathbb{C}_\infty)$ corresponds to the Drinfeld A'-module associated to the A'-lattice

$$\Lambda := \omega'(F'^{r'} \cap h'\widehat{A'}^{r'}) \subset \mathbb{C}_\infty$$

of rank r' with I'-level structure given by the composition

$$(I'^{-1}/A')^{r'} \xrightarrow{h'} I'^{-1} \cdot (F'^{r'} \cap h'\widehat{A'}^{r'})/(F'^{r'} \cap h'\widehat{A'}^{r'}) \xrightarrow{\omega'} I'^{-1} \cdot \Lambda/\Lambda.$$

The $\mathbb{C}_\infty$-valued point $\iota^{F'}_{F,\,b}(p) \in S(\mathbb{C}_\infty)$ corresponds to the restriction of this Drinfeld module to A, which is associated to the same $\Lambda \subset \mathbb{C}_\infty$ (considered as A-lattice), together with the composition of $(I^{-1}/A)^r \xrightarrow{b} (I'^{-1}/A')^{r'}$ with the above level structure (note that $I'^{-1} \cdot \Lambda = I^{-1} \cdot \Lambda \subset \mathbb{C}_\infty$). Because of

$$\Lambda = \omega'(F'^{r'} \cap h'\widehat{A'}^{r'}) = (\omega' \circ \varphi)(F^r \cap (\varphi^{-1} \circ h' \circ b)\widehat{A}^r),$$

we indeed have $\iota^{F'}_{F,\,b}(p) = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)] \in S(\mathbb{C}_\infty)$.

Case (ii): For $b : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ and $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ arbitrary, we choose

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A}')$,
- an $\mathbb{A}_F^f$-linear isomorphism $b' : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ with $b'(\widehat{A}^r) = \widehat{A}'^{r'}$,
- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}_F^f)$.

Then $g' \circ b = b' \circ g^{-1}$, hence

$$g'\mathcal{K}'g'^{-1} = (b'g^{-1}\mathcal{K}gb'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \supset (b'\mathcal{K}(I)b'^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) = \mathcal{K}(IA'),$$

and by case (i) and Theorem 2.1.1, the composition of morphisms

$$S^{r'}_{F',\mathcal{K}(IA')} \xrightarrow{\iota^{F'}_{F,\,b'}} S^r_{F,\mathcal{K}(I)} \xrightarrow{\pi_g} S^r_{F,\mathcal{K}}$$

is defined and finite. Because of

$$(g'\mathcal{K}'g'^{-1})b'g^{-1} = b'g^{-1}(b^{-1}\mathcal{K}'b) \subset b'g^{-1}\mathcal{K}$$

this composition is invariant under the action of $g'\mathcal{K}'g'^{-1}$ on $S^{r'}_{F',\mathcal{K}(IA')}$. Hence, it induces a finite morphism $f : S^{r'}_{F',g'\mathcal{K}'g'^{-1}} \to S^r_{F,\mathcal{K}}$ such that the diagram

$$\begin{array}{ccc} S^{r'}_{F',\mathcal{K}(IA')} & \xrightarrow{\iota^{F'}_{F,\,b'}} & S^r_{F,\mathcal{K}(I)} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_g} \\ S_{F',g'\mathcal{K}'g'^{-1}} & \xrightarrow{f} & S^r_{F,\mathcal{K}} \end{array}$$

commutes. We can now define $\iota^{F'}_{F,\,b} := f \circ \pi_{g'}$, where $\pi_{g'} : S^{r'}_{F',\mathcal{K}'} \to S^{r'}_{F',g'\mathcal{K}'g'^{-1}}$. For $[(\overline{\omega'}, h')] \in S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$ we indeed have

$$\iota^{F'}_{F,\,b}([(\overline{\omega'}, h')]) = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h'g'^{-1} \circ b' \circ g^{-1})] = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)],$$

independently of the choice of $\varphi : F^r \xrightarrow{\sim} F'^{r'}$ and the representative $(\overline{\omega'},\, h') \in \Omega^{r'}_{F'} \times \mathrm{GL}_{r'}(\mathbb{A}^{r'}_{F'})$. This also shows that our definition of $\iota^{F'}_{F,\,b}$ is independent of the choice of g', b' and I.

It remains only to prove that $\iota^{F'}_{F,\,b}$ is injective on $\mathbb{C}_\infty$-valued points, i.e., that the map (2.2.1) is injective. For this, consider two elements $[(\overline{\omega'_1}, h'_1)]$, $[(\overline{\omega'_2}, h'_2)]$ of $S'(\mathbb{C}_\infty)$ with $\overline{\omega'_1}$, $\overline{\omega'_2} \in \Omega^{r'}_{F'}$ associated to $\omega'_1, \omega'_2 : F'^{r'}_\infty \hookrightarrow \mathbb{C}_\infty$ and h'_1, $h'_2 \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ which are mapped to the same element of $S(\mathbb{C}_\infty)$. This means that there exist $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that

(i) $\overline{\omega_1' \circ \varphi \circ T^{-1}} = \overline{\omega_2' \circ \varphi}$,

(ii) $T(\varphi^{-1} \circ h_1' \circ b)k = \varphi^{-1} \circ h_2' \circ b$.

By (i) there is a $\rho \in \mathbb{C}_\infty^*$ such that the diagram

$$\begin{array}{ccccc} F_\infty^r & \xrightarrow[\sim]{\varphi} & F'^{r'}_{\infty'} & \xhookrightarrow{\omega_1'} & \mathbb{C}_\infty \\ \downarrow{\scriptstyle T} & & & & \downarrow{\scriptstyle \rho} \\ F_\infty^r & \xrightarrow[\sim]{\varphi} & F'^{r'}_{\infty'} & \xhookrightarrow{\omega_2'} & \mathbb{C}_\infty \end{array}$$

commutes. Since the maps ω_1', ω_2', ρ are injective and F'-linear, this implies that the F-linear automorphism $T' := \varphi \circ T \circ \varphi^{-1}$ of $F'^{r'}$ is also F'-linear and lies in $\mathrm{GL}_{r'}(F')$. Thus, we have $T' \cdot \overline{\omega_1'} = \overline{\omega_2'}$, i.e., $\overline{\omega_1'}$ and $\overline{\omega_2'}$ lie in the same $\mathrm{GL}_{r'}(F')$-orbit.

Equation (ii) implies that $T' h_1'(b \circ k \circ b^{-1}) = h_2'$ in $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$. Since h_1', h_2' and T' all lie in $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$, we conclude that

$$b \circ k \circ b^{-1} \in \mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f),$$

i.e., $[(\overline{\omega_1'}, h_1')] = [(\overline{\omega_2'}, h_2')]$ in $S'(\mathbb{C}_\infty)$. □

Since the morphism $\iota_{F,\,b}^{F'} : S' \to S$ is injective on $\mathbb{C}_\infty$-valued points, we call it an *inclusion* of Drinfeld modular varieties (by a slight abuse of terminology). If $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is amply small (in the sense of Definition 2.1.2), we can show that it is in fact a closed immersion:

Proposition 2.2.2. *Let $\iota_{F,\,b}^{F'} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ be an inclusion of Drinfeld modular varieties with $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ amply small. Then $\mathcal{K}' \subset \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ is also amply small and $\iota_{F,\,b}^{F'}$ is a closed immersion of varieties.*

Before giving the proof of Proposition 2.2.2, we summarize the description of the tangent spaces at the closed points of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ with $\mathcal{K} = \mathcal{K}(I)$ for a proper ideal I of A given in [17].

We use for $a \in A$ the notation

$$\deg a := \log_q(|A/(a)|)$$

and denote by $\mathbb{C}_\infty\{\{\tau\}\}$ the ring of formal non-commutative power series in the variable τ with coefficients in $\mathbb{C}_\infty$ and the commutator rule $\tau\lambda = \lambda^q\tau$ for $\lambda \in \mathbb{C}_\infty$.

Definition 2.2.3. *Let $\varphi : A \to \mathbb{C}_\infty\{\tau\}$ be a Drinfeld module over $\mathbb{C}_\infty$ of rank r. An $\mathbb{F}_q$-linear map $\eta : A \to \tau\mathbb{C}_\infty\{\tau\}$ is called a* derivation *with respect to φ if, for all $a, b \in A$, the derivation rule*

$$\eta_{ab} = a\eta_b + \eta_a \circ \varphi_b$$

is satisfied. Such a derivation is called reduced *resp.* strictly reduced *if it satisfies $\deg_\tau \eta_a \leq r \cdot \deg a$ resp. $\deg_\tau \eta_a < r \cdot \deg a$ for all $a \in A$. The space of reduced resp. strictly reduced derivations $A \to \tau\mathbb{C}_\infty\{\tau\}$ with respect to φ is denoted by $D_r(\varphi)$ resp. $D_{sr}(\varphi)$.*

Theorem 2.2.4. *Let x be a $\mathbb{C}_\infty$-valued point of $S^r_{F,\mathcal{K}(I)}$ corresponding to a Drinfeld A-module φ with I-level structure α. Then there is a natural isomorphism*

$$T_x(S^r_{F,\mathcal{K}(I)}) \xrightarrow{\sim} D_{sr}(\varphi) \tag{2.2.3}$$

of vector spaces over $\mathbb{C}_\infty$.

Proof. This follows from the discussion in the proof of Theorem 6.11 in [17] and the lemmata before this proof. □

The isomorphism (2.2.3) is given as follows: A tangent vector $\xi \in T_x(S^r_{F,\mathcal{K}(I)})$ is an element of $S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2))$ which projects to $x \in S^r_{F,\mathcal{K}(I)}(\mathbb{C}_\infty)$ under the canonical projection $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2) \to \mathbb{C}_\infty$. It corresponds to the isomorphism class of a Drinfeld A-module over $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2)$ with I-level structure which projects to (φ, α) under the canonical projection $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2) \to \mathbb{C}_\infty$. There is a unique Drinfeld A-module $\tilde{\varphi}$ in this isomorphism class such that, for all $a \in A$,

$$\tilde{\varphi}_a = \varphi_a + \varepsilon \cdot \eta_a$$

where $a \mapsto \eta_a$ is a strictly reduced derivation with respect to φ. The tangent vector ξ is mapped to this strictly reduced derivation under (2.2.3).

Theorem 2.2.5. *Let φ be the Drinfeld A-module over $\mathbb{C}_\infty$ associated to an A-lattice $\Lambda \subset \mathbb{C}_\infty$. Then there is a natural isomorphism*

$$D_r(\varphi) \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty). \tag{2.2.4}$$

The $\mathbb{C}_\infty$-linear subspace $D_{sr}(\varphi) \subset D_r(\varphi)$ is mapped to a subspace of $\mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty)$ which is a complement of $\mathbb{C}_\infty \cdot \mathrm{id}$, where $\mathrm{id} : \Lambda \hookrightarrow \mathbb{C}_\infty$ is the canonical inclusion.

Proof. See Theorem 5.14 in [16] and Theorem 6.10 in [16]. □

The isomorphism (2.2.4) is called *de Rham isomorphism* and can be described as follows: Let η be a reduced derivation with respect to φ. Then, for all non-constant $a \in A$, there is a unique solution $F_\eta \in \mathbb{C}_\infty\{\{\tau\}\}$ satisfying the difference equation

$$F_\eta(az) - aF_\eta(z) = \eta_a(e_\Lambda(z)) \tag{2.2.5}$$

where

$$e_\Lambda(z) = z \cdot \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda)$$

denotes the exponential function associated to the lattice Λ. This solution is independent of the choice of $a \in A$ and defines an entire function $\mathbb{C}_\infty \to \mathbb{C}_\infty$ which restricts to an A-linear map $\Lambda \to \mathbb{C}_\infty$. The reduced derivation η is mapped to $F_\eta|_\Lambda$ under (2.2.4).

Proof of Proposition 2.2.2. As in the proof of Theorem 2.2.1, we denote the integral closure of A in F' by A'.

We first show that $\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ is amply small. Since $\mathcal{K}$ is amply small, there is a proper ideal I of A and a $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ such that $g\mathcal{K}g^{-1} \subset \mathcal{K}(I)$. Therefore $\mathcal{K}'$ is contained in the subgroup

$$(bg^{-1}\mathcal{K}(I)gb^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$$

of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$. This subgroup exactly consists of all elements of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ which stabilize $\Lambda := bg^{-1}(\hat{A}^r) \subset (\mathbb{A}_{F'}^f)^{r'}$ and induce the identity on $\Lambda/I \cdot \Lambda$. Since these elements are $\mathbb{A}_{F'}^f$-linear, they also stabilize the $\widehat{A}'$-lattice $\Lambda' := \widehat{A}' \cdot \Lambda$ and induce the identity on $\Lambda'/I \cdot \Lambda'$. Since Λ' is a finitely generated $\widehat{A}'$-submodule of $(\mathbb{A}_{F'}^f)^{r'}$ with $\mathbb{A}_{F'}^f \cdot \Lambda' = (\mathbb{A}_{F'}^f)^{r'}$ and $\widehat{A}'$ is a direct product of principal ideal domains, Λ' is a free $\widehat{A}'$-module of rank r'. Hence, there is a $g' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ such that $\Lambda' = g'\widehat{A}'^{r'}$ and

$$\mathcal{K}' \subset (bg^{-1}\mathcal{K}(I)gb^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \subset g'\mathcal{K}(I')g'^{-1}$$

for $I' := IA'$. This implies that $\mathcal{K}'$ is amply small.

To show that $\iota_{F,b}^{F'}$ is a closed immersion, we use the following criterion given in Proposition 12.94 of [21]:

A proper morphism $f : X \to Y$ of varieties over an algebraically closed field K is a closed immersion if and only if the map $X(K) \to Y(K)$ induced by f is injective and, for all $x \in X(K)$, the induced map on Zariski tangent spaces $T_x(X) \to T_{f(x)}(Y)$ is injective.

Since the morphism $\iota_{F,b}^{F'}$ is finite and injective on $\mathbb{C}_\infty$-valued points by Theorem 2.2.1, it is proper and we therefore only have to show that, for all $x \in$

$S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$, the induced map on Zariski tangent spaces $\iota^{F'}_{F,\,b_*} : T_x(S^{r'}_{F',\mathcal{K}'}) \to T_{\iota^{F'}_{F,\,b}(x)}(S^r_{F,\mathcal{K}})$ is injective.

Case (i): As in the proof of Theorem 2.2.1, we first consider the case where $b(\hat{A}^r) = \widehat{A}'^{r'}$ and $\mathcal{K} = \mathcal{K}(I)$ is a principal congruence subgroup modulo a proper ideal I of A. In this case, we have $\mathcal{K}' = \mathcal{K}(I')$ with $I' := IA'$. We can therefore use the description of the tangent spaces given above.

Let $x \in S^{r'}_{F',\mathcal{K}(I')}(\mathbb{C}_\infty)$ be a point corresponding to the Drinfeld A'-module φ associated to an A'-lattice $\Lambda \subset \mathbb{C}_\infty$ of rank r' with I'-level structure. Since we defined $\iota^{F'}_{F,\,b}$ by restricting Drinfeld A'-modules to Drinfeld A-modules, the point $\iota^{F'}_{F,\,b}(x) \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ corresponds to the Drinfeld A-module $\varphi|_A$ associated to the same $\Lambda \subset \mathbb{C}_\infty$ considered as A-lattice of rank r with some I-level structure. We can therefore consider the following diagram

$$\begin{array}{ccccc} T_x(S^{r'}_{F',\mathcal{K}(I')}) & \xrightarrow[\sim]{(2.2.3)} & D_{sr}(\varphi) & \xhookrightarrow{(2.2.4)} & \mathrm{Hom}_{A'}(\Lambda, \mathbb{C}_\infty) \\ \downarrow{\scriptstyle \iota^{F'}_{F,\,b_*}} & & \downarrow & & \downarrow \\ T_{\iota^{F'}_{F,\,b}(x)}(S^r_{F,\mathcal{K}}) & \xrightarrow[\sim]{(2.2.3)} & D_{sr}(\varphi|_A) & \xhookrightarrow{(2.2.4)} & \mathrm{Hom}_A(\Lambda, \mathbb{C}_\infty) \end{array}$$

where the vertical arrow in the middle denotes the restriction of derivations from A' to A and the one at the right the canonical inclusion. The left square of the diagram commutes by the definition of (2.2.3) because $\iota^{F'}_{F,\,b_*}$ has the modular interpretation of restricting Drinfeld A'-modules over $\mathbb{C}_\infty[\varepsilon]/(\varepsilon^2)$ to A. The right square also commutes because the unique solution of (2.2.5) is independent of $a \in A'$ and Λ as an A'-lattice has the same exponential function as Λ as an A-lattice.

Hence, the diagram commutes and, since the right vertical arrow is an injective map, also the other two are injective maps. In particular, the induced map $\iota^{F'}_{F,\,b_*}$ between tangent spaces is injective.

Case (ii): Let $b : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ be arbitrary and $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ be an arbitrary amply small subgroup. Then, by the construction in the proof of Theorem 2.2.1, there is

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A}')$,
- an $\mathbb{A}_F^f$-linear isomorphism $b' : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ with $b'(\widehat{A}^r) = \widehat{A}'^{r'}$,
- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}_F^f)$

such that the diagram

$$\begin{array}{ccc} S^{r'}_{F',\mathcal{K}(I')} & \xrightarrow{\iota^{F'}_{F,\,b'}} & S^{r}_{F,\mathcal{K}(I)} \\ \downarrow{\scriptstyle \pi_{g'^{-1}}} & & \downarrow{\scriptstyle \pi_g} \\ S^{r'}_{F',\mathcal{K}'} & \xrightarrow{\iota^{F'}_{F,\,b}} & S^{r}_{F,\mathcal{K}} \end{array}$$

with $I' := IA'$ commutes. By Proposition 2.1.3 and Corollary 2.1.4, the projection maps $\pi_{g'^{-1}}$ and π_g in this diagram are étale morphisms between non-singular varieties because $\mathcal{K}'$ and $\mathcal{K}$ are amply small. Hence, they induce isomorphisms on tangent spaces of closed points (Proposition 10.4 in [22]). By case (i), the upper horizontal arrow $\iota^{F'}_{F,\,b'}$ induces injections on tangent spaces of closed points. Therefore, the commutativity of the diagram implies that, for all $x \in S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$, the induced map $\iota^{F'}_{F,\,b*} : T_x(S^{r'}_{F',\mathcal{K}'}) \to T_{\iota^{F'}_{F,\,b}(x)}(S^{r}_{F,\mathcal{K}})$ is injective. □

The image of an inclusion $\iota^{F'}_{F,\,b} : S' \to S$ of Drinfeld modular varieties is a subvariety of S because finite morphisms are proper.

Definition 2.2.6. *A subvariety of S of the form $X = \iota^{F'}_{F,\,b}(S')$ for an inclusion $\iota^{F'}_{F,\,b}$ is called a* Drinfeld modular subvariety *of S. An irreducible component of a Drinfeld modular subvariety over $\mathbb{C}_\infty$ is called a* special subvariety *and a special subvariety of dimension 0 a* special point*.*

Proposition 2.2.7. *Let $\iota^{F'}_{F,\,b_1} : S^{r'}_{F',\mathcal{K}'} \to S$ and $\iota^{F''}_{F,\,b_2} : S^{r''}_{F'',\mathcal{K}''} \to S$ be two inclusions of Drinfeld modular varieties with $F'' \subset F'$. Then for a $\mathbb{A}^f_{F''}$-linear isomorphism $c : (\mathbb{A}^f_{F''})^{r''} \to (\mathbb{A}^f_{F'})^{r'}$ with*

$$b_1 = c \circ b_2 \circ k \tag{2.2.6}$$

for some $k \in \mathcal{K}$, the diagram

$$\begin{array}{ccc} S^{r'}_{F',\mathcal{K}'} & & \\ \downarrow{\scriptstyle \iota^{F'}_{F'',\,c}} & \searrow^{\iota^{F'}_{F,\,b_1}} & \\ & & S \\ & \nearrow_{\iota^{F''}_{F,\,b_2}} & \\ S^{r''}_{F'',\mathcal{K}''} & & \end{array}$$

commutes.

Proof. First note that we have

$$\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) = (c\mathcal{K}''c^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$$

by the definition of $\mathcal{K}'$ and $\mathcal{K}''$ and equation (2.2.6). Therefore, there is an inclusion $\iota^{F'}_{F'',\,c} : S^{r'}_{F',\mathcal{K}'} \to S^{r''}_{F'',\mathcal{K}''}$.

Now we choose an F-linear isomorphism $\varphi : F^r \xrightarrow{\sim} F''^{r''}$ and an F''-linear isomorphism $\psi : F''^{r''} \xrightarrow{\sim} F'^{r'}$. The following calculation on $\mathbb{C}_\infty$-valued points shows the commutativity of the depicted diagram:

$$\begin{aligned}
\iota^{F''}_{F,\,b_2}(\iota^{F'}_{F'',\,c}([(\overline{\omega'}, h')])) &= [(\overline{\omega' \circ \psi \circ \varphi},\ \varphi^{-1} \circ \psi^{-1} \circ h' \circ c \circ b_2] \\
&= [(\overline{\omega' \circ \psi \circ \varphi},\ \varphi^{-1} \circ \psi^{-1} \circ h' \circ b_1 \circ k^{-1})] \\
&= [(\overline{\omega' \circ (\psi \circ \varphi)},\ (\psi \circ \varphi)^{-1} \circ h' \circ b_1)] = \iota^{F'}_{F,\,b_1}([(\overline{\omega'}, h')]). \quad \square
\end{aligned}$$

Lemma 2.2.8. *Let $\tilde{K} \subset \mathcal{K}$ be an open subgroup and $\pi_1 : S^r_{F,\tilde{\mathcal{K}}} \to S^r_{F,\mathcal{K}}$ the corresponding canonical projection. Then the following holds:*

(i) For each Drinfeld modular subvariety $X' \subset S^r_{F,\tilde{\mathcal{K}}}$, the image $\pi_1(X')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.

(ii) For each Drinfeld modular subvariety $X = \iota^{F'}_{F,\,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$, the preimage $\pi_1^{-1}(X)$ is a finite union of Drinfeld modular subvarieties of $S^r_{F,\tilde{\mathcal{K}}}$.

Proof. For (i), assume that X' is the image of the inclusion $\tilde{\iota}^{F'}_{F,\,b} : S^{r'}_{F',\tilde{\mathcal{K}}'} \to S^r_{F,\tilde{\mathcal{K}}}$ associated to the datum $(F',\, b)$ and consider the inclusion morphism $\iota^{F'}_{F,\,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ associated to the same datum. The diagram

$$\begin{array}{ccc}
S^{r'}_{F',\tilde{\mathcal{K}}'} & \xrightarrow{\tilde{\iota}^{F'}_{F,\,b}} & S^r_{F,\tilde{\mathcal{K}}} \\
\downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle \pi_1} \\
S^{r'}_{F',\mathcal{K}'} & \xrightarrow{\iota^{F'}_{F,\,b}} & S^r_{F,\mathcal{K}}
\end{array}$$

with π_1' and π_1 the respective canonical projections as defined in Section 2.1 commutes by definition of the inclusion morphisms. Hence,

$$\pi_1(X') = \iota^{F'}_{F,\,b}(\pi_1'(S^{r'}_{F',\tilde{\mathcal{K}}'})) = \iota^{F'}_{F,\,b}(S^{r'}_{F',\mathcal{K}'})$$

is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.

For (ii), choose a set of representatives $k_1, \ldots, k_l \in \mathcal{K}$ for the left cosets $\mathcal{K}/\tilde{\mathcal{K}}$ and consider the inclusion morphisms $\iota^{F'}_{F,\,b\circ k_i} : S^{r'}_{F',\tilde{\mathcal{K}}'_i} \to S^r_{F,\tilde{\mathcal{K}}}$ associated to $(F',\, b \circ k_i)$ for $i = 1, \ldots, l$. By the definition of the inclusion morphisms we have

$$\pi_1^{-1}(X) = \bigcup_{i=1}^{l} \iota^{F'}_{F,\,b\circ k_i}(S^{r'}_{F',\tilde{\mathcal{K}}'_i}),$$

hence $\pi_1^{-1}(X)$ is a finite union of Drinfeld modular subvarieties of $S^r_{F,\tilde{\mathcal{K}}}$. $\square$

Lemma 2.2.9. *For an inclusion $\iota_{F,b}^{F'} : S' \to S$, we have*

$$\mathrm{End}(p') = \mathrm{End}(\iota_{F,b}^{F'}(p'))$$

for all $p' \in S'(\mathbb{C}_\infty)$.

Remark: This is an equality of subfields of $\mathbb{C}_\infty$ and not just an abstract isomorphism of fields.

Proof. This follows from the definition of $\iota_{F,b}^{F'}$ because $(\omega' \circ \varphi)(F^r) = \omega'(F'^{r'})$ for $\overline{\omega'} \in \Omega_{F'}^{r'}$ and an F-linear isomorphism $\varphi : F^r \xrightarrow{\cong} F'^{r'}$. □

Now we give a criterion under which two Drinfeld modular subvarieties are contained in each other.

Proposition 2.2.10. *Let $X' = \iota_{F,b_1}^{F'}(S_{F',\mathcal{K}'}^{r'})$ and $X'' = \iota_{F,b_2}^{F''}(S_{F'',\mathcal{K}''}^{r''})$ be two Drinfeld modular subvarieties of S. The following statements are equivalent:*

(i) *X' is contained in X''.*

(ii) *There is an irreducible component of X' over $\mathbb{C}_\infty$ which is contained in X''.*

(iii) *$F'' \subset F'$ and there exist $k \in \mathcal{K}$ and an $\mathbb{A}_{F''}^f$-linear isomorphism $c : (\mathbb{A}_{F''}^f)^{r''} \to (\mathbb{A}_{F'}^f)^{r'}$ such that $b_1 = c \circ b_2 \circ k$.*

Proof. We write $S' = S_{F',\mathcal{K}'}^{r'}$ and $S'' = S_{F'',\mathcal{K}''}^{r''}$.

The implication (i) ⇒ (ii) is trivial.

For (ii) ⇒ (iii) assume that $\iota_{F,b_1}^{F'}(Y') \subset \iota_{F,b_2}^{F''}(S'')$ for an irreducible component Y' of S' over $\mathbb{C}_\infty$. By Lemma 1.1.6 there is a $p' = [(\overline{\omega'}, h')] \in Y'(\mathbb{C}_\infty)$ with $\mathrm{End}(p') = F'$. Now let $\iota_{F,b_1}^{F'}(p') = \iota_{F,b_2}^{F''}(p'')$ for a suitable $p'' = [(\overline{\omega''}, h'')] \in S''(\mathbb{C}_\infty)$. Lemma 1.1.5 and 2.2.9 yield

$$F' = \mathrm{End}(p') = \mathrm{End}(\iota_{F,b_1}^{F'}(p')) = \mathrm{End}(\iota_{F,b_2}^{F''}(p'')) = \mathrm{End}(p'') \supset F''.$$

Because of $\iota_{F,b_1}^{F'}(p') = \iota_{F,b_2}^{F''}(p'')$ we have

$$[(\overline{\omega' \circ \varphi_1}, \varphi_1^{-1} \circ h' \circ b_1)] = [(\overline{\omega'' \circ \varphi_2}, \varphi_2^{-1} \circ h'' \circ b_2)]$$

for F-linear isomorphisms $\varphi_1 : F^r \xrightarrow{\sim} F'^{r'}$ and $\varphi_2 : F^r \xrightarrow{\sim} F''^{r''}$. Hence, there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that

1. $\overline{\omega' \circ \varphi_1} = \overline{\omega'' \circ \varphi_2 \circ T^{-1}}$,

2. $\varphi_1^{-1} \circ h' \circ b_1 = T(\varphi_2^{-1} \circ h'' \circ b_2)k$.

Because of 1. and $F'' \subset F'$, one concludes as in the proof of Theorem 2.2.1 that the F-linear isomorphism $\psi := \varphi_1 \circ T \circ \varphi_2^{-1} : F''^{r''} \to F'^{r'}$ is F''-linear.

We set $c := b_1 \circ k^{-1} \circ b_2^{-1} : (\mathbb{A}_{F''}^f)^{r''} \to (\mathbb{A}_{F'}^f)^{r'}$. By 2. this is equal to

$$c = h'^{-1} \circ \varphi_1 \circ T \circ \varphi_2^{-1} \circ h'' = h'^{-1} \circ \psi \circ h''.$$

Since ψ is F''-linear and $F'' \subset F'$ we conclude that c is an $\mathbb{A}_{F''}^f$-linear isomorphism. Furthermore, we have $b_1 = c \circ b_2 \circ k$ by the definition of c, which shows (iii).

The implication (iii) $\Rightarrow$ (i) follows from Proposition 2.2.7. □

Corollary 2.2.11. *Let $X' = \iota_{F,\,b'}^{F''}(S_{F'',\mathcal{K}''}^{r''})$ be a fixed Drinfeld modular subvariety of S. Then the assignment*

$$X \longmapsto \iota_{F,\,b'}^{F''}(X)$$

is a bijection from the set of Drinfeld modular subvarieties of $S_{F'',\mathcal{K}''}^{r''}$ to the set of Drinfeld modular subvarieties of S contained in X'.

Proof. Since $\iota_{F,\,b'}^{F''}$ is injective on $\mathbb{C}_\infty$-valued points, it is enough to show that

(i) $\iota_{F,\,b'}^{F''}(X)$ is a Drinfeld modular subvariety of S for each Drinfeld modular subvariety X of $S_{F'',\mathcal{K}''}^{r''}$,

(ii) $(\iota_{F,\,b'}^{F''})^{-1}(X)$ is a Drinfeld modular subvariety of $S_{F'',\mathcal{K}''}^{r''}$ for every Drinfeld modular subvariety $X \subset X'$ of S.

For (i), let $X = \iota_{F'',\,c}^{F'}(S_{F',\mathcal{K}'}^{r'})$ be a Drinfeld modular subvariety of $S_{F'',\mathcal{K}''}^{r''}$. The map

$$b := c \circ b' : (\mathbb{A}_F^f)^r \to (\mathbb{A}_{F'}^f)^{r'}$$

is an $\mathbb{A}_F^f$-linear isomorphism, hence we can apply Proposition 2.2.7 to conclude that

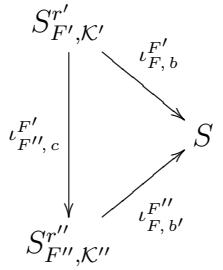
$$\iota_{F,\,b'}^{F''}(X) = \iota_{F,\,b'}^{F''}(\iota_{F'',\,c}^{F'}(S_{F',\mathcal{K}'}^{r'})) = \iota_{F,\,b}^{F'}(S_{F',\mathcal{K}'}^{r'})$$

is a Drinfeld modular subvariety of $S_{F,\mathcal{K}}^{r}$.

For (ii), let $X = \iota_{F,\,b}^{F'}(S_{F',\mathcal{K}'}^{r'})$ be a Drinfeld modular subvariety of S which is contained in X'. By Proposition 2.2.10, we have $F \subset F'' \subset F'$ and there are an $\mathbb{A}_{F''}^f$-linear isomorphism $c : (\mathbb{A}_{F''}^f)^{r''} \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ and a $k \in \mathcal{K}$ such that

$$b = c \circ b' \circ k.$$

By Proposition 2.2.7 the diagram



commutes. Therefore, $(\iota^{F''}_{F,\,b'})^{-1}(X) = \iota^{F'}_{F'',\,c}(S^{r'}_{F',\mathcal{K}'})$ is a Drinfeld modular subvariety of $S^{r''}_{F'',\mathcal{K}''}$. □

From Proposition 2.2.10, the following criterion for equality of Drinfeld modular subvarieties follows:

Corollary 2.2.12. *Let $X' = \iota^{F'}_{F,\,b_1}(S^{r'}_{F',\mathcal{K}'})$ and $X'' = \iota^{F''}_{F,\,b_2}(S^{r''}_{F'',\mathcal{K}''})$ be two Drinfeld modular subvarieties of S. The following statements are equivalent:*

(i) $X' = X''$.

(ii) *X' and X'' have a common irreducible component over $\mathbb{C}_\infty$.*

(iii) *$F' = F''$ (hence $r' = r''$) and there exist $s \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ and $k \in \mathcal{K}$ such that $b_1 = s \circ b_2 \circ k$.*

In particular, each special subvariety of S is an irreducible component over $\mathbb{C}_\infty$ of a unique Drinfeld modular subvariety of S. □

Corollary 2.2.13. *For a Drinfeld modular subvariety $X' \subset S$ there is a unique extension $F' \subset \mathbb{C}_\infty$ of F and a unique conjugacy class C of compact open subgroups of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ with $r' = r/[F'/F]$ such that $F'' = F'$ and $\mathcal{K}'' \in C$ for all inclusions $\iota^{F''}_{F,\,c} : S^{r''}_{F'',\mathcal{K}''} \to S$ with image X'.*

Proof. By definition, X' is the image of some inclusion $\iota^{F'}_{F,\,b} : S^{r'}_{F',\mathcal{K}'} \to S$. For any other inclusion $\iota^{F''}_{F,\,c} : S^{r''}_{F'',\mathcal{K}''} \to S$ with image X', Corollary 2.2.12 implies $F'' = F'$ and $b = s \circ c \circ k$ for suitable $s \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ and $k \in \mathcal{K}$. The latter implies $\mathcal{K}' = s\mathcal{K}''s^{-1}$, i.e., $\mathcal{K}''$ lies in the conjugacy class of $\mathcal{K}'$ in $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. □

The preceding corollary allows us to make the following definition:

Definition 2.2.14. *For a Drinfeld modular subvariety $X' = \iota_{F,\,b}^{F'}(S_{F',\mathcal{K}'}^{r'})$ of S, the extension $F' \subset \mathbb{C}_\infty$ of F is called the* reflex field *of X' and the index of $\mathcal{K}'$ in a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ is called the* index *of X' and is denoted by $i(X')$. Furthermore, the product*

$$D(X') := |\mathrm{Cl}(F')| \cdot i(X'),$$

where $\mathrm{Cl}(F')$ denotes the class group of $A' \subset F'$, is called the predegree *of X'.*

By Corollary 2.2.12, each special subvariety of S is an irreducible component of a unique Drinfeld modular subvariety of S. This allows us to define the reflex field of a special subvariety:

Definition 2.2.15. *For a special subvariety V of S which is an irreducible component of a Drinfeld modular subvariety X' of S, the reflex field of V is defined to be the reflex field of X'.*

If $\mathcal{K} = \mathrm{GL}_r(\hat{A})$, we can characterize the set of Drinfeld modular subvarieties of S with a given reflex field F' in terms of $\hat{A}$-lattices in $(\mathbb{A}_{F'}^f)^{r'}$:

Proposition 2.2.16. *Assume that $S = S_{F,\mathcal{K}}^r$ with $\mathcal{K} = \mathrm{GL}_r(\hat{A})$ and let $F' \subset \mathbb{C}_\infty$ be an extension of F of degree r/r' for some integer $r' \geq 1$ with only one place ∞' lying over ∞. Then the set of Drinfeld modular subvarieties of S with reflex field F' is in bijective correspondence with the set of orbits of the action of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ on the set of free $\hat{A}$-submodules of rank r of $(\mathbb{A}_{F'}^f)^{r'}$ via the assignment*

$$\iota_{F,\,b}^{F'}(S') \longmapsto \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \cdot b(\hat{A}^r).$$

Proof. For two Drinfeld modular subvarieties $\iota_{F,\,b_1}^{F'}(S_1')$ and $\iota_{F,\,b_2}^{F''}(S_2')$ of S with reflex field F' we have

$$\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \cdot b_1(\hat{A}^r) = \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \cdot b_2(\hat{A}^r)$$

if and only if there exist $s \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ and $k \in \mathcal{K} = \mathrm{GL}_r(\hat{A})$ with $b_1 = s \circ b_2^{-1} \circ k$ because the stabilizer of $\hat{A}^r$ in $\mathrm{GL}_r(\mathbb{A}_F^f)$ is exactly $\mathrm{GL}_r(\hat{A})$. Hence, the assignment $\iota_{F,\,b}^{F'}(S') \mapsto \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f) \cdot \varphi(m^{-1}\hat{A}^r)$ is well-defined and injective by Corollary 2.2.12. Furthermore, its image is exactly equal to the set of orbits of the action of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ on free $\hat{A}$-submodules of rank r of $(\mathbb{A}_{F'}^f)^{r'}$. □

Proposition 2.2.17. *The natural action of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ defined in Proposition 0.2.1 on the set of subvarieties of $S = S_{F,\mathcal{K}}^r$ which are defined over $\overline{F}$ restricts to an action on the set of Drinfeld modular subvarieties of S. For $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ and a Drinfeld modular subvariety $X = \iota_{F,\,b}^{F'}(S_{F',\mathcal{K}'}^{r'})$, the Galois conjugate $\sigma(X)$ is given by $\iota_{F,\,\sigma\circ b}^{\sigma(F')}(S_{\sigma(F'),\sigma\circ\mathcal{K}'\circ\sigma^{-1}}^{r'})$.*

Remark: In the above formula for the Galois conjugate $\sigma(X)$, the $\mathbb{A}_F^f$-linear isomorphism $(\mathbb{A}_{F'}^f)^{r'} \xrightarrow{\sim} (\mathbb{A}_{\sigma(F')}^f)^{r'}$ obtained by tensoring $\sigma : F' \xrightarrow{\sim} \sigma(F')$ with $(\mathbb{A}_F^f)^{r'}$ over F is also denoted by σ.

Proof. As explained in Section 0.1, we identify $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ with $\mathrm{Aut}_F(\overline{F})$ via the unique extension of the elements of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ to $\overline{F}$.

Case (i): We first consider the case where $S = S^r_{F,\mathcal{K}(I)}$ for a proper ideal I of A and $X = \iota^{F'}_{F,\,b}(S^{r'}_{F',\mathcal{K}'})$ for an inclusion morphism $\iota^{F'}_{F,\,b}$ associated to a datum $(F',\, b)$ satisfying $b(\widehat{A}^r) = \widehat{A'}^{r'}$ with A' the ring of elements of F' regular away from ∞'. As explained in the proof of Theorem 2.2.1, in this case we have $\mathcal{K}' = \mathcal{K}(I')$ with $I' = IA'$ and $\iota^{F'}_{F,\,b}$ is defined by the morphism (2.2.2) of functors from $\mathcal{F}^{r'}_{F',I'}$ to $\mathcal{F}^r_{F,I}$ (restricted to the subcategory of F'-schemes) using the modular interpretation of $S^{r'}_{F',\mathcal{K}(I')}$ and $S^r_{F,\mathcal{K}(I)}$.

Note that, for any Drinfeld A'-module $\varphi : A' \to \overline{F}\{\tau\}$ over $\overline{F}$,

$$\varphi^\sigma : \begin{array}{ccc} \sigma(A') & \longrightarrow & \overline{F}\{\tau\} \\ \sigma(a') & \longmapsto & (\varphi_{a'})^\sigma \end{array},$$

where $(\varphi_{a'})^\sigma$ is obtained from $\varphi_{a'}$ by applying σ to its coefficients, is a Drinfeld $\sigma(A')$-module over $\overline{F}$. Furthermore, for any I'-level structure $\alpha : (I'^{-1}/A')^{r'} \xrightarrow{\sim} \varphi_{I'} \subset \overline{F}$ on φ, the composition

$$(\sigma(I')^{-1}/\sigma(A'))^{r'} \xrightarrow{\sigma^{-1}} (I'^{-1}/A')^{r'} \xrightarrow{\alpha} \varphi_{I'} \xrightarrow{\sigma} (\varphi^\sigma)_{\sigma(I')}$$

is an $\sigma(I')$-level structure on φ^σ. Using the modular interpretation of $S^{r'}_{F',\mathcal{K}(I')}$ and $S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}$, the assignment

$$(\varphi,\, \alpha) \longmapsto (\varphi^\sigma,\, \sigma \circ \alpha \circ \sigma^{-1})$$

defines a map $g_\sigma : S^{r'}_{F',\mathcal{K}(I')}(\overline{F}) \to S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}(\overline{F})$. By construction, the map g_σ is bijective with inverse $g_{\sigma^{-1}}$.

Note that we have $(\sigma \circ b)(\hat{A}^r) = \widehat{\sigma(A')}^{r'}$ (where the $\mathbb{A}_F^f$-linear isomorphism $\sigma : (\mathbb{A}_{F'}^f)^{r'} \xrightarrow{\sim} (\mathbb{A}_{\sigma(F')}^f)^{r'}$ obtained by tensoring $\sigma : F \xrightarrow{\sim} \sigma(F)$ with $(\mathbb{A}_F^f)^{r'}$ over F is also denoted by σ). Hence the datum $(\sigma(F'), \sigma \circ b)$ defines an inclusion map

$$\iota^{\sigma(F')}_{F,\,\sigma\circ b} : S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))} \longrightarrow S^r_{F,\mathcal{K}(I)},$$

which is defined by a morphism of functors from $\mathcal{F}^{r'}_{\sigma(F'),\sigma(I')}$ to $\mathcal{F}^r_{F,I}$ (restricted

to the subcategory of $\sigma(F')$-schemes). We now show that the diagram

$$\begin{array}{ccc} S^{r'}_{F',\mathcal{K}(I')}(\overline{F}) & \xrightarrow{\iota^{F'}_{F,\,b}} & S^{r}_{F,\mathcal{K}(I)}(\overline{F}) \\ \downarrow{\scriptstyle g_\sigma} & & \downarrow{\scriptstyle \sigma} \\ S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}(\overline{F}) & \xrightarrow{\iota^{\sigma(F')}_{F,\,\sigma\circ b}} & S^{r}_{F,\mathcal{K}(I)}(\overline{F}) \end{array}$$

commutes, where the right vertical map is given by the action of σ on the closed points of $S^r_{F,\mathcal{K}(I)}$ defined over $\overline{F}$ defined in Proposition 0.2.1 (note that we identify $S^r_{F,\mathcal{K}(I)}(\overline{F})$ with the closed points of $S^r_{F,\mathcal{K}(I)}$ defined over $\overline{F}$ as discussed in Section 0.1).

Recall that we defined $S^r_{F,\mathcal{K}(I)}(\overline{F}) := M^r_{F,I}(\mathrm{Spec}(\overline{F}))$ where $M^r_{F,I}$ is the moduli scheme over F representing the functor $\mathcal{F}^r_{F,I}$ and satisfying $S^r_{F,\mathcal{K}(I)} = (M^r_{F,I})_{\mathbb{C}_\infty}$. For an $\overline{F}$-valued point $p : \mathrm{Spec}(\overline{F}) \to M^r_{F,I}$ in $S^r_{F,\mathcal{K}(I)}(\overline{F})$, by our definition of the Galois action in Section 0.2 and our conventions in Section 0.1, $\sigma(p)$ is the composition $p \circ \mathrm{Spec}(\sigma)$. Hence, the modular interpretation of $\sigma : S^r_{F,\mathcal{K}(I)}(\overline{F}) \to S^r_{F,\mathcal{K}(I)}(\overline{F})$ is given by pulling back Drinfeld A-modules over $\mathrm{Spec}(\overline{F})$ with I-level structure along $\mathrm{Spec}(\sigma)$. This is given by the assignment

$$(\varphi,\, \alpha) \longmapsto (\varphi^\sigma,\, \sigma \circ \alpha)$$

on Drinfeld A-modules $\varphi : A \to \overline{F}\{\tau\}$ of rank r with I-level structure $\alpha : (I^{-1}/A)^r \xrightarrow{\sim} \varphi_I \subset \overline{F}$.

From this, the commutativity of the above diagram follows together with the modular interpretation (2.2.2) of the inclusions $\iota^{F'}_{F,\,b}$ and $\iota^{\sigma(F')}_{F,\,\sigma\circ b}$. Indeed, for a Drinfeld A'-module $\varphi : A' \to \overline{F}\{\tau\}$ of rank r' over $\overline{F}$ together with I'-level structure $\alpha : (I'^{-1}/A')^{r'} \xrightarrow{\sim} \varphi_{I'} \subset \overline{F}$ corresponding to $p \in S^{r'}_{F',\mathcal{K}(I')}(\overline{F})$, the point $\sigma(\iota^{F'}_{F,\,b}(p))$ corresponds to

$$((\varphi|_A)^\sigma, \sigma \circ \alpha \circ b)$$

and $\iota^{\sigma(F')}_{F,\,\sigma\circ b}(g_\sigma(p))$ to

$$(\varphi^\sigma|_A,\, (\sigma \circ \alpha \circ \sigma^{-1}) \circ (\sigma \circ b)) = ((\varphi|_A)^\sigma, \sigma \circ \alpha \circ b).$$

Since, for any subvariety $Y \subset S$ defined over $\overline{F}$, the set $Y(\overline{F})$ of $\overline{F}$-valued points (viewed as a subset of the closed points of $Y \subset S$) is Zariski dense in Y (see, e.g., Corollary AG. 13.3 in [3]), the commutativity of the above diagram implies that

$$\sigma(X) = \iota^{\sigma(F')}_{F,\,\sigma\circ b}(g_\sigma(S^{r'}_{F',\mathcal{K}(I')})) = \iota^{\sigma(F')}_{F,\,\sigma\circ b}(S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))}).$$

for $X = \iota_{F,\,b}^{F'}(S^{r'}_{F',\mathcal{K}(I')})$. Hence, $\sigma(X)$ is a Drinfeld modular subvariety of S and it is of the desired form because of $\sigma \circ \mathcal{K}' \circ \sigma^{-1} = \mathcal{K}(\sigma(I'))$.

Case (ii): For a general $X = \iota_{F,\,b}^{F'}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$, by the construction in the proof of Theorem 2.2.1, there is

- a $g' \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$ with $g'\mathcal{K}'g'^{-1} \subset \mathrm{GL}_{r'}(\widehat{A}')$,
- an $\mathbb{A}_F^f$-linear isomorphism $b' : (\mathbb{A}_F^f)^r \xrightarrow{\sim} (\mathbb{A}_{F'}^f)^{r'}$ with $b'(\widehat{A}^r) = \widehat{A}'^{r'}$,
- a proper ideal I of A with $\mathcal{K}(I) \subset g^{-1}\mathcal{K}g$, where $g := b^{-1} \circ g'^{-1} \circ b' \in \mathrm{GL}_r(\mathbb{A}_F^f)$

such that the diagram

$$\begin{array}{ccc} S^{r'}_{F',\mathcal{K}(I')} & \xrightarrow{\iota_{F,\,b'}^{F'}} & S^r_{F,\mathcal{K}(I)} \\ \downarrow{\scriptstyle \pi_{g'^{-1}}} & & \downarrow{\scriptstyle \pi_g} \\ S^{r'}_{F',\mathcal{K}'} & \xrightarrow{\iota_{F,\,b}^{F'}} & S^r_{F,\mathcal{K}} \end{array}$$

with $I' := IA'$ commutes where π_g and $\pi_{g'^{-1}}$ are surjective and defined over F. This implies together with case (i)

$$\begin{aligned} \sigma(X) &= \sigma(\iota_{F,\,b}^{F'}(\pi_{g'^{-1}}(S^{r'}_{F',\mathcal{K}(I')}))) = \sigma(\pi_g(\iota_{F,\,b'}^{F'}(S^{r'}_{F',\mathcal{K}(I')}))) \\ &= \pi_g(\sigma(\iota_{F,\,b'}^{F'}(S^{r'}_{F',\mathcal{K}(I')}))) = \pi_g(\iota_{F,\,\sigma\circ b'}^{\sigma(F')}(S^{r'}_{\sigma(F'),\mathcal{K}(\sigma(I'))})). \end{aligned}$$

By a similar commutative diagram, this is equal to

$$\iota_{F,\,\sigma\circ b}^{\sigma(F')}(S^{r'}_{\sigma(F'),\sigma\circ\mathcal{K}'\circ\sigma^{-1}}),$$

hence a Drinfeld modular subvariety of S of the desired form. □

2.3 Determinant map and irreducible components

For a general Drinfeld modular variety $S^r_{F,\mathcal{K}}$, we denote by $\det \mathcal{K} \subset (\mathbb{A}_F^f)^*$ the image of $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ under the determinant map. Since the determinant map is a group homomorphism and maps principal congruence subgroups of $\mathrm{GL}_r(\mathbb{A}_F^f)$ to principal congruence subgroups of $(\mathbb{A}_F^f)^*$, the subgroup $\det \mathcal{K} \subset (\mathbb{A}_F^f)^*$ is open and compact.

Definition 2.3.1. *The map* $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty) \to S^1_{F,\det \mathcal{K}}(\mathbb{C}_\infty)$ *given by*

$$\begin{array}{rcl} \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f) \,/\, \mathcal{K}) & \longrightarrow & F^* \setminus (\mathbb{A}_F^f)^* \,/\, \det \mathcal{K} \\ [(\overline{\omega}, h)] & \longmapsto & [\det h]. \end{array}$$

is called determinant map *and is denoted by* det.

Remark: The determinant map can be described in terms of the modular interpretation, using the construction of exterior powers of Drinfeld modules in [23, Theorem 3.3]. We refrain from doing so because we do not need that.

Proposition 2.3.2. *The determinant map is surjective and its fibers are exactly the irreducible components of* $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$.

Proof. The surjectivity is immediate because $\det : \mathrm{GL}_r(\mathbb{A}_F^f) \to (\mathbb{A}_F^f)^*$ is surjective.

We know by Proposition 1.1.3 that the irreducible components of $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ are in bijective correspondence with the double quotient $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$. A point $[(\overline{\omega}, h)] \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ lies in the irreducible component corresponding to a double coset $[g] \in \mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ if and only if $h \in [g]$.

We show that, for every $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$, the fiber of $[\det g] \in S^1_{F,\det \mathcal{K}}(\mathbb{C}_\infty)$ is equal to the irreducible component corresponding to $[g] \in \mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$. By the above remarks, this is equivalent to

$$h \in \mathrm{GL}_r(F) \cdot g \cdot \mathcal{K} \Longleftrightarrow \det h \in F^* \cdot (\det g) \cdot (\det \mathcal{K})$$

for all $h \in \mathrm{GL}_r(\mathbb{A}_F^f)$.

If $h \in \mathrm{GL}_r(F) \cdot g \cdot \mathcal{K}$, then we have $\det h \in F^* \cdot (\det g) \cdot (\det \mathcal{K})$ by the multiplicativity of the determinant. Conversely, assume that $\det h \in F^* \cdot (\det g) \cdot (\det \mathcal{K})$. Then there are $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ such that

$$\det h = \det(T \cdot g \cdot k),$$

hence $Tgkh^{-1} \in \mathrm{SL}_r(\mathbb{A}_F^f)$. By the strong approximation theorem [33] for semi-simple simply connected groups over function fields, $\mathrm{SL}_r(F)$ is dense in $\mathrm{SL}_r(\mathbb{A}_F^f)$. Since $h\mathcal{K}h^{-1}$ is an open subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$, we therefore have

$$\mathrm{SL}_r(\mathbb{A}_F^f) = \mathrm{SL}_r(F) \cdot ((h\mathcal{K}h^{-1}) \cap \mathrm{SL}_r(\mathbb{A}_F^f)).$$

So there are $T' \in \mathrm{SL}_r(F)$ and $k' \in \mathcal{K} \cap \mathrm{SL}_r(\mathbb{A}_F^f)$ such that $Tgkh^{-1} = T'hk'h^{-1}$. This implies

$$h = T'^{-1}Tgkk'^{-1} \in \mathrm{GL}_r(F) \cdot g \cdot \mathcal{K}. \qquad \square$$

By Proposition 2.3.2, the determinant map induces a bijection

$$\det_* : \pi_0(S^r_{F,\mathcal{K}}) \xrightarrow{\sim} S^1_{F,\det\mathcal{K}}$$

between the set $\pi_0(S^r_{F,\mathcal{K}})$ of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$ and the set $S^1_{F,\det\mathcal{K}}$ (we identify the latter set with $S^1_{F,\det\mathcal{K}}(\mathbb{C}_\infty)$ as explained in Section 1.2). We now consider the natural action defined in Proposition 0.2.1 of the absolute Galois group $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ on these two sets.

Proposition 2.3.3. *The bijection* $\det_*$ *is* G_F*-equivariant.*

Proof. We consider separable extensions $F' \subset \mathbb{C}_\infty$ of F of degree r with only one place ∞' above ∞. The intersection F'' of all these extensions is equal to F. This follows by induction over r:

Assume by contradiction that $F'' \supsetneq F$ with $[F''/F] = r' > 1$. By Eisenstein's Criterion (Proposition III.1.14 in [35]) we find a second extension $F_2'' \neq F''$ of F of degree r' with only one place ∞_2'' above ∞. By induction hypothesis, the intersection of all separable extensions of F_2'' of degree r/r' with only one place above ∞_2'' is equal to F_2''. These extensions of F_2'' are all separable extensions of F of degree r with only one place above ∞, hence its intersection F_2'' contains F''. This is not possible because $F_2'' \neq F''$ and $[F_2''/F] = [F''/F] = r'$.

The equality $F'' = F$ implies that the subgroups $\mathrm{Gal}(F^{\mathrm{sep}}/F') \subset G_F$ where F' runs over all separable extensions of F of degree r with only one place above ∞ generate the whole absolute Galois group G_F. Therefore it is enough to show that $\det_*$ is $\mathrm{Gal}(F^{\mathrm{sep}}/F')$-equivariant for all these extensions F'.

Let now F'/F be a fixed extension of the above form, Y an irreducible component of $S^r_{F,\mathcal{K}}$ and $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F')$. We have to show that $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$. We assume that Y corresponds to the class of $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ in $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$ via the bijective correspondence from Proposition 1.1.3. We choose an F-linear isomorphism

$$\varphi : F^r \xrightarrow{\sim} F',$$

and define

$$b := \varphi \circ g : (\mathbb{A}_F^f)^r \xrightarrow{\sim} \mathbb{A}_{F'}^f.$$

The datum $(F',\, b)$ defines an inclusion morphism $\iota_{F,\,b}^{F'} : S^1_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$. By its definition, the point $p' := [1] \in S^1_{F',\mathcal{K}'} = F'^* \setminus (\mathbb{A}_{F'}^f)^*/\mathcal{K}'$ is mapped to the closed point

$$p := \iota_{F,\,b}^{F'}([1]) = [(\overline{i \circ \varphi},\, \varphi^{-1} \circ 1 \circ b)] = [(\overline{i \circ \varphi},\, g)]$$

of $S^r_{F,\mathcal{K}}$, where i denotes the canonical inclusion $F'_{\infty'} \hookrightarrow \mathbb{C}_\infty$. This point lies in the irreducible component Y, which corresponds to the class of g in $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f) \,/\, \mathcal{K}$.

By Proposition 0.2.2, the point $p' \in S^1_{F',\mathcal{K}'}$ is defined over $F'^{\mathrm{sep}} = F^{\mathrm{sep}}$. Since $\iota_{F,\,b}^{F'}$ is defined over F', the closed point $p = \iota_{F,\,b}^{F'}(p') \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ is also defined over F^{sep} and we have

$$\iota_{F,\,b}^{F'}(\sigma(p')) = \sigma(p) \in \sigma(Y),$$

i.e., $\sigma(Y)$ is the unique irreducible component of $S^r_{F,\mathcal{K}}$ containing $\iota_{F,\,b}^{F'}(\sigma(p'))$. The equality $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$ is therefore equivalent to

$$\det(\iota_{F,\,b}^{F'}(\sigma(p'))) = \sigma(\det p). \tag{2.3.1}$$

We use the description of the Galois action on $S^1_{F',\mathcal{K}'}$ and $S^1_{F,\det\mathcal{K}}$ given by Theorem 1.2.1 to calculate both sides of (2.3.1). For this, let H/F resp. H'/F' be the finite abelian extensions corresponding to the closed finite index subgroups $F^* \cdot \det\mathcal{K} \subset (\mathbb{A}_F^f)^*$ resp. $F'^* \cdot \mathcal{K}' \subset (\mathbb{A}_{F'}^f)^*$ in class field theory, and let E be the compositum of H and H'. Then the diagram of Artin maps

$$\begin{array}{ccc}
(\mathbb{A}_F^f)^* & \xrightarrow{\psi_{H/F}} & \mathrm{Gal}(H/F) \\
\uparrow{\scriptstyle N_{F'/F}} & & \uparrow{\scriptstyle r_{E/H}} \\
(\mathbb{A}_{F'}^f)^* & \xrightarrow{\psi_{E/F'}} & \mathrm{Gal}(E/F') \\
& \searrow^{\psi_{H'/F'}} & \downarrow{\scriptstyle r_{E/H'}} \\
& & \mathrm{Gal}(H'/F')
\end{array}$$

commutes with $N_{F'/F}$ the norm map and $r_{E/H}$, $r_{E/H'}$ the restriction maps. Therefore, if $h' \in (\mathbb{A}_{F'}^f)^*$ is chosen such that $\psi_{E/F'}(h') = \sigma|_E$, then we have

$$\begin{aligned}
\psi_{H'/F'}(h') &= \sigma|_{H'}, \\
\psi_{H/F}(N_{F'/F}(h')) &= \sigma|_H.
\end{aligned}$$

With Theorem 1.2.1 this implies

$$\begin{aligned}\det(\iota_{F,b}^{F'}(\sigma(p'))) &= \det(\iota_{F,b}^{F'}([h'^{-1}])) = \det(\varphi^{-1}\circ h'^{-1}\circ b)\\ &= \det(\varphi^{-1}\circ h'^{-1}\circ\varphi)\cdot\det g = [N_{F'/F}(h')^{-1}\cdot\det g]\\ &= \sigma([\det g]) = \sigma(\det p).\end{aligned}$$

So we have shown (2.3.1), which is equivalent to $\det_*(\sigma(Y)) = \sigma(\det_*(Y))$. □

Corollary 2.3.4. *The determinant map is induced by a unique morphism $S^r_{F,\mathcal{K}} \to S^1_{F,\det\mathcal{K}}$ defined over F.*

Proof. By Proposition 2.3.2, the determinant map is constant on the irreducible components of $S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$. Since these irreducible components and all closed points of $S^1_{F,\det\mathcal{K}}$ are defined over F^{sep}, the determinant map is therefore induced by a unique morphism defined over F^{sep}.

By Proposition 2.3.3, this morphism over F^{sep} is G_F-equivariant. Hence, by [3, AG 14.3] it is defined over F. □

Corollary 2.3.5. *$S^r_{F,\mathcal{K}}$ is F-irreducible and has exactly*

$$|S^1_{F,\det\mathcal{K}}| = |F^*\setminus(\mathbb{A}_F^f)^*/\det\mathcal{K}| = |\mathrm{Cl}(F)|\cdot|\hat{A}^*/(\mathbb{F}_q^*\cdot\det\mathcal{K})|$$

irreducible components over $\mathbb{C}_\infty$.

Proof. By Corollary 1.2.2 and Proposition 2.3.3, it follows that the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts transitively on the set of irreducible components of $S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$. Hence, $S^r_{F,\mathcal{K}}$ is F-irreducible by Proposition 0.2.2.

It only remains to show the second equality. Note that

$$\begin{array}{ccc}(\mathbb{A}_F^f)^* & \longrightarrow & I(A)\\ (x_{\mathfrak{p}}) & \longmapsto & \prod_{\mathfrak{p}}\mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}\end{array},$$

with $v_{\mathfrak{p}}$ the normalized discrete valuation associated to $\mathfrak{p}$ and $I(A)$ the group of fractional ideals of A, is a surjective homomorphism with kernel $\hat{A}^*$. Therefore there are isomorphisms of abelian groups

$$(\mathbb{A}_F^f)^*/(F^*\cdot\hat{A}^*) \cong F^*\setminus((\mathbb{A}_F^f)^*/\hat{A}^*) \cong F^*\setminus I(A) = \mathrm{Cl}(F).$$

The compact open subgroup $\det\mathcal{K}$ of $(\mathbb{A}_F^f)^*$ lies in the unique maximal compact open subgroup $\hat{A}^*$. Hence

$$|F^*\setminus(\mathbb{A}_F^f)^*/\det\mathcal{K}| = |(\mathbb{A}_F^f)^*/(F^*\cdot\det\mathcal{K})| = |\mathrm{Cl}(F)|\cdot|(F^*\cdot\hat{A}^*)/(F^*\cdot\det\mathcal{K})|.$$

The claim now follows from

$$(F^* \cdot \hat{A}^*)/(F^* \cdot \det \mathcal{K}) \cong \hat{A}^* \,/\, ((F^* \cdot \det \mathcal{K}) \cap \hat{A}^*)$$

and

$$(F^* \cdot \det \mathcal{K}) \cap \hat{A}^* = (F^* \cap \hat{A}^*) \cdot \det \mathcal{K} = \mathbb{F}_q^* \cdot \det \mathcal{K}. \qquad \square$$

Corollary 2.3.6. *Each Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ with reflex field F' is F'-irreducible.*

Proof. A Drinfeld modular subvariety X of $S^r_{F,\mathcal{K}}$ with reflex field F' is the image of an inclusion morphism $\iota^{F'}_{F,\,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$. Since $\iota^{F'}_{F,\,b}$ is defined over F' by Theorem 2.2.1, Corollary 2.3.5 immediately implies the F'-irreducibility of X. $\square$

Chapter 3

Degree of subvarieties

3.1 Compactification of Drinfeld modular varieties

In [30] Pink constructs the *Satake compactification* $\overline{S}^r_{F,\mathcal{K}}$ of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ with $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$. It is a normal projective variety which contains $S^r_{F,\mathcal{K}}$ as an open dense subvariety.

If $\mathcal{K}$ is amply small, $\overline{S}^r_{F,\mathcal{K}}$ is endowed with a natural ample invertible sheaf $\mathcal{L}^r_{F,\mathcal{K}}$. In [30], the space of global sections of its k-th power is defined to be the space of algebraic modular forms of weight k on $S^r_{F,\mathcal{K}}$.

If $\mathcal{K} \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ is arbitrary (not necessarily contained in $\mathrm{GL}_r(\hat{A})$) and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ is chosen such that $g\mathcal{K}g^{-1} \subset \mathrm{GL}_r(\hat{A})$, we can define

$$\overline{S}^r_{F,\mathcal{K}} := \overline{S}^r_{F,g\mathcal{K}g^{-1}} \tag{3.1.1}$$

and, if $\mathcal{K}$ is amply small,

$$\mathcal{L}^r_{F,\mathcal{K}} := \mathcal{L}^r_{F,g\mathcal{K}g^{-1}}. \tag{3.1.2}$$

As in Step (v) of the proof of Theorem 1.1.2, one can show, using part (i) of the following proposition for $\mathcal{K} \subset \mathrm{GL}_r(\hat{A})$, that this defines $\overline{S}^r_{F,\mathcal{K}}$ and $\mathcal{L}^r_{F,\mathcal{K}}$ up to isomorphism.

Proposition 3.1.1. *(i) For $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and a compact open subgroup $\mathcal{K}' \subset g^{-1}\mathcal{K}g$ the morphism $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ defined in Section 2.1 extends uniquely to a finite morphism $\overline{\pi_g} : \overline{S}^r_{F,\mathcal{K}'} \to \overline{S}^r_{F,\mathcal{K}}$ defined over F. If $\mathcal{K}$ is amply small, then there is a canonical isomorphism*

$$\mathcal{L}^r_{F,\mathcal{K}'} \cong \overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}}.$$

(ii) *Any inclusion $\iota_{F,\mathfrak{b}}^{F'} : S_{F',\mathcal{K}'}^{r'} \to S_{F,\mathcal{K}}^{r}$ of Drinfeld modular varieties extends uniquely to a finite morphism $\overline{\iota_{F,\mathfrak{b}}^{F'}} : \overline{S}_{F',\mathcal{K}'}^{r'} \to \overline{S}_{F,\mathcal{K}}^{r}$ defined over F'. If $\mathcal{K}$ is amply small, then there is a canonical isomorphism*

$$\mathcal{L}_{F',\mathcal{K}'}^{r'} \cong \overline{\iota_{F,\mathfrak{b}}^{F'}}^{*} \mathcal{L}_{F,\mathcal{K}}^{r}.$$

Proof. This follows from Proposition 4.11 and 4.12 and Lemma 5.1 in [30]. Note that these statements automatically hold for arbitrary levels $\mathcal{K}$ and $\mathcal{K}'$ (not necessarily contained in $\mathrm{GL}_r(\hat{A})$ respectively $\mathrm{GL}_{r'}(\widehat{A'})$) because the equations (3.1.1) and (3.1.2) define the Satake compactification of a general Drinfeld modular variety as the Satake compactification of a Drinfeld modular variety with level contained in $\mathrm{GL}_r(\hat{A})$ resp. $\mathrm{GL}_{r'}(\widehat{A'})$. □

3.2 Degree of subvarieties

In this section, $S_{F,\mathcal{K}}^{r}$ always denotes a Drinfeld modular variety with $\mathcal{K}$ amply small.

Definition 3.2.1. *The* degree *of an irreducible subvariety $X \subset S_{F,\mathcal{K}}^{r}$ is defined to be the degree of its Zariski closure $\overline{X}$ in $\overline{S}_{F,\mathcal{K}}^{r}$ with respect to $\mathcal{L}_{F,\mathcal{K}}^{r}$, i.e., the integer*

$$\deg X := \deg_{\mathcal{L}_{F,\mathcal{K}}^{r}} \overline{X} = \int_{\overline{S}_{F,\mathcal{K}}^{r}} c_1(\mathcal{L}_{F,\mathcal{K}}^{r})^{\dim X} \cap [\overline{X}],$$

where $c_1(\mathcal{L}_{F,\mathcal{K}}^{r}) \in A^1\overline{S}_{F,\mathcal{K}}^{r}$ denotes the first Chern class of $\mathcal{L}_{F,\mathcal{K}}^{r}$, the cycle class of $\overline{X}$ in $A_{\dim X}\overline{S}_{F,\mathcal{K}}^{r}$ is denoted by $[\overline{X}]$ and $\cap$ is the cap-product between $A^{\dim X}\overline{S}_{F,\mathcal{K}}^{r}$ and $A_{\dim X}\overline{S}_{F,\mathcal{K}}^{r}$.

The degree of a reducible subvariety $X \subset S_{F,\mathcal{K}}^{r}$ is the sum of the degrees of all irreducible components of X.

Remark: Note that our definition of degree for reducible subvarieties differs from the one used in many textbooks where only the sum over the irreducible components of maximal dimension is taken.

Lemma 3.2.2. *The degree of a subvariety $X \subset S_{F,\mathcal{K}}^{r}$ is at least the number of irreducible components of X.*

Proof. This follows by our definition of degree because $\mathcal{L}_{F,\mathcal{K}}^{r}$ is ample and the degree of an irreducible subvariety of a projective variety with respect to an ample invertible sheaf is a positive integer (see, e.g., Lemma 12.1 in [15]). □

Proposition 3.2.3. *(i) Let $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ be the morphism defined in Section 2.1 for $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and $\mathcal{K}' \subset g^{-1}\mathcal{K}g$. Then*

$$\deg \pi_g^{-1}(X) = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \deg X \tag{3.2.1}$$

for subvarieties $X \subset S^r_{F,\mathcal{K}}$ and

$$\deg \pi_g(X') \leq \deg X' \tag{3.2.2}$$

for subvarieties $X' \subset S^r_{F,\mathcal{K}'}$. In particular, we have

$$\deg T_g(X) \leq [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] \cdot \deg X \tag{3.2.3}$$

for subvarieties $X \subset S^r_{F,\mathcal{K}}$.

(ii) For any inclusion $\iota^{F'}_{F,\,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ of Drinfeld modular varieties and for any subvariety $X \subset S^{r'}_{F',\mathcal{K}'}$, we have

$$\deg X = \deg \iota^{F'}_{F,\,b}(X). \tag{3.2.4}$$

Proof. We use the *projection formula* for Chern classes (see, e.g., Proposition 2.5 (c) in [15]):

If $f : X \to Y$ is a proper morphism of varieties and $\mathcal{L}$ is an invertible sheaf on Y, then, for all k-cycles $\alpha \in A_k(X)$, we have the equality

$$f_*(c_1(f^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap f_*(\alpha) \tag{3.2.5}$$

of $(k-1)$-cycles in $A_{k-1}(Y)$.

For the proof of (3.2.1) and (3.2.2), we first assume that $X \subset S^r_{F,\mathcal{K}}$ and $X' \subset S^r_{F,\mathcal{K}'}$ are irreducible. For this, note that $\pi_g : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is finite of degree $[g^{-1}\mathcal{K}g : \mathcal{K}']$ by Theorem 2.1.1 and étale by Proposition 2.1.3 because $\mathcal{K}$ is amply small. The latter implies that the scheme-theoretic preimage of X under π_g is a reduced closed subscheme of $S^r_{F,\mathcal{K}'}$ (by Theorem III.10.2 in [22] the fibers of the generic points of the irreducible components of X are regular). Hence the scheme-theoretic preimage is equal to the subvariety $\pi_g^{-1}(X)$. Note that the degree of a finite, flat surjective morphism of varieties $f : V \to W$ is preserved under base extension by our characterisation of the degree as rank of the locally free $\mathcal{O}_W$-module $f_*(\mathcal{O}_V)$ in Section 0.1. Therefore the restriction of π_g to the subvariety $\pi_g^{-1}(X)$ is also finite of degree $[g^{-1}\mathcal{K}g : \mathcal{K}']$ and we have the equality

$$\overline{\pi_g}_*[\overline{\pi_g^{-1}(X)}] = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot [\overline{X}]$$

of cycles on $\overline{S}^r_{F,\mathcal{K}}$. For $d := \dim X$, with Proposition 3.1.1 (i) and the above projection formula we get

$$\begin{aligned}
\deg \pi_g^{-1}(X) &= \deg_{\overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}}} \overline{\pi_g^{-1}(X)} = \int_{\overline{S}^r_{F,\mathcal{K}'}} c_1(\overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\pi_g^{-1}(X)}] \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} \overline{\pi_g}_* \left(c_1(\overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\pi_g^{-1}(X)}] \right) \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap \overline{\pi_g}_* [\overline{\pi_g^{-1}(X)}] \\
&= [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{X}] = [g^{-1}\mathcal{K}g : \mathcal{K}'] \cdot \deg X.
\end{aligned}$$

For the proof of (3.2.2), we note that

$$\overline{\pi_g}_* [\overline{X'}] = \deg(\pi_g|_{X'}) \cdot [\overline{\pi_g(X')}]$$

as cycles on $\overline{S}^r_{F,\mathcal{K}}$. Again with the projection formula and Proposition 3.1.1 (i), we get

$$\begin{aligned}
\deg(\pi_g|_{X'}) \cdot \deg \pi_g(X') &= \deg(\pi_g|_{X'}) \cdot \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\pi_g(X')}] \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap \overline{\pi_g}_* [\overline{X'}] \\
&= \int_{\overline{S}^r_{F,\mathcal{K}}} \overline{\pi_g}_* \left(c_1(\overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{X'}] \right) \\
&= \int_{\overline{S}^r_{F,\mathcal{K}'}} c_1(\overline{\pi_g}^* \mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{X'}] = \deg X'
\end{aligned}$$

with $d := \dim X'$ and in particular

$$\deg \pi_g(X') \leq \deg X'.$$

If $X \subset S^r_{F,\mathcal{K}}$ is reducible with irreducible components $X_1, \ldots, X_n$, we have

$$\deg \pi_g^{-1}(X) = \sum_{i=1}^{n} \deg \pi_g^{-1}(X_i)$$

because the set of irreducible components of $\pi_g^{-1}(X)$ is exactly equal to

$$\bigcup_{i=1}^{n} \left\{ \text{irreducible components of } \pi_g^{-1}(X_i) \right\}$$

and this union is disjoint. Therefore, the formula (3.2.1) follows from the irreducible case.

If $X' \subset S^r_{F,\mathcal{K}'}$ is reducible with irreducible components $X'_1, \dots, X'_k$, then the set of irreducible components of $\pi_g(X')$ is a subset of $\{\pi_g(X'_1), \dots, \pi_g(X'_k)\}$, hence we have

$$\deg \pi_g(X') \leq \sum_{i=1}^{k} \deg \pi_g(X'_i)$$

and the inequality (3.2.2) follows from the irreducible case.

The inequality (3.2.3) immediately follows from (3.2.1) and (3.2.2) because

$$T_g(X) = \pi_g(\pi_1^{-1}(X))$$

where π_1 and π_g are projection morphisms $S^r_{F,\mathcal{K}_g} \to S^r_{F,\mathcal{K}}$ with $\mathcal{K}_g := \mathcal{K} \cap g^{-1}\mathcal{K}g$ and

$$\deg \pi_1 = [\mathcal{K} : \mathcal{K}_g] = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g].$$

Finally, for the proof of (3.2.4) we use that $\iota^{F'}_{F,b} : S^{r'}_{F',\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ is a closed immersion by Proposition 2.2.2 because $\mathcal{K}$ is amply small. For an irreducible subvariety $X \subset S^{r'}_{F',\mathcal{K}'}$, we therefore have the equality

$$\overline{\iota^{F'}_{F,b}}_*[\overline{X}] = [\overline{\iota^{F'}_{F,b}(X)}]$$

of cycles on $\overline{S}^r_{F,\mathcal{K}}$. The same calculation as in the proof of (3.2.2) with $d := \dim X$ therefore gives

$$\begin{aligned} \deg \iota^{F'}_{F,b}(X) &= \int_{\overline{S}^r_{F,\mathcal{K}}} c_1(\mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{\iota^{F'}_{F,b}(X)}] \\ &= \int_{\overline{S}^{r'}_{F',\mathcal{K}'}} c_1(\overline{\iota^{F'}_{F,b}}^* \mathcal{L}^r_{F,\mathcal{K}})^d \cap [\overline{X}] = \deg X \end{aligned}$$

because $\overline{\iota^{F'}_{F,b}}^* \mathcal{L}^r_{F,\mathcal{K}} \cong \mathcal{L}^{r'}_{F',\mathcal{K}'}$ by Proposition 3.1.1 (ii).

If $X \subset S^{r'}_{F',\mathcal{K}'}$ is reducible with irreducible components $X_1, \dots, X_l$, then $\iota^{F'}_{F,b}(X)$ has exactly the irreducible components $\iota^{F'}_{F,b}(X_1), \dots, \iota^{F'}_{F,b}(X_l)$ because $\iota^{F'}_{F,b}$ is a closed immersion. Therefore, the formula (3.2.4) for X reducible follows from the irreducible case. □

We will use the following two consequences of Bézout's theorem to get an upper bound for the degree of the intersection of two subvarieties of $S^r_{F,\mathcal{K}}$:

Lemma 3.2.4. *For subvarieties V, W of a projective variety U and an ample invertible sheaf $\mathcal{L}$ on U, we have*

$$\deg V \cap W \leq \deg V \cdot \deg W,$$

where deg *denotes the degree with respect to $\mathcal{L}$.*

Proof. See Example 8.4.6 in [15] in the case that V and W are irreducible.

If $V = V_1 \cup \cdots \cup V_k$ and $W = W_1 \cup \cdots \cup W_l$ are decompositions into irreducible components, then

$$V \cap W = \bigcup_{i,j} V_i \cap W_j.$$

Therefore, each irreducible component of $V \cap W$ is an irreducible component of some $V_i \cap W_j$. By our definition of degree for reducible varieties this implies

$$\deg V \cap W \leq \sum_{i,j} \deg(V_i \cap W_j).$$

Hence by the case that V and W are irredubible, we get

$$\deg V \cap W \leq \sum_{i,j} \deg V_i \cdot \deg W_j = \left(\sum_i \deg V_i \right) \cdot \left(\sum_j \deg W_j \right) = \deg V \cdot \deg W.$$

□

Lemma 3.2.5. *For subvarieties V, W of $S^r_{F,\mathcal{K}}$ we have*

$$\deg V \cap W \leq \deg V \cdot \deg W.$$

Proof. Recall that we defined the degree of a subvariety of $S^r_{F,\mathcal{K}}$ as the degree of its Zariski closure in the compactification $\overline{S}^r_{F,\mathcal{K}}$ with respect to the line bundle $\mathcal{L}^r_{F,\mathcal{K}}$. In view of the previous lemma, it is therefore enough to show the following inequality of degrees of Zariski closures in $\overline{S}^r_{F,\mathcal{K}}$ with respect to $\mathcal{L}^r_{F,\mathcal{K}}$:

$$\deg \overline{V \cap W} \leq \deg \overline{V} \cap \overline{W}.$$

For this, note that $\overline{V \cap W} \subset \overline{V} \cap \overline{W}$ and

$$\overline{V \cap W} \cap S^r_{F,\mathcal{K}} = V \cap W = (\overline{V} \cap S^r_{F,\mathcal{K}}) \cap (\overline{W} \cap S^r_{F,\mathcal{K}}) = (\overline{V} \cap \overline{W}) \cap S^r_{F,\mathcal{K}}$$

because $S^r_{F,\mathcal{K}}$ is Zariski open in $\overline{S}^r_{F,\mathcal{K}}$. Therefore

$$\overline{V} \cap \overline{W} = \overline{V \cap W} \ \cup \ (Y \cap (\overline{V} \cap \overline{W})) \tag{3.2.6}$$

where $Y := \overline{S}^r_{F,\mathcal{K}} \setminus S^r_{F,\mathcal{K}}$ denotes the boundary of the compactification. Since the irreducible components of $\overline{V \cap W}$ are the Zariski closures of the irreducible components of $V \cap W$, they all have non-empty intersection with $S^r_{F,\mathcal{K}}$. Hence, every irreducible component of $\overline{V \cap W}$ is not contained in $Y \cap (\overline{V} \cap \overline{W})$ and therefore by (3.2.6) an irreducible component of $\overline{V} \cap \overline{W}$. The desired inequality

$$\deg \overline{V \cap W} \leq \deg \overline{V} \cap \overline{W}$$

follows. □

3.3 Degree of Drinfeld modular subvarieties

We let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety.

Proposition 3.3.1. *If $\mathcal{K}$ is amply small, there is a constant $C > 0$ only depending on F and r such that*

$$\deg(X) \geq C \cdot D(X)$$

for all Drinfeld modular subvarieties $X \subset S^r_{F,\mathcal{K}}$ with $D(X)$ the predegree of X from Definition 2.2.14.

Remark: We expect that one could also prove an upper bound for $\deg(X)$ of the form $\deg(X) \leq C' \cdot D(X)$ with a constant C' depending on F, $\mathcal{K}$ and r. Because of this expectation we call $D(X)$ the predegree of X. We refrain from proving an upper bound because we only need a lower bound in the following.

Proof. Let $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ be an arbitrary Drinfeld modular subvariety of S. We fix primes $\mathfrak{p}$ of F and $\mathfrak{p}'$ of F' with $\mathfrak{p}'|\mathfrak{p}$ and let $\mathcal{K}(\mathfrak{p}') \subset \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ be the principal congruence subgroup modulo $\mathfrak{p}'$. By Proposition 2.2.2, the subgroup $\mathcal{K}' \subset \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ is amply small and, by definition, also $\mathcal{K}(\mathfrak{p}')$ is amply small. Therefore, by Proposition 3.2.3, we have

$$\begin{aligned} \deg(X) &= \deg(S^{r'}_{F',\mathcal{K}'}) = \frac{\deg(S^{r'}_{F',\mathcal{K}(\mathfrak{p}')\cap\mathcal{K}'})}{[\mathcal{K}' : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']} \\ &= \deg(S^{r'}_{F',\mathcal{K}(\mathfrak{p}')}) \cdot \frac{[\mathcal{K}(\mathfrak{p}') : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']}{[\mathcal{K}' : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']}. \end{aligned}$$

Now let $\mathcal{K}'_{\max}$ be a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ which contains $\mathcal{K}'$. Then $\mathcal{K}(\mathfrak{p}') \cap \mathcal{K}'$ is a subgroup of the same index in the maximal compact subgroups $\mathcal{K}'_{\max}$ and $\mathrm{GL}_{r'}(\hat{A}')$. Hence, we get

$$\frac{[\mathcal{K}(\mathfrak{p}') : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']}{[\mathcal{K}' : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']} = \frac{\frac{[\mathrm{GL}_{r'}(\hat{A}') : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']}{[\mathrm{GL}_{r'}(\hat{A}') : \mathcal{K}(\mathfrak{p}')]}}{\frac{[\mathcal{K}'_{\max} : \mathcal{K}(\mathfrak{p}') \cap \mathcal{K}']}{[\mathcal{K}'_{\max} : \mathcal{K}']}} = \frac{[\mathcal{K}'_{\max} : \mathcal{K}']}{[\mathrm{GL}_{r'}(\hat{A}') : \mathcal{K}(\mathfrak{p}')]}.$$

Since $S^{r'}_{F',\mathcal{K}(\mathfrak{p}')}$ has at least $|\mathrm{Cl}(F')|$ irreducible components over $\mathbb{C}_\infty$ by Corollary 2.3.5, we have $\deg(S^{r'}_{F',\mathcal{K}(\mathfrak{p}')}) \geq |\mathrm{Cl}(F')|$ by Lemma 3.2.2. With

$$[\mathrm{GL}_{r'}(\hat{A}') : \mathcal{K}(\mathfrak{p}')] < |k(\mathfrak{p}')|^{r'^2} \leq \left(|k(\mathfrak{p})|^{r/r'}\right)^{r'^2} \leq |k(\mathfrak{p})|^{r^2},$$

and $i(X) = [\mathcal{K}'_{\max} : \mathcal{K}']$ we therefore get the desired estimate

$$\deg(X) \geq |\mathrm{Cl}(F')| \cdot \frac{i(X)}{|k(\mathfrak{p})|^{r^2}} = \frac{1}{|k(\mathfrak{p})|^{r^2}} \cdot D(X).$$

Since we can choose the same prime $\mathfrak{p}$ of F in the above estimates for all Drinfeld modular subvarieties $X \subset S^r_{F,\mathcal{K}}$, the constant $C := \frac{1}{|k(\mathfrak{p})|^{r^2}}$ only depends on F and r. □

Theorem 3.3.2. *For each sequence (X_n) of pairwise distinct Drinfeld modular subvarieties of S, the sequence of predegrees $(D(X_n))$ is unbounded. In particular, if $\mathcal{K}$ is amply small, the degrees $\deg(X_n)$ are unbounded.*

Proof. By Proposition 3.3.1, it is enough to show that the sequence

$$D(X_n) = i(X_n) \cdot |\mathrm{Cl}(F_n)|$$

where F_n is the reflex field of X_n is unbounded.

The following two propositions imply that there are only finitely many extensions F' of F of degree dividing r and bounded class number:

Proposition 3.3.3. *There are only finitely many extensions $F' \subset \mathbb{C}_\infty$ of F of fixed genus g' and fixed constant field $\mathbb{F}_{q'}$.*

Proof. See the proof of Theorem 8.23.5. in [18]. □

Proposition 3.3.4. *Let F' be a function field of genus g' with field of constants $\mathbb{F}_{q'}$. Then*

$$|\mathrm{Cl}(F')| \geq \frac{(q'-1)(q'^{2g'} - 2g'q'^{g'} + 1)}{2g'(q'^{g'+1} - 1)}.$$

Proof. Proposition 3.1 in [4]. □

Therefore, the sequence $D(X_n)$ is unbounded if the set of reflex fields F_n is infinite. So it suffices to show unboundedness of the predegree $D(X_n)$ in a sequence of pairwise distinct Drinfeld modular subvarieties of S with fixed reflex field. This follows from the next theorem. □

Theorem 3.3.5. *For each sequence (X_n) of pairwise distinct Drinfeld modular subvarieties of S with fixed reflex field F', the indices $i(X_n)$ are unbounded.*

Proof. We first note that we can assume w.l.o.g. that the given compact subgroup $\mathcal{K}$ equals $\mathrm{GL}_r(\hat{A})$. Indeed, if $\mathcal{K}$ is replaced by a compact open subgroup $\mathcal{L} \supset \mathcal{K}$ and the X_n by their images under the canonical projection $\pi_1 : S^r_{F,\mathcal{K}} \to S^r_{F,\mathcal{L}}$, the indices $i(X_n)$ decrease by Definition 2.2.14. Hence, we can assume that $\mathcal{K}$ is a maximal compact open subgroup and therefore some conjugate $h\mathrm{GL}_r(\hat{A})h^{-1}$ of $\mathrm{GL}_r(\hat{A})$. If we further replace the X_n by their images under the isomorphism $\pi_{h^{-1}} : S^r_{F,\, h\mathrm{GL}_r(\hat{A})h^{-1}} \to S^r_{F,\,\mathrm{GL}_r(\hat{A})}$, then the $i(X_n)$ do obviously not change because the X_n are the image of an inclusion from the same $S^{r'}_{F',\mathcal{K}'}(\mathbb{C}_\infty)$. Therefore, we can w.l.o.g. assume $\mathcal{K} = \mathrm{GL}_r(\hat{A})$.

For the following considerations, we assume that $X_n = \iota^{F'}_{F,\,b_n}(S^{r'}_{F',\mathcal{K}'_n}(\mathbb{C}_\infty))$ with $\mathbb{A}^f_F$-linear isomorphisms $b_n : (\mathbb{A}^f_F)^r \to (\mathbb{A}^f_{F'})^{r'}$. We denote by Λ_n the $\hat{A}$-lattices $b_n(\hat{A}^r)$ in $(\mathbb{A}^f_{F'})^{r'}$. By Proposition 2.2.16, they are determined up to and only up to the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ and their orbits under the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ are pairwise distinct.

We have the product decomposition $\Lambda_n = \prod_{\mathfrak{p}\neq\infty} \Lambda_{n,\mathfrak{p}} := \prod_{\mathfrak{p}\neq\infty} b_{n,\mathfrak{p}}(A^r_\mathfrak{p})$, where $\Lambda_{n,\mathfrak{p}} \subset {F'_\mathfrak{p}}^{r'}$ are free $A_\mathfrak{p}$-submodules of rank r. The $A'_\mathfrak{p}$-modules $A'_\mathfrak{p} \cdot \Lambda_{n,\mathfrak{p}}$ are finitely generated submodules of ${F'_\mathfrak{p}}^{r'}$ with $F'_\mathfrak{p} \cdot \Lambda_{n,\mathfrak{p}} = {F'_\mathfrak{p}}^{r'}$, hence free of rank r' because $A'_\mathfrak{p}$ is a direct product of principal ideal domains. This implies that $\hat{A}' \cdot \Lambda_n = \prod_\mathfrak{p} A'_\mathfrak{p} \cdot \Lambda_{n,\mathfrak{p}}$ is a free $\hat{A}'$-submodule of $(\mathbb{A}^f_{F'})^{r'}$ of rank r'. Since the Λ_n are determined up to and only up to the action of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$, we may therefore assume w.l.o.g. that $\hat{A}' \cdot \Lambda_n = {\hat{A}'}^{r'}$ for all n.

Note that we have

$$\mathcal{K}'_n = (b_n^{-1}\mathrm{GL}_r(\hat{A})b_n) \cap \mathrm{GL}_{r'}(\mathbb{A}^f_{F'}) = \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})}\Lambda_n.$$

Since $\hat{A}' \cdot \Lambda_n = {\hat{A}'}^{r'}$, these compact open subgroups of $\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$ are all contained in the maximal compact subgroup $\mathrm{GL}_{r'}(\hat{A}') = \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})}{\hat{A}'}^{r'}$. Hence, we can write the indices $i(X_n)$ as

$$i(X_n) = [\mathrm{GL}_{r'}(\hat{A}') : \mathrm{Stab}_{\mathrm{GL}_{r'}(\mathbb{A}^f_{F'})}\Lambda_n]$$

and, using the above product decompositions, as $i(X_n) = \prod_{\mathfrak{p}\neq\infty} i_{n,\mathfrak{p}}$, where

$$i_{n,\mathfrak{p}} = [\mathrm{GL}_{r'}(A'_\mathfrak{p}) : \mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}\Lambda_{n,\mathfrak{p}}].$$

For each n, almost all factors of this product are 1 because $\Lambda_{n,\mathfrak{p}} = {A'_\mathfrak{p}}^{r'}$ for almost all $\mathfrak{p}$.

Since we assumed that $A'_{\mathfrak{p}} \cdot \Lambda_{n,\mathfrak{p}} = {A'_{\mathfrak{p}}}^{r'}$, by the Proposition 3.3.6 below, we get the estimates $i_{n,\mathfrak{p}} \geq C \cdot [{A'_{\mathfrak{p}}}^{r'} : \Lambda_{n,\mathfrak{p}}]^{1/r}$, where the constant C is independent of n and $\mathfrak{p}$.

We now finish the proof by assuming (by contradiction) that the sequence $(i(X_n))$ is bounded. This implies by the above product decomposition of $i(X_n)$ and estimates of $i_{n,\mathfrak{p}}$ that $[{A'_{\mathfrak{p}}}^{r'} : \Lambda_{n,\mathfrak{p}}] \leq D$ for all n and $\mathfrak{p}$ for some uniform constant D.

But, note that, as finite $A_{\mathfrak{p}}$-module, ${A'_{\mathfrak{p}}}^{r'}/\Lambda_{n,\mathfrak{p}}$ is isomorphic to some product

$$A_{\mathfrak{p}}/\mathfrak{p}^{m_1}A_{\mathfrak{p}} \times \cdots \times A_{\mathfrak{p}}/\mathfrak{p}^{m_l}A_{\mathfrak{p}}.$$

If $\Lambda_{n,\mathfrak{p}} \not\supseteq \mathfrak{p}^N \cdot {A'_{\mathfrak{p}}}^{r'}$, we have $m_i \geq N+1$ for some i and therefore

$$|k(\mathfrak{p})|^{N+1} \leq [{A'_{\mathfrak{p}}}^{r'} : \Lambda_{n,\mathfrak{p}}] \leq D.$$

In particular, we have $|k(\mathfrak{p})| \leq D$ whenever $\Lambda_{n,\mathfrak{p}} \neq {A'_{\mathfrak{p}}}^{r'}$. Since there are only finitely many primes $\mathfrak{p}$ with $|k(\mathfrak{p})| \leq D$, we conclude:

- There are finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that $\Lambda_{n,\mathfrak{p}} = {A'_{\mathfrak{p}}}^{r'}$ for all n and $\mathfrak{p} \neq \mathfrak{p}_1, \ldots, \mathfrak{p}_k$.
- There is a $N \in \mathbb{N}$ such that, for all $\mathfrak{p}$ and n, the $A_{\mathfrak{p}}$-lattice $\Lambda_{n,\mathfrak{p}}$ contains $\mathfrak{p}^N {A'_{\mathfrak{p}}}^{r'}$.

Since the quotients ${A'_{\mathfrak{p}}}^{r'}/\mathfrak{p}^N {A'_{\mathfrak{p}}}^{r'}$ are finite, the second statement implies that for all $1 \leq i \leq k$ there are only finitely many possibilities for $\Lambda_{n,\mathfrak{p}_i}$. As for $\mathfrak{p} \neq \mathfrak{p}_1, \ldots, \mathfrak{p}_k$ the lattices $\Lambda_{n,\mathfrak{p}}$ are independent of n, this implies that only finitely many $\hat{A}$-lattices $\Lambda_n \subset (\mathbb{A}_{F'}^f)^{r'}$ occur, a contradiction to our assumptions. □

Proposition 3.3.6. *Let K be a complete field with respect to a discrete valuation v with finite residue field containing $\mathbb{F}_q$ and let R be the corresponding discrete valuation ring with maximal ideal $\mathfrak{m}$. Let $K' := L_1 \times \cdots \times L_m$ with L_i finite field extensions of K and $R' := S_1 \times \cdots \times S_m$ with $S_i \subset L_i$ the discrete valuation ring associated to the unique extension of v to L_i. Suppose that $r' \geq 1$ and set $r := r' \cdot \sum_{i=1}^m [L_i : K]$.*

There is a constant $C > 0$ only depending on q and r such that, for any free R-submodule $\Lambda \subset {K'}^{r'}$ of rank r, we have

$$[\mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(R' \cdot \Lambda) : \mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(\Lambda)] \geq C \cdot [R' \cdot \Lambda : \Lambda]^{1/r}.$$

Proof. For a free R-submodule $\Lambda \subset K'^{r'}$ of rank r, the R'-submodule $R' \cdot \Lambda \subset K'^{r'}$ is free of finite rank because it is torsionfree and finitely generated and R' is a principal ideal domain as direct product of discrete valuation rings. Furthermore, we have $K \cdot \Lambda = K'^{r'}$ because $K \cdot \Lambda$ is a K-linear subspace of $K'^{r'}$ of dimension

$$r = r' \cdot \sum_{i=1}^{m} [L_i : K] = \dim_K(K'^{r'}).$$

Therefore we also have $K' \cdot \Lambda = K'^{r'}$ and $R' \cdot \Lambda$ is of full rank r'. So there is a $t' \in \mathrm{GL}_{r'}(R')$ with

$$R' \cdot \Lambda = t' \cdot R'^{r'}.$$

Since the inequality in the conclusion of the lemma is invariant under replacing Λ by $t'^{-1} \cdot \Lambda$, we can from now on assume that $R' \cdot \Lambda = R'^{r'}$ and need to show

$$[\mathrm{GL}_{r'}(R') : \mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(\Lambda)] \geq C \cdot [R'^{r'} : \Lambda]^{1/r}$$

for some constant C only depending on q and r.

We introduce the notation

$$H := \{T \in \mathrm{Mat}_{r'}(R') : T \cdot \Lambda \subseteq \Lambda\}.$$

This set of matrices is an R-subalgebra of $\mathrm{Mat}_{r'}(R')$ with $H^* = \mathrm{Stab}_{\mathrm{GL}_{r'}(R')}(\Lambda)$.

Note that, if $g_1, \ldots, g_r$ is a R-basis of Λ, then $\Lambda = \xi(R^r) \subset K'^{r'}$ for

$$\xi : \begin{array}{ccc} K^r & \longrightarrow & K'^{r'} \\ (x_1, \ldots, x_r) & \longmapsto & x_1 g_1 + \cdots + x_r g_r \end{array}.$$

Since K is complete, ξ is a homeomorphism (cf. Proposition 4.9 in [28]). This implies that $\Lambda \subset R'^{r'}$ is open.

Hence, there is a $k \in \mathbb{N}$ such that $\mathfrak{m}^k R'^{r'} \subseteq \Lambda$. Therefore $\mathrm{Mat}_{r'}(\mathfrak{m}^k R') \subseteq H$ and

$$H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R') = \{\overline{T} \in \mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R') : \overline{T} \cdot (\Lambda/\mathfrak{m}^k R'^{r'}) \subseteq \Lambda/\mathfrak{m}^k R'^{r'}\}$$

if we identify $\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')$ with $\mathrm{Mat}_{r'}(R')/\mathrm{Mat}_{r'}(\mathfrak{m}^k R')$. For the stabilizer of $\Lambda/\mathfrak{m}^k R'^{r'}$ under the action of $\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$, this means that

$$(H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R'))^* = \mathrm{Stab}_{\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')}(\Lambda/\mathfrak{m}^k R').$$

Now note that $T \cdot \Lambda \supseteq \mathfrak{m}^k R'^{r'}$ for all $T \in \mathrm{GL}_{r'}(R')$. Hence, we have

$$\overline{T} \cdot (\Lambda/\mathfrak{m}^k R'^{r'}) = (T \cdot \Lambda)/\mathfrak{m}^k R'^{r'}$$

for all $T \in \mathrm{GL}_{r'}(R')$, where $\overline{T}$ denotes the reduction of T modulo $\mathfrak{m}^k R'$. Therefore,

$$T \cdot \Lambda \longmapsto (T \cdot \Lambda)/\mathfrak{m}^k R'^{r'} \tag{3.3.1}$$

is an injective map from the orbit of Λ under $\mathrm{GL}_{r'}(R')$ to the orbit of $\Lambda/\mathfrak{m}^k R'^{r'}$ under $\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$. It is also surjective because the reduction map modulo $\mathfrak{m}^k R'$ from $\mathrm{GL}_{r'}(R')$ to $\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$ is surjective. Indeed, if $\overline{T} \in \mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')$ is the reduction modulo $\mathfrak{m}^k R'$ of some $T \in \mathrm{Mat}_{r'}(R')$, there is a $T' \in \mathrm{Mat}_{r'}(R')$ such that $TT' \in 1_{r'} + \mathrm{Mat}_{r'}(\mathfrak{m}^k R')$. This implies $\det T \cdot \det T' \in 1 + \mathfrak{m}^k R' \subseteq R'^*$, hence $\det T \in R'^*$ and $T \in \mathrm{GL}_{r'}(R')$.

The above bijection of orbits and formulas for the corresponding stabilizers give us the following estimate:

$$[\mathrm{GL}_{r'}(R') : H^*] = [\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R') : (H/\mathrm{Mat}_{r'}(\mathfrak{m}^k R'))^*] \geq \frac{|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')|}{[H : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]}$$

Lemma 3.3.7. *There is a constant C only depending on q and r such that*

$$|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')| \geq C \cdot |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')|.$$

Proof. By definition of R' the quotient $R'/\mathfrak{m}^k R'$ is isomorphic to $S_1/\mathfrak{m}^k S_1 \times \cdots \times S_m/\mathfrak{m}^k S_m$, hence

$$\begin{aligned} \mathrm{GL}_{r'}(R'/\mathfrak{m}^k R') &\cong \mathrm{GL}_{r'}(S_1/\mathfrak{m}^k S_1) \times \cdots \times \mathrm{GL}_{r'}(S_m/\mathfrak{m}^k S_m) \\ \mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R') &\cong \mathrm{Mat}_{r'}(S_1/\mathfrak{m}^k S_1) \times \cdots \times \mathrm{Mat}_{r'}(S_m/\mathfrak{m}^k S_m). \end{aligned}$$

Now note that, for any $l \geq 1$ and any discrete valuation ring U with maximal ideal $\mathfrak{n}$ and residue field $\mathbb{F}_{q'}$ containing $\mathbb{F}_q$, a matrix $T \in \mathrm{Mat}_{r'}(U)$ is invertible if and only if its reduction modulo $\mathfrak{n}^l$ is invertible in $\mathrm{Mat}_{r'}(U/\mathfrak{n}^l)$. This follows by the same argument as in the proof of the bijection of orbits (3.3.1). In particular, $\mathrm{GL}_{r'}(U/\mathfrak{n}^l)$ exactly consists of the matrices with reduction modulo $\mathfrak{n}$ lying in $\mathrm{GL}_{r'}(U/\mathfrak{n})$. As the fibers of the projection $\mathrm{Mat}_{r'}(U/\mathfrak{n}^l) \to \mathrm{Mat}_{r'}(U/\mathfrak{n})$ have all cardinality $|\mathfrak{n}/\mathfrak{n}^l|^{r'^2} = q'^{(l-1)r'^2}$, we get

$$\begin{aligned} |\mathrm{GL}_{r'}(U/\mathfrak{n}^l)| &= q'^{(l-1)r'^2} |\mathrm{GL}_{r'}(\mathbb{F}_{q'})| \\ &= q'^{(l-1)r'^2} (q'^{r'} - 1)(q'^{r'} - q') \cdots (q'^{r'} - q'^{r'-1}) \\ &\geq q'^{lr'^2} \left(1 - \frac{1}{q}\right)^{r'} \\ &= \left(1 - \frac{1}{q}\right)^{r'} |\mathrm{Mat}_{r'}(U/\mathfrak{n}^l)|. \end{aligned}$$

Since $m \leq r/r'$, we have in total

$$|\mathrm{GL}_{r'}(R'/\mathfrak{m}^k R')| \geq \left(1 - \frac{1}{q}\right)^{mr'} |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')| \geq C \cdot |\mathrm{Mat}_{r'}(R'/\mathfrak{m}^k R')|$$

with $C = \left(1 - \frac{1}{q}\right)^r$. □

By Lemma 3.3.7 and the estimate before, we have

$$[\mathrm{GL}_{r'}(R') : H^*] \geq C \cdot \frac{[\mathrm{Mat}_{r'}(R') : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]}{[H : \mathrm{Mat}_{r'}(\mathfrak{m}^k R')]} = C \cdot [\mathrm{Mat}_{r'}(R') : H].$$

To finish the proof of Proposition 3.3.6, we consider a R-basis $g_1, \ldots, g_r$ of Λ and the R-module homomorphism

$$\begin{array}{ccc} \mathrm{Mat}_{r'}(R')^r & \longrightarrow & R' \cdot \Lambda/\Lambda \\ (T_1, \ldots, T_r) & \longmapsto & (T_1 \cdot g_1 + \cdots + T_r \cdot g_r) \mod \Lambda. \end{array}$$

It is surjective and its kernel contains H^r. Therefore, we have

$$[\mathrm{Mat}_{r'}(R') : H]^r = [\mathrm{Mat}_{r'}(R')^r : H^r] \geq [R' \cdot \Lambda : \Lambda]$$

and in total

$$[\mathrm{GL}_{r'}(R') : \mathrm{Stab}_{\mathrm{GL}_{r'}(K')}(\Lambda)] \geq C \cdot [\mathrm{Mat}_{r'}(R') : H] \geq C \cdot [R' \cdot \Lambda : \Lambda]^{1/r}. \quad \square$$

Chapter 4

Zariski density of Hecke orbits

In the whole chapter, $S = S^r_{F,\mathcal{K}}$ denotes a Drinfeld modular variety and C a set of representatives in $\mathrm{GL}_r(\mathbb{A}_F^f)$ for $\mathrm{GL}_r(F) \setminus \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}$. We use the description of the irreducible components of S over $\mathbb{C}_\infty$ given in Proposition 1.1.3: We let Y_h be the irreducible component of S over $\mathbb{C}_\infty$ corresponding to $h \in C$ and identify its $\mathbb{C}_\infty$-valued points $Y_h(\mathbb{C}_\infty) \subset \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K})$ with $\Gamma_h \setminus \Omega_F^r$ where $\Gamma_h := h\mathcal{K}h^{-1} \cap \mathrm{GL}_r(F)$ via the isomorphism from Proposition 1.1.3.

4.1 $T_g + T_{g^{-1}}$-orbits

For $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ and closed subvarieties $Z \subset S$ we define

$$(T_g + T_{g^{-1}})(Z) := T_g(Z) \cup T_{g^{-1}}(Z),$$

and recursively

$$\begin{aligned} (T_g + T_{g^{-1}})^0(Z) &:= Z \\ (T_g + T_{g^{-1}})^n(Z) &:= (T_g + T_{g^{-1}})\big((T_g + T_{g^{-1}})^{n-1}(Z)\big),\ n \geq 1. \end{aligned}$$

Definition 4.1.1. *For a geometric point $x \in S(\mathbb{C}_\infty)$ and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$, the union*

$$T_g^\infty(x) := \bigcup_{n \geq 0} (T_g + T_{g^{-1}})^n(x) \subset S(\mathbb{C}_\infty)$$

*is called the $T_g + T_{g^{-1}}$-*orbit *of x.*

Note that $T_g^\infty(x)$ is the smallest subset of $S(\mathbb{C}_\infty)$ containing x which is mapped into itself under T_g and $T_{g^{-1}}$.

We now give an explicit description of the intersection of $T_g^\infty(x)$ with the irreducible components of S over $\mathbb{C}_\infty$ for $x \in S(\mathbb{C}_\infty)$ and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$.

Proposition 4.1.2. *Let $h_1, h_2 \in C$ and assume that $x \in Y_{h_1}(\mathbb{C}_\infty)$ with $x = [\omega] \in \Gamma_{h_1} \setminus \Omega_F^r$. Then the intersection of $T_g^\infty(x)$ with $Y_{h_2}(\mathbb{C}_\infty)$ is given by*

$$T_g^\infty(x) \cap Y_{h_2}(\mathbb{C}_\infty) = \{[T\omega] \in \Gamma_{h_2} \setminus \Omega_F^r \,:\, T \in h_2 \langle \mathcal{K} g \mathcal{K} \rangle h_1^{-1} \cap \mathrm{GL}_r(F)\},$$

where $\langle \mathcal{K} g \mathcal{K} \rangle$ denotes the subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$ generated by the double coset $\mathcal{K} g \mathcal{K}$.

Proof. By assumption, we have $x = [(\omega, h_1)] \in \mathrm{GL}_r(F) \setminus \big(\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}\big)$. Hence, by Definition 2.1.5

$$T_g(x) = \{[(\omega, h_1 k g^{-1})] : k \in \mathcal{K}\}$$

and similarly

$$T_{g^{-1}}(x) = \{[(\omega, h_1 k g)] : k \in \mathcal{K}\}.$$

Therefore, by the recursive definition of $(T_g + T_{g^{-1}})^n(x)$, the elements of $T_g^\infty(x)$ are exactly those of the form $[(\omega, h_1 k_1 g_1 k_2 g_2 \cdots k_n g_n)]$ with $n \geq 0$, $k_i \in \mathcal{K}$ and $g_i \in \{g, g^{-1}\}$, i.e., we have

$$T_g^\infty(x) = \{[(\omega, h_1 s)] : s \in \langle \mathcal{K} g \mathcal{K} \rangle\}.$$

Hence, an element $y \in T_g^\infty(x) \cap Y_{h_2}(\mathbb{C}_\infty)$ can be written as $y = [(\omega, h_1 s)]$ with $s \in \langle \mathcal{K} g \mathcal{K} \rangle$. Since y lies in Y_{h_2}, there exist $T \in \mathrm{GL}_r(F)$ and $k \in \mathcal{K}$ with

$$T h_1 s k = h_2 \iff T = h_2 k^{-1} s^{-1} h_1^{-1}.$$

Therefore, $y = [(\omega, h_1 s)] = [(T\omega, T h_1 s k)] = [(T\omega, h_2)]$ is equal to $[T\omega] \in \Gamma_{h_2} \setminus \Omega_F^r$, where $T \in h_2 \langle \mathcal{K} g \mathcal{K} \rangle h_1^{-1} \cap \mathrm{GL}_r(F)$.

Conversely, an element $[T\omega] \in \Gamma_{h_2} \setminus \Omega_F^r$ with $T = h_2 s h_1^{-1} \in h_2 \langle \mathcal{K} g \mathcal{K} \rangle h_1^{-1} \cap \mathrm{GL}_r(F)$ is equal to

$$[(T\omega, h_2)] = [(\omega, T^{-1} h_2)] = [(\omega, h_1 s^{-1} h_2^{-1} h_2)] = [(\omega, h_1 s^{-1})]$$

with $s^{-1} \in \langle \mathcal{K} g \mathcal{K} \rangle$, hence lies in $T_g^\infty(x) \cap Y_{h_2}(\mathbb{C}_\infty)$. □

4.2 Zariski density

We give a sufficient condition for a subset $M \subset S(\mathbb{C}_\infty)$ to be Zariski dense in one irreducible component Y_h of S over $\mathbb{C}_\infty$. Recall that, for a place $\mathfrak{p} \neq \infty$ of F, we denote the adeles outside ∞ and $\mathfrak{p}$ by $\mathbb{A}_F^{f,\mathfrak{p}}$.

Proposition 4.2.1. *Let M be a subset of $S(\mathbb{C}_\infty)$ contained in an irreducible component Y_h of S over $\mathbb{C}_\infty$ for $h \in C$ and suppose that M contains an element $x = [\omega] \in Y_h(\mathbb{C}_\infty) = \Gamma_h \setminus \Omega_F^r$ such that there exists a place $\mathfrak{p} \neq \infty$ of F and an open subgroup $\mathcal{K}' \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$ with*

$$M' := \{[T\omega] \in \Gamma_h \setminus \Omega_F^r \ : \ T \in (\mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)\} \subset M.$$

Then M is Zariski dense in Y_h.

Proof. We denote the Zariski closure of M' by Y. It is enough to show that $Y(\mathbb{C}_\infty) = Y_h(\mathbb{C}_\infty)$. As the nonsingular locus Y^{ns} of Y over $\mathbb{C}_\infty$ is Zariski open in Y (Theorem I.5.3 in [22]), the intersection $Y^{\mathrm{ns}}(\mathbb{C}_\infty) \cap M'$ is non-empty, say $[\omega'] \in Y^{\mathrm{ns}}(\mathbb{C}_\infty) \cap M'$. Since $(\mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)$ is a subgroup of $\mathrm{GL}_r(F)$, after replacing $[\omega]$ by $[\omega']$, we still have

$$M' = \{[T\omega] \in Y_h(\mathbb{C}_\infty) \ : \ T \in (\mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)\}.$$

Hence, we can assume that $x = [\omega]$ lies in $Y^{\mathrm{ns}}(\mathbb{C}_\infty)$ and it is enough to show that the tangent space T_xY of Y at x is of dimension $r - 1 = \dim S$.

Since $\mathcal{K}'$ is open in $\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$, there is an $N \in A$ with $N \notin \mathfrak{p}$ such that $K'(N) \subset \mathcal{K}'$, where $K'(N)$ denotes the principal congruence subgroup modulo N of $\mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$. Now let $l \geq 1$ such that $\mathfrak{p}^l = (\pi)$ is a principal ideal of A and consider for $1 \leq i \leq r-1$ and $k \geq 1$ the matrices

$$A_{ik} := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \frac{N}{\pi^k} & & 1 \end{pmatrix} \in \mathrm{SL}_r(F),$$

with the entry $\frac{N}{\pi^k}$ in the ith column. As elements of $\mathrm{GL}_r(\mathbb{A}_F^f)$ (diagonally embedded) they lie in $\mathrm{SL}_r(F_\mathfrak{p}) \times K'(N) \subset \mathrm{SL}_r(F_\mathfrak{p}) \times \mathcal{K}'$. Hence, for all $1 \leq i \leq r-1$ and $k \geq 1$, $[A_{ik}\omega]$ lies in $M' \subset Y(\mathbb{C}_\infty)$.

We now view Ω_F^r as a subset of $\mathbb{A}^{r-1}(\mathbb{C}_\infty)$ by identifying $[\omega_1 : \cdots : \omega_{r-1} : 1]$ with $(\omega_1, \ldots, \omega_{r-1})$ (note that the r-th projective coordinate ω_r of an arbitrary

element of Ω_F^r can be assumed to be 1 because the F_∞-rational hyperplane $\omega_r = 0$ does not belong to Ω_F^r). Assume that we have $\omega = (\omega_1, \ldots, \omega_{r-1})$ in this identification. Then, using (1.1.2), we see that

$$A_{ik}\omega = (\omega_1, \ldots, \omega_i - \frac{N}{\pi^k}, \ldots, \omega_{r-1})$$

for all $1 \leq i \leq r-1$ and $k \geq 1$. Note that $\omega_i - \frac{N}{\pi^k}$ converges to ω_i in $\mathbb{C}_\infty$ for $k \to \infty$ and that $\{[A_{ik}\omega]\}_{k\geq 1} \subset Y(\mathbb{C}_\infty)$ for all $1 \leq i \leq r-1$. Since $Y(\mathbb{C}_\infty) \subset Y_h(\mathbb{C}_\infty) = \Gamma_h \setminus \Omega_F^r$ is closed in the rigid analytic topology, it follows that there is an $\varepsilon > 0$ such that for all $1 \leq i \leq r-1$ and $c \in \mathbb{C}_\infty$ with $|c|_\infty < \varepsilon$

$$[(\omega_1, \ldots, \omega_i + c, \ldots, \omega_{r-1})] \in Y(\mathbb{C}_\infty).$$

This implies $\dim T_x Y = r-1$ and $Y(\mathbb{C}_\infty) = Y_h(\mathbb{C}_\infty)$. □

Now let $\mathfrak{p} \neq \infty$ be a place of F and $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ trivial outside $\mathfrak{p}$, i.e., $g := (1, \ldots, g_\mathfrak{p}, \ldots, 1)$ for some $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$. Using Proposition 4.2.1, we prove a sufficient condition for the $T_g + T_{g^{-1}}$-orbit $T_g^\infty(x)$ to be Zariski dense in the irreducible component of S over $\mathbb{C}_\infty$ containing x. This result is a generalization of Theorem 4.11 in [5].

Theorem 4.2.2. *Assume that the image of the cyclic subgroup $\langle g_\mathfrak{p} \rangle \subset \mathrm{GL}_r(F_\mathfrak{p})$ in $\mathrm{PGL}_r(F_\mathfrak{p})$ is unbounded and, for $x \in S(\mathbb{C}_\infty)$, let Y_x be the irreducible component of S over $\mathbb{C}_\infty$ containing x. Then, for all $x \in S(\mathbb{C}_\infty)$, the intersection of the $T_g + T_{g^{-1}}$-orbit $T_g^\infty(x)$ with $Y_x(\mathbb{C}_\infty)$ is Zariski dense in Y_x.*

Proof. We assume that $Y_x = Y_h$ for some $h \in C$. Then, by Proposition 4.1.2, we have

$$T_g^\infty(x) \cap Y_x(\mathbb{C}_\infty) = \{[T\omega] \in \Gamma_h \setminus \Omega_F^r \ : \ T \in h\langle \mathcal{K}g\mathcal{K}\rangle h^{-1} \cap \mathrm{GL}_r(F)\}.$$

Since $\mathcal{K}$ is an open subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$, there is an $N \in A$ such that the principal congruence subgroup $K(N) \subset \mathrm{GL}_r(\mathbb{A}_F^f)$ modulo N is contained in $h\mathcal{K}h^{-1}$. If the principal ideal $(N) \subset A$ is equal to $\prod_\mathfrak{q} \mathfrak{q}^{\nu_\mathfrak{q}}$, then we can write $K(N) = \prod_\mathfrak{q} K_\mathfrak{q}(N)$ with

$$K_\mathfrak{q}(N) = \begin{cases} \{t_\mathfrak{q} \in \mathrm{GL}_r(A_\mathfrak{q}) \ : \ t_\mathfrak{q} \equiv 1 \pmod{\mathfrak{q}^{v_\mathfrak{q}}}\} & , \ v_\mathfrak{q} > 0 \\ \mathrm{GL}_r(A_\mathfrak{q}) & , \ v_\mathfrak{q} = 0 \end{cases}.$$

Hence, we have

$$\langle K_\mathfrak{p}(N) h_\mathfrak{p} g_\mathfrak{p} h_\mathfrak{p}^{-1} K_\mathfrak{p}(N)\rangle \times \prod_{\mathfrak{q}\neq\mathfrak{p}} K_\mathfrak{q}(N) = \langle K(N) h g h^{-1} K(N)\rangle \subset h\langle \mathcal{K}g\mathcal{K}\rangle h^{-1}.$$

We set $U_{\mathfrak{p}} := \langle K_{\mathfrak{p}}(N) h_{\mathfrak{p}} g_{\mathfrak{p}} h_{\mathfrak{p}}^{-1} K_{\mathfrak{p}}(N) \rangle$ and $\mathcal{K}' := \prod_{\mathfrak{q} \neq \mathfrak{p}} K_{\mathfrak{q}}(N)$. Since the image of $\langle g_{\mathfrak{p}} \rangle \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ in $\mathrm{PGL}_r(F_{\mathfrak{p}})$ is unbounded, the same applies to the subgroup $U_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$.

We now consider the subgroup $U_{\mathfrak{p}} \cap \mathrm{SL}_r(F_{\mathfrak{p}})$ of $\mathrm{SL}_r(F_{\mathfrak{p}})$. It is open in $\mathrm{SL}_r(F_{\mathfrak{p}})$ because $U_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ is open, and is normalized by the image of $U_{\mathfrak{p}}$ in $\mathrm{PGL}_r(F_{\mathfrak{p}})$, which is unbounded. Since PGL_r is an absolutely simple linear algebraic group over the local field $F_{\mathfrak{p}}$ and $\mathrm{SL}_r \hookrightarrow \mathrm{GL}_r \to \mathrm{PGL}_r$ is its universal covering, we conclude by Theorem 2.2 of [32] that $U_{\mathfrak{p}} \cap \mathrm{SL}_r(F_{\mathfrak{p}})$ is equal to $\mathrm{SL}_r(F_{\mathfrak{p}})$.

Hence, $\mathrm{SL}_r(F_{\mathfrak{p}})$ is contained in $U_{\mathfrak{p}}$ and we have

$$\{[T\omega] \in \Gamma_h \setminus \Omega_F^r \ : \ T \in (\mathrm{SL}_r(F_{\mathfrak{p}}) \times \mathcal{K}') \cap \mathrm{GL}_r(F)\} \subset T_g^{\infty}(x) \cap Y_x(\mathbb{C}_{\infty}).$$

Therefore, we can apply Proposition 4.2.1 to the subset $T_g^{\infty}(x) \cap Y_x(\mathbb{C}_{\infty})$ of $S(\mathbb{C}_{\infty})$ and conclude that $T_g^{\infty}(x) \cap Y_x(\mathbb{C}_{\infty})$ is Zariski dense in Y_x. $\square$

Chapter 5

Geometric criterion for being a Drinfeld modular subvariety

Proposition 5.1.1. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $Z \subset S$ an irreducible subvariety over $\mathbb{C}_\infty$ such that $Z = T_g Z = T_{g^{-1}} Z$ for some $g = (1, \ldots, g_\mathfrak{p}, \ldots, 1)$ with $g_\mathfrak{p} \in \mathrm{GL}_r(F_\mathfrak{p})$. If the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $g_\mathfrak{p}$ is unbounded, then Z is an irreducible component of S over $\mathbb{C}_\infty$.*

Proof. Let $x \in Z(\mathbb{C}_\infty)$ be a geometric point of Z. By assumption we have $T_g(x) \subset T_g Z = Z$ and $T_{g^{-1}}(x) \subset T_{g^{-1}} Z = Z$, hence

$$(T_g + T_{g^{-1}})(x) \subset Z.$$

Iterating we get for all $n \geq 1$

$$(T_g + T_{g^{-1}})^n(x) \subset Z,$$

so the $(T_g + T_{g^{-1}})$-orbit $T_g^\infty(x)$ of x is contained in Z. Since Z is irreducible over $\mathbb{C}_\infty$, the orbit $T_g^\infty(x)$ is contained in one irreducible component Y of S over $\mathbb{C}_\infty$. So $T_g^\infty(x)$ is Zariski dense in Y by Theorem 4.2.2. Since Z is Zariski closed in S, it follows that $Z = Y$ is an irreducible component of S over $\mathbb{C}_\infty$. □

Definition 5.1.2. *A subvariety X defined over $\overline{F}$ of a Drinfeld modular subvariety $S^r_{F,\mathcal{K}}$ is called* Hodge-generic *if each of its irreducible components over $\mathbb{C}_\infty$ is not contained in a proper Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$.*

Theorem 5.1.3. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety with $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ amply small where $\mathcal{K}_\mathfrak{p} \subset \mathrm{GL}_r(F_\mathfrak{p})$ and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$. Suppose that $Z \subset S$ is an F-irreducible Hodge-generic subvariety with $\dim Z \geq 1$ such that $Z \subset T_g Z$*

for some $g = (1, \ldots, g_{\mathfrak{p}}, \ldots, 1)$ *with* $g_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$. *If, for all* $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}}$, *the cyclic subgroup of* $\mathrm{PGL}_r(F_{\mathfrak{p}})$ *generated by the image of* $k_1 \cdot g_{\mathfrak{p}} \cdot k_2$ *is unbounded, then* $Z = S$.

Proof. In this proof, for simplicity of notation, we identify $\mathrm{GL}_r(F_{\mathfrak{p}})$ as a subgroup of $\mathrm{GL}_r(\mathbb{A}_F^f)$ via the inclusion

$$h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}}) \longmapsto (1, \ldots, h_{\mathfrak{p}}, \ldots, 1) \in \mathrm{GL}_r(\mathbb{A}_F^f).$$

Let $Z = Z_1 \cup \cdots \cup Z_s$ be a decomposition of Z into irreducible components over $\mathbb{C}_\infty$. Since Z is defined over F, the irreducible component Z_1 is defined over some finite, separable extension E of F. By the F-irreducibility of S and Z, it is enough to show that Z_1 is an irreducible component of S over $\mathbb{C}_\infty$. We divide the proof into two steps:

(i) We show that there is an open subgroup $\mathcal{K}' \subset \mathcal{K}$ with associated canonical projection $\pi : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}$ and an E-irreducible component Z_1' of $\pi^{-1}(Z_1)$ which is also irreducible over $\mathbb{C}_\infty$ such that $T_{h_{\mathfrak{p}}} Z_1'$ is E-irreducible for all $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$.

(ii) Using Proposition 5.1.1, we prove that Z_1' is an irreducible component of $S^r_{F,\mathcal{K}'}$ over $\mathbb{C}_\infty$.

Steps (i) and (ii) imply that $Z_1 = \pi(Z_1')$ is an irreducible component of $S = S^r_{F,\mathcal{K}}$ over $\mathbb{C}_\infty$.

Step (i): Note that, by Proposition 2.1.3, the canonical projections

$$\pi_{U_{\mathfrak{p}}} : S^r_{F,\, U_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}} \longrightarrow S$$

where $U_{\mathfrak{p}}$ runs over all open normal subgroups of $\mathcal{K}_{\mathfrak{p}}$ form a projective system of finite étale Galois covers defined over F with Galois groups $\mathcal{K}_{\mathfrak{p}}/U_{\mathfrak{p}}$. Hence, by Proposition 2.1.3

$$\pi_{\mathfrak{p}} : S^{(\mathfrak{p})} := \varprojlim_{U_{\mathfrak{p}}} S^r_{F, U_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}} \longrightarrow S$$

is a pro-étale Galois cover with group $\varprojlim_{U_{\mathfrak{p}}} \mathcal{K}_{\mathfrak{p}}/U_{\mathfrak{p}}$. Since $\mathcal{K}_{\mathfrak{p}}$ is a profinite group, this group is isomorphic to $\mathcal{K}_{\mathfrak{p}}$. Therefore there are bijections

$$\varprojlim_{U_{\mathfrak{p}}} \mathrm{GL}_r(F_{\mathfrak{p}})/U_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{GL}_r(F_{\mathfrak{p}}), \ \varprojlim_{U_{\mathfrak{p}}} \mathrm{GL}_r(\mathbb{A}_F^f)/(U_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}) \xrightarrow{\sim} \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}^{(\mathfrak{p})}$$

of sets. Hence we have the following isomorphisms of rigid-analytic spaces:

$$\begin{aligned} S^{(\mathfrak{p})}(\mathbb{C}_\infty) &\cong \varprojlim_{U_{\mathfrak{p}}} \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/(U_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})})) \\ &\cong \mathrm{GL}_r(F) \setminus (\Omega_F^r \times \mathrm{GL}_r(\mathbb{A}_F^f)/\mathcal{K}^{(\mathfrak{p})}). \end{aligned}$$

By Proposition 2.1.3 and the above isomorphisms, the automorphism of the $\mathcal{K}_\mathfrak{p}$-cover $\pi_\mathfrak{p}$ corresponding to a $k_\mathfrak{p} \in \mathcal{K}_\mathfrak{p}$ is given by

$$\varprojlim_{U_\mathfrak{p}} \pi_{k_\mathfrak{p}} : [(\overline{\omega}, h)] \mapsto [(\overline{\omega}, h k_\mathfrak{p}^{-1})]$$

on $\mathbb{C}_\infty$-valued points of $S^{(\mathfrak{p})}$.

We now denote by Y the nonsingular locus of the variety Z_1 over $\mathbb{C}_\infty$. By Theorem I.5.3. in [22], Y is a non-empty open subset of Z_1 and Y is also defined over E.

Let $y \in Y(\mathbb{C}_\infty) \subset S(\mathbb{C}_\infty)$ be a geometric point of Y. We denote by $\pi_1^{\mathrm{arithm}}(Y, y)$ the arithmetic fundamental group of the variety Y over E, i.e., $\pi_1^{\mathrm{arithm}}(Y, y) := \pi_1(Y_0, y)$ if $Y = (Y_0)_{\mathbb{C}_\infty}$ for a scheme Y_0 over E. Furthermore we fix a geometric point $x = [(\overline{\omega}, h)] \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ with $\pi_\mathfrak{p}(x) = y$ and consider the monodromy representation

$$\rho : \pi_1^{\mathrm{arithm}}(Y, y) \longrightarrow \mathcal{K}_\mathfrak{p}$$

associated to $x \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ and the $\mathcal{K}_\mathfrak{p}$-cover $\pi_\mathfrak{p}$.

By Theorem 4 in [6] the image of ρ is open in $\mathrm{GL}_r(F_\mathfrak{p})$ under the assumptions

- $\mathcal{K}$ is amply small,
- Y is a smooth irreducible locally closed subvariety of S with $\dim Y \geq 1$,
- The Zariski closure of Y in S is Hodge-generic.

These assumptions are satisfied in our case, hence $\mathcal{K}'_\mathfrak{p} := \rho(\pi_1^{\mathrm{arithm}}(Y, y))$ is open in $\mathcal{K}_\mathfrak{p}$.

Now we set $\mathcal{K}' := \mathcal{K}'_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ and consider the canonical projection

$$\pi : S^r_{F,\mathcal{K}'} \to S^r_{F,\mathcal{K}}.$$

The orbit of the point $x' := [(\overline{\omega}, h)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty)$ lying between our base points $x \in S^{(\mathfrak{p})}(\mathbb{C}_\infty)$ and $y \in S^r_{F,\mathcal{K}}(\mathbb{C}_\infty)$ under the action of $\pi_1^{\mathrm{arithm}}(Y, y)$ on the fiber $\pi^{-1}(y)$ equals

$$\{[(\overline{\omega}, h k_\mathfrak{p}'^{-1})] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty) \ : \ k'_\mathfrak{p} \in \rho(\pi_1^{\mathrm{arithm}}(Y, y)) = \mathcal{K}'_\mathfrak{p}\}$$

and is therefore of cardinality 1. Hence, the E-irreducible component Y' of $\pi^{-1}(Y)$ containing x' is mapped isomorphically onto Y by π. Since Y is irreducible over $\mathbb{C}_\infty$, it follows that Y' is also irreducible over $\mathbb{C}_\infty$.

Note furthermore, for any open subgroup $\tilde{\mathcal{K}}'_{\mathfrak{p}} \subset \mathcal{K}'_{\mathfrak{p}}$ and $\tilde{\mathcal{K}}' := \tilde{\mathcal{K}}'_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with canonical projection $\pi' : S^r_{F,\tilde{\mathcal{K}}'} \to S^r_{F,\mathcal{K}'}$ that

$$\pi'^{-1}(x') = \{[(\overline{\omega}, hk'_{\mathfrak{p}})] \in S^r_{F,\tilde{\mathcal{K}}'}(\mathbb{C}_\infty) \ : \ k'_{\mathfrak{p}} \in \mathcal{K}'_{\mathfrak{p}}\}$$

is exactly one orbit under the action of $\pi_1^{\mathrm{arithm}}(Y, y)$ on $\pi'^{-1}(\pi^{-1}(y))$. Therefore, $\pi'^{-1}(Y')$ is E-irreducible. Since this holds for every open subgroup $\tilde{\mathcal{K}}'_{\mathfrak{p}} \subset \mathcal{K}'_{\mathfrak{p}}$, this implies that $T_{h_{\mathfrak{p}}}Y'$ is E-irreducible for all $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$.

We now define Z'_1 to be the Zariski closure of Y' in $S^r_{F,\mathcal{K}'}$. Since Y' is irreducible over $\mathbb{C}_\infty$, its Zariski closure Z'_1 is also irreducible over $\mathbb{C}_\infty$ and moreover, by dimension reasons, an irreducible component of $\pi^{-1}(Z_1)$ over $\mathbb{C}_\infty$. Since Y' is also E-irreducible, we similarly conclude that Z'_1 is an E-irreducible component of $\pi^{-1}(Z_1)$.

Note that, for all $h_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$, the projections π_1 and $\pi_{h_{\mathfrak{p}}}$ in the definition of the Hecke correspondence $T_{h_{\mathfrak{p}}}$ on $S^r_{F,\mathcal{K}'}$ are open and closed because they are finite and étale. By the E-irreducibility of $T_{h_{\mathfrak{p}}}Y'$ this implies that

$$T_{h_{\mathfrak{p}}}Z'_1 = \pi_{h_{\mathfrak{p}}}(\pi_1^{-1}(\overline{Y'})) = \overline{\pi_{h_{\mathfrak{p}}}(\pi_1^{-1}(Y'))} = \overline{T_{h_{\mathfrak{p}}}Y'}$$

is E-irreducible and concludes step (i).

Step (ii): By the assumption $Z \subset T_g Z$, the irreducible component Z_1 of Z is contained in $T_g Z_i$ for some i. Since Z is F-irreducible, there is an element $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ with $Z_i = \sigma(Z_1)$. This gives for $Z'_1 \subset S^r_{F,\mathcal{K}'}$

$$Z'_1 \subset \pi^{-1}(Z_1) \subset \pi^{-1}(T_g\sigma(Z_1)) = \sigma(\pi^{-1}(T_g Z_1)), \tag{5.1.1}$$

where the last equality holds because all our projection morphisms are defined over F.

Lemma 5.1.4. *Let $\{k_1, \ldots, k_l\}$ be a set of representatives for the left cosets in $\mathcal{K}_{\mathfrak{p}}/\mathcal{K}'_{\mathfrak{p}}$. Then we have*

$$\pi^{-1}(T_g Z_1) = \bigcup_{i,j=1}^{l} T_{k_i^{-1} g_{\mathfrak{p}} k_j} Z'_1.$$

Proof of Lemma 5.1.4. We show both inclusions on $\mathbb{C}_\infty$-valued points.

First consider a $\mathbb{C}_\infty$-valued point x of $\pi^{-1}(T_g Z_1)$. Because of $Z_1 = \pi(Z'_1)$ there is a $p' = [(\overline{\omega}, h)] \in Z'_1(\mathbb{C}_\infty)$ with $x \in \pi^{-1}(T_g \pi(p'))$. Therefore there are $l_1, l_2 \in \mathcal{K}$ such that

$$x = [(\overline{\omega}, h l_1 g_{\mathfrak{p}}^{-1} l_2)] \in S^r_{F,\mathcal{K}'}(\mathbb{C}_\infty).$$

Since $\{k_1, \cdots, k_l\}$ is a set of representatives for the left cosets in $\mathcal{K}_\mathfrak{p}/\mathcal{K}'_\mathfrak{p} \cong \mathcal{K}/\mathcal{K}'$, there are i and j and k'_1, $k'_2 \in \mathcal{K}'$ with $l_1^{-1} = k_i k'_1$ and $l_2 = k_j k'_2$. Hence we have

$$x = [(\overline{\omega}, hk_1'^{-1}(k_j^{-1} g_\mathfrak{p} k_i)^{-1} k'_2)] \in T_{k_j^{-1} g_\mathfrak{p} k_i} p' \subset T_{k_j^{-1} g_\mathfrak{p} k_i} Z'_1(\mathbb{C}_\infty).$$

For the other inclusion, let x be a $\mathbb{C}_\infty$-valued point of $T_{k_i^{-1} g_\mathfrak{p} k_j} Z'_1$ for some i and j. Then there is a $p' = [(\overline{\omega}, h)] \in Z'_1(\mathbb{C}_\infty)$ and a $k' \in \mathcal{K}'$ with

$$x = [(\overline{\omega}, hk'(k_i^{-1} g_\mathfrak{p} k_j)^{-1})].$$

It follows that

$$\pi(x) = [(\overline{\omega}, hk' k_j^{-1} g_\mathfrak{p}^{-1})] \in T_g(\pi(p')) \subset T_g Z_1(\mathbb{C}_\infty),$$

hence indeed $x \in \pi^{-1}(T_g Z_1)(\mathbb{C}_\infty)$. □

Since Z'_1 is E-irreducible, Lemma 5.1.4 and (5.1.1) imply the existence of k_1, $k_2 \in \mathcal{K}_\mathfrak{p}$ with

$$Z'_1 \subset \sigma(T_{k_1^{-1} g_\mathfrak{p} k_2} Z'_1).$$

By (i), the subvariety $T_{h_\mathfrak{p}} Z'_1 \subset S^r_{F,\mathcal{K}'}$ with $h_\mathfrak{p} := k_1^{-1} g_\mathfrak{p} k_2$ is also E-irreducible, therefore we even have

$$Z'_1 = \sigma(T_{h_\mathfrak{p}} Z'_1).$$

Iterating this gives the inclusion

$$Z'_1 = \sigma(T_{h_\mathfrak{p}} \sigma(T_{h_\mathfrak{p}} Z'_1)) = \sigma^2(T_{h_\mathfrak{p}}(T_{h_\mathfrak{p}} Z'_1)) \supset \sigma^2(T_{h_\mathfrak{p}^2} Z'_1),$$

which also must be an equality because both sides are of the same dimension and Z'_1 is E-irreducible. Repeating the same argument gives

$$Z'_1 = \sigma^i(T_{h_\mathfrak{p}^i} Z'_1)$$

for all $i \geq 1$. There is an $n \geq 1$ with $\sigma^n \in \mathrm{Gal}(F^{\mathrm{sep}}/E)$. Since $T_{h_\mathfrak{p}^n} Z'_1$ is defined over E, we conclude the relations

$$Z'_1 = \sigma^n(T_{h_\mathfrak{p}^n} Z'_1) = T_{h_\mathfrak{p}^n} Z'_1$$

and

$$T_{h_\mathfrak{p}^{-n}} Z'_1 = T_{h_\mathfrak{p}^{-n}}(T_{h_\mathfrak{p}^n} Z'_1) \supset Z'_1.$$

Again, the latter relation must be an equality because $T_{h_\mathfrak{p}^{-n}} Z'_1$ is E-irreducible and of the same dimension as Z'_1. Hence we have

$$Z'_1 = T_{h_\mathfrak{p}^n} Z'_1 = T_{h_\mathfrak{p}^{-n}} Z'_1.$$

Note that the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $h_\mathfrak{p}^n = (k_1^{-1} g_\mathfrak{p} k_2)^n$ is unbounded by our assumption. So we can apply Proposition 5.1.1 and conclude that Z'_1 is an irreducible component of $S^r_{F,\mathcal{K}'}$ over $\mathbb{C}_\infty$. □

Chapter 6

Existence of good primes and suitable Hecke operators

6.1 Good primes

In this section, $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ denotes a Drinfeld modular subvariety of a Drinfeld modular variety $S^r_{F,\mathcal{K}}$ associated to the datum (F', b).

Definition 6.1.1. *For a prime $\mathfrak{p}$ of F, a free $A_\mathfrak{p}$-submodule $\Lambda_\mathfrak{p} \subset F_\mathfrak{p}^r$ of rank r is called an $A_\mathfrak{p}$-*lattice*.*

Definition 6.1.2. *A prime $\mathfrak{p}$ is called* good *for $X \subset S^r_{F,\mathcal{K}}$ if there exists an $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} \subset F_\mathfrak{p}^r$ such that*

(i) $\mathcal{K} = \mathcal{K}_\mathfrak{p} \times \mathcal{K}^{(\mathfrak{p})}$ *with $\mathcal{K}_\mathfrak{p}$ the kernel of the natural map*

$$\mathrm{Stab}_{\mathrm{GL}_r(F_\mathfrak{p})}(\Lambda_\mathfrak{p}) \to \mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_\mathfrak{p}/\pi_\mathfrak{p} \cdot \Lambda_\mathfrak{p})$$

for a uniformizer $\pi_\mathfrak{p}$ and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$,

(ii) *there is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_\mathfrak{p}] = 1$,*

(iii) *$b_\mathfrak{p}(\Lambda_\mathfrak{p})$ is an $A'_\mathfrak{p}$-submodule of $F_\mathfrak{p}'^{r'}$.*

Remarks:

- The definition is independent of the datum (F', b) describing X because F' is uniquely determined by X and $b'_\mathfrak{p} = s_\mathfrak{p} \circ b_\mathfrak{p} \circ k_\mathfrak{p}$ with $s_\mathfrak{p} \in \mathrm{GL}_{r'}(F'_\mathfrak{p})$ and $k_\mathfrak{p} \in \mathcal{K}_\mathfrak{p} \subset \mathrm{Stab}_{\mathrm{GL}_r(F_\mathfrak{p})}(\Lambda_\mathfrak{p})$ for a second datum (F', b') describing X by Corollary 2.2.12.

- The existence of a good prime $\mathfrak{p}$ for X implies that the reflex field F' of X is separable over F because there exists a prime $\mathfrak{p}'$ of F' which is unramified over F.
- If $\Lambda_{\mathfrak{p}} = s_{\mathfrak{p}} A_{\mathfrak{p}}^r$ for an $s_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$, then condition (i) is equivalent to

$$\mathcal{K} = s_{\mathfrak{p}} \mathcal{K}(\mathfrak{p}) s_{\mathfrak{p}}^{-1} \times \mathcal{K}^{(\mathfrak{p})},$$

  where $\mathcal{K}(\mathfrak{p}) \subset \mathrm{GL}_r(A_{\mathfrak{p}})$ is the principal congruence subgroup modulo $\mathfrak{p}$.

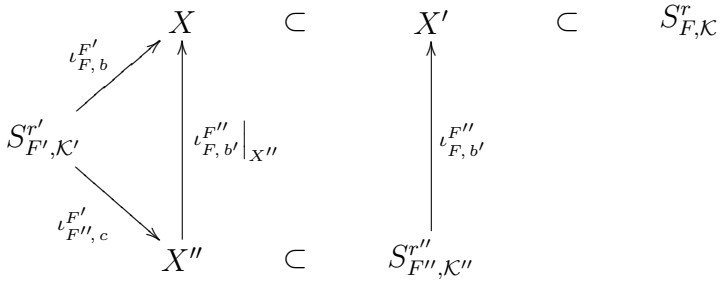
Proposition 6.1.3. *Let $\mathfrak{p}$ be a good prime for X. Suppose that X is contained in a Drinfeld modular subvariety $X' = \iota_{F,\,b'}^{F''}(S_{F'',\mathcal{K}''}^{r''}) \subset S_{F,\mathcal{K}}^{r}$.*

Then $X'' := (\iota_{F,\,b'}^{F''})^{-1}(X)$ is a Drinfeld modular subvariety of $S_{F'',\mathcal{K}''}^{r''}$ and there is a prime $\mathfrak{p}''$ of F'' above $\mathfrak{p}$ with $k(\mathfrak{p}) = k(\mathfrak{p}'')$ such that $\mathfrak{p}''$ is good for $X'' \subset S_{F'',\mathcal{K}''}^{r''}$.

Proof. By Corollary 2.2.11, $X'' = (\iota_{F,\,b'}^{F''})^{-1}(X)$ is a Drinfeld modular subvariety of $S_{F'',\mathcal{K}''}^{r''}$. In the proof of Corollary 2.2.11 we saw that $F \subset F'' \subset F'$ and there are an $\mathbb{A}_F^f$-linear isomorphism $c : (\mathbb{A}_{F''}^f)^{r''} \xrightarrow{\cong} (\mathbb{A}_{F'}^f)^{r'}$ and a $k \in \mathcal{K}$ such that

$$b = c \circ b' \circ k \tag{6.1.1}$$

and $X'' = \iota_{F'',\,c}^{F'}(S_{F',\mathcal{K}'}^{r'})$. The situation is summarized in the following commutative diagram where all arrows are bijections on $\mathbb{C}_\infty$-valued points:



Since $\mathfrak{p}$ is a good prime for $X = \iota_{F,\,b}^{F'}(S_{F,\mathcal{K}'}^{r'}) \subset S_{F,\mathcal{K}}^{r}$, there is an $A_{\mathfrak{p}}$-lattice $\Lambda_{\mathfrak{p}} \subset A_{\mathfrak{p}}^r$ for which the conditions (i)-(iii) of Definition 6.1.2 are satisfied. We now show the existence of a good prime $\mathfrak{p}''$ for $X'' = \iota_{F'',\,c}^{F'}(S_{F',\mathcal{K}'}^{r'}) \subset S_{F'',\mathcal{K}''}^{r''}$.

By (ii), there is a prime $\mathfrak{p}'$ of F' lying over $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$. We define $\mathfrak{p}''$ to be the prime of F'' lying between $\mathfrak{p}$ and $\mathfrak{p}'$. By construction, $k(\mathfrak{p}) = k(\mathfrak{p}'')$ and $\mathfrak{p}'$ is also of local degree 1 over F'', i.e., $[F'_{\mathfrak{p}'}/F''_{\mathfrak{p}''}] = 1$.

By (iii), $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of $F_{\mathfrak{p}}'^{r'}$. So we can write

$$b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = \Lambda'_{\mathfrak{p}''} \times \Lambda'^{(\mathfrak{p}'')}$$

with $\Lambda'_{\mathfrak{p}''} \subset {F'_{\mathfrak{p}''}}^{r'}$ an $A'_{\mathfrak{p}''}$-submodule (recall that $A'_{\mathfrak{p}''} = A' \otimes_{A''} A''_{\mathfrak{p}''}$ by our conventions). Since c is $\mathbb{A}_{F''}^f$-linear and $A'' \subset A'$, it follows that we can also write

$$c_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})) = c_{\mathfrak{p}}^{-1}(\Lambda'_{\mathfrak{p}''} \times \Lambda'^{(\mathfrak{p}'')}) = \Lambda''_{\mathfrak{p}''} \times \Lambda''^{(\mathfrak{p}'')}$$

with $\Lambda''_{\mathfrak{p}''}$ an $A''_{\mathfrak{p}''}$-submodule of $F''^{r''}_{\mathfrak{p}}$ for which $c_{\mathfrak{p}''}(\Lambda''_{\mathfrak{p}''}) = \Lambda'_{\mathfrak{p}''}$ is an $A'_{\mathfrak{p}''}$-submodule of ${F'_{\mathfrak{p}''}}^{r'}$. By (6.1.1) and since $k_{\mathfrak{p}}$ stabilizes $\Lambda_{\mathfrak{p}}$ by (i), we furthermore have

$$b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = b'_{\mathfrak{p}}(k_{\mathfrak{p}}\Lambda_{\mathfrak{p}}) = c_{\mathfrak{p}}^{-1}(b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})) = \Lambda''_{\mathfrak{p}''} \times \Lambda''^{(\mathfrak{p}'')}.$$

So we have shown that the conditions (ii) and (iii) of Definition 6.1.2 are satisfied for the lattice $\Lambda''_{\mathfrak{p}''} \subset {A''_{\mathfrak{p}''}}^{r''}$ and the datum (F', c) describing $X'' \subset S^{r''}_{F'',\mathcal{K}''}$. To show also (i), we note that $\mathcal{K}'' = \mathcal{K}''_{\mathfrak{p}} \times \mathcal{K}''^{(\mathfrak{p})}$ with

$$\mathcal{K}''_{\mathfrak{p}} = (b'_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}{b'_{\mathfrak{p}}}^{-1}) \cap \mathrm{GL}_{r''}(F''_{\mathfrak{p}}).$$

Since $\mathcal{K}_{\mathfrak{p}}$ is the kernel of the natural map

$$\mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}) \to \mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\pi_{\mathfrak{p}} \cdot \Lambda_{\mathfrak{p}})$$

for a uniformizer $\pi_{\mathfrak{p}} \in A_{\mathfrak{p}}$, it follows that $\mathcal{K}''_{\mathfrak{p}}$ is the kernel of the natural map

$$\mathrm{Stab}_{\mathrm{GL}_{r''}(F''_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}) \to \mathrm{Aut}_{k(\mathfrak{p})}(b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})/\pi_{\mathfrak{p}} \cdot b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})).$$

Since $\pi_{\mathfrak{p}}$ is also a uniformizer at $\mathfrak{p}''$ and because of $b'_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = \Lambda''_{\mathfrak{p}''} \times \Lambda''^{(\mathfrak{p}'')}_{\mathfrak{p}}$, we see that (i) is satisfied for the lattice $\Lambda''_{\mathfrak{p}''}$. □

6.2 Suitable Hecke correspondences

Proposition 6.2.1. *Let $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$ be a Drinfeld modular subvariety and $g' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. Then, we have*

$$X \subset T_g X$$

for $g := b^{-1} \circ g' \circ b \in \mathrm{GL}_r(\mathbb{A}^f_F)$.

Proof. Let $p = \iota^{F'}_{F,b}([(\overline{\omega'}, h')]) \in X(\mathbb{C}_\infty)$ for some $\overline{\omega'} \in \Omega^{r'}_{F'}$ and $h' \in \mathrm{GL}_{r'}(\mathbb{A}^f_{F'})$. Then we have

$$p = [(\overline{\omega' \circ \varphi}, \varphi^{-1} \circ h' \circ b)] = [(\overline{\omega' \circ \varphi}, (\varphi^{-1} \circ h'g' \circ b \circ g^{-1})]$$

for an F-linear isomorphism $\varphi : F^r \xrightarrow{\cong} F'^{r'}$, therefore p lies in $T_g\left(\iota^{F'}_{F,b}([(\overline{\omega'}, h'g')])\right)$ and therefore in $T_g X(\mathbb{C}_\infty)$. Since $p \in X(\mathbb{C}_\infty)$ was arbitrary, we conclude $X \subset T_g X$. □

Theorem 6.2.2. *Let $\mathfrak{p}$ be a good prime for a Drinfeld modular subvariety $X = \iota_{F,b}^{F'}(S_{F',\mathcal{K}'}^{r'}) \subset S_{F,\mathcal{K}}^{r}$ and let $\mathfrak{p}'$ be a prime of F' above $\mathfrak{p}$ with local degree 1 over F. Then there is a*

$$g' = (1, \ldots, 1, g'_{\mathfrak{p}'}, 1, \ldots, 1) \in \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$$

with $g'_{\mathfrak{p}'} \in \mathrm{GL}_{r'}(F'_{\mathfrak{p}'})$ such that the following holds for $g := b^{-1} \circ g' \circ b \in \mathrm{GL}_r(\mathbb{A}_F^f)$:

(i) $X \subset T_g X$,

(ii) $\deg T_g = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] = |k(\mathfrak{p})|^{r-1}$,

(iii) For all $k_1, k_2 \in \mathcal{K}_{\mathfrak{p}}$, the cyclic subgroup of $\mathrm{PGL}_r(F_{\mathfrak{p}})$ generated by the image of $k_1 \cdot g_{\mathfrak{p}} \cdot k_2$ is unbounded.

Proof. Suppose that the conditions (i)-(iii) of Definition 6.1.2 are satisfied for the $A_{\mathfrak{p}}$-lattice $\Lambda_{\mathfrak{p}} \subset F_{\mathfrak{p}}^r$.

By (iii), we can write

$$b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}}) = \Lambda'_{\mathfrak{p}'} \times {\Lambda'_{\mathfrak{p}}}^{(\mathfrak{p}')}$$

with $\Lambda'_{\mathfrak{p}'} \subset {F'_{\mathfrak{p}'}}^{r'}$ a free $A'_{\mathfrak{p}'}$-submodule of rank r' and ${\Lambda'_{\mathfrak{p}}}^{(\mathfrak{p}')} \subset (F'^{(\mathfrak{p}')}_{\mathfrak{p}})^{r'}$. Let $g'_{\mathfrak{p}'} : {F'_{\mathfrak{p}'}}^{r'} \to {F'_{\mathfrak{p}'}}^{r'}$ be given by

$$\mathrm{diag}(\pi_{\mathfrak{p}'}, 1, \ldots, 1)$$

for a uniformizer $\pi_{\mathfrak{p}'} \in A'_{\mathfrak{p}'}$ with respect to an $A'_{\mathfrak{p}'}$-basis of $\Lambda'_{\mathfrak{p}'}$.

We now check the conditions (i)-(iii) for $g'_{\mathfrak{p}'}$. Statement (i) follows by Proposition 6.2.1.

For (ii) and (iii), note that each $A'_{\mathfrak{p}'}$-basis of $\Lambda'_{\mathfrak{p}'}$ is also an $A_{\mathfrak{p}}$-basis of $\Lambda'_{\mathfrak{p}'}$ and can be extended to an $A_{\mathfrak{p}}$-basis of $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ because the local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}]$ is equal to 1. In particular, $g'_{\mathfrak{p}} = b_{\mathfrak{p}} \circ g_{\mathfrak{p}} \circ b_{\mathfrak{p}}^{-1} : F_{\mathfrak{p}}^{r'} \to F_{\mathfrak{p}}^{r'}$ is given by the diagonal matrix

$$D_{\mathfrak{p}} := \mathrm{diag}(\pi_{\mathfrak{p}}, 1, \ldots, 1)$$

with respect to some $A_{\mathfrak{p}}$-basis B' of $b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ for a uniformizer $\pi_{\mathfrak{p}} \in A_{\mathfrak{p}}$. It follows that $g_{\mathfrak{p}} : F_{\mathfrak{p}}^r \to F_{\mathfrak{p}}^r$ is also given by $D_{\mathfrak{p}}$ with respect to the $A_{\mathfrak{p}}$-basis $b_{\mathfrak{p}}^{-1}(B')$ of $\Lambda_{\mathfrak{p}}$. Hence, there is a $s_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$ such that

$$\begin{aligned} g_{\mathfrak{p}} &= s_{\mathfrak{p}} D_{\mathfrak{p}} s_{\mathfrak{p}}^{-1}, \\ \Lambda_{\mathfrak{p}} &= s_{\mathfrak{p}} A_{\mathfrak{p}}^r. \end{aligned}$$

By the remark after Definition 6.1.2, we therefore have

$$\mathcal{K}_{\mathfrak{p}} = s_{\mathfrak{p}} \mathcal{K}(\mathfrak{p}) s_{\mathfrak{p}}^{-1}$$

with $\mathcal{K}(\mathfrak{p})$ the principal congruence subgroup of $\mathrm{GL}_r(A_\mathfrak{p})$ modulo $\mathfrak{p}$.

Hence, we can and do assume $\mathcal{K}_\mathfrak{p} = \mathcal{K}(\mathfrak{p})$ and $g_\mathfrak{p} = D_\mathfrak{p}$ because (ii) and (iii) are invariant under conjugation.

For the proof of (ii) consider the map

$$\alpha : \begin{array}{ccc} \mathcal{K}(\mathfrak{p}) & \longrightarrow & (A_\mathfrak{p}/(\pi_\mathfrak{p}))^{r-1} \\ h & \longmapsto & ([\pi_\mathfrak{p}^{-1} \cdot h_{21}], \ldots, [\pi_\mathfrak{p}^{-1} \cdot h_{r1}]) \end{array}.$$

For $h, h' \in \mathcal{K}(\mathfrak{p})$, we have for $2 \leq i \leq r$

$$\begin{aligned} \pi_\mathfrak{p}^{-1} \cdot (hh')_{i1} &= (\pi_\mathfrak{p}^{-1} h_{i1}) h'_{11} + h_{ii}(\pi_\mathfrak{p}^{-1} h'_{i1}) + \sum_{j \neq i,1} (\pi_\mathfrak{p}^{-1} h_{ij}) h'_{j1} \\ &\equiv \pi_\mathfrak{p}^{-1} h_{i1} + \pi_\mathfrak{p}^{-1} h'_{i1} + 0 \pmod{\mathfrak{p}}, \end{aligned}$$

therefore α is a homomorphism of groups. It is furthermore surjective and its kernel is exactly equal to $\mathcal{K}(\mathfrak{p}) \cap D_\mathfrak{p} \mathcal{K}(\mathfrak{p}) D_\mathfrak{p}^{-1}$. Hence, we have

$$[\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] = [\mathcal{K}_\mathfrak{p} : \mathcal{K}_\mathfrak{p} \cap g_\mathfrak{p}^{-1} \mathcal{K}_\mathfrak{p} g_\mathfrak{p}] = |k(\mathfrak{p})|^{r-1}$$

and

$$\deg T_g = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g]$$

as explained at the end of Section 2.1.

For (iii), let $k_1, k_2 \in \mathcal{K}_\mathfrak{p} = \mathcal{K}(\mathfrak{p})$ be arbitrary. We prove that the eigenvalues of $(k_1 g_\mathfrak{p} k_2)^{-1} = k_2^{-1} D_\mathfrak{p}^{-1} k_1^{-1}$ do not all have the same $\mathfrak{p}$-valuation by showing that the Newton polygon of the characteristic polynomial

$$\chi(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1 \lambda + a_0$$

of $k_2^{-1} D_\mathfrak{p}^{-1} k_1^{-1}$ consists at least of two line segments. This implies that the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $k_1 g_\mathfrak{p} k_2$ is unbounded.

Since k_1, k_2 are elements of $\mathrm{GL}_r(A_\mathfrak{p})$, we have $\det(k_1), \det(k_2) \in A_\mathfrak{p}^*$ and hence

$$v_\mathfrak{p}(a_0) = v_\mathfrak{p}(\det(k_2^{-1} D_\mathfrak{p}^{-1} k_1^{-1})) = 0 - v_\mathfrak{p}(\det(D_\mathfrak{p})) + 0 = -1.$$

The coefficient a_{r-1} can be expressed as

$$a_{r-1} = -\mathrm{tr}(k_2^{-1} D_\mathfrak{p}^{-1} k_1^{-1}) = -\sum_i (k_2^{-1})_{i1} \pi_\mathfrak{p}^{-1} (k_1^{-1})_{1i} - \sum_i \sum_{j \neq 1} (k_2^{-1})_{ij} (k_1^{-1})_{ji}.$$

Because of $k_1, k_2 \in \mathcal{K}(\mathfrak{p})$ we have $v_\mathfrak{p}((k_1^{-1})_{ij}), v_\mathfrak{p}((k_2^{-1})_{ij} \geq 0$ with equality exactly for $i = j$. Therefore, in the above expression for a_{r-1}, the summand for $i = 1$ in

the first sum has $\mathfrak{p}$-valuation -1 and all the other summands have $\mathfrak{p}$-valuation at least 0. We conclude

$$v_{\mathfrak{p}}(a_{r-1}) = -1.$$

Hence, the point $(r-1, v_{\mathfrak{p}}(a_{r-1}))$ lies below the line through $(0, v_{\mathfrak{p}}(a_0))$ and $(r, 0)$. This implies that the Newton polygon of χ consists at least of two line segments. □

6.3 Existence of good primes

Proposition 6.3.1. *Let $X = \iota_{F,b}^{F'}(S^{r'}_{F',\mathcal{K}'}) \subset S^r_{F,\mathcal{K}}$ be a Drinfeld modular subvariety and $\mathfrak{p}$ a prime of F such that the following holds:*

(i) *there is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$,*

(ii) *$\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ a maximal compact subgroup and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$,*

(iii) *$\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}} \mathcal{K}_{\mathfrak{p}} b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$.*

Then there is a subgroup $\tilde{\mathcal{K}} \subset \mathcal{K}$ and a Drinfeld modular subvariety $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$ such that

(i) *$\pi_1(\tilde{X}) = X$ for the canonical projection $\pi_1 : S^r_{F,\tilde{\mathcal{K}}} \to S^r_{F,\mathcal{K}}$,*

(ii) *$\mathfrak{p}$ is good for $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$,*

(iii) *$[\mathcal{K} : \tilde{\mathcal{K}}] < |k(\mathfrak{p})|^{r^2}$.*

Proof. As $\mathcal{K}_{\mathfrak{p}}$ is a maximal compact subgroup of $\mathrm{GL}_r(F_{\mathfrak{p}})$, there is an $s_{\mathfrak{p}} \in \mathrm{GL}_r(F_{\mathfrak{p}})$ with $\mathcal{K}_{\mathfrak{p}} = s_{\mathfrak{p}} \mathrm{GL}_r(A_{\mathfrak{p}}) s_{\mathfrak{p}}^{-1}$. We define $\Lambda_{\mathfrak{p}}$ to be the lattice $s_{\mathfrak{p}} \cdot A_{\mathfrak{p}}^r$, for which we have

$$\mathcal{K}_{\mathfrak{p}} = \mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}).$$

Now, we let $\tilde{\mathcal{K}}_{\mathfrak{p}}$ be the kernel of the natural map

$$\mathrm{Stab}_{\mathrm{GL}_r(F_{\mathfrak{p}})}(\Lambda_{\mathfrak{p}}) \to \mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\pi_{\mathfrak{p}} \cdot \Lambda_{\mathfrak{p}})$$

for a uniformizer $\pi_{\mathfrak{p}} \in A_{\mathfrak{p}}$ and define $\tilde{\mathcal{K}} := \tilde{\mathcal{K}}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$.

By construction, we get the upper bound (iii) for the index of $\tilde{\mathcal{K}}$ in $\mathcal{K}$:

$$[\mathcal{K} : \tilde{\mathcal{K}}] = [\mathcal{K}_{\mathfrak{p}} : \tilde{\mathcal{K}}_{\mathfrak{p}}] = \left|\mathrm{Aut}_{k(\mathfrak{p})}(\Lambda_{\mathfrak{p}}/\pi_{\mathfrak{p}} \cdot \Lambda_{\mathfrak{p}})\right| = |\mathrm{GL}_r(k(\mathfrak{p}))| < |k(\mathfrak{p})|^{r^2}.$$

We denote by $\tilde{\iota}^{F'}_{F,\,b}$ the inclusion $S^{r'}_{F',\tilde{\mathcal{K}}'} \to S^{r}_{F,\tilde{\mathcal{K}}}$ associated to the same datum $(F',\, b)$ as $\iota^{F'}_{F,\,b}$. By definition of the inclusion morphisms, the diagram

$$\begin{array}{ccc} S^{r'}_{F',\tilde{\mathcal{K}}'} & \xrightarrow{\tilde{\iota}^{F'}_{F,\,b}} & S^{r}_{F,\tilde{\mathcal{K}}} \\ \downarrow{\scriptstyle \pi'_1} & & \downarrow{\scriptstyle \pi_1} \\ S^{r'}_{F,\mathcal{K}'} & \xrightarrow{\iota^{F'}_{F,\,b}} & S^{r}_{F,\mathcal{K}} \end{array}$$

with π'_1 and π_1 projection morphisms as defined in Section 2.1 commutes. Therefore, the Drinfeld modular subvariety $\tilde{X} := \tilde{\iota}^{F'}_{F,\,b}(S^{r'}_{F',\tilde{\mathcal{K}}'}) \subset S^{r}_{F,\tilde{\mathcal{K}}}$ satisfies

$$\pi_1(\tilde{X}) = \pi_1(\tilde{\iota}^{F'}_{F,\,b}(S^{r'}_{F',\tilde{\mathcal{K}}'})) = \iota^{F'}_{F,\,b}(\pi'_1(S^{r'}_{F',\tilde{\mathcal{K}}'})) = \iota^{F'}_{F,\,b}(S^{r'}_{F',\mathcal{K}'}) = X.$$

It remains to show that $\mathfrak{p}$ is good for $\tilde{X} \subset S^{r}_{F,\tilde{\mathcal{K}}}$. By assumption, there is a prime $\mathfrak{p}'|\mathfrak{p}$ of F' with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$ and by construction $\tilde{\mathcal{K}}$ satisfies condition (i) of Definition 6.1.2. So we only have to check that $\Lambda'_{\mathfrak{p}} := b_{\mathfrak{p}}(\Lambda_{\mathfrak{p}})$ is an $A'_{\mathfrak{p}}$-submodule of $F'^{r'}_{\mathfrak{p}}$. Since $\mathcal{K}_{\mathfrak{p}}$ is the stabilizer of $\Lambda_{\mathfrak{p}}$ in $\mathrm{GL}_r(F_{\mathfrak{p}})$, the stabilizer of $\Lambda'_{\mathfrak{p}}$ in $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is exactly

$$\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}}),$$

which is a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ by assumption. Therefore we have

$$\mathrm{Stab}_{F'^{*}_{\mathfrak{p}}}(\Lambda'_{\mathfrak{p}}) = \mathcal{K}'_{\mathfrak{p}} \cap F'^{*}_{\mathfrak{p}} = A'^{*}_{\mathfrak{p}}$$

because $A'^{*}_{\mathfrak{p}}$ is the unique maximal compact subgroup of $F'^{*}_{\mathfrak{p}}$. In particular $A'^{*}_{\mathfrak{p}}$ is contained in the subring

$$R := \{\lambda \in F'_{\mathfrak{p}} \,:\, \lambda \cdot \Lambda'_{\mathfrak{p}} \subset \Lambda'_{\mathfrak{p}}\}$$

of $F'_{\mathfrak{p}}$. By the lemma below, it follows that this subring contains $A'_{\mathfrak{p}}$. Therefore $\Lambda'_{\mathfrak{p}}$ is an $A'_{\mathfrak{p}}$-submodule of $F'^{r'}_{\mathfrak{p}}$. □

Lemma 6.3.2. *$A'^{*}_{\mathfrak{p}}$ generates $A'_{\mathfrak{p}}$ as a ring.*

Proof. The ring

$$A'_{\mathfrak{p}} = \prod_{\mathfrak{q}'|\mathfrak{p}} A'_{\mathfrak{q}'}$$

is generated by $A'^{*}_{\mathfrak{p}}$ and all the elements

$$\lambda_{\mathfrak{q}'} := (0, \ldots, 0, \pi_{\mathfrak{q}'}, 0, \ldots, 0)$$

with $\mathfrak{q}'|\mathfrak{p}$, where $\pi_{\mathfrak{q}'}$ denotes a uniformizer at $\mathfrak{q}'$. So we only have to show that all $\lambda_{\mathfrak{q}'}$ lie in the subring of $A'_{\mathfrak{p}}$ generated by $A'^*_{\mathfrak{p}}$. This is true because of

$$\lambda_{\mathfrak{q}'} = \underbrace{(1,\ldots,1)}_{\in A'^*_{\mathfrak{p}}} + \underbrace{(-1,\ldots,-1,\pi_{\mathfrak{q}'}-1,-1,\ldots,-1)}_{\in A'^*_{\mathfrak{p}}}. \qquad \square$$

Theorem 6.3.3. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $N > 0$. For every prime $\mathfrak{p}$ of F, denote by $\mathcal{K}_{\mathfrak{p}}$ the projection of $\mathcal{K}$ to $\mathrm{GL}_r(F_{\mathfrak{p}})$. Then, for almost all Drinfeld modular subvarieties $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ with separable reflex field over F, there is a prime $\mathfrak{p}$ with the following properties:*

(i) *there is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$,*

(ii) *$\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ is a maximal compact subgroup and $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$,*

(iii) *$\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}}\mathcal{K}_{\mathfrak{p}}b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$,*

(iv) *$|k(\mathfrak{p})|^N < D(X)$ where $D(X)$ denotes the predegree of X from Definition 2.2.14.*

Before giving the proof of this theorem, we show two lemmas.

Lemma 6.3.4. *There are absolute constants C_1, $C_2 > 0$ such that for all global function fields F' with field of constants containing $\mathbb{F}_q$*

$$g(F') \le C_1 + C_2 \cdot \log_q(|\mathrm{Cl}(F')|)$$

where $g(F')$ denotes the genus of F' and $|\mathrm{Cl}(F')|$ the class number of F'.

Proof. Let F' be a global function field with field of constants $\mathbb{F}_{q'} \supset \mathbb{F}_q$. Then, with Proposition 3.3.4 we get the estimate

$$\begin{aligned} |\mathrm{Cl}(F')| &\ge \frac{(q'-1)(q'^{2g(F')} - 2g(F')q'^{g(F')} + 1)}{2g(F')(q'^{g(F')+1} - 1)} \ge (q'-1)\cdot\left(\frac{q'^{g(F')-1}}{2g(F')} - \frac{1}{q'}\right) \\ &\ge (q-1)\cdot\left(\frac{q^{g(F')-1}}{2g(F')} - \frac{1}{q}\right), \end{aligned}$$

which implies

$$\frac{q^{g(F')-1}}{g(F')} \le \frac{2|\mathrm{Cl}(F')|}{q-1} + \frac{2}{q} \le 4|\mathrm{Cl}(F')|.$$

There is a constant $C > 0$ such that for all F' with $g(F') \ge C$

$$q^{g(F')/2} \le \frac{q^{g(F')-1}}{g(F')} \le 4|\mathrm{Cl}(F')|$$

and hence

$$g(F') \leq 2 \cdot \log_q(4|\mathrm{Cl}(F')|) \leq 4 + 2\log_q(|\mathrm{Cl}(F')|).$$

So the bound

$$g(F') \leq C_1 + C_2 \cdot \log_q(|\mathrm{Cl}(F')|)$$

holds with $C_1 := \max\{4, C\}$ and $C_2 := 2$. □

Lemma 6.3.5. *There are constants C_3, $C_4 > 0$ only depending on r such that for all finite separable extensions F'/F of global function fields with $[F'/F] \leq r$*

$$g(E') \leq C_3 + C_4 \cdot g(F')$$

where E' denotes the normal closure of the extension F'/F.

Proof. Let F'/F be a finite separable extension of global function fields of degree $r' \leq r$. Its normal closure E' is the compositum of all Galois conjugates $F'_1, \ldots, F'_{r'}$ of F' over F. We use Castelnuovo's inequality (Theorem III.10.3 in [35]) to bound its genus:

If a global function field K is the compositum of two subfields K_1 and K_2 with $n_i := [K/K_i] < \infty$ for $i = 1, 2$, then

$$g(K) \leq n_1 \cdot g(K_1) + n_2 \cdot g(K_2) + (n_1 - 1)(n_2 - 1).$$

For $K_1 = F'_1$ and $K_2 = F'_2$ this gives

$$g(F'_1F'_2) \leq r' \cdot g(F') + r' \cdot g(F') + (r' - 1)^2 \leq 2r' \cdot g(F') + r'^2$$

because all Galois conjugates of F' over F have the same genus and $[F'_1F'_2/F'_1] \leq [F'_2/F] = r'$ and $[F'_1F'_2/F'_2] \leq [F'_1/F] = r'$. With induction over k we get

$$g(F'_1 \cdots F'_k) \leq kr'^{k-1} \cdot g(F') + (k-1)r'^k$$

and with $k = r'$

$$g(E') \leq r'^{r'} \cdot g(F') + (r' - 1) \cdot r'^{r'} \leq (r-1)r^r + r^r \cdot g(F').$$ □

Proof of Theorem 6.3.3. For a Drinfeld modular subvariety $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$ with separable reflex field over F, we denote by $n(X)$ the number of primes of F for which (ii) and (iii) do not both hold and by $m(X, N)$ the number of primes of F with (i) and (iv). We show the following statements for Drinfeld modular subvarieties X of S with separable reflex field:

(a) $n(X) \leq C_5 + C_6 \cdot \log_q(i(X))$ for constants C_5, C_6 independent of X where $i(X)$ denotes the index of X as defined in Definition 2.2.14,

(b) there is an $M > 0$ such that $m(X, N) > n(X)$ for all X with $D(X) > M$.

Statement (b) implies the theorem because $D(X) > M$ for almost all Drinfeld modular subvarieties X of S by Theorem 3.3.2.

Proof of (a): For a Drinfeld modular subvariety $X = \iota_{F,b}^{F'}(S_{F',\mathcal{K}'}^{r'})$ of S we have

$$\mathcal{K}' = (b\mathcal{K}b^{-1}) \cap \mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$$

and the index $i(X)$ is the index of $\mathcal{K}'$ in a maximal compact subgroup of $\mathrm{GL}_{r'}(\mathbb{A}_{F'}^f)$.

For a prime $\mathfrak{p}$ for which (ii) holds we can write $\mathcal{K}_\mathfrak{p} = \mathrm{Stab}_{\mathrm{GL}_r(F_\mathfrak{p})}(\Lambda_\mathfrak{p})$ for some $A_\mathfrak{p}$-lattice $\Lambda_\mathfrak{p} \subset F_\mathfrak{p}^r$ and

$$\mathcal{K}'_\mathfrak{p} = (b_\mathfrak{p}\mathcal{K}_\mathfrak{p}b_\mathfrak{p}^{-1}) \cap \mathrm{GL}_{r'}(F'_\mathfrak{p}) = \mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(\Lambda'_\mathfrak{p})$$

with $\Lambda'_\mathfrak{p} := b_\mathfrak{p}(\Lambda_\mathfrak{p})$. By Proposition 3.3.6 we have the estimates

$$[\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}) : \mathcal{K}'_\mathfrak{p}] \geq C \cdot [A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p} : \Lambda'_\mathfrak{p}]^{1/r}$$

for some constant $C > 0$ only depending on q and r. Note that $A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}$ is a free $A'_\mathfrak{p}$-submodule of rank r' (see, e.g., the proof of Proposition 3.3.6). Therefore, $\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p})$ is conjugate to $\mathrm{GL}_{r'}(A'_\mathfrak{p})$ and a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_\mathfrak{p})$. Hence, if $\mathcal{K}'_\mathfrak{p}$ is not a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_\mathfrak{p})$ (i.e., (iii) does not hold for $\mathfrak{p}$), then $\Lambda'_\mathfrak{p} \subsetneq A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}$ and

$$[\mathrm{Stab}_{\mathrm{GL}_{r'}(F'_\mathfrak{p})}(A'_\mathfrak{p} \cdot \Lambda'_\mathfrak{p}) : \mathcal{K}'_\mathfrak{p}] \geq C \cdot |k(\mathfrak{p})|^{1/r}$$

because each finite non-trivial $A_\mathfrak{p}$-module has at least $|k(\mathfrak{p})|$ elements.

Since, for each prime $\mathfrak{p}$ satisfying (ii), we have $\mathcal{K}' = \mathcal{K}'_\mathfrak{p} \times \mathcal{K}'^{(\mathfrak{p})}$ for some subgroup $\mathcal{K}'^{(\mathfrak{p})} \subset \mathrm{GL}_{r'}(F' \otimes \mathbb{A}_F^{f,\mathfrak{p}})$, we conclude that

$$i(X) \geq C \cdot |k(\mathfrak{p})|^{n_3(X)/r} \geq C \cdot q^{n_3(X)/r},$$

where $n_3(X)$ is the number of primes of F for which (ii) holds, but (iii) does not hold. If n_2 is the number of primes of F, for which (ii) does not hold, then we conclude

$$n(X) = n_2 + n_3(X) \leq n_2 - r \cdot \log_q(C) + r \cdot \log_q(i(X)).$$

This finishes the proof of (a) because n_2 is independent of X.

Proof of (b): Let X be a Drinfeld modular subvariety of S with separable reflex field F' over F. We denote the normal closure of the extension F'/F by E'. To give a lower bound for $m(X, N)$ we note that all primes $\mathfrak{p}$ of F which

completely split in E' satisfy condition (i). We bound the number of such primes with fixed degree using an effective version of Čebotarev's theorem.

For the application of Čebotarev's theorem we fix some notations. We denote the constant extension degree of E'/F by n and its geometric extension degree by k. Since we assumed F to have field of constants $\mathbb{F}_q$, the field of constants of E' is $\mathbb{F}_{q^n}$ and $k = [E'/\mathbb{F}_{q^n} \cdot F]$. We furthermore fix a separating transcendence element θ of $F/\mathbb{F}_q$ (i.e., an element θ of F such that $F/\mathbb{F}_q(\theta)$ is finite and separable) and set $d := [F/\mathbb{F}_q(\theta)]$.

The effective version of Čebotarev's theorem in [14] (Proposition 6.4.8) says that for all $i \geq 1$ with $n|i$

$$\left| |C_i(E'/F)| - \frac{q^i}{ik} \right| < \frac{2}{ik} \left((k + g(E'))q^{i/2} + k(2g(F)+1)q^{i/4} + g(E') + dk \right)$$

where

$$\begin{aligned} C_i(E'/F) \;:=\; & \{\mathfrak{p} \text{ place of } F \mid k(\mathfrak{p}) = \mathbb{F}_{q^i},\ \mathfrak{p} \text{ completely splits in } E' \text{ and} \\ & \mathfrak{p} \text{ is unramified over } \mathbb{F}_q(\theta)\}. \end{aligned}$$

We apply this for all X with predegree $D(X) \geq q^{4Nr!}$. Because of $n \leq [E'/F] \leq r!$, for these X we have $q^n \leq D(X)^{\frac{1}{4N}}$. Therefore there are $j \geq 1$ with $n|j$ and $q^j < D(X)^{\frac{1}{N}}$ and we can define

$$i := \max\{j \geq 1 \,:\, n|j,\, q^j < D(X)^{\frac{1}{N}}\}.$$

Our choice of i ensures that

$$m(X, N) \geq |C_i(E'/F)|.$$

By our choice of i and X we have $q^i < D(X)^{\frac{1}{N}}$, $q^{n+i} \geq D(X)^{\frac{1}{N}}$ and $q^n \leq D(X)^{\frac{1}{4N}}$. Hence we have the bounds

$$q^i < D(X)^{\frac{1}{N}}, \quad q^i = \frac{q^{n+i}}{q^n} \geq D(X)^{\frac{3}{4N}}.$$

Furthermore Lemma 6.3.4 and 6.3.5 imply

$$\begin{aligned} g(F') &\leq C_1 + C_2 \cdot \log_q(D(X)), \\ g(E') &\leq C_3 + C_4 \cdot g(F'). \end{aligned}$$

Since d is independent of X and $1 \leq n, k \leq r!$ for all X, the above conclusion of Čebotarev's theorem and these bounds imply

$$m(X, N) \geq \frac{C_1' \cdot D(X)^{\frac{3}{4N}}}{\log_q(D(X))} - \frac{C_2' + C_3' \log_q(D(X))}{\log_q(D(X))} \left(D(X)^{\frac{1}{2N}} + D(X)^{\frac{1}{4N}} + 1 \right)$$

with C_1', C_2', $C_3' > 0$ independent of X. On the other hand, our statement (a) gives the bound

$$n(X) \leq C_5 + C_6 \cdot \log_q(D(X))$$

with C_5, C_6 independent of X. Since $x^{\frac{1}{2N}}(\log_q(x))^2 = o(x^{\frac{3}{4N}})$ for $x \to \infty$, these bounds imply the existence of an $M > 0$ such that $m(X, N) > n(X)$ for all X with $D(X) > M$. □

Chapter 7

The André-Oort Conjecture for Drinfeld modular varieties

7.1 Statement and first reduction

Conjecture 7.1.1 (André-Oort Conjecture for Drinfeld modular varieties)**.** *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and Σ a set of special points of S. Then each irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of Σ is a special subvariety of S.*

Our main result is the following theorem:

Theorem 7.1.2. *Conjecture 7.1.1 is true if the reflex fields of all special points in Σ are separable over F.*

Since the reflex field of a special point in $S^r_{F,\mathcal{K}}$ is of degree r over F, special points with inseparable reflex field over F can only occur if r is divisible by $p = \mathrm{char}(F)$. Hence, Theorem 7.1.2 implies

Corollary 7.1.3. *Conjecture 7.1.1 is true if r is not a multiple of $p = \mathrm{char}(F)$.*

Theorem 7.1.2 follows from the following crucial statement whose proof we give in the next section:

Theorem 7.1.4. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $Z \subset S$ an F-irreducible subvariety of S. Suppose that Σ is a set of Drinfeld modular subvarieties of S, all of the same dimension $d < \dim Z$ and with separable reflex field over F, whose union is Zariski dense in Z. Then, for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.*

Proposition 7.1.5. *Theorem 7.1.4 implies Theorem 7.1.2.*

Proof of Proposition 7.1.5. Theorem 7.1.2 can be seen as a special case of the following more general statement:

Proposition 7.1.6. *Let S be a Drinfeld modular variety and $Z \subset S$ an F-irreducible subvariety. Suppose that Σ is a set of Drinfeld modular subvarieties of S, all of the same dimension $d \leq \dim Z$ and with separable reflex field over F, whose union is Zariski dense in Z. Then each irreducible component of Z over $\mathbb{C}_\infty$ is a special subvariety.*

Indeed, Proposition 7.1.6 for $d = 0$ implies Theorem 7.1.2:

Assume that Y is an irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of a set Σ of special points of a Drinfeld modular variety $S = S^r_{F,\mathcal{K}}$ with separable reflex field over F. Note that each special point in Σ is defined over the separable closure F^{sep} of F because it is an irreducible component over $\mathbb{C}_\infty$ of a Drinfeld modular subvariety of dimension 0 defined over a separable extension of F. Therefore, the Zariski closure of Σ and hence Y are also defined over F^{sep}. We set $Z := \text{Gal}(F^{\text{sep}}/F) \cdot Y$. This is a finite union of Galois conjugates of Y over F because Y is defined over some finite separable extension of F (as variety defined over F^{sep}). By Proposition 0.2.2, the subvariety $Z \subset S$ is F-irreducible.

Let now Σ' be the set of special points in Σ contained in Y and Σ'' the union of the $\text{Gal}(F^{\text{sep}}/F)$-conjugates of the elements of Σ'. By our assumption, the Zariski closure of Σ' is equal to Y and the Zariski closure of Σ'' is Z.

Note that Σ'' is a union of Drinfeld modular subvarieties of dimension 0 with separable reflex field over F. Indeed, for a geometric point $p \in \Sigma''$, there is a $\sigma \in \text{Gal}(F^{\text{sep}}/F)$ with $p = \sigma(q) \subset \sigma(\text{Gal}(F^{\text{sep}}/F') \cdot q)$ for some special point $q \in \Sigma'$ with reflex field F'. Since q is a special point with reflex field F', Corollary 2.3.6 implies that the F'-irreducible subvariety $\text{Gal}(F^{\text{sep}}/F') \cdot q$ of S is a Drinfeld modular subvariety. Hence by Proposition 2.2.17, $\sigma(\text{Gal}(F^{\text{sep}}/F') \cdot q) \subset \Sigma''$ is a Drinfeld modular subvariety with separable reflex field $\sigma(F')$ over F.

So Proposition 7.1.6 for $d = 0$ implies that all irreducible components of Z over $\mathbb{C}_\infty$ and in particular Y are special subvarieties of S. Hence, Proposition 7.1.6 for $d = 0$ implies Theorem 7.1.2. It remains to show that Theorem 7.1.4 implies Proposition 7.1.6:

Proof of Proposition 7.1.6 assuming Theorem 7.1.4. We give a proof by descending induction over d. For $d = \dim Z$ the statement is true because each irreducible component of Z over $\mathbb{C}_\infty$ must be an irreducible component over $\mathbb{C}_\infty$ of a Drinfeld modular subvariety lying in Σ.

So we let $0 \leq d < \dim Z$ and assume that Proposition 7.1.6 is true for d' with $d < d' \leq \dim Z$. By Theorem 7.1.4, there is a finite subset $\tilde{\Sigma} \subset \Sigma$ such that for all $X \in \Sigma \setminus \tilde{\Sigma}$, there is a Drinfeld modular subvariety X' with $X \subsetneq X' \subset Z$. We denote the set of these Drinfeld modular subvarieties X' by Σ'.

Since the union of the finitely many elements of $\tilde{\Sigma}$ is Zariski closed of dimension $d < \dim Z$, the union of the Drinfeld modular subvarieties in $\Sigma \setminus \tilde{\Sigma}$ is still Zariski dense in Z. Therefore, the union of all subvarieties in Σ' is also Zariski dense in Z.

Note that Proposition 2.2.10 implies that all elements X' of Σ' are of dimension $d' > d$. Therefore there is a d' with $d < d' \leq \dim Z$ such that the Zariski closure of the union of all subvarieties of dimension d' in Σ' is of codimension 0 in Z. We let Σ'' be the set of all $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-conjugates of the subvarieties of dimension d' in Σ'. Since Z is F-irreducible, this is a set of Drinfeld modular subvarieties of S, all of the same dimension $d' > d$, whose union is Zariski dense in Z. Hence, we can apply the induction hypothesis and conclude that each irreducible component of Z over $\mathbb{C}_\infty$ is special. □

7.2 Inductive proof in the separable case

The proof of Theorem 7.1.4 requires the results from Section 6.3 about the existence of good primes and the following theorem. We first give an inductive proof of the latter theorem using our results about existence of suitable Hecke correspondences from Section 6.2 and our geometric criterion in Theorem 5.1.3.

Theorem 7.2.1. *Let $S = S^r_{F,\mathcal{K}}$ be a Drinfeld modular variety and $X \subset S$ a Drinfeld modular subvariety over F which is contained in an F-irreducible subvariety $Z \subset S$ with $\dim Z > \dim X$. Suppose that $\mathfrak{p}$ is a good prime for $X \subset S$ and*

$$\deg(X) > |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)} \cdot \deg(Z)^{2^s}$$

for $s := \dim Z - \dim X$. Then there is a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Z$.

Remark: The degree $\deg(X)$ makes sense here because $\mathcal{K}$ is amply small by condition (i) in Definition 6.1.2.

Proof. In this proof, by "irreducible component" we always mean an irreducible component over $\mathbb{C}_\infty$. We assume that $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'})$. Note that F' is separable over F by the remark after Definition 6.1.2.

We prove the following statements for all $n \geq 1$:

(i) If the theorem is true for $s = n$ and Z Hodge-generic (i.e., all irreducible components of Z do not lie in a proper Drinfeld modular subvariety of S, see Definition 5.1.2), then it is true for $s = n$ and general Z.

(ii) If the theorem is true for all s with $1 \leq s < n$ and general Z, then it is true for $s = n$ and Z Hodge-generic.

These two statements imply the theorem by induction over s.

Proof of (i): We assume that the theorem is true for $s = n$ and Z Hodge-generic and have to show that it is true for $s = n$ if Z is not Hodge-generic. In this case, there is an irreducible component of Z which is contained in a proper Drinfeld modular subvariety of S. Since $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts transitively on the irreducible components of Z by the F-irreduciblity of Z (Proposition 0.2.2) and $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on the set of Drinfeld modular subvarieties of S (Proposition 2.2.17), also the other irreducible components of Z are contained in a proper Drinfeld modular subvariety of S. In particular, this is the case for some chosen irreducible component Z' of Z which contains an irreducible component V of X.

We now consider a minimal Drinfeld modular subvariety $Y = \iota_{F,\,b'}^{F''}(S_{F'',\mathcal{K}''}^{r''})$ of S with $Z' \subset Y \subsetneq S$. By Proposition 2.2.10, the reflex field F'' of Y is contained in F' and is therefore also separable over F. Since Y is defined over F'', the F''-irreducible component $Z'' := \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot Z'$ of Z is contained in Y. Furthermore, the F'-irreducibility of X (see Corollary 2.3.6) implies

$$X = \mathrm{Gal}(F^{\mathrm{sep}}/F') \cdot V \subset \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot V \subset \mathrm{Gal}(F^{\mathrm{sep}}/F'') \cdot Z' = Z'' \subset Y.$$

We now set $\tilde{X} := (\iota_{F,\,b'}^{F''})^{-1}(X)$ and $\tilde{Z} := (\iota_{F,\,b'}^{F''})^{-1}(Z'')$. These are subvarieties of $S_{F'',\mathcal{K}''}^{r''}$ with

$$\tilde{X} \subset \tilde{Z} \subset S_{F'',\mathcal{K}''}^{r''}.$$

and

$$\dim \tilde{Z} - \dim \tilde{X} = \dim Z - \dim X = n.$$

The subvariety $\tilde{Z} = (\iota_{F,\,b'}^{F''})^{-1}(Z'')$ is F''-irreducible because $Z'' \subset \iota_{F,\,b'}^{F''}(S_{F'',\mathcal{K}''}^{r''})$ is F''-irreducible and $\iota_{F,\,b'}^{F''}$ is a closed immersion defined over F'' by Proposition 2.2.2.

By Corollary 2.2.11 and minimality of Y, the subvariety $\tilde{Z} \subset S_{F'',\mathcal{K}''}^{r''}$ is Hodge-generic and $\tilde{X}$ is a Drinfeld modular subvariety of $S_{F'',\mathcal{K}''}^{r''}$ with separable reflex field F' over F''. Furthermore, by Proposition 6.1.3, there is a prime $\mathfrak{p}''$ of F'' above $\mathfrak{p}$ with $k(\mathfrak{p}) = k(\mathfrak{p}'')$ such that $\mathfrak{p}''$ is good for $\tilde{X} \subset S_{F'',\mathcal{K}''}^{r''}$.

Proposition 3.2.3 (ii) implies

$$\begin{aligned} \deg \tilde{X} &= \deg X, \\ \deg \tilde{Z} &= \deg Z'' \leq \deg Z. \end{aligned}$$

Because of $k(\mathfrak{p}) = k(\mathfrak{p}'')$ and $r'' < r$ the assumption

$$\deg(\tilde{X}) > |k(\mathfrak{p}'')|^{(r''-1)\cdot(2^n-1)} \cdot \deg(\tilde{Z})^{2^n}$$

is satisfied. So if Theorem 7.2.1 is true for Z Hodge-generic and $s = n$ then there is a Drinfeld modular subvariety $\tilde{X}'$ of $S^{r''}_{F'',\mathcal{K}''}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$ and $X' := \iota^{F''}_{F,b'}(\tilde{X}')$ is the desired Drinfeld modular subvariety of S with $X \subsetneq X' \subset Z$. This concludes the proof of (i).

Proof of (ii): We assume that the theorem is true for all s with $1 \leq s < n$ and have to show that it is true for Z Hodge-generic and $\dim Z - \dim X = n$. Since $\mathfrak{p}$ is a good prime for X, we can apply Theorem 6.2.2 and find a $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ with the following properties:

(a) $X \subset T_g X$,

(b) $\deg T_g = [\mathcal{K} : \mathcal{K} \cap g^{-1}\mathcal{K}g] = |k(\mathfrak{p})|^{r-1}$,

(c) For all $k_1, k_2 \in \mathcal{K}_\mathfrak{p}$, the cyclic subgroup of $\mathrm{PGL}_r(F_\mathfrak{p})$ generated by the image of $k_1 \cdot g_\mathfrak{p} \cdot k_2$ is unbounded.

Because of (a) and $X \subset Z$ we have

$$X \subset Z \cap T_g Z.$$

Lemma 3.2.5 together with Proposition 3.2.3 and property (b) of our $g \in \mathrm{GL}_r(\mathbb{A}_F^f)$ give us the upper bound

$$\deg(Z \cap T_g Z) \leq \deg Z \cdot \deg T_g Z \leq (\deg Z)^2 \cdot \deg T_g = (\deg Z)^2 \cdot |k(\mathfrak{p})|^{r-1}.$$

With the assumption on $\deg X$ and $n = \dim Z - \dim X \geq 1$ we conclude

$$\deg X > |k(\mathfrak{p})|^{(r-1)\cdot(2^n-1)} \cdot \deg(Z)^{2^n} \geq \deg(Z \cap T_g Z).$$

Therefore X cannot be a union of irreducible components of $Z \cap T_g Z$. Note that $Z \cap T_g Z$ is defined over F, hence also over the reflex field F' of X. Since X is F'-irreducible, there is an F'-irreducible component Y' of $Z \cap T_g Z$ with $X \subset Y'$. We have $X \subsetneq Y'$ because X is not a union of irreducible components (over $\mathbb{C}_\infty$) of $Z \cap T_g Z$.

Now we set $Y := \mathrm{Gal}(F^{\mathrm{sep}}/F) \cdot Y'$. This is an F-irreducible component of $Z \cap T_g Z$ which contains X with $\dim X < \dim Y$. We distinguish two cases:

Case 1: $Y = Z$
Because of $Y \subset Z \cap T_g Z$ this is only possible if $Z \subset T_g Z$. Since Z is F-irreducible and Hodge-generic, property (c) from above holds and $\mathcal{K}$ is amply small, we can

apply our geometric criterion (Theorem 5.1.3) and conclude that $Z = S$. So $X' := Z = S$ satisfies the conclusion of the theorem.

Case 2: $Y \subsetneq Z$
Set $s' := \dim Y - \dim X$. Since Y and Z are F-irreducible, we have $1 \leq s' < n = \dim Z - \dim X$. Hence, by our assumption, we can apply the theorem to $X \subset Y \subset S$ and the prime $\mathfrak{p}$ provided that the inequality of degrees

$$\deg X > |k(\mathfrak{p})|^{(r-1)\cdot(2^{s'}-1)} \cdot \deg(Y)^{2^{s'}}$$

holds.

To check the latter, note that Y is a union of irreducible components (over $\mathbb{C}_\infty$) of $Z \cap T_g Z$ because it is an F-irreducible component of $Z \cap T_g Z$, whence

$$\deg Y \leq \deg(Z \cap T_g Z) \leq |k(\mathfrak{p})|^{r-1} \cdot (\deg Z)^2.$$

Therefore we indeed have

$$\begin{aligned} |k(\mathfrak{p})|^{(r-1)\cdot(2^{s'}-1)} \cdot \deg(Y)^{2^{s'}} &\leq |k(\mathfrak{p})|^{(r-1)\cdot(2^{n-1}-1)} \cdot \deg(Y)^{2^{n-1}} \\ &\leq |k(\mathfrak{p})|^{(r-1)\cdot(2^{n-1}-1)} \cdot |k(\mathfrak{p})|^{(r-1)\cdot 2^{n-1}} \cdot (\deg Z)^{2^n} \\ &= |k(\mathfrak{p})|^{(r-1)\cdot(2^n-1)} \cdot (\deg Z)^{2^n} < \deg X. \end{aligned}$$

So we find a Drinfeld modular subvariety X' of S with $X \subsetneq X' \subset Y \subset Z$ as desired. □

Proof of Theorem 7.1.4. We first reduce ourselves to the case that $\mathcal{K}$ is amply small. If $\mathcal{K}$ is not amply small, there is a amply small open subgroup $\mathcal{L} \subset \mathcal{K}$ with corresponding canonical projection $\pi_1 : S^r_{F,\mathcal{L}} \to S^r_{F,\mathcal{K}}$. We choose an F-irreducible component $\tilde{Z}$ of $\pi_1^{-1}(Z)$ with $\dim Z = \dim \tilde{Z}$ and set

$$\tilde{\Sigma} := \{\tilde{X} \subset \tilde{Z} \, F'\text{-irreducible component of } \pi_1^{-1}(X) \,|\, X \in \Sigma \text{ with reflex field } F'\}.$$

Since Drinfeld modular subvarieties with reflex field F' are F'-irreducible by Corollary 2.3.6, all $\tilde{X} \in \tilde{\Sigma}$ are Drinfeld modular subvarieties of $S^r_{F,\mathcal{L}}$ by Lemma 2.2.8. They are all contained in $\tilde{Z}$ and their union is Zariski dense in $\tilde{Z}$ by our assumption on Σ. If Theorem 7.1.4 is true for $\mathcal{K}$ amply small, we conclude that for almost all $\tilde{X} \in \tilde{\Sigma}$, there is a Drinfeld modular subvariety $\tilde{X}'$ of $S^r_{F,\mathcal{L}}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$. For such an $\tilde{X}'$, again by Lemma 2.2.8, $X' := \pi_1(\tilde{X}')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$. Hence for almost all $X \in \Sigma$, there is a Drinfeld modular subvariety X' with $X \subsetneq X' \subset Z$.

So we now assume that $\mathcal{K}$ is amply small. By Theorem 6.3.3 with $N = 2(r-1)\cdot(2^s-1)+r^2\cdot 2^{s+1}$ for $s := \dim Z - d$, for almost all $X = \iota^{F'}_{F,b}(S^{r'}_{F',\mathcal{K}'}) \in \Sigma$, there exists a prime $\mathfrak{p}$ of F with the properties

(i) there is a prime $\mathfrak{p}'$ of F' above $\mathfrak{p}$ with local degree $[F'_{\mathfrak{p}'}/F_{\mathfrak{p}}] = 1$,

(ii) $\mathcal{K} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{(\mathfrak{p})}$ with $\mathcal{K}_{\mathfrak{p}} \subset \mathrm{GL}_r(F_{\mathfrak{p}})$ a maximal compact subgroup and $\mathcal{K}^{(\mathfrak{p})} \subset \mathrm{GL}_r(\mathbb{A}_F^{f,\mathfrak{p}})$,

(iii) $\mathcal{K}'_{\mathfrak{p}} := (b_{\mathfrak{p}} \mathcal{K}_{\mathfrak{p}} b_{\mathfrak{p}}^{-1}) \cap \mathrm{GL}_{r'}(F'_{\mathfrak{p}})$ is a maximal compact subgroup of $\mathrm{GL}_{r'}(F'_{\mathfrak{p}})$,

(iv) $|k(\mathfrak{p})|^{2(r-1)\cdot(2^s-1)+r^2\cdot 2^{s+1}} < D(X)$ for $s := \dim Z - d$.

Furthermore, by Theorem 3.3.2 we have

(v) $D(X) > \frac{\deg(Z)^{2^{s+1}}}{C^2}$

for almost all $X \in \Sigma$ with C the constant from Proposition 3.3.1.

By Proposition 6.3.1, for all $X = \iota_{F,b}^{F'}(S^{r'}_{F',\mathcal{K}'})$ and $\mathfrak{p}$ with (i)-(v) there is a subgroup $\tilde{\mathcal{K}} \subset \mathcal{K}$ and a Drinfeld modular subvariety $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$ such that

(vi) $\pi_1(\tilde{X}) = X$ for the canonical projection $\pi_1 : S^r_{F,\tilde{\mathcal{K}}} \to S^r_{F,\mathcal{K}}$,

(vii) $\mathfrak{p}$ is good for $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$,

(viii) $[\mathcal{K} : \tilde{\mathcal{K}}] < |k(\mathfrak{p})|^{r^2}$.

Furthermore, for such an $\tilde{X} \subset S^r_{F,\tilde{\mathcal{K}}}$, we choose an F-irreducible component $\tilde{Z}$ of $\pi_1^{-1}(Z)$ with $\tilde{X} \subset \tilde{Z}$. Since π_1 is finite of degree $[\mathcal{K} : \tilde{\mathcal{K}}]$ by Theorem 2.1.1, we have $\dim \tilde{Z} = \dim Z > \dim X = \dim \tilde{X}$ and

$$\begin{aligned} \deg \tilde{Z} &\leq \deg \pi_1^{-1} Z = [\mathcal{K} : \tilde{\mathcal{K}}] \cdot \deg Z < |k(\mathfrak{p})|^{r^2} \cdot \deg Z, \\ \deg \tilde{X} &\geq \deg \pi_1(\tilde{X}) = \deg X \end{aligned}$$

by Proposition 3.2.3. Therefore, using Proposition 3.3.1, we get the inequality

$$\begin{aligned} \deg \tilde{X} \quad &\geq \quad \deg X \geq C \cdot D(X) = D(X)^{1/2} \cdot (C \cdot D(X)^{1/2}) \\ &\overset{(iv),(v)}{>} \quad |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)+r^2\cdot 2^s} \cdot \deg(Z)^{2^s} \geq |k(\mathfrak{p})|^{(r-1)\cdot(2^s-1)} \cdot \deg(\tilde{Z})^{2^s}. \end{aligned}$$

Therefore $\tilde{X} \subset \tilde{Z} \subset S^r_{F,\tilde{\mathcal{K}}}$ together with $\mathfrak{p}$ satisfy the assumptions of Theorem 7.2.1. So we find a Drinfeld modular subvariety $\tilde{X}'$ of $S^r_{F,\tilde{\mathcal{K}}}$ with $\tilde{X} \subsetneq \tilde{X}' \subset \tilde{Z}$ and $X' := \pi_1(\tilde{X}')$ is a Drinfeld modular subvariety of $S^r_{F,\mathcal{K}}$ with $X \subsetneq X' \subset Z$. □

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