# Euler-Poincaré Formula in equal characteristic under ordinariness assumptions

by

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**Abstract**

Let $X$ be an irreducible smooth projective curve over an algebraically closed field of characteristic $p > 0$. Let $\mathbb{F}$ be either a finite field of characteristic $p$ or a local field of residue characteristic $p$. Let $F$ be a constructible étale sheaf of $\mathbb{F}$-vector spaces on $X$. Suppose that there exists a finite Galois covering $\pi : Y \to X$ such that the generic monodromy of $\pi^*F$ is pro-$p$ and $Y$ is ordinary. Under these assumptions we derive an explicit formula for the Euler-Poincaré characteristic $\chi(X, F)$ in terms of easy local and global numerical invariants, much like the formula of Grothendieck-Ogg-Shafarevich in the case of different characteristic. Although the ordinariness assumption imposes severe restrictions on the local ramification of the covering $\pi$, it is satisfied in interesting cases such as Drinfeld modular curves.[1]

[1]MR 1991 classification: 11G20, 14F20, 14F30

## Contents



# 0 Introduction

Let $X$ be an irreducible smooth projective curve over an algebraically closed field $k$. Let $\mathbb{F}$ be either a finite field of characteristic $\ell$ or a local field of residue characteristic $\ell$. Let $F$ be a constructible étale sheaf of $\mathbb{F}$-vector spaces on $X$. The Euler-Poincaré characteristic of $F$ is defined as

$$\chi(X, F) := \sum_i (-1)^i \dim_{\mathbb{F}} H^i(X, F).$$

The cohomology groups in this formula are known to have finite dimension and to vanish for almost all $i$; hence this invariant is a well-defined integer.

If $\ell$ is different from the characteristic of $k$, the fundamental formula of Grothendieck-Ogg-Shafarevich ([SGA5] exp. X, [11]) expresses this invariant as a sum of an easy global term with local terms at finitely many closed points of $X$. More precisely, let $g_X$ denote the genus of $X$, and let $F_{\bar\eta}$ denote the stalk of $F$ at a geometric point $\bar\eta$ above the generic point of $X$. The formula asserts

$$\chi(X, F) = 2 \cdot (1 - g_X) \cdot \dim_{\mathbb{F}} F_{\bar\eta} + \sum_x \big(\dim_{\mathbb{F}} F_x - \mathrm{LT}_x^{\mathbb{F}} F_{\bar\eta}\big). \tag{0.1}$$

Here the local term $\mathrm{LT}_x^{\mathbb{F}} F_{\bar\eta}$ depends only on the action of the inertia group at $x$ on the generic stalk $F_{\bar\eta}$ and can be written explicitly in the form

$$\mathrm{LT}_x^{\mathbb{F}} F_{\bar\eta} = \dim_{\mathbb{F}} F_{\bar\eta} + \mathrm{Swan}_x^{\mathbb{F}} F_{\bar\eta},$$

where $\mathrm{Swan}_x^{\mathbb{F}} F_{\bar\eta}$ is the Swan conductor of the local Galois representation (see [loc. cit.]) In particular, we have $\mathrm{LT}_x^{\mathbb{F}} F_{\bar\eta} = \dim_{\mathbb{F}} F_x$ whenever $F$ is lisse at $x$, so the sum is really finite. The importance of this formula stems from the fact that it separates clearly the global and local contributions and that both of them have easy expressions in terms of appropriate numerical invariants.

Let us now suppose that $\ell = p$, where $p$ is the characteristic of $k$. Then there cannot exist a universal formula as simple as 0.1, because the $p$-rank of a curve in characteristic $p$ depends on algebraic invariants in addition to the usual numerical invariants of algebraic geometry. To illustrate this phenomenon, consider an elliptic curve $E$ over an algebraically closed field $k$ of characteristic $p \neq 2$. Every such elliptic curve is a double covering $\pi : E \to \mathbb{P}^1_k$ of the projective line which is (tamely) ramified in exactly 4 points. Thus the local numerical invariants of the étale sheaf $\pi_* \mathbb{F}_p$ on $\mathbb{P}^1_k$, as well as its generic rank, are independent of $E$. But the Euler-Poincaré characteristic $\chi(\mathbb{P}^1_k, \pi_* \mathbb{F}_p) = \chi(E, \mathbb{F}_p)$ can be 0 or 1, according to whether $E$ is ordinary or not. Thus there cannot exist a general Euler-Poincaré formula involving the usual kind of numerical invariants alone.

As this failure is related to the fact that the $p$-rank of a curve can decrease under specialization, one may still expect a numerical formula under suitable genericity assumptions. To explain this let us assume for the moment that $\mathbb{F}$ is finite. Then the sheaf $F$ has finite monodromy, and we may fix an irreducible finite Galois covering $\pi : Y \to X$ with Galois group $G$ such that $\pi^* F$ is constant over some open dense subscheme of $Y$. Apart from some easy local terms the Euler-Poincaré characteristic $\chi(X, F)$ will depend on this covering together with the $\mathbb{F}[G]$-module $F_{\bar\eta}$. A long exact cohomology sequence shows that this mysterious contribution of $F_{\bar\eta}$ is additive in short exact sequences.

If $G$ is a $p$-group, every $\mathbb{F}[G]$-module is an extension of copies of the trivial representation $\mathbb{F}$. Thus in this case $\chi(X, F)$ is equal to $\chi(X, \mathbb{F}_p) \cdot \dim_{\mathbb{F}} F_{\bar\eta}$ plus a finite number of explicit local terms. This observation goes back essentially to Deuring [4] and Shafarevich [13] and has been used by other authors (e.g., Madan [8], Crew [3], Hawkins [6]).

In this article we pursue a different direction and assume that *$Y$ (sic!) is ordinary.* Then the $p$-rank of $Y$ is $g_Y$; hence $\chi(X, \pi_* \mathbb{F}) = \chi(Y, \mathbb{F}) = 1 - g_Y$. By the Hurwitz genus formula, this is equal to $|G| \cdot (1 - g_X)$ minus a sum of local terms depending only on the ramification of $\pi$. This expression is certainly in the spirit of 0.1. Since the generic stalk of $\pi_* \mathbb{F}$ is free of rank 1 over $\mathbb{F}[G]$, one can view this as an explicit formula for some positive linear combination of the mysterious contributions from all irreducible $\mathbb{F}[G]$-modules. While this argument falls, of course, much short of determining the contributions for all $\mathbb{F}[G]$-modules, it nevertheless suggests that their behavior might be predictable.

The main result of this paper gives an explicit formula for them. It involves a certain local term $\mathrm{LT}^{\mathbb{F}}_x F_{\bar\eta}$, which is a rational number depending only on the action of the inertia group at $x$ on the generic stalk $F_{\bar\eta}$, and which is equal to $\dim_{\mathbb{F}} F_x$ whenever $F$ is lisse at $x$. Curiously, in contrast to the case $\ell \neq p$, this local term depends only on the action of the tame part of inertia, disregarding wild inertia entirely. For its explicit definition see Setion 5. By combining methods relying on ordinariness assumptions with the $p$-group argument sketched above we arrive at the following main result:

**Theorem 0.2** *Let $X$ be an irreducible smooth projective curve over an algebraically closed field of characteristic $p > 0$. Let $\mathbb{F}$ be either a finite field* *of characteristic $p$ or a local field of residue characteristic $p$. Let $F$ be a constructible étale sheaf of $\mathbb{F}$-vector spaces on $X$. Assume that there exists an irreducible finite Galois covering $\pi : Y \to X$ such that*

*(a) the generic monodromy of $\pi^* F$ is a pro-$p$-group, and*

*(b) $Y$ is ordinary.*

*Then*

$$\chi(X, F) = (1 - g_X) \cdot \dim_{\mathbb{F}} F_{\bar{\eta}} + \sum_x \big(\dim_{\mathbb{F}} F_x - \mathrm{LT}_x^{\mathbb{F}} F_{\bar{\eta}}\big).$$

It is natural to wonder whether the ordinariness assumption should be viewed as very restrictive. The existence of at least one interesting application (see below) suggests that this is not necessarily so. On the other hand, it was observed already by Nakajima that this assumption imposes severe restrictions on the local ramification of the covering $\pi$. Namely, any automorphism of $Y$ which fixes a point $y$ and acts trivially on $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^3$ must be the identity: see Section 1 or [10] Th. 2 (i). In other words, all the wild inertia must be concentrated in the first possible step for the lower numbering filtration. (Note that this forces wild inertia to be an elementary abelian $p$-group, which is already a purely group theoretical restriction.) In a different form (see Section 3) such local properties actually play an essential role in our proof of Theorem 0.2. If the global ordinariness condition is replaced by this weaker ramification condition, our methods yield an inequality: see Proposition 7.1.

The assumptions of Theorem 0.2 hold in the following interesting case. Suppose that $X$ is the smooth compactification of a Drinfeld modular curve in characteristic $p$. (For the general background in this area see [1] and the references therein.) There is a global function field underlying the definition of $X$; let $\mathbb{F}$ denote its completion at some finite place $\wp$. The $\wp$-adic Tate modules for the universal family of Drinfeld modules combine to an étale sheaf $F$ of $\mathbb{F}$-vector spaces on $X$. This sheaf is, say, zero at a finite number of so-called cusps of $X$ and lisse of rank 2 elsewhere, provided that the "level" of $X$ is sufficiently high. Of special interest are sheaves deduced from $F$ by linear algebra, say be taking symmetric powers. To satisfy the above assumptions let $Y \to X$ be the finite Galois covering obtained by imposing an additional principal level structure at $\wp$. Then condition 0.2 (a) holds for $F$ and any sheaf deduced from it. Moreover, analytic uniformization shows that $Y$ specializes to a semistable curve composed of rational curves only; this implies condition 0.2 (b).

Finally, the author would like to thank the referee for some useful suggestions.

# 1 Ordinariness and ramification

Throughout this paper we consider the following situation. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\pi : Y \to X$ be a finite Galois covering of irreducible smooth projective curves over $k$, with Galois group $G$. In this section we review a theorem of Nakajima on local obstructions to $Y$ being ordinary. This result will not be used in the proof of Theorem 0.2.

For any closed point $y \in Y$ we let $G_y$ denote its stabilizer in $G$ and $\mathfrak{m}_{Y,y} \subset \mathcal{O}_{Y,y}$ the maximal ideal of the associated local ring. The kernel of the action of $G_y$ on $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^{i+1}$ is denoted $G_{y,i}$ and called the $i^{\text{th}}$ *ramification subgroup* (with respect to the lower numbering). As the field of constants is algebraically closed, the residue field extension is trivial and $G_y = G_{y,0}$ is the inertia group at $y$. The wild inertia subgroup is $G_{y,1}$; the factor group $G_{y,0}/G_{y,1}$ consists of tame inertia and is cyclic of order prime to $p$.

**Definition 1.1** *We call the ramification at $y$ "*of type $i$*" if $G_y$ acts faithfully on $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^{i+1}$, that is, if $G_{y,i} = 1$.*

Thus, type 0 means unramified, type 1 means at most tamely ramified, and type 2 allows both tame and the simplest kind of wild ramification. The following theorem was proved essentially in [10] Th. 2 (i).

**Theorem 1.2** *If $G$ is a $p$-group, the following assertions are equivalent:*

*(a) $Y$ is ordinary.*

*(b) $X$ is ordinary and the ramification of $\pi$ at every closed point $y \in Y$ is of type 2.*

*Without assumption on $G$ one still has the implication (a)⇒(b).*

*Proof.* We first consider the case that $G$ is a $p$-group. Under this assumption the $p$-rank of $Y$ can be calculated explicitly via the Deuring-Shafarevich formula ([4], [13], [8]). We review the necessary arguments. Note that, since $Y$ is a curve, the étale cohomology group $H^i(Y, \mathbb{F}_p)$ vanishes for $i \neq 0, 1$ (cf. Section 2). Moreover, we have $H^0(Y, \mathbb{F}_p) = \mathbb{F}_p$, because $Y$ is connected. Thus the Euler characteristic is $\chi(Y, \mathbb{F}_p) = 1 - h^1(Y, \mathbb{F}_p)$. To determine it, choose a $G$-invariant effective divisor $D \subset Y$ containing all ramification points of $\pi$. The open and closed embeddings

$$Y \smallsetminus D \overset{\tilde{j}}{\hookrightarrow} Y \overset{\tilde{i}}{\hookleftarrow} D$$

induce a short exact sequence of étale sheaves on $Y$:

$$0 \longrightarrow \tilde{j}_! \mathbb{F}_p \longrightarrow \mathbb{F}_p \longrightarrow \tilde{i}_* \mathbb{F}_p \longrightarrow 0.$$

Letting $|D|$ denote the cardinality of $D$, the associated long exact cohomology sequence implies

$$\begin{aligned} \chi_c(Y \smallsetminus D, \mathbb{F}_p) &= \chi(Y, \tilde{j}_! \mathbb{F}_p) \\ &= \chi(Y, \mathbb{F}_p) - \chi(Y, \tilde{i}_* \mathbb{F}_p) \\ &= 1 - h^1(Y, \mathbb{F}_p) - |D|. \end{aligned} \tag{1.3}$$

Similarly, we obtain

$$\chi_c(X \smallsetminus \pi(D), \mathbb{F}_p) \;=\; 1 - h^1(X, \mathbb{F}_p) - |\pi(D)|. \tag{1.4}$$

On the open subscheme $X \smallsetminus \pi(D)$, the étale sheaf $\pi_*\mathbb{F}_p$ is the locally constant sheaf associated to the natural representation of $G$ on its group ring $\mathbb{F}_p[G]$. As $G$ is a $p$-group, this representation is a successive extension of $|G|$ copies of the identity representation $\mathbb{F}_p$. Therefore $\pi_*\mathbb{F}_p$ on $X \smallsetminus \pi(D)$ is a successive extension of $|G|$ copies of the constant sheaf $\mathbb{F}_p$. The additivity of the Euler characteristic in short exact sequences implies

$$\begin{aligned} \chi_c(Y \smallsetminus D, \mathbb{F}_p) &= \chi_c(X \smallsetminus \pi(D), \pi_*\mathbb{F}_p) \\ &= \chi_c(X \smallsetminus \pi(D), \mathbb{F}_p) \cdot |G|. \end{aligned} \tag{1.5}$$

Combining formulas 1.3, 1.4, and 1.5 yields:

$$1 - h^1(Y, \mathbb{F}_p) - |D| = \big(1 - h^1(X, \mathbb{F}_p) - |\pi(D)|\big) \cdot |G|. \tag{1.6}$$

Since

$$|\pi(D)| \cdot |G| \;=\; \sum_{y \in D \bmod G} |G| \;=\; \sum_{y \in D \bmod G} [G : G_y] \cdot |G_y| \;=\; \sum_{y \in D} |G_y|,$$

we may rewrite 1.6 in the form (cf. [3] Cor. 1.8)

$$1 - h^1(Y, \mathbb{F}_p) = \big(1 - h^1(X, \mathbb{F}_p)\big) \cdot |G| - \sum_{y \in D} \big(|G_y| - 1\big). \tag{1.7}$$

We will compare this with the Hurwitz genus formula ([5] Ch. IV Cor. 2.4)

$$2 \cdot \big(1 - g_Y\big) = 2 \cdot \big(1 - g_X\big) \cdot |G| - \sum_{y \in D} \delta_y, \tag{1.8}$$

where the ramification term is ([12] ch. III §7 Prop. 14, ch. IV §2 Prop. 4)

$$\delta_y = \sum_{i \geq 0} \big(|G_{y,i}| - 1\big).$$

As $G$ is a $p$-group, all inertia is wild; hence $G_y = G_{y,0} = G_{y,1}$. Therefore

$$\delta_y = 2 \cdot \big(|G_y| - 1\big) + \sum_{i \geq 2} \big(|G_{y,i}| - 1\big).$$

Substituting this expression into 1.8 and dividing by 2 we obtain

$$1 - g_Y = \big(1 - g_X\big) \cdot |G| - \sum_{y \in D} \left( |G_y| - 1 + \sum_{i \geq 2} \frac{|G_{y,i}| - 1}{2} \right). \tag{1.9}$$

Subtracting 1.9 from 1.7 yields

$$g_Y - h^1(Y, \mathbb{F}_p) = \big(g_X - h^1(X, \mathbb{F}_p)\big) \cdot |G| + \sum_{y \in D} \sum_{i \geq 2} \frac{|G_{y,i}| - 1}{2}. \tag{1.10}$$

Here the left hand side is always $\geq 0$, and it is $= 0$ if and only if $Y$ is ordinary. Likewise, the first summand on the right hand side is always $\geq 0$, and it is $= 0$ if and only if $X$ is ordinary. The remaining summands on the right hand side are also $\geq 0$, and they vanish for all $i \geq 2$ if and only if the ramification is of type 2. Thus the equality 1.10 implies the desired equivalence (a)$\Leftrightarrow$(b).

It remains to prove the implication (a)$\Rightarrow$(b) for arbitrary $G$. Assume (a) that $Y$ is ordinary. Then so is its Jacobian $\mathrm{Jac}_Y$. It is known that ordinariness for abelian varieties is an isogeny invariant (see [9] §15). Since, moreover, a direct product of abelian varieties is ordinary if and only if both factors are ordinary, any subquotient of an ordinary abelian variety is ordinary. In our case the pullback and pushforward maps induce an isogeny between $\mathrm{Jac}_X$ and a subquotient of $\mathrm{Jac}_Y$; hence $\mathrm{Jac}_X$ is ordinary. Thus $X$ is ordinary, proving the first part of (b). To prove the second part fix any closed point $y \in Y$. We must show that $G_{y,2}$ vanishes. This assertion does not change if we replace $X = Y/G$ by $Y/G_{y,2}$ and $G$ by $G_{y,2}$. After this replacement $G$ is a $p$-group, so the equivalence already proved implies $G_{y,2} = 1$, as desired. This finishes the proof of Theorem 1.2. **q.e.d.**

# 2 Artin-Schreier theory

This section contains a brief review of Artin-Schreier theory in étale cohomology and some preparatory remarks for use in the following sections.

Quasi-coherent sheaves on a scheme $X$ are usually viewed as sheaves with respect to the Zariski topology. But they also induce sheaves in the étale topology by pullback (see [SGA4] exp. VII §4). It is a fundamental fact that the quasi-coherent cohomology of the former is canonically isomorphic to the étale cohomology of the latter ([loc. cit.] Prop. 4.3). Therefore we can use the two standpoints interchangeably and will not distinguish them in our notation.

Suppose now that $X$ is a scheme over $\mathbb{F}_p$. Then the structure sheaf carries the natural Frobenius endomorphism $\sigma: \mathcal{O}_X \to \mathcal{O}_X$, $s \mapsto s^p$. Consider a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $X$. By a *$\sigma$-linear endomorphism* $\tau: \mathcal{F} \to \mathcal{F}$ we mean a homomorphism of sheaves of abelian groups satisfying $\tau(s \cdot f) = s^p \cdot \tau(f)$ for all local sections $s$ of $\mathcal{O}_X$ and $f$ of $\mathcal{F}$. For example, every ideal sheaf in $\mathcal{O}_X$ carries a canonical such endomorphism, namely the restriction of $\sigma$. Whenever $\tau$ is clear from the context, we call it the associated Frobenius endomorphism.

One natural way that such pairs $(\mathcal{F}, \tau)$ arise is as follows (cf. [SGA4$\frac{1}{2}$], Fonctions $L \ldots$, §3). Let $F$ be a locally constant étale sheaf of $\mathbb{F}_p$-modules of finite type on $X$. Set $\mathcal{F} := F \otimes \mathcal{O}_X$ and $\tau := \mathrm{id} \otimes \sigma$. (All tensor products will be taken over $\mathbb{F}_p$ unless otherwise specified.) Then $\mathcal{F}$ is, actually, a locally free coherent sheaf, and the sequence

$$0 \longrightarrow F \xrightarrow{\mathrm{id}\otimes 1} \mathcal{F} \xrightarrow{1-\tau} \mathcal{F} \longrightarrow 0$$

is exact. Consider an open embedding $j: X \hookrightarrow \bar{X}$ where $\bar{X}$ is a noetherian scheme over $\mathbb{F}_p$. Let $\mathcal{I} \subset \mathcal{O}_{\bar{X}}$ be an ideal sheaf whose zero locus is the complement $\bar{X} \setminus X$. The extension by zero on the étale side corresponds to the following construction on the $\sigma$-linear side. We give a formulation that is slightly stronger than in [SGA4$\frac{1}{2}$], Fonctions $L \ldots$, Lemme 3.3, but the proof remains almost literally the same:

**Proposition 2.1** *(a) There exists a coherent sheaf $\bar{\mathcal{F}}$ on $\bar{X}$ extending $\mathcal{F}$ and a $\sigma$-linear endomorphism $\bar{\tau}: \bar{\mathcal{F}} \to \bar{\mathcal{F}}$ extending $\tau$ such that the induced endomorphism on $\bar{\mathcal{F}}/\mathcal{I}\bar{\mathcal{F}}$ is nilpotent.*

*(b) For any $\bar{\mathcal{F}}$ and $\bar{\tau}$ as in (a) the following sequence is exact:*

$$0 \longrightarrow j_! F \longrightarrow \bar{\mathcal{F}} \xrightarrow{1-\bar{\tau}} \bar{\mathcal{F}} \longrightarrow 0.$$

For example, if $F$ is the constant sheaf $\mathbb{F}_p$, we can take $\bar{\mathcal{F}} := \mathcal{I}$ and obtain the exact sequence

$$0 \longrightarrow j_! \mathbb{F}_p \longrightarrow \mathcal{I} \xrightarrow{1-\sigma} \mathcal{I} \longrightarrow 0. \tag{2.2}$$

Suppose now that $\bar{X}$ is proper over an algebraically closed field $k$. Then the cohomology groups of the above $\bar{\mathcal{F}}$ are finite dimensional over $k$. This fact, together with the long exact cohomology sequence associated to 2.1 (b), are the basic tools in studying étale cohomology with torsion coefficients in equal characteristic. The following elementary statements summarize what we need from [SGA7] exp. XXII §1.

**Proposition 2.3** *Let $H$ be a finite dimensional vector space over an algebraically closed field $k$ of characteristic $p > 0$, and $\tau : H \to H$ a $\sigma$-linear endomorphism. Put*

$$H^\tau := \{\, h \in H \mid \tau h = h \,\}.$$

*(a) The following sequence is exact:*

$$0 \longrightarrow H^\tau \longrightarrow H \xrightarrow{1-\tau} H \longrightarrow 0.$$

*(b) $\dim_{\mathbb{F}_p} H^\tau \le \dim_k H$, with equality if and only if $\tau$ is an isomorphism.*

Applying this to the Frobenius endomorphism of $H^i(\bar{X}, \bar{\mathcal{F}})$ induced by $\bar{\tau}$ one deduces an isomorphism (cf. [SGA4$\frac{1}{2}$], Fonctions $L \ldots$, 3.7)

$$H^i_c(X, F) \xrightarrow{\sim} H^i(\bar{X}, \bar{\mathcal{F}})^\tau. \tag{2.4}$$

It follows that $h^i_c(X, F) \le h^i(\bar{X}, \bar{\mathcal{F}})$, with equality if and only if the Frobenius map on $H^i(\bar{X}, \bar{\mathcal{F}})$ is an isomorphism. In any case $h^i_c(X, F)$ vanishes whenever $h^i(\bar{X}, \bar{\mathcal{F}})$ vanishes; this happens in particular for all $i$ outside the interval $[0, \dim \bar{X}]$.

We finish this section with a technical result in the curve case.

**Proposition 2.5** *Let $Z$ be an irreducible smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the ideal sheaf of an effective divisor $E \subset Z$. Then the following assertions are equivalent:*

(a) *The Frobenius map on* $H^1(Z, \mathcal{I})$ *is an isomorphism.*

(b) $Z$ *is ordinary and* $E$ *is reduced.*

*Proof.* The short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_Z \to \mathcal{O}_E \to 0$ induces a long exact cohomology sequence

$$H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Z, \mathcal{O}_E) \longrightarrow H^1(Z, \mathcal{I}) \longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow 0.$$

All these maps commute with the respective Frobenius endomorphisms. On the $H^0$-terms Frobenius is given by $s \mapsto s^p$. Since the leftmost term is $k$, Frobenius is an isomorphism there. It follows that Frobenius on $H^1(Z, \mathcal{I})$ is an isomorphism if and only if it is an isomorphism on both $H^0(Z, \mathcal{O}_E)$ and $H^1(Z, \mathcal{O}_Z)$. As $H^0(Z, \mathcal{O}_E)$ is just the affine ring of $E$, Frobenius is an isomorphism there if and only if $E$ is reduced. On the other hand Frobenius on $H^1(Z, \mathcal{O}_Z)$ is an isomorphism if and only if $h^1(Z, \mathbb{F}_p) = h^1(Z, \mathcal{O}_Z) = g_Z$, that is, if $Z$ is ordinary. This proves the desired equivalence. **q.e.d.**

# 3 Local Freeness

We return to the setup and the notations of Section 1. The aim of this section is to relate ordinariness of $Y$ to a somewhat different local condition, by taking a closer look at the relevant coherent sheaves.

Let $D \subset Y$ be a $G$-invariant effective divisor containing all ramification points of $\pi$. Let $\mathcal{J} \subset \mathcal{O}_Y$ denote the associated ideal sheaf. We assume that $D$ is *reduced.* Then the stalk $\mathcal{J}_y$ of $\mathcal{J}$ at any closed point $y \in Y$ is either $\mathcal{O}_{Y,y}$ or $\mathfrak{m}_{Y,y}$. The second case occurs whenever $G_y \neq 1$, but possibly also at some other points. Since $D$ is $G$-invariant, the coherent sheaf $\pi_*\mathcal{J}$ carries a natural (left) action of $G$. Thus we can view $\pi_*\mathcal{J}$ as a sheaf of (left) modules over the "sheaf of group rings" $\mathcal{O}_X[G] := \mathcal{O}_X \otimes \mathbb{F}_p[G]$. The main result of this section is the following. (But see also the corresponding local result at the end of this section.)

**Theorem 3.1** *If $G$ is a $p$-group, the following assertions are equivalent:*

(a) $Y$ *is ordinary.*

(b) $X$ *is ordinary and the sheaf* $\pi_*\mathcal{J}$ *is locally free of rank* 1 *over* $\mathcal{O}_X[G]$.

*Without assumption on $G$ one still has the implication (a)⇒(b).*

The proof will extend over most of this section. First we note a technical lemma concerning passage to completion.

**Lemma 3.2** *Consider a closed point $x \in X$ and let* $\hat{}$ *denote completion with respect to* $\mathfrak{m}_{X,x}$. *The following assertions are equivalent:*

(a) $\pi_*\mathcal{J}$ *is locally free of rank* 1 *over* $\mathcal{O}_X[G]$ *in a neighborhood of* $x$.

*(b)* $(\pi_*\mathcal{J})^{\widehat{}}_x$ *is free of rank* 1 *over* $\widehat{\mathcal{O}}_{X,x}[G]$.

Its proof, by standard arguments of commutative algebra, is left to the reader. For any closed point $x \in X$ which is unramified in $Y$ we have

$$(\pi_*\mathcal{J})^{\widehat{}}_x = \bigoplus_{y\in\pi^{-1}(x)} \mathcal{J}^{\widehat{}}_y\,.$$

This is clearly free of rank 1 over $\widehat{\mathcal{O}}_{X,x}[G]$. Thus Lemma 3.2 shows that $\pi_*\mathcal{J}$ is locally free of rank 1 over $\mathcal{O}_X[G]$ outside the ramification locus. In particular, this holds outside $\pi(D)$.

We now assume that $G$ is a $p$-group and begin with some technical preparations. Consider the group ring $A := \mathbb{F}_p[G]$ as a right module over itself. The assumption on $G$ implies that $A$ is a successive extension of $n := |G|$ copies of the trivial 1-dimensional representation. In other words, there exists a flag

$$0 \;=\; A_0 \;\subset\; A_1 \;\subset\; \ldots \;\subset\; A_{n-1} \;\subset\; A_n \;=\; A$$

of right $A$-submodules with $\dim_{\mathbb{F}_p} A_i = i$ for all $i$. We fix such a flag.

Let $I_G$ denote the augmentation ideal of $A$. Since $G$ acts trivially on the 1-dimensional subquotient $\mathrm{gr}_i\, A := A_i/A_{i-1}$, we have $A_i\cdot I_G \subset A_{i-1}$, for every $1 \le i \le n$. In particular, we have $I_G = A_n\cdot I_G \subset A_{n-1}$, which for dimension reasons implies $I_G = A_{n-1}$. Similarly, we have $A_1\cdot I_G \subset A_0 = 0$, so $A_1$ is contained in the subspace of right $G$-invariants in $A$. This subspace is generated by $N_G := \sum_{g\in G} g$, so again for dimension reasons we must have $A_1 = \mathbb{F}_p\cdot N_G$.

**Lemma 3.3** *There exist* $\mathbb{F}_p$*-bases* $\{a_1,\ldots,a_n\}$ *and* $\{a'_1,\ldots,a'_n\}$ *of* $A$ *such that, for all* $1\le i\le n$*, left multiplication by* $a_i$*, resp.* $a'_i$*, induce isomorphisms*

$$\mathrm{gr}_n\, A \xrightarrow[\sim]{[a_i\cdot\ ]} \mathrm{gr}_i\, A \xrightarrow[\sim]{[a'_i\cdot\ ]} \mathrm{gr}_1\, A = A_1.$$

*Proof.* Choosing any $a_i \in A_i \smallsetminus A_{i-1}$, we clearly obtain an $\mathbb{F}_p$-basis of $A$. Moreover, we have $a_i\cdot A_n \subset A_i$ since the latter is a right $A$-submodule, and $a_i\cdot A_{n-1} = a_i\cdot I_G \subset A_i\cdot I_G \subset A_{i-1}$, as seen above. Thus the map on the left hand side is well-defined. As $a_i \notin A_{i-1}$, this map is non-zero. It is therefore an isomorphism, as desired.

For the right hand side we will dualize this argument. Consider the anti-automorphism

$$(\ )^*: A \longrightarrow A,\ \textstyle\sum \alpha_g g \mapsto \sum \alpha_g g^{-1}$$

and the linear form

$$\ell\colon A \longrightarrow \mathbb{F}_p,\ \textstyle\sum \alpha_g g \mapsto \alpha_1.$$

The pairing

$$A\times A \longrightarrow A,\ (a,a') \mapsto \ell(a^*\cdot a')$$

comes out explicitly as

$$\big(\sum \alpha_g g, \sum \alpha'_g g\big) \mapsto \sum \alpha_g \alpha'_g,$$

so it is the standard $G$-invariant non-degenerate symmetric $\mathbb{F}_p$-bilinear pairing on $A$. The orthogonal complements $A_i^\perp$ thus form another flag of right $A$-submodules, with $\dim_{\mathbb{F}_p} A_i^\perp = n - i$. Choose any $\tilde{a}_i \in A_{i-1}^\perp \setminus A_i^\perp$. Then we have $\ell(\tilde{a}_i^* \cdot A_{i-1}) = 0$. As $A_{i-1}$ and hence $\tilde{a}_i^* \cdot A_{i-1}$ is right invariant under $G$, from the definition of $\ell$ we see at once that $\tilde{a}_i^* \cdot A_{i-1} = 0$. On the other hand $\tilde{a}_i \notin A_i^\perp$ implies $\tilde{a}_i^* \cdot A_i \neq 0$. But $\big(\tilde{a}_i^* \cdot A_i\big) \cdot I_G = \tilde{a}_i^* \cdot \big(A_i \cdot I_G\big) \subset \tilde{a}_i^* \cdot A_{i-1} = 0$, so $\tilde{a}_i^* \cdot A_i$ is contained in the subspace of right $G$-invariants $A_1$. Setting $a'_i := \tilde{a}_i^*$, it follows that the map on the right hand side is well-defined and non-zero. It is therefore an isomorphism, as desired. **q.e.d.**

We now use the above setup to decompose $\mathcal{F} := \pi_* \mathcal{J}$ as a sheaf of $\mathcal{O}_X$-modules. For every $0 \leq i \leq n$ let $\mathcal{F}_i \subset \mathcal{F}$ denote the saturation of the subsheaf $A_i \cdot \mathcal{F}$, that is, the unique coherent subsheaf of $\mathcal{O}_X$-modules such that $\mathcal{F}_i = A_i \cdot \mathcal{F}$ generically and $\mathcal{F}/\mathcal{F}_i$ is torsion free. Wherever $\mathcal{F}$ is locally isomorphic to $\mathcal{O}_X \otimes A$, we have $A_i \cdot \mathcal{F} \cong \mathcal{O}_X \otimes A_i$ and hence $A_i \cdot \mathcal{F} = \mathcal{F}_i$. In particular, we have $\operatorname{rank}_{\mathcal{O}_X} \mathcal{F}_i = i$, and $\operatorname{gr}_i \mathcal{F} := \mathcal{F}_i/\mathcal{F}_{i-1}$ is an invertible sheaf for every $1 \leq i \leq n$. From Lemma 3.3 we deduce homomorphisms

$$\text{(3.4)} \qquad \operatorname{gr}_n \mathcal{F} \xrightarrow{[a_i \cdot\ ]} \operatorname{gr}_i \mathcal{F} \xrightarrow{[a'_i \cdot\ ]} \operatorname{gr}_1 \mathcal{F} = \mathcal{F}_1.$$

Wherever $\mathcal{F}$ is locally free with respect to $\mathcal{O}_X \otimes A$, these maps are isomorphisms. In particular, they are isomorphisms outside $\pi(D)$, and they are monomorphisms everywhere.

Next observe that $A_1 = \mathbb{F}_p \cdot N_G$ coincides with the space of left $G$-invariants in $A$. In particular $I_G \cdot A_1 = 0$. This implies $I_G \cdot \mathcal{F}_1 = 0$; hence $\mathcal{F}_1$ is contained in the subsheaf of $G$-invariants $\mathcal{F}^G$. Now $\mathcal{F}^G = \mathcal{O}_X \cap \pi_* \mathcal{J}$ is the ideal sheaf of the reduced divisor $\pi(D)^{\mathrm{red}}$. Let us denote it by $\mathcal{I}$. Since both of $\mathcal{F}_1 \subset \mathcal{I}$ have rank 1 over $\mathcal{O}_X$, and $\mathcal{F}_1$ is saturated in $\mathcal{F}$, we deduce $\mathcal{F}_1 = \mathcal{I}$. Using the right hand side of 3.4 we may now identify each $\operatorname{gr}_i \mathcal{F}$ with some non-zero ideal sheaf contained in $\mathcal{I}$. Let $E_i \subset X$ denote the associated effective divisor. As the homomorphisms 3.4 are isomorphisms outside $\pi(D)$, and $E_i$ contains $\pi(D)^{\mathrm{red}}$, we have $E_i^{\mathrm{red}} = \pi(D)^{\mathrm{red}}$. We will be concerned with the respective multiplicities.

**Lemma 3.5** *The following assertions are equivalent:*

*(a) For every $1 \leq i \leq n$ the divisor $E_i$ is reduced.*

*(b) For every $1 \leq i \leq n$ the sheaf $\operatorname{gr}_i \mathcal{F}$ is isomorphic to $\mathcal{I}$.*

*(c) For every $1 \leq i \leq n$ the homomorphism on the left hand side of 3.4 is an isomorphism.*

*(d) $\mathcal{F}$ is locally free of rank 1 over $\mathcal{O}_X[G]$.*

*Proof.* The equivalence (a)⇔(b) follows from the equality $E_i^{\rm red} = \pi(D)^{\rm red}$. The implication (b)⇒(c) results from the fact that any non-zero endomorphism of an invertible sheaf is an isomorphism. For (c)⇒(d) let $s$ be any local section of $\mathcal{F}$ whose image generates $\operatorname{gr}_n \mathcal{F}$. If (c) holds, the image of $a_i \cdot s$ generates $\operatorname{gr}_i \mathcal{F}$; hence all sections $a_i \cdot s$ together form a local basis of $\mathcal{F}$ over $\mathcal{O}_X$. As the $a_i$ form a basis of $A$ over $\mathbb{F}_p$, the map $\mathcal{O}_X \otimes A \longrightarrow \mathcal{F}$, $f \otimes a \mapsto f \cdot a \cdot s$ is then a local isomorphism, proving (d). The remaining implication (d)⇒(b) follows from the fact that the homomorphisms in 3.4 are isomorphisms wherever $\mathcal{F}$ is locally free over $\mathcal{O}_X[G]$. **q.e.d.**

Since $\mathcal{J}$ is an ideal sheaf in $\mathcal{O}_Y$, it carries a natural Frobenius endomorphism. This induces a Frobenius endomorphism on $\pi_*\mathcal{J}$. This, in turn, obviously commutes with the $G$-action and is therefore compatible with all the constructions above. In particular, it induces endomorphisms of all $\mathcal{F}_i$ and $\operatorname{gr}_i \mathcal{F}$, which commute with the homomorphisms in 3.4. Moreover, on $\mathcal{F}_1 = \mathcal{I}$ it coincides with the natural Frobenius endomorphism from being an ideal sheaf in $\mathcal{O}_X$. The same follows for every $\operatorname{gr}_i \mathcal{F}$ when viewed as an ideal sheaf of $\mathcal{O}_X$.

After all these preparations, we can now prove the desired equivalence in 3.1. We have already seen that $\pi_*\mathcal{J}$ is locally free of rank 1 over $\mathcal{O}_X[G]$ outside the ramification locus of $\pi$. Thus the condition 3.1 (b) does not change when $D$ is enlarged by throwing in some unramified points. Therefore we may, and do, assume without loss of generality that $D$ is non-empty.

We first apply Proposition 2.5 to $(Z, E) = (Y, D)$. By assumption $D$ is reduced, so it follows that $Y$ is ordinary if and only if the Frobenius map on $H^1(Y, \mathcal{J})$ is an isomorphism. Since $H^1(Y, \mathcal{J}) = H^1(X, \pi_*\mathcal{J})$, we can study this condition over $X$. As $D$ is non-empty, so is each $E_i$, and this implies $H^0(X, \operatorname{gr}_i \mathcal{F}) = 0$. Thus $\operatorname{gr}_i \mathcal{F}$ can have non-zero cohomology only in degree 1. The usual long exact cohomology sequences show that $H^1(X, \pi_*\mathcal{J})$ is a successive extension of $H^1(X, \operatorname{gr}_i \mathcal{F})$ for all $1 \leq i \leq n$. All this is compatible with Frobenius; hence Frobenius is an isomorphism on $H^1(X, \pi_*\mathcal{J})$ if and only if it is an isomorphism on every $H^1(X, \operatorname{gr}_i \mathcal{F})$. Applying Proposition 2.5 to $(Z, E) = (X, E_i)$ we find that this is equivalent to saying that $X$ is ordinary and every $E_i$ reduced. By Lemma 3.5 this, in turn, is equivalent to condition 3.1 (b). This shows the desired equivalence (a)⇔(b).

It remains to prove the implication 3.1 (a)⇒(b) for arbitrary $G$. Assume (a) that $Y$ is ordinary. At the end of Section 1 we already proved that $X$ is ordinary. To show the local freeness we consider stalks at any fixed closed point $x \in X$.

We first suppose that $x$ is totally ramified in $Y$. Then $G$ coincides with the inertia group above $x$, and the wild inertia group $G_1$ is the unique $p$-Sylow subgroup of $G$. Thus $\pi$ factors through the two Galois coverings $Y \longrightarrow Z := Y/G_1 \longrightarrow X$. Let $y \in Y$ and $z \in Z$ denote the unique points above $x$. The upper covering has $p$-power order, so by the earlier case of the theorem $\mathcal{J}_y$ is free of rank 1 over $\mathcal{O}_{Z,z}[G_1]$. Clearly this algebra is local, and $\mathfrak{m}_{Y,y} \cdot \mathcal{J}_y$ is a maximal submodule of $\mathcal{J}_y$. It is therefore the unique maximal submodule and any element $v \in \mathcal{J}_y \smallsetminus \mathfrak{m}_{Y,y} \cdot \mathcal{J}_y$ is a generator. The lower covering is totally and tamely ramified at $x$, so its Galois group is cyclic of order prime to $p$. Choose a complement $H$ of $G_1$, so that $G = G_1 \rtimes H$ and $H \xrightarrow{\sim} G/G_1$. Since $H$ acts semisimply on $\mathcal{J}_y$, we can choose the above generator $v$ in an eigenspace for $H$. Then for some abelian character $\chi$ we have ${}^h v = \chi(h) \cdot v$ for all $h \in H$. Similarly, the stalk $\mathcal{O}_{Z,z}$ decomposes under $H$ into torsion free $\mathcal{O}_{X,x}$-modules of rank 1, indexed by the abelian characters of $H$. Therefore $\mathcal{O}_{Z,z}$ is free of rank 1 over $\mathcal{O}_{X,x}[H]$. Choose any generator $w$. We claim that $w \cdot v$ is a basis of $\mathcal{J}_y$ over $\mathcal{O}_{X,x}[G]$. In fact, for all $g_1 \in G_1$ and $h \in H$ we have

$$ {}^{g_1 h}(w \cdot v) = {}^{g_1}({}^h w \cdot \chi(h) \cdot v) = {}^h w \cdot \chi(h) \cdot {}^{g_1} v, $$

where we have used the fact that $G_1$ acts trivially on $\mathcal{O}_{Z,z}$. Now, as $g_1$ varies, the elements ${}^{g_1} v$ form a basis of $\mathcal{J}_y$ over $\mathcal{O}_{Z,z}$. As $h$ varies, the elements ${}^h w$ form a basis of $\mathcal{O}_{Z,z}$ over $\mathcal{O}_{X,x}$. The factor $\chi(h)$ being a unit in $\mathcal{O}_{X,x}$, it follows that the elements ${}^{g_1 h}(w \cdot v)$ form a basis of $\mathcal{J}_y$ over $\mathcal{O}_{X,x}$. This proves the claim and hence the remaining implication at $x$ in the totally ramified case.

In the general case choose any point $y \in Y$ above $x$. Applying the preceding case to the covering $Y \longrightarrow Y/G_y$ and passing to completions we find that $\widehat{\mathcal{J}}_y$ is free of rank 1 over $\widehat{\mathcal{O}}_{X,x}[G_y]$. Now $(\pi_* \mathcal{J})\widehat{}_x$ as a representation of $G$ over $\widehat{\mathcal{O}}_{X,x}$ is induced from the representation $\widehat{\mathcal{J}}_y$ of $G_y$. It is therefore free of rank 1 over $\widehat{\mathcal{O}}_{X,x}[G]$. By Lemma 3.2 this implies that $\pi_* \mathcal{J}$ is locally free of rank 1 over $\mathcal{O}_X[G]$ near $x$, as desired. This finishes the proof of Theorem 3.1. **q.e.d.**

We finish this section with the following local corollary. Given Theorem 1.2, one could deduce Theorem 3.1 essentially from that local result, which must surely possess an entirely local proof. But we proceed in the opposite direction and deduce it from our global results. We feel justified in this because, among the results of Sections 1 and 3, only Theorem 3.1 is really needed to prove Theorem 0.2. Besides, our direct proof for 3.1 is very much in the spirit of the remaining sections.

**Corollary 3.6** *Consider any closed point $x \in X$. If $\pi$ is totally wildly ramified at $x$, the following assertions are equivalent:*

*(a) The ramification of $\pi$ above $x$ is of type 2.*

*(b) The stalk $(\pi_* \mathcal{J})_x$ is locally free of rank 1 over $\mathcal{O}_{X,x}[G]$.*

*Without assumption one still has the implication (a)⇒(b).*

*Proof.* Assume first that $\pi$ is totally wildly ramified at $x$. By Lemma 3.2 we may pass to the respective completions. Let $L/K$ denote the resulting extension of local fields, and choose any identification $K = k((T^{-1}))$. By a theorem of Katz ([7] Thm. 1.4.1) the extension $L/K$ is the completion at $T = \infty$ of some Galois extension of $k(T)$ with the same Galois group $G = \mathrm{Gal}(L/K)$ and which is unramified outside $T = \infty$. Let $\pi' : Y' \to X' := \mathbb{P}^1_k$ denote the associated Galois covering of the projective line. Using Lemma 3.2 again it suffices to prove the equivalence for this new global covering instead of the old one. Thus after replacing $Y \to X$ by $Y' \to X'$ we may assume that $X = \mathbb{P}^1_k$ and $\pi$ is unramified outside $x = \infty$.

As seen earlier the local freeness condition in Theorem 3.1 is then automatically satisfied outside $\infty$. Note also that the total wild ramification implies that $G$ is a $p$-group. Since $X$ is ordinary, Theorems 1.2 and 3.1 thus imply that each of (a) and (b) is equivalent to $Y$ being ordinary. Thus they are equivalent to each other, as desired. (Curiously, the formula 1.7 implies that the $p$-rank of $Y$ is always zero in this situation; hence $Y$ is ordinary if and only if it is rational.)

The implication (a)$\Rightarrow$(b) in the general case follows from the totally wildly ramified case in exactly the same way as in the proof of Theorem 3.1. **q.e.d.**

# 4 Reduction to coherent cohomology

Keeping the previous notations, we now consider the commutative diagram

$$\begin{array}{ccc} V := Y \smallsetminus D & \overset{\tilde{j}}{\hookrightarrow} & Y \\ \varpi \downarrow & & \downarrow \pi \\ U := X \smallsetminus \pi(D) & \overset{j}{\hookrightarrow} & X. \end{array}$$

Let $F$ be a locally constant étale sheaf of $\mathbb{F}_p$-modules on $U$ which becomes constant over $V$. Since $V$ is connected, this constant value is $M := H^0(V, \varpi^* F)$ and it carries a natural representation of $G$. The given isomorphism between $\varpi^* F$ and the constant sheaf $M$ induces an isomorphism $\varpi_* \varpi^* F \cong M \otimes \varpi_* \mathbb{F}_p$, which is compatible with the natural action of $G$ on all terms. Taking $G$-invariants, we deduce

$$F \cong (\varpi_* \varpi^* F)^G \cong (M \otimes \varpi_* \mathbb{F}_p)^G.$$

Conversely, we can begin with any (left) $\mathbb{F}_p[G]$-module $M$. Then

$$F_M := (M \otimes \varpi_* \mathbb{F}_p)^G \tag{4.1}$$

is a locally constant étale $\mathbb{F}_p$-sheaf on $U$ which becomes constant over $V$ and whose associated representation is $M$. Clearly the functor $M \mapsto F_M$ is exact.

Recall that $\mathcal{J} \subset \mathcal{O}_Y$ denotes the ideal sheaf of $D$. It therefore carries a canonical Frobenius endomorphism giving rise to a short exact sequence (cf. 2.2)

$$0 \longrightarrow \tilde{j}_! \mathbb{F}_p \longrightarrow \mathcal{J} \xrightarrow{1-\sigma} \mathcal{J} \longrightarrow 0.$$

If $\tau$ denotes the induced Frobenius endomorphism of $\pi_* \mathcal{J}$, we deduce a short exact sequence

$$0 \longrightarrow j_! \varpi_* \mathbb{F}_p \longrightarrow \pi_* \mathcal{J} \xrightarrow{1-\tau} \pi_* \mathcal{J} \longrightarrow 0.$$

For every finite $\mathbb{F}_p[G]$-module $M$ we consider the coherent sheaf

$$\bar{\mathcal{F}}_M := (M \otimes \pi_* \mathcal{J})^G. \tag{4.2}$$

From Corollary 3.6 we immediately deduce:

**Proposition 4.3** *If the ramification of $\pi$ is everywhere of type 2, the functor $M \mapsto \bar{\mathcal{F}}_M$ is exact.*

Next, the sheaf $\bar{\mathcal{F}}_M$ inherits a Frobenius endomorphism from $\pi_*\mathcal{J}$, denoted again by $\tau$.

**Lemma 4.4** *There is a natural short exact sequence*

$$0 \longrightarrow j_! F_M \longrightarrow \bar{\mathcal{F}}_M \xrightarrow{1-\tau} \bar{\mathcal{F}}_M \longrightarrow 0.$$

*Proof.* Since $\varpi$ is étale, we have $\varpi_*\mathbb{F}_p \otimes \mathcal{O}_U \xrightarrow{\sim} \varpi_*\mathcal{O}_V$. Therefore

$$j^*\bar{\mathcal{F}}_M = (M \otimes \varpi_*\mathcal{O}_V)^G \cong (M \otimes \varpi_*\mathbb{F}_p \otimes \mathcal{O}_U)^G = F_M \otimes \mathcal{O}_U.$$

In other words, the restriction of $\bar{\mathcal{F}}_M$ to $U$ is the coherent sheaf corresponding to the locally constant étale sheaf $F_M$ by Artin-Schreier theory. Let $\mathcal{I} \subset \mathcal{O}_X$ denote the ideal sheaf of $\pi(D)^{\mathrm{red}}$. We will show that $\bar{\mathcal{F}}_M$ together with its Frobenius endomorphism satisfies the condition 2.1 (a). Since $\sigma(\mathcal{J}) \subset \mathcal{J}^p$, there exists an integer $m > 0$ such that $\tau^m(\pi_*\mathcal{J}) \subset \mathcal{I}\cdot(\pi_*\mathcal{J})$. This implies

$$\tau^m(\bar{\mathcal{F}}_M) = \tau^m(M \otimes \pi_*\mathcal{J})^G \subset \big(M \otimes \tau^m(\pi_*\mathcal{J})\big)^G \subset \big(M \otimes \mathcal{I}\cdot(\pi_*\mathcal{J})\big)^G = \mathcal{I}\cdot\bar{\mathcal{F}}_M.$$

The desired assertion now follows from Proposition 2.1 (b). **q.e.d.**

We can now compare certain étale and coherent Euler characteristics.

**Proposition 4.5** *For every $M$ we have $\chi(X, j_!F_M) \geq \chi(X, \bar{\mathcal{F}}_M)$, with equality if $Y$ is ordinary.*

*Proof.* By Lemma 4.4, Proposition 2.3, and the isomorphy 2.4 for every $i$ we have $h^i(X, j_!F_M) \leq h^i(X, \bar{\mathcal{F}}_M)$, with equality if and only if $\tau$ induces an isomorphism on $H^i(X, \bar{\mathcal{F}}_M)$. In degree 0 we have

$$H^0(X, \bar{\mathcal{F}}_M) = H^0(X, M \otimes \pi_*\mathcal{J})^G = \big(M \otimes H^0(Y, \mathcal{J})\big)^G.$$

Note that $H^0(Y, \mathcal{J}) = k$ or 0, depending on whether $D = \varnothing$ or not. Accordingly, we deduce that

$$H^0(X, \bar{\mathcal{F}}_M) = \big(M \otimes k\big)^G = M^G \otimes k, \text{ resp. } = 0.$$

In both cases $\tau$ is an isomorphism on this group. The only other possibly non-zero cohomology group occurs in degree 1. Thus the inequality of Euler characteristics follows from the inequality of $h^1$s, and it remains to show that $\tau$ induces an isomorphism on $H^1(X, \bar{\mathcal{F}}_M)$ if $Y$ is ordinary.

In this case, Theorem 3.1 implies that the functor $M \mapsto \bar{\mathcal{F}}_M$ is exact. Let us assume for simplicity that $D$ is non-empty. (It is easy to extend the proof to the general case.) Then $H^1(X, \bar{\mathcal{F}}_M)$ is a successive extension of $H^1(X, \bar{\mathcal{F}}_N)$ where $N$ runs through the simple subquotients of $M$ in any chosen Jordan-Hölder series. Thus $\tau$ is an isomorphism on $H^1(X, \bar{\mathcal{F}}_M)$ if and only if the same holds for every simple subquotient of $M$. As every irreducible representation is a subquotient of the regular representation, it suffices to consider the case $M = \mathbb{F}_p[G]$. In this case $\bar{\mathcal{F}}_M = (\mathbb{F}_p[G] \otimes \pi_*\mathcal{J})^G \cong \pi_*\mathcal{J}$; hence $H^1(X, \bar{\mathcal{F}}_M) \cong H^1(Y, \mathcal{J})$. As $Y$ is ordinary and $D$ reduced, the desired isomorphy now results from Proposition 2.5.

**q.e.d.**

# 5 Local Terms

In the section we define the local terms occurring in Theorem 0.2. As before let $X$ be an irreducible smooth projective curve over an algebraically closed field $k$ of characteristic $p > 0$. For any closed point $x \in X$ let $I_x$ denote the absolute Galois group of $\mathrm{Quot}(\widehat{\mathcal{O}}_{X,x})$ and $I_x^{\mathrm{tame}}$ its maximal tame quotient. Let $\mathbb{Z}_{(p)}$ denote the ring of rational numbers without $p$ in the denominator. There is a well-known canonical isomorphism

$$\mathbb{Z}_{(p)}/\mathbb{Z} \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\mathrm{cont}}(I_x^{\mathrm{tame}}, k^*).$$

To recall its definition, for any positive integer $n$ which is not divisible by $p$ let $u_n$ denote an $n^{\mathrm{th}}$ root of any uniformizer in $\widehat{\mathcal{O}}_{X,x}$. For $\frac{m}{n} \in \mathbb{Z}_{(p)}$ and $\sigma \in I_x^{\mathrm{tame}}$ set

$$\chi_{\frac{m}{n}}(\sigma) := \left(\frac{{}^{\sigma}u_n}{u_n}\right)^m \in k^*. \tag{5.1}$$

This defines a continuous homomorphism $I_x^{\mathrm{tame}} \to k^*$ which depends only on $\frac{m}{n}$ modulo $\mathbb{Z}$ and not on the choice of $u_n$. The following notation will be convenient.

**Definition 5.2** *For any continuous character $\chi : I_x^{\mathrm{tame}} \to k^*$ we let $\langle\chi\rangle$ denote the unique rational number $\alpha \in \mathbb{Z}_{(p)}$ with $\chi = \chi_\alpha$ and $0 \leq \alpha < 1$.*

Now suppose that $\mathbb{F}$ is a finite field of order $p^r$, and consider a finite dimensional continuous representation $M$ of $I_x$ over $\mathbb{F}$. As $k$ is algebraically closed, there are precisely $r$ different embeddings $\iota\colon \mathbb{F} \hookrightarrow k$. For each of them, the representation $M \otimes_{\mathbb{F},\iota} k$ is a successive extension of 1-dimensional representations of $I_x^{\mathrm{tame}}$. Let $\mu_{\iota,\chi}$ denote the multiplicity of $\chi$ as subquotient of $M \otimes_{\mathbb{F},\iota} k$.

**Definition 5.3** *If $\mathbb{F}$ is finite of order $p^r$, we set*

$$\mathrm{LT}_x^{\mathbb{F}} M := \frac{1}{r} \cdot \sum_{\iota} \sum_{\chi} \mu_{\iota,\chi} \cdot \big(1 - \langle\chi\rangle\big) \in \mathbb{Q}.$$

Note that this local term depends only on the semisimplification of $M$; hence it is additive in short exact sequences. Note also that $\mathrm{LT}_x^{\mathbb{F}} M = \dim_{\mathbb{F}} M$ whenever $I_x$ acts trivially on $M$. Furthermore, the local terms of $M$ as a representation over $\mathbb{F}$, respectively over $\mathbb{F}_p$, are related. Namely, since $M \otimes_{\mathbb{F}_p} k \cong \bigoplus_\iota M \otimes_{\mathbb{F},\iota} k$, one easily sees that

$$\mathrm{LT}_x^{\mathbb{F}_p} M = r \cdot \mathrm{LT}_x^{\mathbb{F}} M. \tag{5.4}$$

Now suppose that $\mathbb{F}$ is a local field of residue characteristic $p$, and consider a finite dimensional continuous representation $M$ of $I_x$ over $\mathbb{F}$. Let $\mathfrak{m} \subset R$ denote the maximal ideal and the ring of integers of $\mathbb{F}$, and set $\mathbb{F}_0 := R/\mathfrak{m}$. The representation of $I_x$ stabilizes some $R$-lattice $\Lambda \subset M$ of maximal rank. It is well-known that the semisimplification of $\Lambda/\mathfrak{m}\Lambda$ is, up to isomorphism, independent of the choice of $\Lambda$. Thus the following definition makes sense.

**Definition 5.5** *If $\mathbb{F}$ is a local field of residue characteristic $p$, we set*

$$\mathrm{LT}_x^{\mathbb{F}} M := \mathrm{LT}_x^{\mathbb{F}_0}(\Lambda/\mathfrak{m}\Lambda) \in \mathbb{Q}.$$

Again this is clearly additive in short exact sequences. Note that again $\mathrm{LT}_x^{\mathbb{F}} M = \dim_{\mathbb{F}} M$ whenever $I_x$ acts trivially on $M$.

We finish this section by giving simple formulas in certain special cases.

**Proposition 5.6** *(a) If $I_x$ acts on $M$ through a pro-$p$-group, then*

$$\mathrm{LT}_x^{\mathbb{F}} M = \dim_{\mathbb{F}} M.$$

*(b) If $M$ is a self-dual representation of $I_x$, then*

$$\mathrm{LT}_x^{\mathbb{F}} M = \frac{1}{2}\cdot\big(\dim_{\mathbb{F}} M + \mu_0\big),$$

*where $\mu_0$ is the multiplicity of the trivial 1-dimensional representation of $I_x$ as subquotient of $M$ if $\mathbb{F}$ is finite, resp. of $\Lambda/\mathfrak{m}\Lambda$ if $\mathbb{F}$ is local.*

*Proof.* Part (a) is immediate from the definitions. In (b) the local case follows from the finite case, so we assume that $\mathbb{F}$ is finite of order $p^r$. As $M$ is self-dual, we have $\mu_{\iota,\chi^{-1}} = \mu_{\iota,\chi}$ for all $\iota$ and $\chi$. In the case of the trivial character $\mathbb{1}$, we have $\mu_{\iota,\mathbf{1}} = \mu_0$ for every $\iota$. Thus from Definition 5.3 we obtain

$$\begin{aligned}
\mathrm{LT}_x^{\mathbb{F}} M &:= \frac{1}{r}\cdot\sum_\iota\sum_\chi \mu_{\iota,\chi}\cdot\big(1-\langle\chi\rangle\big)\\
&= \frac{1}{2r}\cdot\sum_\iota\sum_\chi \mu_{\iota,\chi}\cdot\big(2-\langle\chi\rangle-\langle\chi^{-1}\rangle\big)\\
&= \frac{1}{2r}\cdot\sum_\iota\sum_\chi \mu_{\iota,\chi}\cdot\begin{cases}2 & \text{if } \chi = \mathbb{1},\\ 1 & \text{if } \chi \neq \mathbb{1},\end{cases}\\
&= \frac{1}{2r}\cdot\sum_\iota\big(\dim_k(M\otimes_{\mathbb{F},\iota} k) + \mu_{\iota,\mathbf{1}}\big)\\
&= \frac{1}{2}\cdot\big(\dim_{\mathbb{F}} M + \mu_0\big),
\end{aligned}$$

as desired. **q.e.d.**

## 6 Degree calculation

Now we return to the situation of Section 4, where $M$ is a finite dimensional representation of $G$ over the field $\mathbb{F}_p$. For any closed point $x \in X$ there is a canonical conjugacy class of continuous homomorphisms $I_x \to G$ whose images are precisely the stabilizers $G_y$ for all points $y \in Y$ above $x$. Thus $M$ can be viewed as a representation of $I_x$, unique up to isomorphism, and hence the local terms $\mathrm{LT}_x^{\mathbb{F}_p} M$ are defined. The aim of this section is to prove the following formula. (Compare the formula of Chevalley-Weil [2] for Riemann surfaces.)

**Proposition 6.1** *If the ramification of $\pi$ is everywhere of type 2, for any finite $\mathbb{F}_p[G]$-module $M$ we have*

$$\deg \bar{\mathcal{F}}_M = -\sum_{x \in \pi(D)} \mathrm{LT}_x^{\mathbb{F}_p} M.$$

This will result from a series of reduction steps. To begin with, observe that

$$\deg \bar{\mathcal{F}}_M = \frac{\deg \pi^* \bar{\mathcal{F}}_M}{|G|}. \tag{6.2}$$

The definition 4.2 of $\bar{\mathcal{F}}_M$ implies

$$\pi^* \bar{\mathcal{F}}_M \;=\; \pi^*(M \otimes \pi_* \mathcal{J})^G \;\cong\; (M \otimes \pi^* \pi_* \mathcal{J})^G \;\subset\; M \otimes \pi^* \pi_* \mathcal{J},$$

so the adjunction map $\pi^* \pi_* \mathcal{J} \longrightarrow \mathcal{J} \hookrightarrow \mathcal{O}_Y$ induces a homomorphism

$$\varphi_M : \; \pi^* \bar{\mathcal{F}}_M \longrightarrow M \otimes \mathcal{O}_Y .$$

Its local properties are described by the following lemma.

**Lemma 6.3** *$\varphi_M$ is an isomorphism outside $D$ and a monomorphism everywhere. Taking stalks at any point $y \in D$ we have*

$$(\operatorname{Coker} \varphi_M)_y \cong \frac{M \otimes \widehat{\mathcal{O}}_{Y,y}}{(M \otimes \widehat{\mathcal{J}}_y)^{G_y} \cdot \widehat{\mathcal{O}}_{Y,y}}.$$

*Proof.* For any closed point $y \in Y$ there is a natural isomorphism of $\widehat{\mathcal{O}}_{Y,y}[G]$-modules

$$(\pi^* \pi_* \mathcal{J})\hat{}_y \cong (\mathrm{Ind}_{G_y}^G \widehat{\mathcal{J}}_y) \otimes_{\widehat{\mathcal{O}}_{X,x}} \widehat{\mathcal{O}}_{Y,y},$$

where on the right hand side $G$ acts on the first tensor factor and $\widehat{\mathcal{O}}_{Y,y}$ on the second. With this identification the adjunction map is the composite of two maps

$$(\mathrm{Ind}_{G_y}^G \widehat{\mathcal{J}}_y) \otimes_{\widehat{\mathcal{O}}_{X,x}} \widehat{\mathcal{O}}_{Y,y} \longrightarrow \widehat{\mathcal{J}}_y \otimes_{\widehat{\mathcal{O}}_{X,x}} \widehat{\mathcal{O}}_{Y,y} \longrightarrow \widehat{\mathcal{J}}_y,$$

where the first map comes from projection to the identity coset $G_y \subset G$ and the second is simply the multiplication map. It follows that

$$\begin{aligned}
(\pi^* \bar{\mathcal{F}}_M)\hat{}_y &\cong \big(M \otimes (\pi^* \pi_* \mathcal{J})\hat{}_y\big)^G \\
&\cong \big(M \otimes (\mathrm{Ind}_{G_y}^G \widehat{\mathcal{J}}_y) \otimes_{\widehat{\mathcal{O}}_{X,x}} \widehat{\mathcal{O}}_{Y,y}\big)^G \\
&\cong \big(M \otimes \widehat{\mathcal{J}}_y\big)^{G_y} \otimes_{\widehat{\mathcal{O}}_{X,x}} \widehat{\mathcal{O}}_{Y,y};
\end{aligned}$$

and here the map $\varphi_M$ to $M \otimes \widehat{\mathcal{O}}_{Y,y}$ is given simply by multiplication between $\widehat{\mathcal{J}}_y$ and $\widehat{\mathcal{O}}_{Y,y}$.

If $y \notin D$, we have $G_y = 1$ and $\widehat{\mathcal{J}}_y = \widehat{\mathcal{O}}_{Y,y} = \widehat{\mathcal{O}}_{X,x}$. In this case the above map on completions is visibly an isomorphism. Thus $\varphi_M$ is an isomorphism outside $D$. As $\pi^*\bar{\mathcal{F}}_M$ and $M \otimes \mathcal{O}_Y$ are locally free coherent sheaves of the same rank, it follows that $\varphi_M$ is a monomorphism everywhere and its cokernel is torsion. This proves the first assertion, and for $y \in D$ it also shows that $(\operatorname{Coker}\varphi_M)_y$ does not change on passing to completions. Thus the second assertion follows directly from the calculation above. **q.e.d.**

Since $\deg(M \otimes \mathcal{O}_Y) = 0$, Lemma 6.3 implies

$$\deg \pi^* \bar{\mathcal{F}}_M = -\sum_{y \in D} \dim_k(\operatorname{Coker}\varphi_M)_y. \tag{6.4}$$

Clearly any two points $y$ that are conjugate under $G$ yield the same local contribution. Thus by combining 6.4 with 6.2 we obtain

$$\deg \bar{\mathcal{F}}_M = -\sum_{y \in D \bmod G} \frac{\dim_k(\operatorname{Coker}\varphi_M)_y}{|G_y|}. \tag{6.5}$$

To prove Proposition 6.1 we may thus fix a point $y \in D$ with image $x := \pi(y)$ and must show

$$\frac{\dim_k(\operatorname{Coker}\varphi_M)_y}{|G_y|} = \mathrm{LT}_x^{\mathbb{F}_p} M. \tag{6.6}$$

By Lemma 6.3 and the definition of $\mathrm{LT}_x^{\mathbb{F}_p} M$, both sides of 6.6 depend only on the action of $G_y$ on $Y$ and $M$. Thus we are reduced to an entirely local problem. For ease of notation, let us replace $X = Y/G$ by $Y/G_y$ and $G$ by $G_y$, while keeping $Y$ and $y$ the same. In other words, we now assume that $G_y = G$. Then the next lemma becomes very useful.

**Lemma 6.7** *If the ramification of $\pi$ at $y$ is of type 2, then* $\dim_k(\operatorname{Coker}\varphi_M)_y$ *is additive in short exact sequences.*

*Proof.* Any short exact sequence $0 \to M' \to M \to M'' \to 0$ gives rise to a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\bar{\mathcal{F}}_{M'} & \longrightarrow & M' \otimes \mathcal{O}_Y & \longrightarrow & \operatorname{Coker}\varphi_{M'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\bar{\mathcal{F}}_{M} & \longrightarrow & M \otimes \mathcal{O}_Y & \longrightarrow & \operatorname{Coker}\varphi_{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\bar{\mathcal{F}}_{M''} & \longrightarrow & M'' \otimes \mathcal{O}_Y & \longrightarrow & \operatorname{Coker}\varphi_{M''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here the rows are exact by construction, and the middle column is obviously split exact. The left hand column is exact by Proposition 4.3 and the flatness of $\pi$. Thus the $3\times 3$-lemma implies that the right hand column is exact; whence the lemma. **q.e.d.**

By Lemma 6.7 and the definition of $\mathrm{LT}_x^{\mathbb{F}_p} M$, both sides of 6.6 depend only on the semisimplification of $M$. To prove 6.6 we may thus assume that $M$ is semisimple. Let $G_1 \triangleleft G$ denote the wild inertia group at $x$. Then $M$ comes from a representation of the tame quotient $G/G_1$.

**Lemma 6.8** *For any finite $\mathbb{F}_p[G/G_1]$-module $M$ the two sides of 6.6 do not change when $Y$ is replaced by $Y/G_1$ and $G$ by $G/G_1$.*

*Proof.* For the right hand side of 6.6 this is clear from the definition of $\mathrm{LT}_x^{\mathbb{F}_p} M$. For the left hand side consider the factorization $Y \longrightarrow Z := Y/G_1 \longrightarrow X$, where the second map is again a Galois covering, with group $H := G/G_1$. Let $z \in Z$ denote the unique point between $y$ and $x$. Since $y \in D$, we have $\widehat{\mathcal{J}}_y = \widehat{\mathfrak{m}}_{Y,y}$ and hence $(\widehat{\mathcal{J}}_y)^{G_1} = \widehat{\mathfrak{m}}_{Y,y} \cap \widehat{\mathcal{O}}_{Z,z} = \widehat{\mathfrak{m}}_{Z,z}$. The denominator in Lemma 6.3 is therefore

$$\big(M \otimes \widehat{\mathcal{J}}_y\big)^G \cdot \widehat{\mathcal{O}}_{Y,y} = \big(M \otimes (\widehat{\mathcal{J}}_y)^{G_1}\big)^H \cdot \widehat{\mathcal{O}}_{Y,y} = \big(M \otimes \widehat{\mathfrak{m}}_{Z,z}\big)^H \cdot \widehat{\mathcal{O}}_{Y,y}.$$

Thus both the denominator and the numerator in 6.3 are obtained by base extension from $\widehat{\mathcal{O}}_{Z,z}$ to $\widehat{\mathcal{O}}_{Y,y}$. Since the latter is free of rank $|G_1| = |G|/|H|$ over the former, the left hand side of 6.6 is

$$\begin{aligned}
\frac{\dim_k(\operatorname{Coker}\varphi_M)_y}{|G|} &\overset{6.3}{=} \frac{1}{|G|}\cdot \dim_k\left(\frac{M \otimes \widehat{\mathcal{O}}_{Y,y}}{(M \otimes \widehat{\mathcal{J}}_y)^G \cdot \widehat{\mathcal{O}}_{Y,y}}\right) \\
&= \frac{1}{|G|}\cdot \dim_k\left(\frac{M \otimes \widehat{\mathcal{O}}_{Y,y}}{(M \otimes \widehat{\mathfrak{m}}_{Z,z})^H \cdot \widehat{\mathcal{O}}_{Y,y}}\right) \\
&= \frac{1}{|H|}\cdot \dim_k\left(\frac{M \otimes \widehat{\mathcal{O}}_{Z,z}}{(M \otimes \widehat{\mathfrak{m}}_{Z,z})^H \cdot \widehat{\mathcal{O}}_{Z,z}}\right),
\end{aligned}$$

as desired. **q.e.d.**

To prove 6.6 we may now assume that $G = G_y$ is tame. We can identify the group of characters $\chi : G \to k^*$ with a subgroup of continuous characters $I_x^{\mathrm{tame}} \to k^*$ of the absolute tame inertia group. Consider any finite dimensional representation $V$ of $G$ over $k$, and let $\mu_\chi$ denote the multiplicity of $\chi$ in $V$. Any finite $\mathbb{F}_p[G]$-module $M$ gives rise to such $V := M \otimes k$ via the unique embedding $\mathbb{F}_p \hookrightarrow k$, and Definition 5.3 becomes

$$\mathrm{LT}_x^{\mathbb{F}_p} M = \sum_\chi \mu_\chi \cdot \big(1 - \langle\chi\rangle\big).$$

In view of Lemma 6.3 we must therefore prove

$$\dim_k\left(\frac{V \otimes_k \widehat{\mathcal{O}}_{Y,y}}{(V \otimes_k \widehat{\mathcal{J}}_y)^G \cdot \widehat{\mathcal{O}}_{Y,y}}\right) = |G| \cdot \sum_\chi \mu_\chi \cdot \big(1 - \langle\chi\rangle\big). \tag{6.9}$$

This formula makes sense for any $V$ as above, even if $V$ does not arise from a representation over $\mathbb{F}_p$. On the other hand any such $V$ is a direct sum of 1-dimensional representations. Since both sides of 6.9 are additive in direct sums, it suffices to prove the equality when $\dim_k V = 1$. Assume this from now on.

Put $n := |G|$. There is a unique integer $0 \leq m < n$ such that $G$, and equivalently $I_x^{\mathrm{tame}}$, acts on $V$ through the character $\chi_{\frac{m}{n}}$ from 5.1. Following Definition 5.2 the right hand side of 6.9 is then

$$|G|\cdot\big(1 - \langle\chi_{\frac{m}{n}}\rangle\big) = n\cdot\Big(1 - \frac{m}{n}\Big) = n - m.$$

To determine the left hand side let $u_n \in \widehat{\mathcal{O}}_{Y,y}$ be an $n^{\mathrm{th}}$ root of any uniformizer in $\widehat{\mathcal{O}}_{X,x}$. Then $G$ acts on $u_n$ through the character $\chi_{\frac{1}{n}}$, and we have the $G$-isotypic decomposition

$$\widehat{\mathcal{J}}_y = \widehat{\mathfrak{m}}_{Y,y} = \bigoplus_{i=1}^{n} u_n^i\cdot\widehat{\mathcal{O}}_{X,x}.$$

Since $G$ acts on $V$ through the character $\chi_{\frac{m}{n}} = (\chi_{\frac{1}{n}})^m$, we deduce that

$$(V \otimes_k \widehat{\mathcal{J}}_y)^G = u_n^{n-m}\cdot\widehat{\mathcal{O}}_{X,x}.$$

Thus the left hand side of 6.9 is

$$\dim_k\left(\frac{V \otimes_k \widehat{\mathcal{O}}_{Y,y}}{V \otimes_k u_n^{n-m}\cdot\widehat{\mathcal{O}}_{Y,y}}\right) = n - m.$$

This shows the equality 6.9 in the case $\dim_k V = 1$ and thereby finishes the proof of Proposition 6.1. **q.e.d.**

# 7 Euler-Poincaré formula

With the notations of the preceding sections we can now deduce:

**Proposition 7.1** *If the ramification of $\pi$ is everywhere of type 2, for any finite $\mathbb{F}_p[G]$-module $M$ we have*

$$\chi(X, j_! F_M) \geq (1 - g_X)\cdot\dim_{\mathbb{F}_p} M - \sum_{x\in\pi(D)} \mathrm{LT}_x^{\mathbb{F}_p} M.$$

*If $Y$ is ordinary, we have equality.*

*Proof.* This follows from the Riemann-Roch formula via the calculation

$$\begin{aligned}
\chi(X, j_! F_M) &\overset{4.5}{\geq (=)} \chi(X, \bar{\mathcal{F}}_M) \\
&\overset{\mathrm{RR}}{=} (1 - g_X)\cdot\mathrm{rank}\,\bar{\mathcal{F}}_M + \deg\bar{\mathcal{F}}_M \\
&\overset{6.1}{=} (1 - g_X)\cdot\dim_{\mathbb{F}_p} M - \sum_{x\in\pi(D)} \mathrm{LT}_x^{\mathbb{F}_p} M.
\end{aligned}$$

**q.e.d.**

**Remark 7.2** It can happen that the above inequality is an equality, even when $F_M$ cannot be trivialized by an ordinary covering. The referee suggested that such a sheaf be called *ordinary*. A systematic analysis of such ordinary sheaves is as yet lacking. It would also be interesting to know whether the inequality holds without ramification assumption.

In the remainder of this section, we deduce Theorem 0.2 from Proposition 7.1.

**Lemma 7.3** *Theorem 0.2 holds when* $\mathbb{F} = \mathbb{F}_p$.

*Proof.* Let $\pi : Y \to X$ be as in 0.2. Choose a $G$-invariant reduced effective divisor $D \subset Y$ such that $F$ is lisse outside $\pi(D)$. For simplicity, we suppose that $D$ is non-empty. The open and closed embeddings

$$X \smallsetminus \pi(D) \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} \pi(D)$$

induce a short exact sequence of étale sheaves

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0.$$

As both sides of the desired equality are additive in short exact sequences, we are reduced to the cases $F = j_! j^* F$ and $F = i_* i^* F$. In the second case the equality is obvious. In the first case consider the representation of the fundamental group $\pi_1^{\text{ét}}(X \smallsetminus \pi(D), \bar\eta)$ on the stalk $M := F_{\bar\eta}$. By assumption, the normal subgroup $\pi_1^{\text{ét}}(Y \smallsetminus D, \bar\eta)$ acts through a $p$-group. The coefficients being $\mathbb{F}_p$, it therefore acts trivially on the semisimplification of $M$. As both sides of the desired equality are additive in short exact sequences, we may replace $M$ and $F$ by their semisimplifications. Thereafter, we can view $M$ as a representation of $G$ and identify $F$ with the sheaf $j_! F_M$ (see 4.1). It then remains to show that the right hand side in Proposition 7.1 coincides with that in Theorem 0.2. The contributions of any point $x \in \pi(D)$ are equal, because $(j_! F_M)_x = 0$. For $x \in X \smallsetminus \pi(D)$ the contribution in 0.2 vanishes, because $F_M$ is lisse at $x$. Thus the respective right hand sides do coincide, as desired. **q.e.d.**

**Lemma 7.4** *Theorem 0.2 holds when* $\mathbb{F}$ *is finite.*

*Proof.* Suppose that $[\mathbb{F}/\mathbb{F}_p] = r$. Then for any $\mathbb{F}$-vector space $M$ we have $\dim_{\mathbb{F}_p} M = r \cdot \dim_{\mathbb{F}} M$. If $F$ is viewed as a sheaf of modules over $\mathbb{F}_p$ instead of $\mathbb{F}$, its cohomology groups do not change; hence its Euler characteristic is multiplied by $r$. By 5.4 the same happens to the right hand side of the desired equality. Thus we are reduced to Lemma 7.3. **q.e.d.**

Now suppose that $\mathbb{F}$ is a local field of residue characteristic $p$. For applications to function fields we definitely want to include the case of equal characteristic, where étale sheaves of $\mathbb{F}$-modules are not commonly used. Since all notions and arguments are literal extensions from the case $\mathbb{F} = \mathbb{Q}_p$, we merely give a brief review based on [SGA4$\frac{1}{2}$] Rapport §2 and [SGA5] exp. V, exp. VI, keeping everything as elementary as possible.

Let $\mathfrak{m} \subset R$ denote the maximal ideal and the ring of integers of $\mathbb{F}$, and set $\mathbb{F}_0 := R/\mathfrak{m}$. A constructible sheaf of $\mathbb{F}$-modules on $X$ is given by the following data. For every integer $n \geq 0$ consider a constructible étale sheaf $F_n$ of $R/\mathfrak{m}^{n+1}$-modules on $X$. Consider isomorphisms $F_m/\mathfrak{m}^{n+1}F_m \xrightarrow{\sim} F_n$ for all $m \geq n \geq 0$ which satisfy the obvious cocycle relation. Thus $(F_n)$ is an inverse system of étale sheaves of torsion $R$-modules ([SGA5] exp. V Def. 3.1.1). The category of constructible sheaves of $\mathbb{F}$-modules is obtained from the category of such systems by a certain localization process ([SGA5] exp. VI 1.4.2), which corresponds to the heuristic interpretation

$$F \;=\; \mathbb{F} \otimes_R \text{“}\varprojlim_n F_n\text{”}.$$

In fact, the stalk of $F$ at any geometric point $x$ is formed exactly in this way:

$$F_x := \mathbb{F} \otimes_R \varprojlim_n F_{n,x}.$$

To some extent, the system $(F_n)$ can be modified without changing these stalks. It turns out that the $F_n$ can be chosen flat over $R/\mathfrak{m}^{n+1}$, that is, their stalks are free of finite type over $R/\mathfrak{m}^{n+1}$ ([SGA4$\frac{1}{2}$] Rapport 2.8). We assume this from now on. Then $\Lambda_x := \varprojlim F_{n,x}$ is an $R$-lattice of maximal rank in $F_x$, and $F_{n,x} = \Lambda_x/\mathfrak{m}^{n+1}\Lambda_x$. In particular, this implies $\dim_{\mathbb{F}} F_x = \dim_{\mathbb{F}_0} F_{0,x}$.

Let $j : U \hookrightarrow X$ be the embedding of a dense open subscheme such that $j^*F_0$ is lisse. Then flatness implies that $j^*F_n$ is lisse for every $n$. Giving $j^*F$ is therefore equivalent to giving the continuous $\mathbb{F}$-linear representation of $\pi_1^{\text{ét}}(U, \bar{\eta})$ on the geometric generic stalk $F_{\bar{\eta}}$.

Like the stalks, the cohomology groups are obtained by actually taking the inverse limit:

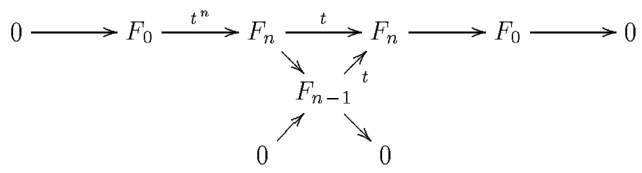
$$H^i(X, F) := \mathbb{F} \otimes_R \varprojlim_n H^i(X, F_n).$$

As $X$ is proper, the groups $H^i(X, F_n)$ are finite. The inverse system thus automatically satisfies the Mittag-Leffler condition; hence the $H^i(X, F)$ form a system of $\delta$-functors on the category of constructible sheaves of $\mathbb{F}$-modules. One can also deduce in general that $H^i(X, F)$ has finite dimension over $\mathbb{F}$ ([SGA5] exp. VI 2.2). Under the flatness assumption above, the Euler characteristics of $F$ (with respect to $\mathbb{F}$) and of $F_0$ (with respect to $\mathbb{F}_0$) coincide:

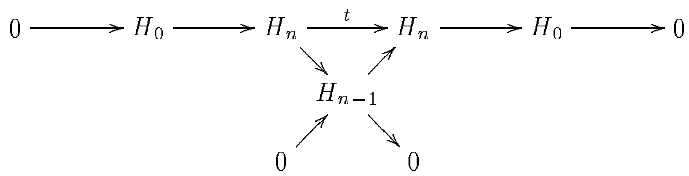
**Lemma 7.5** $\chi(X, F) = \chi(X, F_0)$.

*Proof.* The general way to show this uses the notion of perfect complexes and their invariance under cohomology ([SGA4$\frac{1}{2}$] 4.6, 4.9), and the argument in [loc. cit.] 4.11. But in the curve case the following standard direct argument suffices.

If $F$ is supported on a finite set of points, its cohomology is concentrated in degree 0 and the assertion follows at once from the corresponding equality for stalks. Using long exact sequences, we can thus reduce ourselves to the case that $F = j_! j^* F$ for some dense open subscheme $j : U \hookrightarrow X$ such that $j^* F_0$ is lisse. After shrinking $U$ we may also suppose that $U \neq X$. Then all the cohomology of $F_n = j_! j^* F_n$ is concentrated in degree 1. Let us abbreviate $H_n := H^1(X, F_n)$. Let $h := \dim_{\mathbb{F}_0} H_0$ and choose a uniformizer $t \in \mathfrak{m}$. For any $n \geq 0$ the exact sequences



give rise to the following exact cohomology sequences:



Using any half of this, one shows by induction that $\text{length}_R H_n = (n+1) \cdot h$ for every $n \geq 0$. The right half implies $H_n / \mathfrak{m} H_n \cong H_0$; hence $H_n$ is generated by $h$ elements. Being a module over $R/\mathfrak{m}^{n+1}$, whose own length is $n+1$, it is therefore free of rank $h$ over this ring. We also find that the maps $H_n \to H_{n-1}$ are surjective. It follows that $\varprojlim H_n$ is free of rank $h$ over $R$; whence $\dim_{\mathbb{F}} H^i(X, F) = h$, as desired. **q.e.d.**

**Lemma 7.6** *Theorem 0.2 holds when $\mathbb{F}$ is a local field.*

*Proof.* By Lemma 7.5 the left hand side of the desired equality remains unchanged on passing to $F_0$. By the above remarks the same is true for the dimensions of stalks, and for the local terms it is true by Definition 5.5. Thus we are reduced to Lemma 7.4. **q.e.d.**

This finishes the proof of Theorem 0.2 in all cases.

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