Lecture 12

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§26 Duality and the Dieudonné functor

Let k be a perfect field of characteristik $p > 0$ and $W(k)$ its ring of Witt vectors, and consider the torsion $W(k)$-module

$$T := W(k)\left[\tfrac{1}{p}\right]/W(k).$$

Proposition 26.1. The functor

$$N \mapsto N^* := \mathrm{Hom}_{W(k)}(N, T)$$

defines an anti-equivalence from the category of finite length $W(k)$-modules to itself, and there is a functorial isomorphism

$$N \cong (N^*)^*.$$

Proof. The biduality homomorphism $N \to (N^*)^*$ is obtained by resolving the evaluation pairing $N \times N^* \to T$. It suffices to prove that this homomorphism is an isomorphism; everything else then follows. Since the functor is additive, and every N is a direct sum of cyclic modules, it suffices to prove the isomorphy in the case $N = W(k)/p^n\, W(k)$. But that is straightforward. □

We denote by σ the endomorphism of T that is induced by F, the Frobenius on $W(k)$. Let E be the ring of "noncommutative polynomials" over $W(k)$ in the two variables F and V with the relations as defined in §23. For any left E-module N we define maps $F, V : N^* \to N^*$ by

$$\ell \mapsto F\ell,\ n \mapsto (F\ell)(n) := \sigma(\ell(Vn)),$$

$$\ell \mapsto V\ell,\ n \mapsto (V\ell)(n) := \sigma^{-1}(\ell(Fn)).$$

As F is σ-linear and V is σ^{-1}-linear with respect to $W(k)$, the twists by $\sigma^{\pm 1}$ on the right hand side are precisely those necessary to make $F\ell$ and $V\ell$ again $W(k)$-linear. One easily calculates that together with the usual $W(k)$-action on N^*, this turns N^* into a left E-module.

Proposition 26.2. The functor $N \mapsto N^*$ defines an anti-equivalence from the category of finite length left E-modules to itself, and there is a functorial isomorphism

$$N \cong (N^*)^*.$$

Proof. This is a direct consequence of Proposition 26.1. □

The aim of this section is to show:

Theorem 26.3. For any local-local commutative group scheme G there is a functorial isomorphism of E-modules

$$M(G^*) \cong M(G)^*.$$

Note. The idea behind the proof is to reduce the general case to the case $G = W_n^n$ and to use the isomorphism $(W_n^n)^* \cong W_n^n$ from Theorem 25.3.

We start with the isomorphisms from Proposition 23.3 (a)

$$E_n^n := E/(EF^n + EV^n) \cong \mathrm{End}(W_n^n) \cong M(W_n^n). \tag{26.4}$$

We denote the residue class of $e \in E$ in E_n^n by $[e]$.

Note that E_n^n is an algebra quotient of E, that is noncommutative in general. We will always consider E_n^n as a *left* E-module. Multiplication on the right by any $e \in E$ induces an endomorphism of left E-modules, which we denote by $\rho_e : E_n^n \to E_n^n$. Recall that by definition any $\xi \in W(k)$ acts on W_n^n through multiplication by $\sigma^{-n}(\xi)$; we denote this endomorphism by $\mu_{\sigma^{-n}(\xi)} : W_n^n \to W_n^n$. For the later use we observe that under the isomorphisms (26.4) the following endomorphisms correspond:

(26.5)

action on \ of	$\xi \in W(K)$	F	V
$M(W_n^n)$	$M(\mu_{\sigma^{-n}(\xi)})$	$M(F)$	$M(V)$
≀‖ $\mathrm{End}(W_n^n)$	$(\_) \circ \mu_{\sigma^{-n}(\xi)}$	$(\_) \circ F$	$(\_) \circ V$
≀‖ E_n^n	ρ_ξ	ρ_F	ρ_V

Next we determine the relation with the epimorphism $fr : W_{n+1}^{n+1} \to W_n^n$.

Lemma 26.6. The following diagram commutes:

$$\begin{array}{ccc}
M(W_n^n) & \xrightarrow{M(fr)} & M(W_{n+1}^{n+1}) \\
\wr\| & & \wr\| \\
\mathrm{End}(W_n^n) & \xrightarrow{iv\circ(_)\circ fr} & \mathrm{End}(W_{n+1}^{n+1}) \\
\wr\| & & \wr\| \\
E_n^n & \xrightarrow{[p]:\,[e]\mapsto[pe]} & E_{n+1}^{n+1}.
\end{array}$$

Proof. The top square commutes, because $iv : W_n^n \hookrightarrow W_{n+1}^{n+1}$ induces the transition map in the direct system defining M. For the bottom square, since all arrows are E-module homomorphisms, it suffices to prove the commutativity for the generator [1]. But this follows from:

$$\begin{array}{ccc} \mathrm{id} & \longmapsto & ivfr = VF = p \cdot \mathrm{id} \\ \uparrow & & \uparrow \\ {[1]} & \longmapsto & [p]. \end{array}$$

□

By the self-duality $(W_n^n)^* \cong W_n^n$ and the isomorphisms 26.4, Theorem 26.3 in the special case $G = W_n^n$ amounts to an isomorphism of left E-modules $(E_n^n)^* \cong E_n^n$. Our next job is to construct such an isomorphism directly. First we decompose E_n^n as a left $W(k)$-module as

$$E_n^n = \bigoplus_{|i|<n} W(k)/p^{n-|i|}W(k) \cdot \begin{cases} [F^{|i|}], & i \geq 0, \\ [V^{|i|}], & i \leq 0. \end{cases} \tag{26.7}$$

We define a left $W(k)$-bilinear pairing

$$\langle \_, \_ \rangle_n : E_n^n \times E_n^n \to T,$$

by setting

$$\langle [F^i], [F^i] \rangle_n := \langle [V^i], [V^i] \rangle_n := [p^{-(n-i)}],$$

for any $0 \leq i \leq n$ and mapping all the other pairs of generators to zero.

Lemma 26.8. This is a symmetric, perfect bilinear pairing of left $W(k)$-modules, and it satisfies the following relations for all $e, e' \in E$ and $\xi \in W(k)$:

(a) $\langle [Fe], [e'] \rangle_n = \sigma \left(\langle [e], [Ve'] \rangle_n \right)$

(b) $\langle [eF], [e'] \rangle_n = \langle [e], [e'V] \rangle_n$

(c) $\langle [e\xi], [e'] \rangle_n = \langle [e], [e'\xi] \rangle_n$

(d) $\langle [pe], [e'] \rangle_{n+1} = \langle [e], [e'] \rangle_n$

Proof. The first statement follows directly from the construction. It is enough to prove the remaining formulas when e and e' are $W(k)$-multiples of classes of generators. For example, for $\alpha, \beta \in W(k)$ and $0 \leq i \leq n$ we have

$$\langle [F\alpha F^i], [\beta F^{i+1}] \rangle_n = \langle [\sigma(\alpha) F^{i+1}], [\beta F^{i+1}] \rangle_n = [\sigma(\alpha)\beta p^{-(n-i-1)}] \quad \text{and}$$

$$\sigma\left(\langle[\alpha F^i],[V\beta F^{i+1}]\rangle_n\right)=\sigma\left(\langle[\alpha F^i],[\sigma^{-1}(\beta)pF^i]\rangle_n\right)=\sigma\left([\alpha\sigma^{-1}(\beta)pp^{-(n-i)}]\right),$$

which are equal. Together with similar calculations this proves (a). (b) is proved in the same way, except that no twist by σ occurs, because F and V are multiplied from the right. What happens in (c) is illustrated by the typical case:

$$\begin{aligned}\langle[F^i\xi],[F^i]\rangle_n &= \langle[\sigma^i(\xi)F^i],[F^i]\rangle_n = [\sigma^i(\xi)p^{-(n-i)}]\\ &= \langle[F^i],[\sigma^i(\xi)F^i]\rangle_n = \langle[F^i],[F^i\xi]\rangle_n.\end{aligned}$$

Finally, (d) is also straightforward. □

Lemma 26.9. The pairing $\langle\_,\_\rangle_n$ induces a left E-linear isomorphism

$$E_n^n \cong (E_n^n)^*.$$

Proof. By the first assertion of Lemma 26.8 only the compatility with F and V needs to be checked. But that follows at once from 26.8 (a), from the symmetry of the pairing, and the definition of the action of F and V on $(E_n^n)^*$. □

Now we can construct the isomorphism in Theorem 26.3. Fix a local-local G and take any $n \gg 0$ such that F^n and V^n annihilate G. Then they also annihilate G^* and $M(G^*)$ and $M(G)^*$. We obtain the following sequence of isomorphisms

$$\begin{aligned}M(G^*) &\cong \mathrm{Hom}(G^*,W_n^n)\\ &\overset{25.3}{\cong} \mathrm{Hom}(G^*,(W_n^n)^*)\\ &\overset{\text{Cartier duality}}{\cong} \mathrm{Hom}(W_n^n,G)\\ &\overset{23.2}{\cong} \mathrm{Hom}_E(M(G),M(W_n^n))\\ &\overset{26.4}{\cong} \mathrm{Hom}_E(M(G),E_n^n)\\ &\overset{26.2}{\cong} \mathrm{Hom}_E((E_n^n)^*,M(G)^*)\\ &\overset{26.9}{\cong} \mathrm{Hom}_E(E_n^n,M(G)^*)\\ &\overset{\text{evaluate at }[1]\in E_n^n}{\cong} \{\ell\in M(G)^* | F^n\ell=V^n\ell=0\}\\ &= M(G)^*.\end{aligned}$$

Clearly the composite isomorphism is functorial in G. It remains to show that it is E-linear and independent of n. To prove that it is E-linear we trace the action through the whole sequence of isomorphisms:

action on \ of	$\xi \in W(K)$	F	V	explanation
$\mathrm{Hom}(G^*, W_n^n)$	$\mu_{\sigma^{-n}(\xi)} \circ (\_)$	$F \circ (\_)$	$V \circ (\_)$	
$\wr\Vert$ $\mathrm{Hom}(G^*, (W_n^n)^*)$	$\mu^*_{\sigma^{-n}(\xi)} \circ (\_)$	$V^* \circ (\_)$	$F^* \circ (\_)$	Theorem 25.3 (a,d,e)
$\wr\Vert$ $\mathrm{Hom}(W_n^n, G)$	$(\_) \circ \mu_{\sigma^{-n}(\xi)}$	$(\_) \circ V$	$(\_) \circ F$	Functoriality of Cartier duality
$\wr\Vert$ $\mathrm{Hom}_E(M(G), M(W_n^n))$	$M(\mu_{\sigma^{-n}(\xi)}) \circ (\_)$	$M(V) \circ (\_)$	$M(F) \circ (\_)$	Functoriality of M
$\wr\Vert$ $\mathrm{Hom}_E(M(G), E_n^n)$	$\rho_\xi \circ (\_)$	$\rho_V \circ (\_)$	$\rho_F \circ (\_)$	Table (26.5)
$\wr\Vert$ $\mathrm{Hom}_E((E_n^n)^*, M(G)^*)$	$(\_) \circ \rho^*_\xi$	$(\_) \circ \rho^*_V$	$(\_) \circ \rho^*_F$	Functoriality of $(\_)^*$ from Lemma 26.2
$\wr\Vert$ $\mathrm{Hom}_E(E_n^n, M(G)^*)$	$(\_) \circ \rho_\xi$	$(\_) \circ \rho_F$	$(\_) \circ \rho_V$	Lemma 26.8 (b,c)
$\wr\Vert$ $M(G)^*$	ξ	F	V	explicit calculation, see below

The explicit calculation verifying the last step is the commutativity of the following diagram for any $\varphi \in \mathrm{Hom}_E(E_n^n, M(G)^*)$ and any $e \in E$:

$$\begin{array}{ccc}
\varphi & \longmapsto & \varphi(_ \cdot e) \\
\downarrow & & \downarrow \\
\varphi([1]) & \longmapsto \; e \cdot \varphi([1]) \; = & \varphi([e]).
\end{array}$$

Finally, the following commutative diagram gives the independence of n:

$$\begin{array}{ccccl}
\mathrm{Hom}(G^*, W_n^n) & \xhookrightarrow{iv\circ(\cdot)} & \mathrm{Hom}(G^*, W_{n+1}^{n+1}) & & \\
\wr\Vert & & \wr\Vert & & \text{Theorem 25.3 (b,c)} \\
\mathrm{Hom}(G^*, (W_n^n)^*) & \xhookrightarrow{(fr)^*\circ(\cdot)} & \mathrm{Hom}(G^*, (W_{n+1}^{n+1})^*) & & \\
\wr\Vert & & \wr\Vert & & \text{Functoriality of Cartier duality} \\
\mathrm{Hom}(W_n^n, G) & \xhookrightarrow{(\cdot)\circ fr} & \mathrm{Hom}(W_{n+1}^{n+1}, G) & & \\
\wr\Vert & & \wr\Vert & & \text{Functoriality of } M \\
\mathrm{Hom}_E(M(G), M(W_n^n)) & \xhookrightarrow{M(fr)\circ(\cdot)} & \mathrm{Hom}_E(M(G), M(W_{n+1}^{n+1})) & & \\
\wr\Vert & & \wr\Vert & & \text{Lemma 26.6} \\
\mathrm{Hom}_E(M(G), E_n^n) & \xhookrightarrow{[p]\circ()} & \mathrm{Hom}_E(M(G), E_{n+1}^{n+1}) & & \\
\wr\Vert & & \wr\Vert & & \text{Functoriality of } (_)^* \\
\mathrm{Hom}_E((E_n^n)^*, M(G)^*) & \xhookrightarrow{(\cdot)\circ[p]^*} & \mathrm{Hom}_E((E_{n+1}^{n+1})^*, M(G)^*) & & \\
\wr\Vert & & \wr\Vert & & \text{Lemma 26.8 (d)} \\
\mathrm{Hom}_E(E_n^n, M(G)^*) & \xhookrightarrow{(\cdot)\circ[1]} & \mathrm{Hom}_E(E_{n+1}^{n+1}, M(G)^*) & & \\
\wr\Vert & & \wr\Vert & & \text{evaluation at [1]} \\
M(G)^* & \xrightarrow{\mathrm{id}} & M(G)^* & &
\end{array}$$