Lecture 13

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§27 The Dieudonné functor in the étale case

Let E act on W_n from the left hand side, where F and V act as such and $\xi \in W(k)$ through multiplication by $\sigma^{-n}(\xi)$. Then the monomorphisms $v : W_n \hookrightarrow W_{n+1}$ are E-equivariant (compare Prop. 23.1). Also, the W_n^m form a fundamental system of infinitesimal neighborhoods of zero in all W_n. Thus for G local-local the functor M of §23 can be described equivalently as $M(G) = \varinjlim_n \operatorname{Hom}(G, W_n)$. Using this latter description we now prove a similar result for reduced-local groups:

Theorem 27.1. The functor $G \longmapsto M(G) = \varinjlim_n \operatorname{Hom}(G, W_n)$ induces an anti-equivalence of categories:

$$\left\{\left\{ \begin{array}{l} \text{finite commutative} \\ \text{étale group schemes} \\ \text{over } k \text{ of } p\text{-power order} \end{array} \right\}\right\} \xrightarrow{\sim} \left\{\left\{ \begin{array}{l} \text{left } E\text{-modules of} \\ \text{finite length with} \\ F \text{ an isomorphism} \end{array} \right\}\right\}.$$

Moreover, $\operatorname{length}_{W(k)} M(G) = \log_p |G|$.

Remark. The target category can be identified with the category of finite length $W(k)$-modules N together with a σ-linear automorphism $F : N \to N$, because V is determined by the relation $V = pF^{-1}$.

Remark. In [DG70] and [Fo77] the above theorem is proved jointly with the local-local case and using the same kind of reductions. But it also ties up nicely with descent and Lang's theorem, which have an independent interest, and which I want to describe.

Theorem 27.2 (Lang's Theorem). Let k be an algebraically closed field of positive characteristic. Let G be a connected algebraic group of finite type over k, and $F : G \to G$ a homomorphism with $dF = 0$. Then the map

$$G(k) \longrightarrow G(k), \qquad g \longmapsto g^{-1} \cdot F(g)$$

is surjective.

Proof. For any $g \in G(k)$ the morphism $G \to G$, $h \mapsto h^{-1} g F(h)$ has derivative $-\operatorname{id}$ everywhere, which is surjective; hence this morphism is dominant. As G is connected, the image contains an open dense subset $U_g \subseteq G$. The same holds in particular with $g = 1$. It follows that $U_g \cap U_1 \neq \emptyset$, and therefore there exist $h, \tilde{h} \in G(k)$ with $h^{-1} g F(h) = \tilde{h}^{-1} F(\tilde{h})$. Thus $g = h\tilde{h}^{-1} F(\tilde{h}) F(h)^{-1} = (\tilde{h}h^{-1})^{-1} \cdot F(\tilde{h}h^{-1})$, as desired. ◻

Proposition 27.3. Let k be an algebraically closed field of positive characteristic. Let N be a $W(k)$-module of finite length together with a σ-linear automorphism $F : N \to N$. Then

$$N^F \ := \ \{ n \in N \mid Fn = n \}$$

is a finite commutative p-group, and the natural homomorphism

$$W(k) \otimes_{\mathbb{Z}_p} N^F \ \longrightarrow N\,, \quad x \otimes n \ \longmapsto \ xn$$

is an isomorphism. In particular $\mathrm{length}_{W(k)} N \ = \ \log_p |N^F|$.

Proof. Consider first the special case $N = W_n(k)$ with $F = \sigma$. In this case we have

$$N^F = W_n(k^F) = W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z},$$

from which the claim obviously follows. The same follows for direct sums of modules of this type. In the general case, the proposition amounts to showing that every N is isomorphic to such a direct sum, because the desired isomorphism $W(k) \otimes_{\mathbb{Z}_p} N^F \to N$ is equivariant with respect to $\sigma \otimes \mathrm{id}$ on the source and F on the target.

To identify N with such a direct sum, we begin with any isomorphism of $W(k)$-modules

$$\varphi : \bigoplus_{i=1}^{r} W_{n_i}(k) \xrightarrow{\sim} N.$$

Via this the endomorphism ring

$$\underline{\mathrm{End}}_{W(k)} N \cong \bigoplus_{i,j=1}^{r} W_{\min\{n_i,n_j\}\,,\,k}$$

can be viewed as a unitary ring scheme over k. As a scheme it is isomorphic to an affine space of some dimension over k; in particular it is irreducible. Its group of units $G := \underline{\mathrm{Aut}}_{W(k)} N$ is an open subscheme in it; hence G is a connected algebraic group over k. The given σ-linear automorphism F then has the form $\varphi g \sigma \varphi^{-1}$ for some $g \in G(k)$. By Lang's theorem applied to the Frobenius on G we can write $g = h^{-1} \cdot \sigma(h)$ for some $h \in G(k)$. Thus

$$F = \varphi h^{-1} \sigma(h) \sigma \varphi^{-1} \ = \ (\varphi h^{-1}) \sigma (h \varphi^{-1}) \ = \ (\varphi h^{-1}) \sigma (\varphi h^{-1})^{-1}\,,$$

which means that φh^{-1} is the desired F-equivariant isomorphism. $\square$

Proof of Theorem 27.1 for k algebraically closed: In this case the source category is equivalent to the category of finite commutative p-groups Γ, and the functor gives:

$$\Gamma \longmapsto \underline{\Gamma}_k \longmapsto \varinjlim_n \operatorname{Hom}(\underline{\Gamma}_k, W_n)\,.$$

The latter group is equal to $\varinjlim_n \operatorname{Hom}(\Gamma, W_n(k))$, which in turn is isomorphic to

$$\operatorname{Hom}\big(\Gamma, W(k)\big[\tfrac{1}{p}\big]/W(k)\big) \;\cong\; W(k) \otimes_{\mathbb{Z}_p} \operatorname{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p).$$

We note that $\operatorname{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontrjagin dual of Γ, and the action of F corresponds to the action of $\sigma \otimes \mathrm{id}$ on $W(k) \otimes_{\mathbb{Z}_p} \operatorname{Hom}(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$. By Proposition 27.3 this gives the desired anti-equivalence and the formula for the length.

Proof of Theorem 27.1 in general: Let $\bar{k}$ be an algebraic closure of k. Then we have (anti-)equivalences of categories:

$$\begin{array}{ccc}
\left\{\!\!\left\{\begin{array}{l}\text{finite commutative}\\ \text{étale group schemes}\\ \text{over } k \text{ of } p\text{-power order}\end{array}\right\}\!\!\right\} & \xrightarrow{G \mapsto M(G)} & \left\{\!\!\left\{\begin{array}{l}\text{finite length } W(k)\text{-}\\ \text{modules with a } \sigma\text{-linear}\\ \text{automorphism } F\end{array}\right\}\!\!\right\} \\
\cong \Big\downarrow G \mapsto G_{\bar{k}} & & \cong \Big\uparrow N \mapsto N^{\mathrm{Gal}(\bar{k}/k)} \\
\left\{\!\!\left\{\begin{array}{l}\text{finite commutative étale}\\ \text{group schemes over } \bar{k} \text{ of}\\ p\text{-power order with a con-}\\ \text{tinuous } \mathrm{Gal}(\bar{k}/k)\text{-action}\end{array}\right\}\!\!\right\} & \xrightarrow[G_{\bar{k}} \mapsto M(G_{\bar{k}})]{\cong} & \left\{\!\!\left\{\begin{array}{l}\text{finite length } W(\bar{k})\text{-mod-}\\ \text{ules with a } \sigma\text{-linear auto-}\\ \text{morphism } F \text{ and a con-}\\ \text{tinuous } \mathrm{Gal}(\bar{k}/k)\text{-action}\end{array}\right\}\!\!\right\}.
\end{array}$$

In fact, the vertical arrows are equivalences by descent, and the lower horizontal arrow is an anti-equivalence by Theorem 27.1 for $\bar{k}$, where it is proven already, and the functoriality of $M(\ )$ under automorphisms of $\bar{k}$. Since

$$M(G_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \;=\; \varinjlim_n \operatorname{Hom}(G_{\bar{k}}, W_{n,\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \;=\; \varinjlim_n \operatorname{Hom}(G, W_n) \;=\; M(G),$$

the whole diagram commutes, and therefore the upper horizontal arrow is an anti-equivalence, too. Finally the formula for the length is preserved by descent, because

$$\operatorname{length}_{W(k)} M(G) \;=\; \operatorname{length}_{W(\bar{k})} W(\bar{k}) \otimes_{W(k)} M(G) \;=\; \operatorname{length}_{W(\bar{k})} M(G_{\bar{k}})\,,$$

and we are done.

Caution. In general $\operatorname{length}_{W(k)} M(G) \neq \operatorname{length}_E M(G)$, although for local-local G the equality does hold. The point is that all simple local-local G have order p, but not the simple étale ones.

Example. Let $G(\bar{k}) \cong \mathbb{F}_p^r$ with an irreducible action of the absolute Galois group $\operatorname{Gal}(\bar{k}/k)$. Then $M(G)$ must be a simple E-module, i.e., we have $M(G) \cong k^r$ with an irreducible F-action.

§28 The Dieudonné functor in the general case

Recall from Theorems 15.5 and 17.1 that any finite commutative group scheme of p-power order has a unique decomposition

$$G \,=\, G_{r\ell} \oplus G_{\ell r} \oplus G_{\ell\ell}\,.$$

In §23 and §27 we have already defined $M(G_{\ell\ell})$ and $M(G_{r\ell})$. Since $G_{\ell r}^*$ is of reduced-local type, we can define:

$$M(G) \,:=\, M(G_{r\ell}) \oplus M(G_{\ell r}^*)^* \oplus M(G_{\ell\ell})\,. \tag{28.1}$$

By construction this is a finite length left E-module, and by combining Theorem 27.1 and Propositions 23.3 (b) and 26.2, we deduce that

$$\operatorname{length}_{W(k)} M(G) = \log_p |G|.$$

Also, F and V are nilpotent on $M(G_{\ell\ell})$, and F is an isomorphism on $M(G_{r\ell})$. Since $FV = p$ in E, it follows that V is nilpotent on $M(G_{r\ell})$. The same holds for $M(G_{\ell r}^*)$, and so V is an isomorphism and F is nilpotent on $M(G_{\ell r}^*)^*$. In fact, such a decomposition exists for any finite length E-module:

Lemma 28.2. Every finite length left E-module has a unique and functorial decomposition

	$M =$	$M_{r\ell}$	$\oplus$	$M_{\ell r}$	$\oplus$	$M_{\ell\ell}$
where F is		isom.		nilpot.		nilpot.
where V is		nilpot.		isom.		nilpot.

Proof. The images of $F^n : M \to M$ form a decreasing sequence of E-submodules of M. Since M has finite length, this sequence stabilizes, say with $F^n M = M'$ for all $n \gg 0$. Then $F : M' \to M'$ is an isomorphism; hence $M' \cap \ker(F^n|_M) = 0$; and so by looking at the length we find that $M = M' \oplus \ker(F^n|_M)$. Repeating the same with V on $\ker(F^n|_M)$ we obtain the desired decomposition. Uniqueness and functoriality are clear. □

Recall from Theorem 26.3 that there is a functorial isomorphism $M(G_{\ell\ell}^*) \cong M(G_{\ell\ell})^*$. By construction this isomorphism extends to G. Altogether we have now proven:

Theorem 28.3. The functor M defined by (28.1) induces an anti-equivalence of categories

$$\left\{\!\!\left\{ \begin{array}{c} \text{finite commutative} \\ \text{group schemes over} \\ k \text{ of } p\text{-power order} \end{array} \right\}\!\!\right\} \xrightarrow{\sim} \left\{\!\!\left\{ \begin{array}{c} \text{left } E\text{-modules} \\ \text{of finite length} \end{array} \right\}\!\!\right\}.$$

Moreover $\text{length}_{W(k)} M(G) = \log_p |G|$, and there is a functorial isomorphism $M(G^*) \cong M(G)^*$.

Note. The definition $M(G_{\ell r}) := M(G_{\ell r}^*)^*$ looks somewhat artificial and cheap. But it is a fact that often one does need special arguments for $G_{\ell r}$ or $G_{r\ell}$. Nevertheless Fontaine [Fo77] uses a uniform definition of $M(G)$ for all cases, basically using a combination of the W_n with the formal group scheme $\widehat{W}$ from §25.

In principle, since M is an equivalence of categories, all properties of G can be read off from $M(G)$. We end with an example:

Proposition 28.4. There is a natural isomorphism

$$T_{G,0} \cong \big(M(G)/FM(G)\big)^*.$$

Proof. It suffices to show this in each of the cases $G = G_{r\ell}$, $G_{\ell r}$, and $G_{\ell\ell}$. In the first case $T_{G,0} = 0$ and F is an isomorphism on $M(G)$, and so both sides vanish. In the other two cases we have by Proposition 13.1

$$T_{G,0} \cong \text{Hom}(G^*, \mathbb{G}_{a,k}) = \text{Hom}(G^*, W_1).$$

Since $M(G^*) = \varinjlim_n \text{Hom}(G^*, W_n)$ and $W_1 = \ker(V|_{W_n})$ for all $n \geq 1$, the latter is

$$\ker(V|_{M(G^*)}) = \ker(V|_{M(G)^*}) = \text{coker}(F|_{M(G)})^*,$$

as desired. $\square$