Lecture 3

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§6 Actions and quotients in a category

Our goal is to define the notions of group actions and quotients in a general category. Let $\mathscr{C}$ be a category with arbitrary finite products.

Definition. A *(left) action* of a group object G on an object X is a morphism $m : G \times X \to X$ such that for all objects $Z \in \mathrm{Ob}(\mathscr{C})$, the map

$$G(Z) \times X(Z) = (G \times X)(Z) \xrightarrow{m \circ (\ )} X(Z)$$

is a left action of the group $G(Z)$.

We do not distinguish between the use of m for the group operation in G and for the action of G on X.

Equivalent definition. A *(left) action* is equivalent to the commutativity of the following two diagrams. The first expresses associativity of the action:

$$\begin{tikzcd}
G \times G \times X \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"'] & G \times X \arrow[d, "m"] \\
G \times X \arrow[r, "m"] & X.
\end{tikzcd}$$

The second says that the unit element acts as the identity:

$$\begin{tikzcd}
* \times X \arrow[r, "e\times\mathrm{id}"] \arrow[d, "pr_2"', "\wr"] & G \times X \arrow[dl, "m"] \\
X & .
\end{tikzcd}$$

Now we turn our attention to quotients.

Definition. A morphism $X \to Y$ is *G-invariant* if and only if for all $Z \in \mathrm{Ob}(\mathscr{C})$, the map

$$X(Z) \xrightarrow{f\circ()} Y(Z)$$

is G-invariant.

Fact. The G-invariance is equivalent to requiring the diagram

$$\begin{tikzcd} G \times X \arrow[r, "m"] \arrow[d, "pr_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

to be commutative.

Definition. A *categorical quotient* of X by G is a G-invariant morphism $X \xrightarrow{\pi} Y$, such that for all objects Z and for all G-invariant morphisms $X \xrightarrow{f} Z$, there exists a unique morphism $g : Y \to Z$ such that $f = g \circ \pi$.

Fact. If a categorical quotient exists, it is unique up to unique isomorphism.

We usually call Y the quotient, with the morphism π being tacitly included, although it is really π that matters.

The categorical quotient is the only meaningful concept of quotient in a general category, although it doesn't necessarily have all of the "nice" properties we would like. For examples see the following section.

Next, recall that a morphism $X \xrightarrow{f} Y$ is a *monomorphism* if for all $Z \in \mathrm{Ob}(\mathscr{C})$, the map

$$\mathrm{Hom}(Z, X) \xrightarrow{f\circ()} \mathrm{Hom}(Z, Y)$$

is injective. The morphism f is an *epimorphism* if for all objects Z, the map

$$\mathrm{Hom}(Y, Z) \xrightarrow{()\circ f} \mathrm{Hom}(X, Z)$$

is injective.

Consider the morphism

$$\lambda : G \times X \xrightarrow{(m, pr_2)} X \times X,$$

which sends (g, x) to (gx, x). It is natural to call the action m *free* if λ is a monomorphism. If $X \xrightarrow{\pi} Y$ is a categorical quotient and if $\mathscr{C}$ has fiber products, there is a natural monomorphism $X \times_Y X \longrightarrow X \times X$, and one shows (exercise!) that λ factors through a unique morphism

$$\lambda' : G \times X \longrightarrow X \times_Y X.$$

Definition. Assume that the action is free. Then Y is called a *good quotient* if λ' is an isomorphism.

In the category of sets, the categorical quotient is simply the set of G-orbits. An action is free if and only if all stabilizers are trivial, and in this case the quotient is a good quotient.

§7 Quotients of schemes by finite group schemes, part I

We will assume that all schemes are affine of finite type over a field k. We are actually interested in finite schemes, but this added generality will not make things any more difficult for the time being.

Let $G = \operatorname{Spec} R$ act on $X = \operatorname{Spec} A$, i.e. $m : A \to R \otimes A$ is a unitary k-algebra homomorphism such that the duals of the above diagrams commute:

$$\begin{aligned}(m \otimes id) \circ m &= (id \otimes m) \circ m \\ (\epsilon \otimes 1) \circ m &= id.\end{aligned}$$

Then a function $a \in A = \operatorname{Hom}(X, \mathbb{A}^1_k)$ is G-invariant if and only if

$$m(a) = 1 \otimes a.$$

Set

$$B := A^G := \{a \in A \mid m(a) = 1 \otimes a\}$$

and $Y := \operatorname{Spec} B$. By direct application of the definitions one obtains this easy theorem:

Theorem. $X \to Y$ is a categorical quotient of X by G in the category of affine schemes over k.

Example. Let $G = \mathbb{G}_{m,k}$ act on $\mathbb{A}^n_k$ by $t(x_1, \ldots, x_n) := (tx_1, \ldots, tx_n)$. Then $A = k[X_1, \ldots, X_n]$ implies that $B = k$, so we might write "$\mathbb{A}^n_k/\mathbb{G}_{m,k}$"$=$ $\operatorname{Spec} k$. We use the quotes because this quotient does not have the nice properties we desire. For example, its dimension is smaller than expected. The reason for this is that the orbit structure for the action is "bad": The closure of every orbit contains the origin, and so every fiber of π contains the origin; hence π is constant and Y is a point. Thus this quotient is not good.

Example. Now take $U := \mathbb{G}_{m,k} \times \mathbb{A}^{n-1}_k$, which is a G-invariant open subset of $\mathbb{A}^n_k$. Write

$$U = \operatorname{Spec} k[x_1^{\pm 1}, x_2, \ldots, x_n] = \operatorname{Spec} k\Big[x_1^{\pm 1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\Big].$$

Then "$U/\mathbb{G}_{m,k}$"$=$ $\operatorname{Spec} k[\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}] \cong \mathbb{A}^{n-1}_k$ is a good quotient. In fact, the union of copies of such $\mathbb{A}^{n-1}_k$ make up $\mathbb{P}^{n-1}_k$, the categorical quotient of $\mathbb{A}^n_k \smallsetminus \{0\}$ by $\mathbb{G}_{m,k}$ in the category of all schemes. But although $U \subset \mathbb{A}^n_k$ is open, the induced morphism "$U/\mathbb{G}_{m,k}$"$\longrightarrow$"$\mathbb{A}^n_k/\mathbb{G}_{m,k}$" is no longer an open embedding!

From now on let G be finite, and let $\pi : X \longrightarrow Y$ be as above.

Theorem 7.1. (a) $\pi : X \longrightarrow Y$ is finite and surjective.

(b) The topological space underlying Y is the quotient of X by the equivalence relation induced by G.

(c) $\mathscr{O}_Y \xrightarrow{\sim} (\pi_* \mathscr{O}_X)^G$.

Proof. (See [Mu70] Section 12, Theorem 1) The main point is to show that every element $a \in A$ is integral over B. For this we need to find a monic equation satisfied by a. Define a norm map $N : A \to A$ by

$$N(a) := \mathrm{Nm}_{(R\otimes A)/A}(m(a)),$$

where we identify A with $1 \otimes A$. The right side is defined as the determinant over $1 \otimes A$ of the endomorphism "multiply by $m(a)$" of $R \otimes A$, where we use the fact that $\dim_k R$ is finite.

Lemma. $N(a) \in B$.

Sketch of the proof. To show that $N(a)$ is invariant under translation by $G(k)$, one notes simply that this translation induces an automorphism of A that is compatible with the comultiplication m. In general, one must do the same for translation by $G(Z)$ for all Z, or equivalently for translation by the universal element $\mathrm{id} \in G(G)$ after tensoring with another copy of R. The proof is written out in [Mu70], pp. 112-3.

Lemma. A is integral over B.

Proof. We apply the previous lemma to $X \times \mathbb{A}^1_k$ in place of X, where G acts trivially on $\mathbb{A}^1_k$. For its coordinate ring $A[T]$ we deduce

$$N(A[T]) \subset (A[T])^G = B[T].$$

For all $a \in A$, the element

$$\chi_a(T) := N(T-a) = \det_A\big((T - m(a)\cdot \mathrm{id})|R \otimes A\big) \in B[T]$$

is a monic polynomial of degree $\dim_k R$. The identity map on A decomposes as

$$\begin{tikzcd} A \arrow[r, "m"] & R \otimes A \arrow[r, "\epsilon\otimes\mathrm{id}"] \arrow[loop below, "m(a)"] & A \arrow[loop below, "a"] \end{tikzcd},$$

where the self-maps denote multiplication by $m(a)$ and a, respectively. Thus

$$\chi_a(a) = \det_A\big((\mathrm{id} \otimes a - m(a)) \cdot \mathrm{id}\,|R \otimes A\big) = 0,$$

and so a is integral over B. ∎

Now we can prove (a). Suppose that A is generated by $a_1, \ldots, a_n$ as a k-algebra. Let $C \subset B$ be the subalgebra generated by the coefficients of all $\chi_{a_i}(T)$. Then A is integral over C. Thus A is of finite type as a C-module. Since C is a finitely generated k-algebra, it is noetherian. Therefore the C-submodule $B \subset A$ is itself of finite type as a C-module. This implies that B is a finitely generated k-algebra. Finally A is also a B-module of finite type. Since $B \subset A$, the morphism $X \to Y$ is thus finite surjective, as desired.

We turn to (b). For $x \in X$, the image (as a set) of the map $G \times \{x\} \xrightarrow{m} X$ is the G-orbit Gx of x. Using the commutative diagram for associativity, one can show that any two distinct orbits are disjoint. Let Gx and Gy be two disjoint orbits. After possibly interchanging x and y, none of the points in Gx specializes to a point in Gy. In this case there exists a function $a \in A$ that vanishes identically on Gx but is invertible on Gy. This in turn implies that $N(a) \in B$ vanishes on $\pi(x)$ but is invertible on $\pi(y)$. Thus π separates G-orbits. Since π is finite, hence closed, and is also continuous, this implies that Y has the quotient topology, proving (b).

To show (c) note that for any open subset $V \subset Y$ we have

$$(\pi_*\mathscr{O}_X)(V) = \mathscr{O}_X\big(\pi^{-1}(V)\big) = \mathrm{Hom}\big(\pi^{-1}(V), \mathbb{A}^1_k\big),$$

and a function f in this set is G-invariant if and only if $m(f) = 1 \otimes f$. Thus the subsheaf of all G-invariant functions $(\pi_*\mathscr{O}_X)^G$ is the kernel of the homomorphism of sheaves

$$\pi_*\mathscr{O}_X \to R \otimes_k \pi_*\mathscr{O}_X,\ f \mapsto m(f) - 1 \otimes f.$$

As these sheaves are coherent sheaves of $\mathscr{O}_Y$-modules, the kernel is the coherent sheaf associated to the kernel of the homomorphism of B-modules

$$A \longrightarrow R \otimes A,\ a \mapsto m(a) - 1 \otimes a.$$

By definition this kernel is B; hence its associated sheaf is $\mathscr{O}_Y$, as desired. $\square$