TANNAKA DUALITY FOR COALGEBRAS OVER COMMUTATIVE RINGS

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Introduction

Loosely speaking, Tannaka duality concerns the study of the relationship between a group-like object G (ordinary group, compact topological group, algebraic group, group scheme, quantum group, etc.) and its category of representations $\mathbf{Rep}(G)$. This category is naturally equipped with additional structures, the most basic being the forgetful functor which sends a representation of G to its underlying space. That is, for every group-like object G there is a triple $(\mathbf{Rep}(G), V, S)$ where V denotes the forgetful functor and S stands for unspecified additional structures (e.g. a monoidal structure). The question arises whether or not it is possible to go in the other direction: Is there a way to associate a group-like object to a category equipped with suitable additional structures? In other words, is there a way to assign a group-like object $E_{(\mathscr{A},\omega,S)}$ to a category $\mathscr{A}$ equipped with structures S and a functor ω from $\mathscr{A}$ to the category of spaces:

$$\begin{tikzcd}
\boxed{\text{Group-like objects}} \arrow[r, bend left, "\mathbf{Rep}(-)"] & \boxed{\begin{array}{c}\text{categories equipped with}\\ \text{suitable structures}\end{array}} \arrow[l, bend left, "E_{(-)}"]
\end{tikzcd}$$

If such a construction exists, there are three natural questions one would like to answer.

(1) The reconstruction problem: If one starts with a group-like object G and then applies $E_{(-)}$ to the associated category of representations, is the resulting group-like object isomorphic to G?
(2) The recognition problem: Is it possible to give a characterization of those triples $(\mathscr{A}, \omega, S)$ which are equivalent to $(\mathbf{Rep}(G), V, S)$ for some group-like object G?
(3) The description problem: Given a category $\mathscr{A}$ with structures S, is it possible to find conditions for the existence a functor ω from $\mathscr{A}$ into the category of spaces such that $(\mathscr{A}, \omega, S)$ is equivalent to $(\mathbf{Rep}(G), V, S)$ for some group-like object G?

We are interested in the case where the group-like objects are the affine group schemes over some commutative Ring R. For R a field, the above questions were discussed by Deligne (see [Del90]), and his approach was generalized by Wedhorn (see [Wed04]) to the case of Dedekind rings.

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It turns out that, in this context, it is very convenient to use the duality between spaces and their algebras of functions: the category of affine schemes over R is equivalent to the opposite of the category of R-algebras. In particular, an affine group scheme G over R corresponds to a certain R-algebra H. The multiplication of the group scheme gives a *comultiplication* on H, the unit turns into a *counit*, and the homomorphism of R-algebras which corresponds to the morphism of schemes which sends an element of G to its inverse is called the *antipode*. Such an algebra is called a commutative Hopf algebra. Moreover, an action of G on an R-module M corresponds to a *coaction* of H on M; a module equipped with a coaction will be called an *H-comodule*. For R a field, one has the following correspondence between structures on the vector space H and structures on the category of finite dimensional comodules, where the structure on the left is required for the existence of the structure on the right:

comultiplication and counit $\rightsquigarrow$ necessary for the definition of comodules
multiplication and unit $\rightsquigarrow$ tensor product of comodules
antipode $\rightsquigarrow$ duals

This suggests that, in a first step, one should stick to the minimal structure required for defining comodules, i.e., to *coalgebras*.

Definition. An *R-coalgebra* is an R-module C together with a comultiplication $\delta\colon C \to C \otimes C$ and a counit $\varepsilon\colon C \to R$ such that the diagrams

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes C \arrow[r, "\delta\otimes C"] & (C \otimes C) \otimes C \arrow[d, "a"] \\
C \otimes C \arrow[rr, "C\otimes\delta"'] & & C \otimes (C \otimes C)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& C \arrow[dl, "r^{-1}"'] \arrow[d, "\delta"] \arrow[dr, "l^{-1}"] & \\
C \otimes R & C \otimes C \arrow[l, "C\otimes\varepsilon"] \arrow[r, "\varepsilon\otimes C"'] & R \otimes C
\end{tikzcd}$$

are commutative. A *(left) comodule* of C is an R-module M together with a *coaction* $\rho\colon M \to C \otimes M$, that is, a homomorphism $\rho\colon M \to C \otimes M$ of R-modules such that the diagrams

$$\begin{tikzcd}
M \arrow[rr, "\rho"] \arrow[d, "\rho"'] & & C \otimes M \arrow[d, "\delta\otimes M"] \\
C \otimes M \arrow[r, "M\otimes\rho"'] & C \otimes (C \otimes M) \arrow[r, "a^{-1}"'] & (C \otimes C) \otimes M
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & C \otimes M \arrow[d, "\varepsilon\otimes M"] \\
M & R \otimes M \arrow[l, "l"]
\end{tikzcd}$$

are commutative.

Next we have to decide what our category of representations should be. It turns out that replacing 'finite dimensional' by 'finitely generated' does not give the category we want, for the following reason. We will eventually be interested in the reconstruction of Hopf algebras instead of mere coalgebras, and the existence of duals in the category of representations is crucial for the reconstruction of the antipode map of the Hopf algebra (see [Str07], section 16). But for arbitrary commutative rings, a comodule of a Hopf algebra has a dual if and only if its underlying module is *Cauchy*, i.e., finitely generated and projective (see [Str07], proposition 10.6). In order to apply the reconstruction results for Hopf algebras from [Str07] we should therefore ask the following question: Is it possible to reconstruct a coalgebra C from the category of comodules whose underlying module M is Cauchy? Such a comodule will be called a *Cauchy comodule*, and the category of Cauchy comodules of C will be denoted by $\mathbf{Comod}^c(C)$.

The statements at the beginning of the introduction have precise counterparts in this context. We denote the category of coalgebras by $\mathbf{Coalg}_R$, and we write $\mathbf{cat}_R \,/\, \mathbf{Mod}_R^c$ for the category whose objects are the pairs $(\mathscr{A}, \omega)$ consisting of an

R-linear category $\mathscr{A}$ together with an R-linear functor $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^c$ from $\mathscr{A}$ into the category of Cauchy modules. The morphisms in $\mathbf{cat}_R \,/\, \mathbf{Mod}_R^c$ between $(\mathscr{A}, \omega)$ and $(\mathscr{A}', \omega')$ are the R-linear functors $F\colon \mathscr{A} \to \mathscr{A}'$ which make the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "F"] \arrow[dr, "\omega"'] & & \mathscr{A}' \arrow[dl, "\omega'"] \\
& \mathbf{Mod}_R^c &
\end{tikzcd}$$

commutative. The assignment $C \mapsto (\mathbf{Comod}^c(C), V)$ which sends a coalgebra C to the pair consisting of the category $\mathbf{Comod}^c(C)$ and the forgetful functor $V\colon \mathbf{Comod}^c(C) \to \mathbf{Mod}_R^c$ naturally extends to a functor

$$\mathbf{Coalg}_R \to \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c\,,$$

and it turns out that this functor has a left adjoint $\mathbf{E}_{(-)}$:

$$\begin{tikzcd}
\mathbf{Coalg}_R \arrow[rr, bend left, "\mathbf{Comod}^c(-)"] & & \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c\,. \arrow[ll, bend left, "\mathbf{E}_{(-)}"]
\end{tikzcd}$$

That is, there is a bijection

$$\mathbf{Coalg}_R(\mathbf{E}_{(\mathscr{A},\omega)}, C) \to \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c\big((\mathscr{A},\omega), (\mathbf{Comod}^c(C), V)\big),$$

natural in C and $(\mathscr{A}, \omega)$, between the morphisms of coalgebras $\mathbf{E}_{(\mathscr{A},\omega)} \to C$ and R-linear functors $F\colon \mathscr{A} \to \mathbf{Comod}^c(C)$ with $VF = \omega$. In particular, the identity morphism of $\mathbf{E}_{(\mathscr{A},\omega)}$ corresponds under the above bijection to a functor $N\colon \mathscr{A} \to \mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)})$ such that $VN = \omega$. This functor is called the *unit* of the adjunction. Similarly, the identity functor of $\mathbf{Comod}^c(C)$ corresponds to a morphism of coalgebras $\varepsilon\colon E_{(\mathbf{Comod}^c(C),V)} \to C$, the *counit* of the adjunction. The reconstruction and recognition problems mentioned at the beginning of the introduction can now be turned into precise mathematical statements:

(1) Reconstruction problem: Under which conditions on the R-coalgebra C is the counit

$$\varepsilon\colon\ \mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$$

an isomorphism?

(2) Recognition problem: For which pairs $(\mathscr{A}, \omega)$ is the unit

$$N\colon \mathscr{A} \to \mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)})$$

an equivalence of categories?

The goal of this paper is to give answers to the above questions. We succeed in giving a necessary and sufficient condition for (1), and we can provide a sufficient condition for (2). It is well-known that for $R = k$ a field, the counit morphism ε is always an isomorphism ([Str07], proposition 16.3). The proof uses the fact that any k-coalgebra C, considered as a comodule over itself, is the union of its Cauchy subcomodules. It is not to be expected that the same result holds over arbitrary rings. However, a union is a special case of the more general notion of a colimit, and it turns out that the latter can be used to give a necessary and sufficient condition for ε to be an isomorphism over arbitrary rings.

Namely, instead of considering a diagram which consists only of inclusions of Cauchy subcomodules we consider a diagram built from *all* morphisms of comodules $\varphi\colon M \to C$ whose domain M is a Cauchy comodule. This diagram is called the *diagram of Cauchy comodules over C*. Under certain conditions, C (considered as a C-comodule) is the colimit of the canonical diagram of Cauchy comodules over C, for example if C itself is Cauchy. We say that C has *enough Cauchy comodules* if for every C-comodule M and every element $m \in M$ there is a Cauchy comodule N and a morphism $\varphi \colon N \to M$ such that m lies in the image of φ.

Proposition 2.6.4. *If C is flat and has enough Cauchy comodules, then C is the colimit of the canonical diagram of Cauchy comodules over C.*

Theorem 2.6.5. *The counit $\varepsilon \colon \mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$ is an isomorphism if and only if C, considered as a C-comodule, is the colimit of the canonical diagram of Cauchy comodules over C.*

It follows immediately that the counit ε is an isomorphism if C itself is Cauchy, or if C is flat and has enough Cauchy comodules.

Open question. *Are there any examples of coalgebras for which the counit ε is* not *an isomorphism?*

Next we turn to our result concerning the recognition problem.

Theorem 3.4.3. *Let $\mathscr{A}$ be a small additive R-linear category, and let $\omega \colon \mathscr{A} \to \mathbf{Mod}_R^c$ be an R-linear functor. If*

i) ω is flat, and

ii) ω reflects colimits in $\mathbf{Mod}_R$,

then the unit $N \colon \mathscr{A} \to \mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)})$ is fully faithful. If in addition

iii) ω reflects those colimits in $\mathbf{Mod}_R$ which are Cauchy modules,

then N is an equivalence of categories.

The functor ω is called *flat* if a certain other functor $\operatorname{Lan}_Y \omega$ associated to ω (the left Kan extension of ω, see section 1.5) is left exact, just as a module is flat if the associate functor $M \otimes -$ is flat. This condition is quite strong: it implies for example that the coalgebra $\mathbf{E}_{(\mathscr{A},\omega)}$ is flat.

Proposition 3.4.2. *The forgetful functor $V \colon \mathbf{Comod}^c(C) \to \mathbf{Mod}_R^c$ is flat if C is flat and has enough Cauchy comodules.*

Open question. *Does flatness of the coalgebra C imply flatness of the forgetful functor $V \colon \mathbf{Comod}^c(C) \to \mathbf{Mod}_R^c$?*

On the other hand, the properties *ii)* and *iii)* hold whenever the unit N is an equivalence. Explanations of these properties can be found in section 1.2.

In order to prove these two theorems we use various concepts and results from category theory. Our main source is Kelly's book 'Basic concepts of enriched category theory' [Kel82]. In section 1 we introduce those concepts which we will need later. Since we work with R-linear categories instead of the more general $\mathscr{V}$-categories from [Kel82] we can give elementary proofs for all the facts we need later. Another important tool are pasted composites of natural transformations (see section 1.3), which were introduced in [KS74]. The proof of the recognition theorem uses the concepts of locally presentable categories and accessible categories. We cite the necessary results from [AR94] in section 3. The fact that the above functor has a left adjoint follows from [Str07]; here we give a different construction, which uses an embedding of the category of R-modules in the category of endofunctors $\mathbf{Mod}_R \to \mathbf{Mod}_R$.

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CONTENTS

1. PRELIMINARIES

1.1. **Overview.** This chapter contains the material from category theory which we need for the proofs of our main results. In section 1.2 we introduce some terminology for colimits, and we define R-linear categories. The main purpose of this section is to fix notations. We proceed with introducing the pasted composites and mates from [KS74] in section 1.3. The former is a notation for handling composites of natural transformations, and we frequently use this in subsequent sections. The notion of a mate under adjunction is important for the proof of our reconstruction result.

In the remainder of this chapter we give definitions of some concepts from [Kel82], adapted to the special case of R-linear categories. In section 1.4 we introduce tensor products between modules and objects in an arbitrary objects in an R-linear category. The key result from this section is corollary 1.4.4. To motivate the introduction of left Kan extensions and dense functors we mention the following observation. In section 2.3 we will see that for any coalgebra C the forgetful functor $V\colon \mathbf{Comod}(C) \to \mathbf{Mod}_R$ from the category of *all* comodules to the category of R-modules has a left adjoint $W\colon \mathbf{Mod}_R \to \mathbf{Comod}(C)$; and that VW is isomorphic to $C \otimes -\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ (cf. proposition 2.3.1). In other words, reconstructing the coalgebra C is equivalent to reconstructing a certain left adjoint R-linear functor $L\colon \mathscr{C} \to \mathbf{Mod}_R$ from a subcategory of $\mathscr{C}$. This problem is analogous to the problem of reconstructing a group from a set of generators; and for this, the notion of a free group plays an important role. The goal of the sections 1.5 and 1.6 is the construction of the counterpart of a free group in the context of R-linear categories and R-linear functors. More precisely, we will show that the Yoneda embedding can be interpreted as a 'free cocompletion' (see corollary 1.6.5).

1.2. **Colimits, adjoint functors and R-linear categories.** First we fix some notations. We use script letters $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, … for categories, capital roman letters A, A', … for objects of $\mathscr{A}$ and lower case letters for morphisms. Functors are usually denoted by capital letters F, G, H, etc. A category $\mathscr{A}$ is called *small* if the objects of $\mathscr{A}$ form a set.

Given two functors $F, G \colon \mathscr{A} \to \mathscr{B}$, a *natural transformation* α from F to G is a family $(\alpha_A)_{A \in \mathscr{A}}$ such that for every morphism $f \colon A \to A'$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd}
FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\
FA' \arrow[r, "\alpha_B"'] & GA'
\end{tikzcd}$$

is commutative. We call α_A the *A-component* of the natural transformation α, and we usually denote natural transformations by double arrows $\alpha \colon F \Rightarrow G$. If $\mathscr{A}$ is small, then the natural transformations from F to G constitute a set, which we denote by $\mathrm{Nat}(F, G)$.

In our constructions we will frequently use colimits, and we introduce some terminology for handling them. A functor $D \colon \mathscr{D} \to \mathscr{C}$ is called a *diagram of shape $\mathscr{D}$ in $\mathscr{C}$*. A *cocone* on D is a pair $(A, (\kappa_d)_{d \in \mathscr{D}})$ consisting of an object A of $\mathscr{C}$ and a family of morphisms $\kappa_d \colon D(d) \to A$, $d \in \mathscr{D}$, such that for every morphism $f \colon d \to d'$, the diagram

$$\begin{tikzcd}
D(d) \arrow[rr, "D(f)"] \arrow[dr, "\kappa_d"'] & & D(d') \arrow[dl, "\kappa_{d'}"] \\
& A &
\end{tikzcd}$$

is commutative. A cocone $(A, (\kappa_d)_{d \in \mathscr{D}})$ on D is called a *colimit cocone* if for any other cocone $(X, (\xi_d)_{d \in \mathscr{D}})$ on D there is a unique morphism $\varphi \colon A \to X$ such that $\xi_d = \varphi \kappa_d$ for all $d \in \mathscr{D}$. In this situation we say that A is the *colimit* [1] of D, and we call the κ_d the *structure morphisms* of the colimit. Since a colimit, if it exists, is unique up to unique isomorphism, these structure morphisms are sometimes omitted from the notation, and we write $A = \mathrm{colim}^{\mathscr{D}} D$ or $A = \mathrm{colim}_{d \in \mathscr{D}} D(d)$ to express the fact that there exists a family $(\kappa_d)_{d \in \mathscr{D}}$ such that $(A, (\kappa_d)_{d \in \mathscr{D}})$ is a colimit cocone. On the other hand, if we want to emphasize the role of the structure morphisms we say that the morphisms $\kappa_d \colon D(d) \to A$ *exhibit* A as colimit of D to express the fact that $(A, (\kappa_d)_{d \in \mathscr{D}})$ is a colimit cocone.

A diagram of shape $\mathscr{D}$ is called *small* if the objects of $\mathscr{D}$ form a set. A category $\mathscr{C}$ is called *cocomplete* if for all small diagrams D, the colimit of D exists. A functor $F \colon \mathscr{A} \to \mathscr{B}$ is called *cocontinuous* if F *preserves* colimits of small diagrams, meaning that for every small diagram $D \colon \mathscr{D} \to \mathscr{A}$, if $(A, (\kappa_d)_{d \in \mathscr{D}})$ is a colimit cocone on D, then $(FA, (F(\kappa_d))_{d \in \mathscr{D}})$ is a colimit cocone on FD. The functor F *reflects* colimits (of a certain shape $\mathscr{D}$) if, whenever $(FA, (F(\kappa_d))_{d \in \mathscr{D}})$ is a colimit cocone on FD, then the κ_d exhibit A as colimit of D. We say that F *creates* colimits if, whenever the $\xi_d \colon FD(d) \to B$ exhibit B as colimit of FD, there exists a unique cocone $(A, (\kappa_d)_{d \in \mathscr{D}})$ on D such that $FA = B$, $F(\kappa_d) = \xi_d$ for every $d \in \mathscr{D}$ and $(A, (\kappa_d)_{d \in \mathscr{D}})$ is a colimit cocone. The dual notions of colimits, cocompleteness and cocontinuity are *limits*[2] *completeness* and *continuity.* They are less important for what we will do, hence we do not spell out the definitions explicitly.

A functor $F \colon \mathscr{A} \to \mathscr{B}$ is called *left adjoint* to a functor $G \colon \mathscr{B} \to \mathscr{A}$ if for every

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[1] Colimits are sometimes called *inductive limits* or *direct limits* in the literature.

[2] Limits are also called *projective limits* or *inverse limits.*

$A \in \mathscr{A}$ and every $B \in \mathscr{B}$ there are bijections

$$\varphi_{A,B} \colon \mathscr{B}(FA, B) \longrightarrow \mathscr{A}(A, GB)$$

which are natural in A and B. We say that F and G form an *adjoint pair*, and we denote this by $F \dashv G \colon \mathscr{A} \to \mathscr{B}$. If we let $\eta_A := \varphi_{A,FA}(\mathrm{id}_{FA})$ and $\varepsilon_B := \varphi^{-1}_{GB,B}(\mathrm{id}_{GB})$ we get natural transformations $\eta \colon \mathrm{id} \Rightarrow GF$ and $\varepsilon \colon FG \Rightarrow \mathrm{id}$, called the *unit* and *counit* of the adjunction $F \dashv G$. These natural transformations satisfy the equations

$$\varepsilon_{FA} \circ F(\eta_A) = \mathrm{id}_{FA} \quad \text{and} \quad G(\varepsilon_B) \circ \eta_{GB} = \mathrm{id}_{GB}$$

for every $A \in \mathscr{A}$, $B \in \mathscr{B}$. These are called the *triangular identities*, and giving the natural isomorphism φ is equivalent to giving two natural transformations η and ε satisfying these identities. A full subcategory $\mathscr{C}$ of a category $\mathscr{A}$ is called a *reflective subcategory* if the inclusion functor $i \colon \mathscr{C} \to \mathscr{A}$ has a left adjoint $r \colon \mathscr{A} \to \mathscr{B}$.
From now on we fix a commutative, associative ring R with unit 1. We denote the category of R-modules by $\mathbf{Mod}_R$, and we write $[-,-]$ for the internal hom of $\mathbf{Mod}_R$ (i.e., $[M, N]$ is the R-module of homomorphisms $M \to N$). A category $\mathscr{A}$ is called *R-linear* if the hom-sets $\mathscr{A}(A, B)$ are endowed with the structure of an R-module in such a way that the composition maps

$$\circ_{A,B,C} \colon \mathscr{A}(B, C) \times \mathscr{A}(A, B) \to \mathscr{A}(A, C)$$

are R-bilinear. Note that with our definition biproducts need not exist in an R-linear category $\mathscr{A}$; we say that $\mathscr{A}$ is *additive* if it does have biproducts. An *R-linear functor* $T : \mathscr{A} \to \mathscr{B}$ between R-linear categories $\mathscr{A}, \mathscr{B}$ is a functor $T \colon \mathscr{A} \to \mathscr{B}$ such that the maps

$$T_{A,A'} \colon \mathscr{A}(A, A') \to \mathscr{B}(TA, TA')$$

given by $f \mapsto T(f)$ are homomorphisms of R-modules. An R-linear category $\mathscr{A}$ is called *small* if the objects of $\mathscr{A}$ form a set. We denote the category of small R-linear categories and R-linear functors between them by $\mathbf{cat}_R$.

Definition 1.2.1. An R-module M is called a *Cauchy module* if it is finitely generated and projective. We let $\mathbf{Mod}^c_R$ be the full subcategory of $\mathbf{Mod}_R$ generated by those submodules R^k, $k \in \mathbb{N}$, which are Cauchy modules. We let

$$\mathbf{cat}_R \,/\, \mathbf{Mod}^c_R$$

be the category with objects the pairs $(\mathscr{A}, \omega)$ of small R-linear categories $\mathscr{A}$ together with an R-linear functor $\omega \colon \mathscr{A} \to \mathbf{Mod}^c_R$, and morphisms $(\mathscr{A}, \omega) \to (\mathscr{A}', \omega')$ given by those R-linear functors $F \colon \mathscr{A} \to \mathscr{A}'$ which make the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "F"] \arrow[dr, "\omega"'] & & \mathscr{A}' \arrow[dl, "\omega'"] \\
& \mathbf{Mod}^c_R &
\end{tikzcd}$$

commutative.

Remark 1.2.1. Note that with the above definition, the objects of $\mathbf{Mod}^c_R$ form a set, i.e., $\mathbf{Mod}^c_R$ is a small category. On the other hand, the full subcategory of $\mathbf{Mod}_R$ consisting of all Cauchy comodules is not small: there is already a proper class of modules which are isomorphic to the zero module. The two categories are of course equivalent.

Given two R-linear categories $\mathscr{A}$ and $\mathscr{B}$, we can construct a new R-linear category $\mathscr{A} \otimes \mathscr{B}$, the *tensor product* of $\mathscr{A}$ and $\mathscr{B}$. The objects of $\mathscr{A} \otimes \mathscr{B}$ are the pairs

(A,B) of an object A of $\mathscr{A}$ and an object B of $\mathscr{B}$, and the R-module of morphisms $(A,B) \to (A',B')$ is given by

$$(\mathscr{A} \otimes \mathscr{B})\big((A,B),(A',B')\big) := \mathscr{A}(A,A') \otimes \mathscr{B}(B,B').$$

The composition is given by $(f \otimes g, f' \otimes g') \mapsto ff' \otimes gg'$. Giving an R-linear functor

$$F \colon \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$$

is equivalent to giving an ordinary bifunctor $F_0 \colon \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ which is R-bilinear, meaning that for every $f \colon A \to A'$ and every $g \colon B \to B'$, the induced functions

$$F_0(f,-) \colon \mathscr{B}(B,B') \to \mathscr{C}\big(T(A,B),T(A',B')\big)$$

and

$$F_0(-,g) \colon \mathscr{A}(A,A') \to \mathscr{C}\big(T(A,B),T(A',B')\big)$$

are R-linear.

Let $\mathscr{A}$ be a small R-linear category, let $\mathscr{B}$ be an arbitrary R-linear category and let $F,G \colon \mathscr{A} \to \mathscr{B}$ be two R-linear functors. The set $\mathrm{Nat}(F,G)$ is naturally endowed with the structure of an R-module: given two natural transformations $\alpha \colon F \Rightarrow G$, $\beta \colon F \Rightarrow G$ and an element $r \in R$, we let $(\alpha+\beta)_A := \alpha_A + \beta_A$ and $(r \cdot \alpha)_A := r \cdot \alpha_A$. The fact that the composition in $\mathscr{C}$ is R-linear immediately implies that $\alpha + \beta$ and $r \cdot \alpha$ are natural transformations, and it is clear that this gives the structure of an R-module on $\mathrm{Nat}(F,G)$. We denote this module by

$$[\mathscr{A},\mathscr{B}](F,G)$$

to distinguish it from its underlying set $\mathrm{Nat}(F,G)$. If $H \colon \mathscr{A} \to \mathscr{B}$ is another R-linear functor, the usual composition of natural transformations

$$\circ \colon [\mathscr{A},\mathscr{B}](G,H) \times [\mathscr{A},\mathscr{B}](F,G) \longrightarrow [\mathscr{A},\mathscr{B}](F,H)$$

given by $(\alpha,\beta) \mapsto (\alpha \circ \beta)_A := (\alpha_A \circ \beta_A)$ is clearly R-bilinear. We conclude that the category of R-linear functors $\mathscr{A} \to \mathscr{B}$ and natural transformations between them is again R-linear. We denote this category by $[\mathscr{A},\mathscr{B}]$.

Recall that the Yoneda lemma says that for any functor $F \colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$, the map

$$\mathrm{Nat}(\mathscr{A}(-,A),F) \longrightarrow FA$$

which sends α to $\alpha_A(\mathrm{id}_A)$ is a natural bijection. Given an element $a \in FA$, we denote the unique natural transformation $\alpha \colon \mathscr{A}(-,A) \Rightarrow F$ with $\alpha_A(\mathrm{id}_A) = a$ by $\overline{a} := \alpha$. It is not difficult to see that if $\mathscr{A}$ is a small R-linear category and $F \colon \mathscr{A} \to \mathbf{Mod}_R$ is an R-linear functor, the assignment $\alpha \mapsto \alpha_A(\mathrm{id}_A)$ gives an isomorphism of R-modules

$$[\mathscr{A},\mathbf{Mod}_R](\mathscr{A}(-,A),F) \xrightarrow{\cong} FA.$$

It follows in particular that the Yoneda embedding gives a fully faithful R-linear functor $Y \colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]$, where $Y(A) = \mathscr{A}(-,A)$.

1.3. **Pasted composites and mates under adjunction.** In category theory it is common to summarize the situation 'f is a morphism with domain A and codomain B' by the picture

$$\begin{tikzcd} A \arrow[r, "f"] & B \end{tikzcd}$$

and to denote the composite of a collection of morphisms f_i with domain A_{i-1} and codomain A_i, $i = 1,\ldots,n$, by the chain

$$\begin{tikzcd} A_0 \arrow[r, "f_1"] & A_1 \arrow[r, "f_2"] & A_2 \arrow[r] & \cdots \arrow[r] & A_{n-1} \arrow[r, "f_n"] & A_n. \end{tikzcd}$$

In the special case of a category of functors and natural transformations between them, there is another useful notation. A natural transformation η with domain F and codomain G is an 'arrow between arrows'. The picture

$$F \xrightarrow{\alpha} G$$

does not fully capture the situation, for F and G have domains and codomains, too. Therefore we use the notation

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathscr{B} \arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

to say that α is a natural transformation from the functor $F : \mathscr{A} \to \mathscr{B}$ to the functor $G : \mathscr{A} \to \mathscr{B}$. At first this might seem rather clumsy; it is for example difficult to arrange such expressions in a commutative diagram. The strength of the notation only becomes apparent once we introduce the basic pasting operations.

Definition 1.3.1. Given functors and a natural transformation as in the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] & \mathscr{B} \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathscr{C} \arrow[r, "K"] & \mathscr{D}, \arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

we denote the natural transformation from KGF to KHF with A-component given by $K\alpha_{FA}$ by

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "KGF"{name=U}] \arrow[r, bend right, "KHF"'{name=D}] & \mathscr{D}. \arrow[from=U, to=D, Rightarrow, "K\alpha F"]
\end{tikzcd}$$

We say that $K\alpha F$ is obtained by *whiskering* η by K and F.
If we have functors and natural transformations as in the diagram

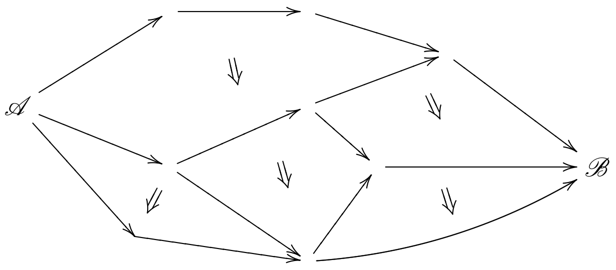
$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M}] \arrow[r, bend right=60, "H"'{name=D}] & \mathscr{B}, \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\beta"]
\end{tikzcd}$$

we call the natural transformation from F to H with A-component given by by the composite $FA \xrightarrow{\alpha_A} GA \xrightarrow{\beta_A} HA$ the *vertical composite* of α and β, and we denote it by

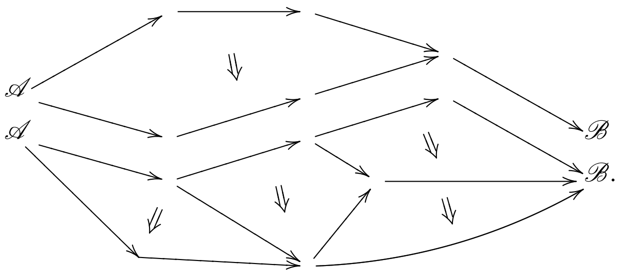
$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathscr{B}. \arrow[from=U, to=D, Rightarrow, "\beta\circ\alpha"]
\end{tikzcd}$$

Whiskering and vertical composition are the two *basic pasting operations.*

Now, given a diagram of functors and natural transformations such as



we can use the basic pasting operations to get a natural transformation going from the composite of the functors on the top of the diagram to the composite of those on the bottom as follows: First, we choose any natural transformation whose domain is contained in the top chain of the diagram. Then we 'split' the diagram along the codomain of this natural transformation:



We proceed by whiskering the diagram on the top to obtain a natural transformation between functors with domain $\mathscr{A}$ and codomain $\mathscr{B}$, and then iterate this whole process with the rest of the diagram. The *pasted composite* of the diagram is the vertical composite of the resulting collection of natural transformations. Usually this process involves choices, namely whenever there are several natural transformations whose domains are contained in the top chain of the diagram. It is a consequence of the following proposition that the resulting composite is independent of these choices.

Proposition 1.3.1. *Given functors and natural transformations as in the diagram*

$$\begin{tikzcd}\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathscr{C},\end{tikzcd}$$

the vertical composites

$$\begin{tikzcd}\mathscr{A} \arrow[rr, bend left=40, "HF"] \arrow[rr, "KF"] \arrow[rr, bend right=40, "KG"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow\mu F"] \arrow[rr, phantom, shift right=3ex, "\Downarrow K\eta"] & & \mathscr{C}\end{tikzcd} = \begin{tikzcd}\mathscr{A} \arrow[r, "F"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathscr{C} \\ & \circ & \\ \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{B} \arrow[r, "K"] & \mathscr{C}\end{tikzcd}$$

and

$$\begin{tikzcd}\mathscr{A} \arrow[rr, bend left=40, "HF"] \arrow[rr, "HG"] \arrow[rr, bend right=40, "KG"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow H\eta"] \arrow[rr, phantom, shift right=3ex, "\Downarrow\mu G"] & & \mathscr{C}\end{tikzcd} = \begin{tikzcd}\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C} \\ & \circ & \\ \mathscr{A} \arrow[r, "G"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathscr{C}\end{tikzcd}$$

are equal. In other words, the pasted composite of the diagram

$$\begin{tikzcd}\mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathscr{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathscr{C}\end{tikzcd}$$

is well-defined. We call this the horizontal composite *of η and μ, and denote it by $\mu * \eta$.*

Proof. We have to check that for every object A of $\mathscr{A}$, the A-components of the two natural transformations are equal. By definition 1.3.1 these are given by $K(\eta_A) \circ \mu_{FA}$ and $\mu_{GA} \circ H(\eta_A)$ respectively. But the diagram

$$\begin{tikzcd}
H(FA) \arrow[r, "H(\eta_A)"] \arrow[d, "\mu_{FA}"'] & H(GA) \arrow[d, "\mu_{GA}"] \\
K(FA) \arrow[r, "K(\eta_A)"'] & K(GA)
\end{tikzcd}$$

is commutative by naturality of μ, which shows that the A-components are indeed equal. □

With this proposition one could prove by induction that for *any* diagram to which the pasting operation described above can be applied, the resulting composite does not depend on any choices. However, in order to do this we would first have to give a formal definition of such 'admissible' diagrams, which would complicate matters needlessly. In all the examples we will consider it will be evident that the general pasting operation can be applied, and using proposition 1.3.1 it will be easy to see that the pasting composite is well-defined.

We now introduce a useful convention from [KS74]: Demanding that a diagram of functors

$$\begin{tikzcd}
\cdot \arrow[r, "F"] \arrow[d, "H"'] & \cdot \arrow[d, "G"] \\
\cdot \arrow[r, "K"'] & \cdot
\end{tikzcd}$$

be commutative is equivalent to demanding that one can place the identity natural transformation in the diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\swarrow 1_{GF}"] & \cdot \arrow[d, "G"] \\
\cdot \arrow[r, "K"'] & \cdot .
\end{tikzcd}$$

Therefore we introduce the following convention: when we compute the pasted composite of a diagram of categories, functors and natural transformations between them, if the diagram has parts containing no natural transformation, these parts will be commutative. Moreover, these parts are treated as if the identity natural transformation would stand there.

Now we are ready to give some applications of the pasting operation. Given an adjoint pair of functors $F \dashv G \colon \mathscr{A} \to \mathscr{B}$ with unit $\eta\colon \text{id} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \text{id}$, the triangular identities are equivalent to the two equations

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "\text{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathscr{A} \arrow[dr, "F"] & \\
& \mathscr{B} \arrow[ur, "G"] \arrow[rr, bend right, "\text{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & \mathscr{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{A} \arrow[r, "\text{id}"] \arrow[d, "F"'] & \mathscr{A} \arrow[d, "F"] \\
\mathscr{B} \arrow[r, "\text{id}"'] & \mathscr{B}
\end{tikzcd}$$

and

$$\begin{tikzcd} & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathscr{A} \\ \mathscr{B} \arrow[ur, "G"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & \mathscr{B} \arrow[ur, "G"'] \end{tikzcd} = \begin{tikzcd} \mathscr{B} \arrow[r, "\mathrm{id}"] \arrow[d, "G"'] & \mathscr{B} \arrow[d, "G"] \\ \mathscr{A} \arrow[r, "\mathrm{id}"'] & \mathscr{A}. \end{tikzcd}$$

between pasted composites.

Proposition 1.3.2. *Given adjunctions $F \dashv G \colon \mathscr{A} \to \mathscr{B}$, $F' \dashv G' \colon \mathscr{A} \to \mathscr{B}$, the assignments*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\sigma"] & \mathscr{B} \end{tikzcd} \mapsto \begin{tikzcd} \mathscr{B} \arrow[r, bend left, "G'"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\overline{\sigma}"] & \mathscr{A} \end{tikzcd} := \begin{tikzcd} & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "F'"'] \arrow[dr, phantom, "\Swarrow\sigma"] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathscr{A} \\ \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\varepsilon'\Downarrow"] & & \mathscr{B} \arrow[ur, bend right, "G"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{B} \arrow[r, bend left, "G'"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\tau"] & \mathscr{A} \end{tikzcd} \mapsto \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\overline{\tau}"] & \mathscr{B} \end{tikzcd} := \begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend right, "F'"'] \arrow[rr, phantom, "\eta'\Downarrow"] & & \mathscr{A} \arrow[dr, bend left, "F"] \\ & \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Searrow\tau"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & \mathscr{B} \end{tikzcd}$$

are mutually inverse, that is, they give a bijection between the set of natural transformations $F \Rightarrow F'$ and the set of natural transformations $G' \Rightarrow G$. The natural transformation $\overline{\sigma}$ is called the mate *of σ, and $\overline{\tau}$ is called the mate of τ.*

Proof. The triangular identities imply that the equalities

$$\begin{tikzcd} \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend right, "F'"'] \arrow[rr, phantom, "\Downarrow\eta'"] & & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "F'"'] \arrow[dr, phantom, "\Swarrow\sigma"] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathscr{A} \arrow[dr, bend left, "F"] \\ & \mathscr{B} \arrow[ur, "G'"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\varepsilon'\Downarrow"] & & \mathscr{B} \arrow[ur, "G"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & \mathscr{B} \end{tikzcd} = \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "\mathrm{id}"] \arrow[dr, bend right, "F'"'] & \mathscr{A} \arrow[r, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "F'"'] \arrow[dr, phantom, "\Swarrow\sigma"] & \mathscr{A} \arrow[dr, bend left, "F"] \\ & \mathscr{B} \arrow[r, bend right, "\mathrm{id}"'] & \mathscr{B} \arrow[r, bend right, "\mathrm{id}"'] & \mathscr{B} \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F'"] \arrow[rr, phantom, "\eta'\Downarrow"] & & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"] \arrow[rr, phantom, "\Downarrow\eta"] & & \mathscr{A} \\ \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon'"] & & \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Searrow\tau"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow\varepsilon"] & & \mathscr{B} \arrow[ur, bend right, "G"'] \end{tikzcd} = \begin{tikzcd} & \mathscr{A} \arrow[r, bend left, "\mathrm{id}"] & \mathscr{A} \arrow[r, bend left, "\mathrm{id}"] & \mathscr{A} \\ \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[r, bend right, "\mathrm{id}"'] & \mathscr{B} \arrow[ur, bend left, "G'"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Searrow\tau"] \arrow[r, bend right, "\mathrm{id}"'] & \mathscr{B} \arrow[ur, bend right, "G"'] \end{tikzcd}$$

hold, which shows that the given maps are indeed mutually inverse. ☐

Proposition 1.3.3. *Let $F \dashv G \colon \mathscr{A} \rightleftarrows \mathscr{B}$ and $F' \dashv G' \colon \mathscr{A} \rightleftarrows \mathscr{B}$ be adjoint pairs, and let $\sigma \colon F \Rightarrow F'$ and $\tau \colon G' \Rightarrow G$ be mates under adjunction (i.e. $\tau = \overline{\sigma}$ and $\sigma = \overline{\tau}$). Then the equations*

$$\tau F' \circ \eta' = G\sigma \circ \eta \quad \textit{and} \quad \varepsilon \circ F\tau = \varepsilon' \circ \sigma G$$

hold, and for every $A \in \mathscr{A}$, $B \in \mathscr{B}$ the diagram

$$\begin{tikzcd}
\mathscr{B}(F'A,B) \arrow[r, "\varphi'"] \arrow[d, "\mathscr{B}(\sigma_A{,}B)"'] & \mathscr{A}(A,G'B) \arrow[d, "\mathscr{A}(A{,}\tau_B)"] \\
\mathscr{B}(FA,B) \arrow[r, "\varphi"'] & \mathscr{A}(A,GB)
\end{tikzcd}$$

is commutative, where φ and φ are the natural isomorphisms induced by the given units and counits. Moreover, if $F'' \dashv G'' \colon \mathscr{A} \rightleftarrows \mathscr{B}$ is another adjoint pair and if $\rho \colon F' \Rightarrow F''$ is a further natural transformation, then

$$\overline{\rho \circ \sigma} = \overline{\sigma} \circ \overline{\rho}.$$

If all the above categories and functors are R linear, then the formation of mates is R-linear, i.e.,

$$\overline{x \cdot \sigma_0 + y \cdot \sigma_1} = x \cdot \overline{\sigma_0} + y \cdot \overline{\sigma_1}$$

for all natural transformations $\sigma_i \colon F \Rightarrow F'$ and all elements $x, y \in R$.

Proof. The equality $\tau = \overline{\sigma}$ means that τ is the pasted composite of the diagram

$$\begin{tikzcd}
& \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left=40, "F", ""{name=F1, below}] \arrow[dr, bend right=40, "F'"', ""{name=Fp1, above}] & & \mathscr{A} \\
\mathscr{B} \arrow[ur, bend left, "G'"] \arrow[rr, bend right, "\mathrm{id}"'] & & \mathscr{B} \arrow[ur, bend right, "G"'] &
\arrow[Rightarrow, from=F1, to=Fp1, "\sigma"]
\arrow[from=1-2, to=1-4, phantom, yshift=-3ex, "\Downarrow\eta"]
\arrow[from=2-1, to=2-3, phantom, yshift=3ex, "\varepsilon'\Downarrow"]
\end{tikzcd}$$

and it follows by a triangular identity that

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F'"'] & & \mathscr{A} \\
& \mathscr{B} \arrow[ur, bend left=40, "G'", ""{name=Gp, below}] \arrow[ur, bend right=40, "G"', ""{name=G, above}] &
\arrow[Rightarrow, from=Gp, to=G, "\tau"]
\arrow[from=1-1, to=1-3, phantom, yshift=-3ex, "\eta'\Downarrow"]
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F'"'] & & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left=40, "F", ""{name=F2, below}] \arrow[dr, bend right=40, "F'"', ""{name=Fp2, above}] & & \mathscr{A} \\
& \mathscr{B} \arrow[ur, "G'"'] \arrow[rr, bend right, "\mathrm{id}"'] & & \mathscr{B} \arrow[ur, bend right, "G"'] &
\arrow[Rightarrow, from=F2, to=Fp2, "\sigma"]
\arrow[from=1-1, to=1-3, phantom, yshift=-3ex, "\eta'\Downarrow"]
\arrow[from=1-3, to=1-5, phantom, yshift=-3ex, "\Downarrow\eta"]
\arrow[from=2-2, to=2-4, phantom, yshift=3ex, "\varepsilon'\Downarrow"]
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left=40, "F", ""{name=F3, below}] \arrow[dr, bend right=40, "F'"', ""{name=Fp3, above}] & & \mathscr{A} \\
& \mathscr{B} \arrow[ur, bend right, "G"'] &
\arrow[Rightarrow, from=F3, to=Fp3, "\sigma"]
\arrow[from=1-1, to=1-3, phantom, yshift=-3ex, "\Downarrow\eta"]
\end{tikzcd}$$

holds. Spelling out the pasted composites of the diagram on the left and on the right of this equation yields

$$\tau F' \circ \eta' = G\sigma \circ \eta,$$

and the equation involving the counit can be derived in a similar fashion. Next we will prove that the diagram

$$\begin{tikzcd}
\mathscr{B}(F'A,B) \arrow[r, "\varphi'"] \arrow[d, "\mathscr{B}(\sigma_A{,}B)"'] & \mathscr{A}(A,G'B) \arrow[d, "\mathscr{A}(A{,}\tau_B)"] \\
\mathscr{B}(FA,B) \arrow[r, "\varphi"'] & \mathscr{A}(A,GB)
\end{tikzcd}$$

is indeed commutative. Recall that the natural transformations φ and φ' send morphisms $f : FA \to B$ and $f' : F'A \to B$ to $\varphi_{A,B}(f) = Gf \circ \eta_A$ and $\varphi'_{A,B}(f') =$

$G'f' \circ \eta'_A$ respectively. It follows that for any object A of $\mathscr{A}$ we have

$$\begin{aligned}\mathscr{A}(A, \tau_{F'A}) \circ \varphi'_{A,F'A}(\mathrm{id}_{FA}) &= \tau_{F'A} \circ \eta'_A \\ &= (\tau F' \circ \eta')_A \\ &= (G\sigma \circ \eta)_A \\ &= G(\sigma_A) \circ \eta_A \\ &= \varphi_{A,FA}(\sigma_A) \\ &= \varphi_{A,FA} \circ \mathscr{B}(\sigma_A, FA)(\mathrm{id}_{FA}),\end{aligned}$$

and by Yoneda it follows that for every $B \in \mathscr{B}$

$$\mathscr{A}(A, \tau_B) \circ \varphi'_{A,B} = \varphi_{A,B} \circ \mathscr{B}(\sigma_A, B),$$

as claimed. To see that forming mates is compatible with vertical composition we consider the composite $\overline{\sigma} \circ \overline{\rho}$. This composite is given by the pasted composite of the diagram

$$\begin{tikzcd}
& \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F'"] \arrow[dr, bend right, "F''"'] \arrow[dr, phantom, "\Swarrow\rho"] & \Downarrow\eta' & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "F'"'] \arrow[dr, phantom, "\Swarrow\sigma"] & \Downarrow\eta & \mathscr{A} \\
\mathscr{B} \arrow[ur, bend left, "G''"] \arrow[rr, bend right, "\mathrm{id}"'] & \varepsilon''\Downarrow & \mathscr{B} \arrow[ur, "G'"] \arrow[rr, bend right, "\mathrm{id}"'] & \varepsilon'\Downarrow & \mathscr{B} \arrow[ur, bend right, "G"'] &
\end{tikzcd}$$

and the triangular identities for F' and G' imply that this is equal to the pasted composite of

$$\begin{tikzcd}
& \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F'"] \arrow[dr, bend right, "F''"'] \arrow[dr, phantom, "\Swarrow\rho"] & & \mathscr{A} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, bend left, "F"] \arrow[dr, bend right, "F'"'] \arrow[dr, phantom, "\Swarrow\sigma"] & \Downarrow\eta & \mathscr{A} \\
\mathscr{B} \arrow[ur, bend left, "G''"] \arrow[rr, bend right, "\mathrm{id}"'] & \varepsilon''\Downarrow & \mathscr{B} \arrow[rr, bend right, "\mathrm{id}"'] & & \mathscr{B} \arrow[ur, bend right, "G"'] & .
\end{tikzcd}$$

On the other hand, the pasted composite of the latter diagram is clearly equal to $\overline{\rho \circ \sigma}$, which gives the desired equality

$$\overline{\rho \circ \sigma} = \overline{\sigma} \circ \overline{\rho}.$$

It remains to show that if all the involved categories and functors are R-linear, then forming mates is R-linear, too. But the mate of σ is given by

$$\overline{\sigma} = G\varepsilon \circ G\sigma G' \circ \eta G',$$

and vertically composing with a fixed natural transformation is clearly R-linear. Thus it suffices to show that the the whiskering operation is also R-linear, i.e., that

$$G(x \cdot \sigma_0 + y \cdot \sigma_1)G' = x \cdot G\sigma_0 G' + y \cdot G\sigma_1 G'$$

for all $x, y \in R$ and all $\sigma_i \colon F \Rightarrow F'$. Since addition and scalar multiplication of natural transformations are defined component wise, this follows directly from definition 1.3.1. $\square$

1.4. **Tensor products.** Given an object A in an R-linear category $\mathscr{C}$, the R-linear functor $\mathscr{C}(A, -) : \mathscr{C} \to \mathbf{Mod}_R$ preserves all limits which exist in $\mathscr{C}$. This is a necessary condition for a functor to have a left adjoint, and in a sufficiently nice setting the fact that a functor preserves all limits is equivalent to the existence of a left adjoint. This motivates part of the following definition.

Definition 1.4.1. i) For any object A in an R-linear category $\mathscr{C}$ and any R-module M we say that the *tensor product* of M and A exists if the functor $[M, \mathscr{C}(A,-)] \colon \mathscr{C} \to \mathbf{Mod}_R$ is representable, i.e., if there is an object $M \otimes A$ in $\mathscr{C}$ together with isomorphisms

$$\mathscr{C}(M \otimes A, C) \cong [M, \mathscr{C}(A, C)]$$

which are natural in C.

ii) An R-linear category is said to have *tensor products* if for all objects A of $\mathscr{C}$ and all R-modules M, the tensor product $M \otimes A$ exists. This implies that all representable functors have left adjoints, and we assume that for each $A \in \mathscr{C}$, a fixed left adjoint $- \otimes A : \mathbf{Mod}_R \to \mathscr{C}$ to $\mathscr{C}(A,-) \colon \mathscr{C} \to \mathbf{Mod}_R$ has been chosen. We denote the unit and counit of this adjunction $-\otimes \dashv \mathscr{C}(A,-)$ by η^A and ε^A respectively.

The notation is motivated by taking $\mathscr{C} = \mathbf{Mod}_R$. Then any representable functor $[A,-]$ is right adjoint to $- \otimes_R A : \mathbf{Mod}_R \to \mathbf{Mod}_R$. So $\mathbf{Mod}_R$ has tensor products, and they are given by the usual tensor product of R-modules.
Let $\mathscr{C}$ be an R-linear category with tensor products. Any morphism $f : A \to A'$ gives a natural transformation $\mathscr{A}(f,-) \colon \mathscr{A}(A',-) \Rightarrow \mathscr{A}(A,-)$, whose mate (see proposition 1.3.2) we denote by $- \otimes f \colon -\otimes A \Rightarrow - \otimes A'$. Thus $- \otimes f$ is given by the pasted composite of the diagram

$$\begin{tikzcd}
\mathbf{Mod}_R \arrow[r, "-\otimes A"] \arrow[rr, bend right=50, "\mathrm{id}"'] \arrow[rr, phantom, bend right=30, "\Downarrow \varepsilon"] & \mathscr{A} \arrow[r, bend left, "\mathscr{A}(A',-)"] \arrow[r, bend right, "\mathscr{A}(A,-)"'] \arrow[r, phantom, "\Downarrow \mathscr{A}(f,-)"] \arrow[rr, bend left=60, "\mathrm{id}"] \arrow[rr, phantom, bend left=35, "\Downarrow \eta'"] & \mathbf{Mod}_R \arrow[r, "-\otimes A'"'] & \mathscr{A}.
\end{tikzcd}$$

Proposition 1.3.3 implies that for $f' \colon A' \to A''$, we have $- \otimes f' \circ - \otimes f = - \otimes (f' \circ f)$, and the triangular identities imply that $- \otimes \mathrm{id}_A$ is the identity natural transformation. Naturality of $- \otimes f$ implies that for any homomorphism $\varphi \colon M \to M'$ of R-modules, the diagram

$$\begin{tikzcd}
M \otimes A \arrow[r, "M\otimes f"] \arrow[d, "\varphi\otimes M"'] & M \otimes A' \arrow[d, "\varphi\otimes A'"] \\
M' \otimes A \arrow[r, "M'\otimes f"'] & M' \otimes A'
\end{tikzcd}$$

is commutative. This shows that, whenever $\mathscr{C}$ has tensor products, we get an R-bifunctor $- \otimes - \colon \mathbf{Mod}_R \otimes \mathscr{C} \to \mathscr{C}$ (R-bilinearity of $- \otimes -$ follows by proposition 1.3.3).
We will later see that cocompleteness of $\mathscr{A}$ is sufficient for the existence of tensor products. For this we need the following results and constructions. Let $\mathscr{F}$ be the full subcategory of $\mathbf{Mod}_R$ generated by the modules R^n, $n \in \mathbb{N}$. For any module M we write $(\mathscr{F} \downarrow M)$ for the category with objects the homomorphisms $\sigma : R^n \to M$, and morphisms from $\sigma : R^n \to M$ to $\sigma' : R^m \to M$ the homomorphisms $\tau : R^n \to R^m$

which make the diagram

$$\begin{tikzcd}
R^n \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & R^m \arrow[dl, "\sigma'"] \\
& M &
\end{tikzcd}$$

commutative. We let D_M be the 'domain functor' $(\mathscr{F} \downarrow M) \to \mathbf{Mod}_R$, which sends an object $\sigma : R^n \to M$ to R^n and a morphism $\tau : \sigma \to \sigma'$ to itself.

Lemma 1.4.1. *The morphisms $\sigma : D_M(\sigma) \to M$ exhibit M as the colimit of the diagram D_M.*

Proof. Given a cocone $\eta_\sigma : D_M(\sigma) \to N$, we have to construct a homomorphism $\varphi : M \to N$ with $\varphi \circ \sigma = \eta_\sigma$ for every object $\sigma : R^n \to M$ of $(\mathscr{F} \downarrow M)$. For $m \in M$, we choose $\sigma : R^n \to M$ such that m lies in the image of σ. The we let $\varphi(m)$ be the element $\eta_\sigma(x)$, where x is any element of R^n with $\sigma(x) = m$. We first have to check that φ is well-defined. If $\sigma' : R^m \to M$ is another homomorphism and if $\sigma'(x') = m$, then the pair (x, x') is in the pullback $E = \{(a, b) \in R^n \times R^m; \sigma(a) = \sigma'(b)\}$. Let $\alpha : R \to E$ denote the morphism which sends 1 to (x, x'). With the notation $\tau = \mathrm{pr}_1 \circ \alpha$, $\tau' = \mathrm{pr}_2 \circ \alpha$ and $\sigma'' = \sigma \circ \tau = \sigma' \circ \tau'$, we get the commutative diagram

$$\begin{tikzcd}
& R^n \arrow[dr, "\sigma"] & \\
R \arrow[ur, "\tau"] \arrow[rr, "\sigma''"] \arrow[dr, "\tau'"'] & & M, \\
& R^m \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

and therefore $\eta_\sigma(x) = \eta_\sigma(\tau(1)) = \eta_{\sigma''}(1) = \eta_{\sigma'}(\tau'(1)) = \eta_{\sigma'}(x')$. This means that φ is a well-defined map, and R-linearity of φ immediately follows from this fact. □

We are now ready to prove that the existence of all colimits in $\mathscr{C}$ is sufficient for the existence of tensor products in $\mathscr{C}$.

Proposition 1.4.2. *Let $\mathscr{C}$ be a cocomplete R-linear category. Then $\mathscr{C}$ has tensor products.*

Proof. We fix an object $A \in \mathscr{C}$. It suffices to show that for each module M there is an object $C \in \mathscr{C}$ together with a morphism $\eta : M \to \mathscr{C}(A, C)$ such that for any $\beta : M \to \mathscr{C}(A, C')$ there is a unique morphism $\widetilde{\beta} : C \to C'$ making the diagram

$$\begin{tikzcd}
M \arrow[r, "\eta"] \arrow[dr, "\beta"'] & \mathscr{C}(A, C) \arrow[d, "\mathscr{C}(A,\widetilde{\beta})"] \\
& \mathscr{C}(A, C')
\end{tikzcd}$$

commutative.

We first consider the special case $M = R^n$ for some $n \in \mathbb{N}$. Let $\eta_n : R^n \to \mathscr{C}(A, A^n)$ be the unique morphism with $\eta_n(e_i) = \mathrm{in}_i$ for $i = 1, \ldots, n$. The pair (A^n, η_n) has the desired universal property; for if $C \in \mathscr{C}$ is any object with a homomorphism $\beta : R^n \to \mathscr{C}(A, C)$, the unique morphism $\widetilde{\beta} : A^n \to C$ with $\widetilde{\beta} \circ \mathrm{in}_i = \beta(e_i)$ makes the diagram

$$\begin{tikzcd}
R^n \arrow[r, "\alpha_n"] \arrow[dr, "\beta"'] & \mathscr{C}(A,A^n) \arrow[d, "\mathscr{C}(A{,}\widetilde{\beta})"] \\
& \mathscr{C}(A,C)
\end{tikzcd}$$

commutative, and it is clearly unique with this property. The assignment $R^n \mapsto A^n$ has a unique extension to a functor $F : \mathscr{F} \to \mathscr{C}$ if we demand that the diagram

$$\begin{tikzcd}
R^n \arrow[r, "\eta_n"] \arrow[d, "\tau"'] & \mathscr{C}(A,FR^n) \arrow[d, "\mathscr{C}(A{,}F\tau)"] \\
R^m \arrow[r, "\eta_m"] & \mathscr{C}(A,FR^m)
\end{tikzcd}$$

be commutative for every homomorphism $\tau : R^n \to R^m$. In fact it is quite simple to give an explicit description of $F\tau$. There are elements $a_{ij} \in R$ such that $\tau(e_i) = \sum_{j=0}^{m} a_{ij}e_j$, and $F\tau$ is the unique morphism $A^n \to A^m$ with $\mathrm{pr}_j \circ F\tau \circ \mathrm{in}_i = a_{ij} \cdot \mathrm{id}_A$. This shows in particular that $F : \mathscr{F} \to \mathscr{C}$ is R-linear.

Using lemma 1.4.1 it is now possible to construct an object C with the above mentioned universal property for an arbitrary module M. Namely, we let C be the colimit of the functor $FD_M : (\mathscr{F} \downarrow M) \to \mathscr{C}$, and we denote the structural morphisms by $\varphi_\sigma \colon FD_M(\sigma) \to C$. We let $\alpha_M : M \to \mathscr{C}(A,C)$ be the unique morphism which makes the diagrams

$$\begin{tikzcd}
R^n \arrow[r, "\eta_n"] \arrow[d, "\sigma"'] & \mathscr{C}(A,A^n) \arrow[d, "\mathscr{C}(FR{,}\varphi_\sigma)"] \\
M \arrow[r, "\eta_M"] & \mathscr{C}(A,C)
\end{tikzcd}$$

commutative for all $\sigma : R^n \to M$. To see that this map has the desired universal property we let $C' \in \mathscr{C}$ be any object with a homomorphism $\beta : M \to \mathscr{C}(A,C')$. Since the universal property is already established for the maps $\eta_n : R^n \to \mathscr{C}(A,A^n)$, we can conclude that for every $\sigma : R^n \to M$ there is a unique morphism $\xi_\sigma : A^n \to C'$ such that $\mathscr{C}(A,\xi_\sigma) \circ \eta_n = \beta \circ \sigma$. If $\sigma' : R^m \to M$ is another homomorphism, and if $\tau : R^n \to R^m$ makes the diagram

$$\begin{tikzcd}
R^n \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & R^m \arrow[dl, "\sigma'"] \\
& M &
\end{tikzcd}$$

commutative, we have $\xi_{\sigma'} \circ F(\tau) = \xi_\sigma$. Indeed, all the quadrilaterals in the diagram

$$\begin{tikzcd}
& R^n \arrow[r, "\eta_n"] \arrow[dl, "\tau"'] \arrow[dd, "\sigma"] & \mathscr{C}(A,A^n) \arrow[dl, "\mathscr{C}(A{,}F\tau)"] \arrow[dd, "\mathscr{C}(A{,}\xi_\sigma)"] \\
R^m \arrow[r, "\eta_m"] \arrow[dr, "\sigma'"'] & \mathscr{C}(A,A^m) \arrow[dr, "\mathscr{C}(A{,}\xi_{\sigma'})"] & \\
& M \arrow[r, "\beta"] & \mathscr{C}(A,C')
\end{tikzcd}$$

are commutative by definition of $F\tau$ and ξ_σ, $\xi_{\sigma'}$. The uniqueness part of the universal property of $\eta_n : R^n \to \mathscr{C}(A,A^n)$ now gives the desired equality $\xi_{\sigma'} \circ F(\tau) = \xi_\sigma$. This shows that the ξ_σ constitute a cocone on the diagram FD_M :

$(\mathscr{F} \downarrow M) \to \mathscr{C}$, so there is a unique morphism $\widetilde{\beta} : C \to C'$ with $\widetilde{\beta} \circ \varphi_\sigma = \xi_\sigma$. For any $\sigma : R^n \to M$, the equalities

$$\begin{aligned} \mathscr{C}(A, \widetilde{\beta}) \circ \eta_M \circ \sigma &= \mathscr{C}(A, \widetilde{\beta}) \circ \mathscr{C}(A, \varphi_\sigma) \circ \eta_n \\ &= \mathscr{C}(A, \xi_\sigma) \circ \eta_n \\ &= \beta \circ \sigma \end{aligned}$$

hold by definition of $\widetilde{\beta}$ and ξ_σ respectively. Since the $\sigma : R^n \to M$ are collectively epimorphic, we find that $\widetilde{\beta} : C \to C'$ is indeed the desired morphism. Uniqueness of $\widetilde{\beta}$ is immediate: if $\overline{\beta}$ is another morphism with $\mathscr{C}(A, \overline{\beta}) \circ \eta_M = \beta$, the diagram

$$\begin{tikzcd}
R^n \arrow[r, "\eta_n"] \arrow[d, "\sigma"'] & \mathscr{C}(A, A^n) \arrow[d, "\mathscr{C}(A,\varphi_\sigma)"] \\
M \arrow[r, "\eta_M"] \arrow[dr, "\beta"'] & \mathscr{C}(A, C) \arrow[d, "\mathscr{C}(A,\overline{\beta})"] \\
 & \mathscr{C}(A, C')
\end{tikzcd}$$

is commutative for all $\sigma \colon R^n \to M$, and it follows that $\overline{\beta} \circ \varphi_\sigma = \xi_\sigma$; and thus that $\overline{\beta} = \widetilde{\beta}$ because the morphism with this property is unique. $\square$

Proposition 1.4.3. *If $\mathscr{C}$ is any R-linear category and if $F : \mathbf{Mod}_R \to \mathscr{C}$ is an R-linear functor which preserves colimits, then F is left adjoint to the functor $\mathscr{C}(FR, -) : \mathscr{C} \to \mathbf{Mod}_R$, with unit $\eta^F : \mathrm{id} \Rightarrow \mathscr{C}(FR, F-)$ such that $\eta^F_{R^n} : R^n \to \mathscr{C}(FR, FR^n)$ sends e_i to $F(\mathrm{in}_i)$ for $i = 1, \ldots, n$.*

Proof. We need to construct a natural transformation $\eta : \mathrm{id} \Rightarrow \mathscr{C}(FR, F-)$ such that each component satisfies the usual universal property (for the sake of brevity we write η for η^F in this proof). First, we let $\alpha_n : R^n \to \mathscr{C}(FR, FR^n)$ be the unique homomorphism which sends e_i to $F(\mathrm{in}_i)$ for $i = 1, \ldots, n$. Given any module M, we let $\eta_M : M \to \mathscr{C}(FR, FM)$ be the unique morphism which makes the diagrams

$$\begin{tikzcd}
R^n \arrow[r, "\alpha_n"] \arrow[d, "\sigma"'] & \mathscr{C}(FR, FR^n) \arrow[d, "\mathscr{C}(FR,F\sigma)"] \\
M \arrow[r, "\eta_M"] & \mathscr{C}(FR, FM)
\end{tikzcd}$$

commutative for all $\sigma : R^n \to M$. Such a morphism exists by lemma 1.4.1. Choosing $M = R^n$ and $\sigma = \mathrm{id} : R^n \to R^n$ we immediately find that $\eta_{R^n} = \alpha_n$. To see that the η_M constitute a natural transformation, we let $\varphi : M \to N$ be an arbitrary homomorphism of modules. Then the upper square and the outer rectangle of the diagram

$$\begin{tikzcd}
R^n \arrow[r, "\alpha_n"] \arrow[d, "\sigma"'] & \mathscr{C}(FR, FR^n) \arrow[d, "\mathscr{C}(FR,F\sigma)"] \\
M \arrow[r, "\eta_M"] \arrow[d, "\varphi"'] & \mathscr{C}(FR, FM) \arrow[d, "\mathscr{C}(FR,F\varphi)"] \\
N \arrow[r, "\eta_N"] & \mathscr{C}(FR, FN)
\end{tikzcd}$$

are commutative for every $\sigma : R^n \to M$, by definition of η_M and η_N respectively. It follows that the equalities

$$\begin{aligned}\eta_N \circ \varphi \circ \sigma &= \mathscr{C}(FR, F\varphi) \circ \mathscr{C}(FR, F\sigma) \circ \alpha_n \\ &= \mathscr{C}(FR, F\varphi) \circ \eta_M \circ \sigma\end{aligned}$$

hold, and thus that $\eta_N \circ \varphi = \mathscr{C}(FR, F\varphi) \circ \eta_M$ since the $\sigma : R^n \to M$ are collectively epimorphic. It remains to show that for any object $C \in \mathscr{C}$ with a homomorphism $\beta : M \to \mathscr{C}(FR, C)$ there is a unique morphism $\widetilde{\beta} : FM \to C$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[rd, "\beta"'] & \mathscr{C}(FR, FM) \arrow[d, "\mathscr{C}(FR, \widetilde{\beta})"] \\ & \mathscr{C}(FR, C) \end{tikzcd}$$

is commutative. For any $\sigma : R^n \to M$, we let ξ_σ be the unique morphism $FR^n \to C$ with $\xi_\sigma \circ F(\mathrm{in}_i) = \beta \circ \sigma(e_i)$ for $i = 1, \ldots, n$. We claim that the ξ_σ constitute a cocone on the diagram $FD_M : (\mathscr{F} \downarrow M) \to \mathscr{C}$. To see this we have to show that for any commutative diagram

$$\begin{tikzcd} R^n \arrow[rr, "\tau"] \arrow[rd, "\sigma"'] & & R^m \arrow[ld, "\sigma'"] \\ & M & \end{tikzcd}$$

we have $\xi_{\sigma'} \circ F(\tau) = \xi_\sigma$. It suffices to check that the equalities

$$\xi_{\sigma'} \circ F(\tau) \circ F(\mathrm{in}_i) = \xi_\sigma \circ F(\mathrm{in}_i)$$

hold for every $i = 1, \ldots, n$. But the right hand side of the above equation is by definition equal to $\beta \circ \sigma(e_i) = \beta \circ \sigma'(\tau(e_i))$. There are $a_{ij} \in R$ such that $\tau(e_i) = \sum_{j=1}^m a_{ij} e_j$. This means that $\tau \circ \mathrm{in}_i = \sum_{j=1}^m a_{ij} \cdot \mathrm{in}_j$ and therefore that

$$F(\tau \circ \mathrm{in}_i) = \sum_{j=1}^m a_{ij} \cdot F(\mathrm{in}_j).$$

By R-linearity of $\mathscr{C}$ it follows that

$$\begin{aligned}\xi_{\sigma'} \circ F(\tau) \circ F(\mathrm{in}_i) &= \xi_{\sigma'} \circ \left(\sum_{j=1}^m a_{ij} \cdot F(\mathrm{in}_j) \right) \\ &= \sum_{j=1}^m a_{ij} \cdot \xi_{\sigma'} \circ F(\mathrm{in}_j) \\ &= \sum_{j=1}^m a_{ij} \cdot \beta \circ \sigma'(e_j) \\ &= \beta \circ \sigma' \left(\sum_{j=1}^m a_{ij} \cdot e_j \right) \\ &= \beta \circ \sigma'\big(\tau(e_i)\big),\end{aligned}$$

so the ξ_σ do indeed form a cocone on FD_M. Since F preserves colimits there is a unique morphism $\widetilde{\beta} : FM \to C$ such that $\widetilde{\beta} \circ F(\sigma) = \xi_\sigma$ for every $\sigma : R^n \to M$.

From this we conclude that

$$\begin{aligned}\mathscr{C}(FR,\widetilde{\beta}) \circ \eta_M \circ \sigma(e_i) &= \mathscr{C}(FR,\widetilde{\beta}) \circ \mathscr{C}(FR,F\sigma) \circ \alpha_n(e_i)\\ &= \mathscr{C}(FR,\widetilde{\beta}) \circ \mathscr{C}(FR,F\sigma)(F(\mathrm{in}_i))\\ &= \widetilde{\beta} \circ F(\sigma) \circ F(\mathrm{in}_i)\\ &= \xi_\sigma \circ F(\mathrm{in}_i)\\ &= \beta \circ \sigma(e_i)\end{aligned}$$

for $i = 1, \ldots, n$. This implies that $\mathscr{C}(FR,\widetilde{\beta}) \circ \eta_M = \beta$, and the result follows since $\widetilde{\beta}$ is clearly unique with this property. ☐

Corollary 1.4.4. *The functor* $\mathrm{ev}_R : \mathrm{Cocts}[\mathbf{Mod}_R, \mathscr{C}] \to \mathscr{C}$ *which sends a cocontinuous R-linear functor F to FR and a natural transformation* $\alpha \colon F \Rightarrow F'$ *to* $\alpha_R : FR \to F'R$ *is fully faithful. In particular, if* $\mathscr{C}$ *is cocomplete, this gives an equivalence of categories* $\mathrm{Cocts}[\mathbf{Mod}_R, \mathscr{C}] \simeq \mathscr{C}$.

Proof. The mate of α is a natural transformation $\overline{\alpha} : \mathscr{C}(F'R,-) \Rightarrow \mathscr{C}(FR,-)$. By Yoneda it must be of the form $\mathscr{C}(\varphi,-)$ for a unique morphism $\varphi : FR \to F'R$. Since the diagram

$$\begin{tikzcd} R \arrow[r, "\eta_R^{F'}"] \arrow[d, "\eta_R^F"'] & \mathscr{C}(F'R,F'R) \arrow[d, "\overline{\alpha}_{F'R}=\mathscr{C}(\varphi,F'R)"] \\ \mathscr{C}(FR,FR) \arrow[r, "\mathscr{C}(FR,\alpha_R)"] & \mathscr{C}(FR,F'R) \end{tikzcd}$$

is commutative (see proposition 1.3.3) it follows that $\varphi = \alpha_R$. This shows that ev_R is faithful. If $\varphi : FR \to F'R$ is any morphism, the natural transformation $\mathscr{C}(\varphi,-) : \mathscr{C}(F'R,-) \Rightarrow \mathscr{C}(FR,-)$ has a mate $\beta : F \to F'$, and the mate $\overline{\beta}$ of β is equal to $\mathscr{C}(\beta_R,-)$ by the above considerations. But the mate of a mate is the natural transformation itself (see proposition 1.3.2), so $\mathscr{C}(\beta_R,-) = \mathscr{C}(\varphi,-)$ and by Yoneda it follows that $\beta_R = \varphi$; in other words, that ev_R is full. ☐

Proposition 1.4.5. *Let* $\mathscr{C}$ *be an R-linear category, and let A be an object of* $\mathscr{C}$ *such that the tensor product* $-\otimes A$ *exists, with unit* $\eta : \mathrm{id} \Rightarrow \mathscr{C}(A,-\otimes A)$. *Then* $\eta_R : R \to \mathscr{C}(A,R\otimes A)$ *sends* $1 \in R$ *to an isomorphism* $A \to R \otimes A$. *We call this the canonical isomorphism* $\varphi^A : A \to R \otimes A$.

Proof. This follows immediately from the fact that the homomorphism $\alpha : R \to \mathscr{C}(A,A)$ which sends 1 to id_R has the same universal property as $\eta_R : R \to \mathscr{C}(A,R\otimes A)$; for this implies that there is a unique isomorphism $\varphi : A \to R \otimes A$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[dr, "\eta_R"'] & \mathscr{C}(A,A) \arrow[d, "\mathscr{C}(A,\varphi)"] \\ & \mathscr{C}(A,R\otimes A) \end{tikzcd}$$

is commutative, and thus that $\eta_R(1) = \varphi \circ \alpha(1) = \varphi$ is an isomorphism. ☐

Corollary 1.4.6. *Let* $\mathscr{C}$ *and* $\mathscr{B}$ *be R-linear categories with tensor products. If an R-linear functor* $F : \mathscr{C} \to \mathscr{B}$ *preserves colimits, then it preserves tensor products. More precisely, whenever* $\eta : \mathrm{id} \Rightarrow \mathscr{C}(A,-\otimes A)$ *is a unit, the composites*

$$M \xrightarrow{\eta_M} \mathscr{C}(A,M\otimes A) \xrightarrow{F} \mathscr{B}(FA,F(M\otimes A))$$

exhibit $F(-\otimes A)$ *as left adjoint of* $\mathscr{B}(FA,-)$.

Proof. For every object A of $\mathscr{A}$, the composite $F \circ - \otimes A$ is an R-linear functor which preserves colimits, so it is left adjoint to $\mathscr{C}(F(R \otimes A), -)$. We write $\varphi : A \to R \otimes A$ for the canonical isomorphism. Let η'_M be the composite

$$\begin{tikzcd}
M \arrow[r, "\eta_M^{F(-\otimes A)}"] & \mathscr{B}(F(R \otimes A), F(M \otimes A)) \arrow[r, "\mathscr{B}(F\varphi, F(M\otimes A))"] & \mathscr{B}(FA, F(M \otimes A)).
\end{tikzcd}$$

The η'_M obviously constitute a natural transformation $\eta' \colon \mathrm{id} \Rightarrow \mathscr{B}(FA, F(- \otimes A))$. This exhibits $F(- \otimes A)$ as left adjoint of $\mathscr{B}(FA, -)$ because φ is an isomorphism. It follows that for every module M there is a unique morphism $\xi_M : F(M \otimes A) \to F(M \otimes A)$ such that the diagram

$$\begin{tikzcd}
M \arrow[r, "\eta'_M"] \arrow[d, "\eta_M"'] & \mathscr{B}(FA, F(M \otimes A)) \arrow[d, "\mathscr{B}(FA{,}\xi_M)"] \\
\mathscr{C}(A, M \otimes A) \arrow[r, "F"] & \mathscr{B}(FA, F(M \otimes A))
\end{tikzcd}$$

is commutative. Since the ξ_M are defined by the universal property of an adjunction it follows easily that they constitute a natural transformation $F(- \otimes A) \Rightarrow F(- \otimes A)$. By Corollary 1.4.4 this natural transformation is entirely determined by the component ξ_R. For $M = R$ the above diagram reduces to $\xi_R \eta'_R(1) = F(\eta_R(1)) = F(\varphi)$. But $\eta'(1) = \mathrm{id} \circ F(\varphi)$, so $\xi_R = \mathrm{id}$ and consequently $\xi_M = \mathrm{id}$ for every module M. The above diagram thus gives the desired result. □

1.5. **Coends and left Kan extensions.** The goal of this section is to prove a 'parametrized' version of the fact that for a cocomplete R-linear category $\mathscr{C}$, the functors $\mathscr{C}(A, -)$ have left adjoints for every $A \in \mathscr{C}$.

Definition 1.5.1. For any R-linear functor $K : \mathscr{A} \to \mathscr{C}$ from a small R-linear category $\mathscr{A}$ to $\mathscr{C}$, we write $\widetilde{K}$ for the functor $\mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ which sends $C \in \mathscr{C}$ to $\mathscr{C}(K-, C) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Mod}_R$ and $\varphi : C \to C'$ to $\mathscr{C}(K-, \varphi) : \mathscr{C}(K-, C) \Rightarrow \mathscr{C}(K-, C')$.

We want to show that for cocomplete R-linear categories $\mathscr{C}$, the functor $\widetilde{K}$ has a left adjoint. In order to this we first have to introduce coends.

Definition 1.5.2. Let $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{C}$ be an R-linear functor. A pair $(X, (\lambda)_{A \in \mathscr{A}})$ consisting of an object X of $\mathscr{C}$ and a family of morphisms $\lambda_A : T(A, A) \to X$ is called a *coend* of T if the diagrams

$$\begin{tikzcd}
T(A', A) \arrow[r, "T(f{,}\mathrm{id})"] \arrow[d, "T(\mathrm{id}{,}f)"'] & T(A, A) \arrow[d, "\lambda_A"] \\
T(A', A') \arrow[r, "\lambda_{A'}"] & X
\end{tikzcd}$$

are commutative for every morphism $f : A \to A'$, and the pair $(X, (\lambda_A)_{A \in \mathscr{A}})$ is universal with this property; that is, if, whenever $\mu_A : T(A, A) \to X'$ make the analogous diagrams commutative, there is a unique morphism $\alpha : X \to X'$ such that $\mu_A = \alpha \circ \lambda_A$ for every object A of $\mathscr{A}$. A category $\mathscr{C}$ is said to *have coends* if for every small R-linear category $\mathscr{A}$ and every functor $T \colon \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{C}$, there is a chosen coend, which we denote by

$$\Big(\int^{A \in \mathscr{A}} T(A, A), (\lambda_A)_{A \in \mathscr{A}} \Big).$$

The morphisms $\lambda_A \colon T(A, A) \to \int^A T(A, A)$ are called the structure morphisms of the coend.

Remark 1.5.1. Let $\mathscr{C}$ be an R-linear category, and let $\mathscr{A}$ be a small category. A pair $(X,(\lambda)_{A\in\mathscr{A}})$ is a coend of $T\colon \mathscr{A}^{\mathrm{op}}\otimes\mathscr{A}\to\mathscr{C}$ if and only if the diagram

$$\begin{tikzcd}
\bigoplus_{f:A\to A'} T(A',A) \arrow[r, shift left, "\left(\mathrm{in}_A \circ T(f,id)\right)_{f:A\to A'}"] \arrow[r, shift right, "\left(\mathrm{in}_{A'} \circ T(\mathrm{id},f)\right)_{f:A\to A'}"'] & \bigoplus_{A\in\mathscr{A}} T(A,A) \arrow[r, "(\lambda_A)_{A\in\mathscr{A}}"] & X
\end{tikzcd}$$

is a coequalizer diagram. In particular, if $\mathscr{C}$ is cocomplete, then $\mathscr{C}$ has coends.

Proof. This is a straightforward reformulation of the universal property of coends. □

Remark 1.5.2. If $\mathscr{A}$ is small and if the R-linear category $\mathscr{C}$ is cocomplete, the coend construction uniquely extends to a functor

$$\int^{A\in\mathscr{A}} : [\mathscr{A}^{\mathrm{op}}\otimes\mathscr{A},\mathscr{C}]\to\mathscr{C}$$

such that for any natural transformation $\alpha : T\to T'$ and any object A in $\mathscr{A}$ the diagram

$$\begin{tikzcd}
T(A,A) \arrow[r, "\lambda_A^T"] \arrow[d, "\alpha_{A,A}"'] & \int^{A\in\mathscr{A}} T(A,A) \arrow[d, "\int^{A\in\mathscr{A}}\alpha_{A,A}"] \\
T'(A,A) \arrow[r, "\lambda_A^{T'}"'] & \int^{A\in\mathscr{A}} T'(A,A)
\end{tikzcd}$$

is commutative.

Proof. This follows immediately from the universal property of coends. □

Definition 1.5.3. Let $K:\mathscr{A}\to\mathscr{C}$ be an R-linear functor, where $\mathscr{A}$ is a small and $\mathscr{C}$ is cocomplete. We write G for the functor $[\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]\to[\mathscr{A}^{\mathrm{op}}\otimes\mathscr{A},\mathscr{C}]$ which sends F to $F-\otimes K- : \mathscr{A}^{\mathrm{op}}\otimes\mathscr{A}\to\mathscr{C}$ and $\alpha : F\Rightarrow A'$ to $\alpha\otimes\mathrm{id} : F-\otimes K-\Rightarrow F'-\otimes K-$. The composite $\int^{A\in\mathscr{A}}\circ G : [\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]\to\mathscr{C}$ is denoted by $L_K=\mathrm{Lan}_Y K : [\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]\to\mathscr{C}$. For any R-linear $F:\mathscr{A}^{\mathrm{op}}\to\mathbf{Mod}_R$, we thus have the formula

$$\mathrm{Lan}_Y K(F)=\int^{A\in\mathscr{A}} FA\otimes KA,$$

and we denote the structure morphisms of this coend by $\lambda_A^F\colon FA\otimes KA\to\int^A FA\otimes KA$.

This functor is called the *left Kan extension* of K along $Y:\mathscr{A}\to[\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]$. Since we only need some basic facts about Kan extensions and the above formula for computing them we simply take that formula as the definition. The general concept of a Kan extension can be found in [Kel82].

Proposition 1.5.3. *With $K:\mathscr{A}\to\mathscr{C}$ as in definition 1.5.3, the functor*

$$\mathrm{Lan}_Y K : [\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]\to\mathscr{C}$$

is left adjoint to

$$\widetilde{K}:\mathscr{C}\to[\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R],$$

with unit η and counit ε given by the unique morphisms such that the diagrams

$$\begin{tikzcd}
FB \arrow[r, "\eta_{FB}^{KB}"] \arrow[dr, "(\eta_F)_B"'] & \mathscr{C}(KB,FB\otimes KB) \arrow[d, "\mathscr{C}(KB{,}\lambda_B^F)"] \\
& \mathscr{C}(KB,\int^A FA\otimes KA)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\int^A \mathscr{C}(KA,C)\otimes KA \arrow[r, "\varepsilon_C"] & C \\
\mathscr{C}(KB,C)\otimes KB \arrow[u, "\lambda_B^{\mathscr{C}(K(-),C)}"] \arrow[ur, "\varepsilon_C^{KB}"'] &
\end{tikzcd}$$

are commutative for every $B \in \mathscr{A}$.

Proof. Note that the definition of ε_C makes sense: For every morphism $f\colon B \to B'$, the diagram

$$\begin{tikzcd}
\mathscr{C}(KB',C) \otimes KB \arrow[r, "\mathscr{C}(KB,C)\otimes Kf"] \arrow[d, "\mathscr{C}(Kf,C)\otimes KB"'] & \mathscr{C}(KB',C) \otimes KB' \arrow[d, "\varepsilon_C^{KB'}"] \\
\mathscr{C}(KB,C) \otimes KB \arrow[r, "\varepsilon_C^{KB}"'] & C
\end{tikzcd}$$

is commutative by proposition 1.3.3, and by definition 1.5.2 it follows that there is a unique arrow ε_C making the desired diagrams commutative. Next we have to check that the $(\eta_F)_B$ really do constitute a natural transformation $F \Rightarrow \mathscr{C}(K-, \int^A FA \otimes KA)$, i.e., that the outer composites of the diagram

$$\begin{tikzcd}
 & & \mathscr{C}(KB, \int^A FA \otimes KA) \arrow[drr, "\mathscr{C}(Kf,\int^A FA\otimes KA)"] & & \\
\mathscr{C}(KB, FB \otimes KB) \arrow[urr, "\mathscr{C}(KB,\lambda_B^F)"] \arrow[drr, "\mathscr{C}(Kf,FB\otimes KB)"'] & & (1) & & \mathscr{C}(KB', \int^A FA \otimes KA) \\
 & & \mathscr{C}(KB', FB \otimes KB) \arrow[urr, "\mathscr{C}(KB',\lambda_B^F)"'] & & \\
 & (2) & & (3) & \\
 & & \mathscr{C}(KB', FB \otimes KB') \arrow[uu, "\mathscr{C}(KB',FB\otimes Kf)"] \arrow[drr, "\mathscr{C}(KB',Ff\otimes KB')"] & & \\
FB \arrow[uuuu, "\eta_{FB}^{KB}"] \arrow[urr, "\eta_{FB}^{KB'}"] \arrow[drr, "Ff"'] & & (4) & & \mathscr{C}(KB', FB' \otimes KB') \arrow[uuuu, "\mathscr{C}(KB',\lambda_{B'}^F)"'] \\
 & & FB' \arrow[urr, "\eta_{FB'}^{KB'}"'] & &
\end{tikzcd}$$

are equal for every morphism $f\colon B' \to B$. But part (1) is evidently commutative, part (2) is commutative since $- \otimes Kf$ is the mate of $\mathscr{C}(Kf, -)$ (see proposition 1.3.3), part (3) is commutative by definition of a coend, and part (4) is commutative by naturality of $\eta^{KB'}$.

We have to show that the η_F and ε_C are natural in F and C respectively. For any natural transformation $\alpha\colon F \Rightarrow F'$, the diagram

$$\begin{tikzcd}
FB \arrow[r, "\eta_{FB}^{KB}"] \arrow[d, "\alpha_B"'] & \mathscr{C}(KB, FB \otimes KB) \arrow[r, "\mathscr{C}(KB,\lambda_B^F)"] \arrow[d, "\mathscr{C}(KB,\alpha_B\otimes KB)"] & \mathscr{C}(KB, \int^A FA \otimes KA) \arrow[d, "\mathscr{C}(KB,\int^A \alpha_A\otimes KA)"] \\
F'B \arrow[r, "\eta_{F'B}^{KB}"'] & \mathscr{C}(KB, F'B \otimes KB) \arrow[r, "\mathscr{C}(KB,\lambda_B^{F'})"'] & \mathscr{C}(KB, \int^A F'A \otimes KA)
\end{tikzcd}$$

is commutative, which shows that the η_F are natural in F; and for any morphism $\varphi\colon C \to C'$, the outer composites in the diagram

$$\begin{tikzcd}
\mathscr{C}(KB,C) \otimes KB \arrow[r, "\lambda_B^{\mathscr{C}(K-,C)}"] \arrow[d, "\mathscr{C}(KB,\varphi)\otimes KB"'] & \int^A \mathscr{C}(KA,C) \otimes KA \arrow[r, "\varepsilon_C"] \arrow[d, "\int^A \mathscr{C}(KA,\varphi)\otimes KA"] & C \arrow[d, "\varphi"] \\
\mathscr{C}(KB,C') \otimes KB \arrow[r, "\lambda_B^{\mathscr{C}(K-,C')}"'] & \int^A \mathscr{C}(KA,C') \otimes KA \arrow[r, "\varepsilon_{C'}"'] & C'
\end{tikzcd}$$

are equal, hence the universal property of coends implies that the ε_C are natural in C. It remains to show that the triangular identities hold. One of these follows from commutativity of the diagram

$$\begin{tikzcd}
& \mathscr{C}(KB,\mathscr{C}(KB,C) \otimes KB) \arrow[dr, "\mathscr{C}(KB{,}\lambda_B^{\mathscr{C}(K-,C)})"] \arrow[dd, "\mathscr{C}(KB{,}\varepsilon_C^{KB})"] & \\
\mathscr{C}(KB,C) \arrow[ur, "\eta^{KB}_{\mathscr{C}(KB,C)}"] \arrow[dr, "\mathrm{id}"'] & & \mathscr{C}(KB,\int^A \mathscr{C}(KA,C) \otimes KA) \arrow[dl, "\mathscr{C}(KB{,}\varepsilon_C)"] \\
& \mathscr{C}(KB,C) &
\end{tikzcd}$$

which is a consequence of the definition of ε_C and the fact that the triangular identities hold for the adjunction $-\otimes KB \dashv \mathscr{C}(KB,-)$. The diagram

$$\begin{tikzcd}
& \int^{A'} \mathscr{C}(KA',\int^A FA \otimes KA) \otimes KA' \arrow[dr, "\varepsilon_{\int FA\otimes KA}"] & \\
\int^A FA \otimes KA \arrow[ur, "\mathrm{Lan}_Y K(\eta_F)"] & \mathscr{C}(KB,\int^A FA \otimes KA) \arrow[u, "\lambda_B^{\mathscr{C}(K-,\int FA\otimes KA)}"'] \arrow[r, "\varepsilon^{KB}_{\int FA\otimes KA}"'] & \int^A FA \otimes KA \\
FB \otimes KB \arrow[u, "\lambda_B"] \arrow[r, "\eta^{KB}_{FB}\otimes KB"'] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "(1)"] & \mathscr{C}(KB,FB \otimes KB) \otimes KB \arrow[u, "\mathscr{C}(KB{,}\lambda^F_B)"'] \arrow[r, "\varepsilon^{KB}_{FB\otimes KB}"'] \arrow[ur, phantom, "(2)"] & FB \otimes KB \arrow[u, "\lambda_B"']
\end{tikzcd}$$

is commutative: part (1) is commutative by definition of $(\eta_F)_B$ and of $\mathrm{Lan}_Y K$ (see definition 1.5.3), and part (2) commutes by naturality of ε^{KB}. This shows that the second triangular identity holds. $\square$

Proposition 1.5.4. *Let $\mathscr{A}$ and $\mathscr{C}$ be R-linear categories such that $\mathscr{A}$ is small and $\mathscr{C}$ is cocomplete. For every R-linear functor $K\colon \mathscr{A} \to \mathscr{C}$ there is a canonical natural isomorphism $\alpha_K\colon K \Rightarrow L_K Y$.*

Proof. Writing $\varphi_{F,C}$ for the natural isomorphism

$$\mathscr{C}(L_K F, C) \cong [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R](F, \mathscr{C}(K-,C))$$

from proposition 1.5.3 we get by Yoneda an isomorphism

$$\mathscr{C}(L_K(\mathscr{A}(-,A)),C) \xrightarrow{\varphi_{\mathscr{A}(-,A),C}} [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R](\mathscr{A}(-,A), \mathscr{C}(K-,C))$$
$$\xrightarrow[\psi_{A,\mathscr{C}(K-,C)}]{} \mathscr{C}(KA,C),$$

where $\psi_{A,\mathscr{C}(K-,C)}$ sends a natural transformation κ to $\kappa_A(\mathrm{id}_A)$. Since these isomorphisms are natural in C, the Yoneda lemma implies that there are unique isomorphisms $(\alpha_K)_A \colon KA \to L_K Y(A)$ such that

$$\psi_{A,\mathscr{C}(K-,C)} \circ \varphi_{\mathscr{A}(-,A),C} = \mathscr{C}((\alpha_K)_A, C).$$

Because the left hand side of the above equation is natural in A, so is the right hand side, and it follows that the $(\alpha_K)_A$ are natural in A. □

1.6. **Dense functors.** The notion of density of a functor is motivated as follows (see [Kel82], chapter 5): A continuous map $f\colon X \to Y$ between Hausdorff topological spaces has dense image if and only if a continuous map $g\colon Y \to Z$ into another Hausdorff space is uniquely determined by the composite gf. The notion of a dense functor is analogous to this property, with 'continuous map' replaced by 'cocontinuous functor'; see proposition 1.6.4 for the precise statement.

Definition 1.6.1. Let $K : \mathscr{A} \to \mathscr{C}$ be an R-linear functor. The functor K is called *dense* if $\widetilde{K}\colon \mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ (see definition 1.5.1) is fully faithful.

Proposition 1.6.1. *For any small R-linear category $\mathscr{A}$, the Yoneda embedding*

$$Y\colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$$

is dense.

Proof. The Yoneda lemma gives a natural isomorphism

$$\psi_F\colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R](Y(-), F) \cong F.$$

The ψ_F are natural in F, hence they give a natural isomorphism

$$\psi\colon \widetilde{Y} \Longrightarrow \mathrm{id}_{[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]}\,.$$

It follows in particular that $\widetilde{Y}$ is fully faithful. □

Lemma 1.6.2. *Let $F \dashv G\colon \mathscr{B} \to \mathscr{C}$ be an adjunction with unit η and counit ε, and let $C \in \mathscr{C}$. Then $\varepsilon_C\colon FGC \to C$ is an isomorphism if and only if $G_{C,D}\colon \mathscr{C}(C,D) \to \mathscr{B}(GC,GD)$ is a bijection for every $D \in \mathscr{C}$. In particular, G is fully faithful if and only if ε_C is an isomorphism for every object C of $\mathscr{C}$.*

Proof. The triangular identities imply that for every $B \in \mathscr{B}$, the map

$$\mathscr{C}(FB,C) \xrightarrow{\varphi_{B,C}} \mathscr{B}(B,GC)$$

given by $\varphi_{B,C}(g) = G(g) \circ \eta_B$ is an isomorphism. For any morphism $f\colon C \to D$ we have

$$\begin{aligned} \varphi_{GC,D} \circ \mathscr{C}(\varepsilon_C, D)(f) &= G(f \circ \varepsilon_C) \circ \eta_{GC} \\ &= G(f) \circ G(\varepsilon_C) \circ \eta_{GC} \\ &= G(f), \end{aligned}$$

where the last equality follows by the triangular identities. It follows that $\varphi_{GC,D} \circ \mathscr{C}(\varepsilon_C, D) = G_{C,D}$. By Yoneda, ε_C is an isomorphism if and only if $\mathscr{C}(\varepsilon_C, D)$ is an isomorphism for every object D; and the above equality shows that this is equivalent to the fact that $G_{C,D}$ is a bijection for every $D \in \mathscr{C}$. □

Proposition 1.6.3. *Let $K : \mathscr{A} \to \mathscr{C}$ be an R-linear functor, where $\mathscr{A}$ is small and $\mathscr{C}$ is cocomplete. Then K is dense if and only if for every object C of $\mathscr{C}$, the counit morphisms $\varepsilon^{KA}\colon \mathscr{C}(KA,C) \otimes KA \to C$ exhibit C as coend of the functor $\mathscr{C}(K-,C) \otimes K- : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{C}$.*

Proof. It follows directly from the definition of the counit $\varepsilon_C \colon L_K \widetilde{K} C \to C$ in proposition 1.5.3 that ε_C is an isomorphism if and only if the counit morphisms $\varepsilon^{KA} \colon \mathscr{C}(KA, C) \otimes KA \to C$ exhibit C as coend of the functor $\mathscr{C}(K-, C) \otimes K- : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{C}$. By lemma 1.6.2 this is equivalent to the fact that $\widetilde{K}$ is fully faithful. □

Proposition 1.6.4. *Let $\mathscr{C}$ be a cocomplete R-linear category, and let $K : \mathscr{A} \to \mathscr{C}$ be a dense functor. Let $F, F' : \mathscr{C} \to \mathscr{B}$ be cocontinuous functors and $\alpha : FK \Rightarrow F'K$ be a natural transformation. Then there is a unique natural transformation $\beta : F \Rightarrow F'$ such that $\alpha = \beta K$.*

Proof. Since F preserves colimits and because K is dense, the morphisms $F(\varepsilon^{KA}_C) : F(\mathscr{C}(KA, C) \otimes KA) \to F(C)$ exhibit $F(C)$ as the coend of $F(\mathscr{C}(K-, C) \otimes K-)$, and the analogous result holds for F' (see proposition 1.6.3). Write γ^{KA}_R for the unique morphism which makes the diagram

$$\begin{tikzcd}
FKA \arrow[r, "F(\varphi^A)"] \arrow[d, "\alpha_A"'] & F(R \otimes KA) \arrow[d, "\gamma^{KA}_R"'] \\
F'KA \arrow[r, "F(\varphi'^A)"] & F'(R \otimes KA)
\end{tikzcd}$$

commutative, where φ^A and φ'^A denote the canonical isomorphisms (see proposition 1.4.5). By corollary 1.4.4 there is a unique natural transformation $\gamma^{KA} : F(- \otimes KA) \Rightarrow F'(- \otimes KA)$ with R-component γ^{KA}_R. If we write $\delta^C_{A,B}$ for the morphism $\gamma^{KB}_{\mathscr{C}(KA,C)} : F(\mathscr{C}(KA, C) \otimes KB) \to F'(\mathscr{C}(KA, C) \otimes KB)$ we get a natural transformation

$$\delta^C \colon F(\mathscr{C}(K-, C) \otimes K-) \Rightarrow F'(\mathscr{C}(K-, C) \otimes K-).$$

Indeed, naturality in the first variable follows directly from naturality of γ^{KA}, and naturality in the second variable follows because for any morphism $f : A \to A'$, $\gamma^{KA'} \circ F(- \otimes Kf)$ and $F'(- \otimes Kf) \circ \gamma^{KA}$ both have the same R-component; hence they are equal by corollary 1.4.4. Therefore there is a unique morphism $\beta_C : FC \to F'C$ such that

$$\begin{tikzcd}
F(\mathscr{C}(KA, C) \otimes KA) \arrow[r, "F(\varepsilon^{KA}_C)"] \arrow[d, "\delta^C_{A,A}"'] & FC \arrow[d, "\beta_C"] \\
F'(\mathscr{C}(KA, C) \otimes KA) \arrow[r, "F'(\varepsilon^{KA}_C)"'] & F'C
\end{tikzcd}$$

is commutative, and since both δ^C and ε are natural in $\mathscr{C}$ it follows that $\beta : F \Rightarrow F'$ is a natural transformation. It remains to show that $\beta K = \alpha$ and that β is unique with this property.

We first consider the diagram

$$\begin{tikzcd}
F(R \otimes KA) \arrow[r, "F(\eta^{KA}_R \otimes KA)"] \arrow[d, "\gamma^{KA}_R"'] & F(\mathscr{C}(KA, R \otimes KA) \otimes KA) \arrow[r, "F(\varepsilon^{KA}_{R \otimes KA})"] \arrow[d, "\gamma^{KA}_{\mathscr{C}(KA, R\otimes KA)} = \delta^{KA}_{A,A}"'] & F(R \otimes KA) \arrow[d, "\beta_{R \otimes KA}"] \\
F'(R \otimes KA) \arrow[r, "F'(\eta^{KA}_R \otimes KA)"'] & F'(\mathscr{C}(KA, R \otimes KA) \otimes KA) \arrow[r, "F'(\varepsilon^{KA}_{R \otimes KA})"'] & F'(R \otimes KA),
\end{tikzcd}$$

which is commutative by naturality of γ^{KA} and by definition of $\beta_{R \otimes KA}$. By the triangular identities for the adjunction $- \otimes KA \dashv \mathscr{C}(KA, -)$ it follows that the top and the bottom morphism in the above diagram are identity morphisms, and therefore that $\beta_{R\otimes KA} = \gamma_R^{KA}$. This and the definition of γ_R^{KA} imply that $\beta_{KA} = \alpha_A$. If $\overline{\beta} : F \Rightarrow F'$ is another natural transformation with this property, we find immediately that $\gamma_R^{KA} = \overline{\beta}_{R\otimes KA}$ and by corollary 1.4.4 that $\gamma^{KA} = \overline{\beta} - \otimes KA$. Therefore $\delta^C_{A,B}$ must be equal to $\overline{\beta}_{\mathscr{C}(KA,C)\otimes KA}$, and naturality of $\overline{\beta}$ implies that the diagram

$$\begin{tikzcd}
F(\mathscr{C}(KA,C)\otimes KA) \arrow[r, "F(\varepsilon_C^{KA})"] \arrow[d, "\delta^C_{A,A}=\beta_{\mathscr{C}(KA,C)\otimes KA}"'] & FC \arrow[d, "\beta_C"] \\
F'(\mathscr{C}(KA,C)\otimes KA) \arrow[r, "F'(\varepsilon_C^{KA})"'] & F'C
\end{tikzcd}$$

is commutative, hence that $\overline{\beta} = \beta$. □

Corollary 1.6.5. *For any cocomplete R-linear category $\mathscr{C}$ and any small R-linear category $\mathscr{A}$, the functor*

$$[Y,\mathscr{C}] : \mathrm{Cocts}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R], \mathscr{C}) \to [\mathscr{A},\mathscr{C}]$$

which sends a cocontinuous functor F to $FY : \mathscr{A} \to \mathscr{C}$ is an equivalence of categories.

Proof. By proposition 1.6.4, the functor $[Y,\mathscr{C}]$ is fully faithful, and in proposition 1.5.4 we have seen that for any $K : \mathscr{A} \to \mathscr{C}$, $\mathrm{Lan}_Y K \circ Y \cong K$. In other words, $[Y,\mathscr{C}]$ is essentially surjective. □

Definition 1.6.2. Let $K\colon \mathscr{A} \to \mathscr{C}$ be an R-linear functor, where $\mathscr{A}$ is a small R-linear category. For any $C \in \mathscr{C}$ we define the category $(K \downarrow C)$ as follows: the objects of $(K \downarrow C)$ are pairs (A,φ), where A is an object of $\mathscr{A}$ and $\varphi\colon KA \to C$ is a morphism of $\mathscr{C}$, and the morphisms $(A,\varphi) \to (A',\varphi')$ are the morphisms $f\colon A \to A'$ in $\mathscr{A}$ for which the diagram

$$\begin{tikzcd}
KA \arrow[rr, "Kf"] \arrow[dr, "\varphi"'] & & KA' \arrow[dl, "\varphi'"] \\
& C &
\end{tikzcd}$$

is commutative. We let $D_C\colon (K \downarrow C) \to \mathscr{C}$ be the functor which sends (A,φ) to KA and $f\colon (A,\varphi) \to (A',\varphi')$ to Kf. The *canonical cocone* on D_C is the cocone

$$\Big(C, \big(\kappa_{(A,\varphi)}\big)_{(A,\varphi)\in(K\downarrow C)}\Big)$$

given by the morphisms $\kappa_{(A,\varphi)} = \varphi$.

Proposition 1.6.6. *Let $K\colon \mathscr{A} \to \mathscr{C}$ be an R-linear functor, where $\mathscr{A}$ is small and additive and $\mathscr{C}$ is cocomplete. For any object C of $\mathscr{C}$, the counit $\varepsilon_C\colon L_K\widetilde{K} \to C$ (see proposition 1.5.3) is an isomorphism if and only if the canonical cocone on D_C (see definition 1.6.2) exhibits C as colimit of D_C. In particular, K is dense if and only if this holds for every object C of $\mathscr{C}$.*

Proof. The second statement follows immediately from the first by lemma 1.6.2. We denote the set of cocones

$$\Big(X, \big(\gamma_{(A,\varphi)}\big)_{(A,\varphi)\in(K\downarrow C)}\Big)$$

on D_C with target $X \in \mathscr{C}$ by S_X. If $\alpha\colon \mathscr{C}(K-,C) \Rightarrow \mathscr{C}(K-,X)$ is a natural transformation, we let $\chi(\alpha)_{(A,\varphi)} = \alpha_A(\varphi)$. If $f\colon (A,\varphi) \to (A',\varphi')$ is a morphism of $(K \downarrow C)$, then $\varphi = \varphi' \circ Kf$, and by naturality of α it follows that

$$\chi(\alpha)_{(A,\varphi)} = \alpha_A(\varphi) = \alpha_A(\varphi' \circ Kf) = \alpha_{A'}(\varphi') \circ Kf,$$

which shows that the $\chi(\alpha)_{(A,\varphi)}$ constitute a cocone. In other words, we get a map

$$\chi\colon \operatorname{Nat}(\mathscr{C}(K-,C),\mathscr{C}(K-,X)) \to S_X.$$

Moreover, the composite

$$\mathscr{C}(C,X) \xrightarrow{\widetilde{K}_{C,X}} \operatorname{Nat}(\mathscr{C}(K-,C),\mathscr{C}(K-,X)) \xrightarrow{\chi} S_X$$

sends a morphism $g\colon C \to X$ to the cocone with components

$$\chi(\widetilde{K}_{C,X}(g))_{(A,\varphi)} = \chi(\mathscr{C}(K-,g))_{(A,\varphi)} = \mathscr{C}(KA,g)(\varphi) = g \circ \varphi.$$

Thus $\chi \circ \widetilde{K}_{C,X}$ is a bijection for every $X \in \mathscr{C}$ if and only if the canonical cocone on D_C exhibits C as colimit of D_C. On the other hand, lemma 1.6.2 shows that ε_C is an isomorphism if and only if $\widetilde{K}_{C,X}$ is a bijection for every $X \in \mathscr{C}$. The statement of the proposition is thus equivalent to the fact that χ is a bijection.
It remains to construct an inverse of χ. Given a cocone

$$\Big(X, \big(\gamma_{(A,\varphi)}\big)_{(A,\varphi)\in(K\downarrow C)}\Big),$$

on D_C we let $\beta(\gamma)_A\colon \mathscr{C}(KA,C) \to \mathscr{C}(KA,X)$ be the map which sends $\varphi\colon KA \to C$ to $\gamma_{(A,\varphi)}$. We claim that $\beta(\gamma)_A$ is a homomorphism of R-modules. For $r \in R$, the diagram

$$\begin{tikzcd}
KA \arrow[rr, "K(r\cdot \mathrm{id})"] \arrow[dr, "r\cdot\varphi"'] & & KA \arrow[dl, "\varphi"] \\
& C &
\end{tikzcd}$$

is commutative. By definition 1.6.2 it follows that $r \cdot \mathrm{id}\colon (A, r\cdot\varphi) \to (A,\varphi)$ is a morphism in $(K \downarrow C)$. Since the $\gamma_{(A,\varphi)}$ constitute a cocone on D_C it follows that

$$\gamma_{(A,r\cdot\varphi)} = \gamma_{(A,\varphi)} \circ K(r\cdot \mathrm{id}) = r \cdot \gamma_{(A,\varphi)},$$

hence that $\beta(\gamma)_A(r\cdot\varphi) = r\cdot\beta(\gamma)_A(\varphi)$. Since $\mathscr{A}$ is additive, the sum $A \oplus A$ exists. We denote the inclusions by $\mathrm{in}_i\colon A \to A\oplus A$, $i = 1,2$. The fact that K is R-linear implies that the $K(\mathrm{in}_i)\colon KA \to K(A\oplus A)$, $i = 1,2$ exhibit $K(A\oplus A)$ as sum of two copies of KA. It follows that for any two morphisms $\varphi_i\colon KA \to C$ in $\mathscr{C}$ there is a unique morphism $(\varphi_1 \quad \varphi_2)\colon K(A\oplus A) \to C$ such that the diagram

$$\begin{tikzcd}
KA \arrow[rr, "K(\mathrm{in}_i)"] \arrow[dr, "\varphi_i"'] & & K(A\oplus A) \arrow[dl, "(\varphi_1 \quad \varphi_2)"] \\
& C &
\end{tikzcd}$$

is commutative for $i = 1,2$. Furthermore, the diagram

$$\begin{tikzcd}
KA \arrow[rr, "K(\mathrm{in}_1+\mathrm{in}_2)"] \arrow[dr, "\varphi_1+\varphi_2"'] & & K(A\oplus A) \arrow[dl, "(\varphi_1 \quad \varphi_2)"] \\
& C &
\end{tikzcd}$$

is commutative because K is R-linear. These three diagrams show that we have morphisms $\mathrm{in}_i\colon (A,\varphi_i) \to (A\oplus A, (\varphi_1 \quad \varphi_2))$ and a morphism $K(\mathrm{in}_1+\mathrm{in}_2)\colon (A,\varphi_1 + \varphi_2) \to (A\oplus A, (\varphi_1 \quad \varphi_2))$ in $(K\downarrow C)$. The fact that the $\gamma_{(A,\varphi)}$ constitute a cocone on D_C therefore implies that

$$\begin{aligned}\gamma_{(A,\varphi_1+\varphi_2)} &= \gamma_{\big(A\oplus A,(\varphi_1\quad\varphi_2)\big)} \circ K(\mathrm{in}_1+\mathrm{in}_2)\\ &= \gamma_{\big(A\oplus A,(\varphi_1\quad\varphi_2)\big)} \circ K(\mathrm{in}_1) + \gamma_{\big(A\oplus A,(\varphi_1\quad\varphi_2)\big)} \circ K(\mathrm{in}_2)\\ &= \gamma_{(A,\varphi_1)} + \gamma_{(A,\varphi_2)}.\end{aligned}$$

This shows that $\beta(\gamma)_A(\varphi_1+\varphi_2) = \beta(\gamma)_A(\varphi_1) + \beta(\gamma)_A(\varphi_2)$, as claimed. For any morphism $f\colon A \to A'$ and any $\varphi\colon KA \to C$, f gives a morphism $(A, \varphi\circ Kf) \to (A,\varphi)$ in $(K \downarrow C)$, which implies that

$$\gamma_{(A,\varphi\circ Kf)} = \gamma_{(A,\varphi)} \circ Kf,$$

i.e, that $\beta(\gamma)_A(\varphi\circ Kf) = \beta(\gamma)_A(\varphi)\circ Kf$. In other words, $\beta(\gamma)$ is natural in A, and we have in fact constructed a map $\beta\colon S_X \to \mathrm{Nat}(\mathscr{C}(K-,C),\mathscr{C}(K-,X))$. For every natural transformation $\alpha\colon \mathscr{C}(K-,C) \Rightarrow \mathscr{C}(K-,X)$ and every $\varphi\colon KA \to C$ we have

$$\Big(\beta\big(\chi(\alpha)\big)\Big)_A(\varphi) = \chi(\alpha)_{(A,\varphi)} = \alpha_A(\varphi),$$

and if the $\gamma_{(A,\varphi)}\colon KA \to X$ constitute a cocone on D_C, the equalities

$$\chi(\beta(\gamma))_{(A,\varphi)} = \beta(\gamma)_A(\varphi) = \gamma_{(A,\varphi)}$$

hold. This shows that χ and β are mutually inverse, which concludes the proof. □

Corollary 1.6.7. *Let $\mathscr{A}$ be a small additive R-linear category. For every R-linear functor $F\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Mod}_R$, the canonical cocone on $D_F\colon (Y\downarrow F) \to [\mathscr{A}^{\mathrm{op}},\mathbf{Mod}_R]$ (see definition 1.6.2) exhibits F as colimit of D_F.*

Proof. This is a direct consequence of proposition 1.6.1 and proposition 1.6.6. □

2. The reconstruction problem

2.1. **Overview.** In this chapter we give an explicit construction of the left adjoint mentioned in the introduction, and we use this construction to prove theorem 2.6.5. In section 2.2 we show that the equivalence of categories from corollary 1.4.4 extends to an equivalence of the category of coalgebras and the category of cocontinuous R-linear comonads (see definition 2.2.2). The proof is straightforward but rather tedious.

In the sections 2.3 and 2.4 we introduce the categories of comodules for coalgebras and for comonads, and we show that these are compatible with the equivalence from section 2.2 (see proposition 2.4.3 for the precise statement). We also cite Beck's monadicity theorem (theorem 2.4.8) which will be used in the proof of our recognition result in section 3.4. We construct the left adjoint to the comodule functor in section 2.5 using left Kan extensions (see definition 1.5.3), and the equivalence mentioned above. This enables us to give an explicit formula for the counit in section 2.6, which can then be used to prove our reconstruction theorem.

2.2. **Comonoids and monoidal functors.** Recall that a monoidal category is a category $\mathscr{M}$ together with a bifunctor $-\otimes-: \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ and an object $I \in \mathscr{M}$ such that $-\otimes-$ is associative up to natural isomorphism $a = a_{A,B,C} : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$ and such that I is a unit up to natural isomorphism $l = l_A : I\otimes A \to A$ and $r = r_A\colon A\otimes I \to A$. Furthermore, these natural isomorphism have to satisfy certain axioms (coherence laws), which then imply that all diagrams containing only instances of a, l, r and $-\otimes-$ applied to such morphisms are commutative (see [ML98]).

Definition 2.2.1. Let $\mathscr{M}$ be a monoidal category. A *comonoid* in $\mathscr{M}$ is an object $C \in \mathscr{M}$ together with a comultiplication $\delta : C \to C \otimes C$ and a counit $\varepsilon : C \to I$ such that the diagrams

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes C \arrow[r, "\delta \otimes C"] & (C \otimes C) \otimes C \arrow[d, "a"] \\
C \otimes C \arrow[rr, "C \otimes \delta"'] & & C \otimes (C \otimes C)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & C \arrow[dl, "r^{-1}"'] \arrow[d, "\delta"] \arrow[dr, "l^{-1}"] & \\
C \otimes I & C \otimes C \arrow[l, "C \otimes \varepsilon"] \arrow[r, "\varepsilon \otimes C"'] & I \otimes C
\end{tikzcd}$$

are commutative. A morphism of comonoids $(C, \delta, \varepsilon) \to (C', \delta', \varepsilon')$ is a morphism $\varphi : C \to C'$ in $\mathscr{M}$ which is compatible with the comultiplications and the counits, that is, such that the diagrams

$$\begin{tikzcd}
C \arrow[r, "\delta"] \arrow[d, "\varphi"'] & C \otimes C \arrow[d, "\varphi \otimes \varphi"] \\
C' \arrow[r, "\delta'"] & C' \otimes C'
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
C \arrow[rr, "\varphi"'] \arrow[dr, "\varepsilon"'] & & C' \arrow[dl, "\varepsilon'"] \\
 & I &
\end{tikzcd}$$

are commutative. The comonoids in $\mathscr{M}$ with comonoid morphisms constitute a category, which we denote by $\mathrm{Comon}(\mathscr{M})$.

We are interested in the following two examples of monoidal categories: The first is $\mathbf{Mod}_R$ itself, with its usual tensor product and unit R. The second example is the category $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ of cocontinuous R-linear functors $\mathbf{Mod}_R \to \mathbf{Mod}_R$, with tensor product given by the composition of functors and unit object the identity functor $\mathrm{id}_{\mathbf{Mod}_R} \colon \mathbf{Mod}_R \to \mathbf{Mod}_R$. This category is *strict* monoidal, i.e., the natural isomorphisms a, l, r are in fact identities.

Definition 2.2.2. A comonoid (C, δ, ε) in $\mathbf{Mod}_R$ is called a *coalgebra*, and the category of coalgebras is denoted by $\mathbf{Coalg}_R$. A comonoid $(T, \delta^T, \varepsilon^T)$ in the category $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ is called a *cocontinuous R-linear comonad*, and the category of cocontinuous R-linear comonads is denoted by $\mathbf{CC}_R$.

The goal of this section is to construct an equivalence between $\mathbf{Coalg}_R$ and $\mathbf{CC}_R$. In order to do this we need a notion of morphism between monoidal categories.

Definition 2.2.3. A *monoidal functor* $F : \mathscr{M} \to \mathscr{M}'$ between monoidal categories $(\mathscr{M}, \otimes, I, a, r, l)$ and $(\mathscr{M}', \otimes', I', a', r', l')$ is a functor $F : \mathscr{M} \to \mathscr{M}'$ together with a natural transformation $\psi_{A,B} : FA \otimes' FB \to F(A \otimes B)$ and a morphism $\psi_0 : I' \to FI$ such that for all objects A, B, C of $\mathscr{M}$, the diagrams

$$\begin{tikzcd}
(FA \otimes' FB) \otimes' FC \arrow[r, "a'"] \arrow[d, "\psi_{A,B} \otimes' FC"'] & FA \otimes' (FB \otimes' FC) \arrow[d, "FA \otimes' \psi_{B,C}"] \\
F(A \otimes B) \otimes' FC \arrow[d, "\psi_{A \otimes B, C}"'] & FA \otimes' F(B \otimes C) \arrow[d, "\psi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(a)"'] & F(A \otimes (B \otimes C))
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \otimes' I' \arrow[r, "r'"] \arrow[d, "FA \otimes' \psi_0"'] & FA \\
FA \otimes' FI \arrow[r, "\psi_{A,I}"'] & F(A \otimes I) \arrow[u, "F(r)"'] \\
I' \otimes' FA \arrow[r, "l'"] \arrow[d, "FA \otimes' \psi_0"'] & FA \\
FI \otimes' FA \arrow[r, "\psi_{I,A}"'] & F(I \otimes A) \arrow[u, "F(l)"']
\end{tikzcd}$$

are commutative. A monoidal functor (F, ψ, ψ_0) is called *strong* if the $\psi_{A,B}$ and ψ_0 are isomorphisms.

Proposition 2.2.1. *Let* $(F, \psi, \psi_0) : \mathscr{M} \to \mathscr{M}'$ *be a strong monoidal functor which is fully faithful and essentially surjective. Then the functor* $\widehat{F} : \mathrm{Comon}(\mathscr{M}) \to \mathrm{Comon}(\mathscr{M}')$ *which sends a comonoid* (C, δ, ε) *in* $\mathscr{M}$ *to the comonoid* FC *with comultiplication*

$$FC \xrightarrow{F\delta} F(C \otimes C) \xrightarrow{\psi^{-1}_{C,C}} FC \otimes' FC$$

and counit $FC \xrightarrow{F\varepsilon} FI \xrightarrow{\psi_0^{-1}} I'$ *and a morphism* $\varphi : C \to C'$ *of comonoids to* $F\varphi : FC \to FC'$ *is an equivalence of categories.*

Proof. First one has to check that $\widehat{F}$ is well-defined, i.e., that FC with the described comultiplication and counit is indeed a comonoid in $\mathscr{M}'$. This follows from the facts that C is a comonoid and that F is strong monoidal. For example, the diagram

$$\begin{tikzcd}
FC \arrow[r, "F\delta"] \arrow[dd, "F\delta"'] \arrow[from=1-1, to=3-2, phantom, "(1)"] & F(C \otimes C) \arrow[r, "\psi^{-1}_{C,C}"] \arrow[d, "F(\delta\otimes C)"] \arrow[from=1-2, to=2-3, phantom, "(2)"] & FC \otimes' FC \arrow[d, "F\delta\otimes' FC"] \\
 & F((C \otimes C) \otimes C) \arrow[r, "\psi^{-1}_{C\otimes C,C}"'] \arrow[d, "F(a)"] \arrow[from=2-2, to=4-3, phantom, "(4)"] & F(C \otimes C) \otimes' FC \arrow[d, "\psi^{-1}_{C,C}\otimes' FC"] \\
F(C \otimes C) \arrow[r, "F(C\otimes\delta)"] \arrow[d, "\psi^{-1}_{C,C}"'] \arrow[from=3-1, to=4-2, phantom, "(3)"] & F(C \otimes (C \otimes C)) \arrow[d, "\psi^{-1}_{C,C\otimes C}"] & (FC \otimes' FC) \otimes' C) \arrow[d, "a'"] \\
F(C) \otimes' F(C) \arrow[r, "FC\otimes' F\delta"'] & FC \otimes' F(C \otimes C) \arrow[r, "FC\otimes'\psi^{-1}_{C,C}"'] & FC \otimes' (FC \otimes' FC)
\end{tikzcd}$$

is commutative. Indeed, commutativity of (2) and (3) follows from the naturality of ψ, (1) is commutative because C is a comonoid and (4) is commutative because (F, ψ, ψ_0) is strong monoidal. The remaining comonoid axioms and the axioms for $F\varphi$ to be a morphism of comonoids can be checked similarly. Next we want to show that $\widehat{F}$ is fully faithful. Clearly $\widehat{F}$ is faithful, because F is. To see that $\widehat{F}$ is full, we let $\varphi' : FC \to FC'$ be a morphism of comonoids, where $(C', \delta', \varepsilon')$ is any comonoid in $\mathscr{M}$. Since F is full, there is a unique morphism $\varphi : C \to C'$ in $\mathscr{M}$ such that $F\varphi = \varphi'$. It remains to check that φ is in fact a morphism of comonoids. But the diagram

$$\begin{tikzcd}
FC \arrow[r, "F\delta"] \arrow[d, "F\varphi=\varphi'"'] & F(C \otimes C) \arrow[r, "\psi^{-1}_{C,C}"] & FC \otimes' FC \arrow[r, "\psi_{C,C}"] \arrow[d, "\varphi'\otimes'\varphi'=F\varphi\otimes' F\varphi"'] & F(C \otimes C) \arrow[d, "F(\varphi\otimes\varphi)"] \\
FC' \arrow[r, "F\delta'"'] & F(C' \otimes C') \arrow[r, "\psi^{-1}_{C',C'}"'] & FC' \otimes' FC' \arrow[r, "\psi_{C',C'}"'] & F(C' \otimes C')
\end{tikzcd}$$

is commutative, hence $F(\varphi \otimes \varphi \circ \delta) = F(\delta' \circ \varphi)$. Since F is faithful, this yields the desired equality $\varphi \otimes \varphi \circ \delta = \delta' \circ \varphi$, and the compatibility with counits follows similarly.

It remains to show that $\widehat{F}$ is essentially surjective. If $(C', \delta', \varepsilon')$ is any comonoid in $\mathscr{M}'$, there is an object C of $\mathscr{M}$ such that C' and FC are isomorphic. There is then a unique comonoid structure on FC such that a chosen isomorphism $FC \to C'$ is an isomorphism of comonoids. Without loss of generality we can therefore assume that $FC = C'$. We let $\delta : C \to C \otimes C$ and $\varepsilon : C \to I$ be the unique morphisms with $F\delta = \psi_{C,C} \circ \delta'$ and $F\varepsilon = \psi_0 \circ \varepsilon$. We are done if we can show that (C, δ, ε) is a comonoid in $\mathscr{M}$, because we then obviously have $\widehat{F}(C, \delta, \varepsilon) = (C', \delta', \varepsilon')$. We illustrate how one can check that (C, δ, ε) is a comonoid by proving the coassociativity axiom. This can be done with the same strategy which was used to prove that $\widehat{F}$ is well-defined: The outer composite in the diagram

$$\begin{tikzcd}
FC \arrow[rr, "F\delta"] \arrow[dd, "F\delta"'] & & F(C \otimes C) \arrow[rr, "\psi^{-1}_{C,C}"] \arrow[d, "F(\delta \otimes C)"] & & FC \otimes' FC \arrow[d, "F\delta \otimes' FC"] \\
 & (1) & F((C \otimes C) \otimes C) \arrow[rr, "\psi^{-1}_{C \otimes C,C}"'] \arrow[d, "F(a)"] & (2) & F(C \otimes C) \otimes' FC \arrow[d, "\psi^{-1}_{C,C} \otimes' FC"] \\
F(C \otimes C) \arrow[rr, "F(C \otimes \delta)"] \arrow[d, "\psi^{-1}_{C,C}"'] & (3) & F(C \otimes (C \otimes C)) \arrow[d, "\psi^{-1}_{C,C \otimes C}"] & (4) & (FC \otimes' FC) \otimes' C) \arrow[d, "a'"] \\
F(C) \otimes' F(C) \arrow[rr, "FC \otimes' F\delta"'] & & FC \otimes' F(C \otimes C) \arrow[rr, "FC \otimes' \psi^{-1}_{C,C}"'] & & FC \otimes' (FC \otimes' FC)
\end{tikzcd}$$

is precisely the coassociativity axiom for δ', because $\psi^{-1}_{C,C} \circ F\delta = \delta'$. Since the parts $(2), (3), (4)$ are commutative, it follows that (1) is commutative, too. Coassociativity of (C, δ, ε) now follows by faithfulness of F. ☐

Proposition 2.2.2. *The functor* $e = \mathrm{ev}_R \colon \mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R] \to \mathbf{Mod}_R$ *is strong monoidal, with* $\psi_0 = \mathrm{id} : R \to \mathrm{id}(R)$ *and* $\psi_{F,G} : e(F) \otimes e(G) \to e(F \circ G)$ *given by* $(\xi_F)_{GR} : FR \otimes GR \to F(GR)$*, where* $\xi_F : FR \otimes - \Rightarrow F$ *is the unique natural transformation with* $(\xi_F)_R = r : FR \otimes R \to FR$.

Proof. The diagrams

$$\begin{tikzcd}
e(F) \otimes R \arrow[r, "r"] \arrow[d, "e(F) \otimes \psi_0"'] & e(F) \\
e(F) \otimes e(\mathrm{id}) \arrow[r, "\psi_{F,\mathrm{id}}"'] & e(F \circ \mathrm{id}) \arrow[u, "e(r)"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \otimes e(F) \arrow[r, "l"] \arrow[d, "e(F) \otimes \psi_0"'] & e(F) \\
e(\mathrm{id}) \otimes e(F) \arrow[r, "\psi_{\mathrm{id},F}"'] & e(\mathrm{id} \circ F) \arrow[u, "e(l)"']
\end{tikzcd}$$

reduce in this context to the diagrams

$$\begin{tikzcd}
FR \otimes R \arrow[r, "r"] \arrow[d, "FR \otimes \mathrm{id}"'] & FR \\
FR \otimes R \arrow[r, "(\xi_F)_R"'] & FR \arrow[u, "\mathrm{id}"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
R \otimes FR \arrow[r, "l"] \arrow[d, "FR \otimes \mathrm{id}"'] & FR \\
R \otimes e(F) \arrow[r, "(\xi_{\mathrm{id}})_{FR}"'] & FR, \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

which are commutative because $(\xi_F)_R = r$ by definition and because $(\xi_{\mathrm{id}})_M = l_M \colon R \otimes M \to M$. The latter follows from the fact that $l_R = r_R \colon R \otimes R \to R$, so $l : R \otimes - \Rightarrow \mathrm{id}_{\mathbf{Mod}_R}$ is indeed the unique natural transformation with R-component $r_R : R \otimes R \to R$.

It remains to show that for cocontinuous functors $F, G, H : \mathbf{Mod}_R \to \mathbf{Mod}_R$, the diagram

$$\begin{tikzcd}
(e(F) \otimes e(G)) \otimes e(H) \arrow[r, "a"] \arrow[d, "\psi_{F,G} \otimes e(H)"'] & e(F) \otimes (e(G) \otimes e(H)) \arrow[d, "e(F) \otimes \psi_{G,H}"] \\
e(F \circ G) \otimes e(H) \arrow[d, "\psi_{F \circ G,H}"'] & e(F) \otimes e(G \circ H) \arrow[d, "\psi_{F,G \circ H}"] \\
e((F \circ G) \circ H) \arrow[r, "e(\mathrm{id})"'] & e(F \circ (G \circ H))
\end{tikzcd}$$

is commutative. Using the definition of e and ψ we find that this is equivalent to commutativity of

$$\begin{tikzcd}
(FR \otimes GR) \otimes HR \arrow[r, "a"] \arrow[d, "(\xi_F)_{GR} \otimes HR"'] & FR \otimes (GR \otimes HR) \arrow[d, "FR \otimes (\xi_G)_{HR}"] \\
FGR \otimes HR \arrow[d, "(\xi_{FG})_{HR}"'] & FR \otimes GHR \arrow[d, "(\xi_F)_{GHR}"] \\
FGHR \arrow[r, "\mathrm{id}"'] & FGHR,
\end{tikzcd}$$

which is an instance of the more general diagram

$$\begin{tikzcd}
(FR \otimes GR) \otimes M \arrow[r, "a"] \arrow[d, "(\xi_F)_{GR} \otimes M"'] & FR \otimes (GR \otimes M) \arrow[d, "FR \otimes (\xi_G)_{M}"] \\
FGR \otimes M \arrow[d, "(\xi_{FG})_{M}"'] & FR \otimes GM \arrow[d, "(\xi_F)_{GM}"] \\
FGM \arrow[r, "\mathrm{id}"'] & FGM,
\end{tikzcd}$$

with $M = HR$. Since both morphisms in the previous diagram are natural in M, corollary 1.4.4 implies that it suffices to check its commutativity for $M = R$. This means that we are done if we can show that the outer composites in the diagram

$$\begin{tikzcd}
(FR \otimes GR) \otimes R \arrow[r, "a"] \arrow[d, "(\xi_F)_{GR} \otimes R"'] \arrow[dr, "r"'] & FR \otimes (GR \otimes R) \arrow[d, "FR \otimes (\xi_G)_{R} = FR \otimes r"] \\
FGR \otimes R \arrow[d, "(\xi_{FG})_{R} = r"'] & FR \otimes GR \arrow[d, "(\xi_F)_{GR}"] \\
FGR \arrow[r, "\mathrm{id}"'] & FGR
\end{tikzcd}$$

are equal. But the lower pentagon of this diagram commutes by naturality of r, and the upper triangle is commutative by Mac Lane's Coherence Theorem for monoidal categories. ☐

Proposition 2.2.3. *The functor* $H\colon \mathbf{Mod}_R \to \mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ *which sends an* R*-module* M *to* $M \otimes -\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ *and* $\varphi\colon M \to M'$ *to the natural transformation* $\varphi \otimes -$ *is strong monoidal, with* $\varphi_0 = l^{-1}\colon \mathrm{id} \Rightarrow R \otimes -$ *and*

$$\varphi_{M,N} = a^{-1}_{M,N,-}\colon H(M) \circ H(N)(-) = M \otimes (N \otimes -) \Rightarrow (M \otimes N) \otimes -.$$

Proof. This is a direct consequence of Mac Lane's coherence theorem for monoidal categories. For example, unraveling the definitions and using corollary 1.4.4 we find that commutativity of the diagram of natural transformations

$$\begin{tikzcd}
(H(M) \circ H(N)) \circ H(L) \arrow[r, "\mathrm{id}"] \arrow[d, "\varphi_{M,N} * H(L)"'] & H(M) \circ (H(N) \circ H(L)) \arrow[d, "H(M) * \varphi_{N,L}"] \\
H(M \otimes N) \circ H(L) \arrow[d, "\varphi_{M \otimes N, L}"'] & H(M) \circ H(N \otimes L) \arrow[d, "\varphi_{M, N \otimes L}"] \\
H((M \otimes N) \otimes L) \arrow[r, "H(a)"'] & H(M \otimes (N \otimes L))
\end{tikzcd}$$

is equivalent to commutativity of

$$\begin{tikzcd}
M \otimes (N \otimes (L \otimes R)) \arrow[r, "\mathrm{id}"] \arrow[d, "a^{-1}_{M,N,L\otimes R}"'] & M \otimes (N \otimes (L \otimes R)) \arrow[d, "M\otimes a^{-1}_{N,L,R}"] \\
(M \otimes N) \otimes (L \otimes R) \arrow[d, "a^{-1}_{M\otimes N,L,R}"'] & M \otimes ((N \otimes L) \otimes R) \arrow[d, "a^{-1}_{M,N\otimes L,R}"] \\
((M \otimes N) \otimes L) \otimes R \arrow[r, "a\otimes R"'] & (M \otimes (N \otimes L)) \otimes R,
\end{tikzcd}$$

which is indeed commutative by the coherence theorem. The remaining axioms follow in a similar fashion. ☐

Lemma 2.2.4. *Let* $(T, \delta^T, \varepsilon^T)$ *be a cocontinuous* R*-linear comonad. With the notations of the propositions 2.2.1, 2.2.2 and 2.2.3, the natural transformation*

$$\xi_T^{-1} \colon T \Rightarrow \widehat{H}\widehat{e}(T)$$

is a morphism of cocontinuous R*-linear comonads.*

Proof. We let $(C, \delta, \varepsilon) = \widehat{e}(T)$. By proposition 2.2.1 it follows that δ is given by the composite

$$TR \xrightarrow{\delta^T_R} TTR \xrightarrow{(\xi_T)^{-1}_{TR}} TR \otimes TR,$$

and that $\varepsilon = \varepsilon^T_R \colon TR \to R$. The same proposition, applied to H, implies that the comultiplication

$$\widehat{H}\widehat{e}(C) \Rightarrow \widehat{H}\widehat{e}(C) \circ \widehat{H}\widehat{e}(C)$$

has as M-component the composite

$$TR \otimes M \xrightarrow{\delta^T_R \otimes M} TTR \xrightarrow{(\xi_T)^{-1}_{TR}\otimes M} (TR \otimes TR) \otimes M \xrightarrow{a_{TR,TR,M}} TR \otimes (TR \otimes M).$$

In order to show that ξ_T^{-1} is a morphism of comonoids we thus have to check that for every R-module M, the diagram

$$\begin{tikzcd}
TM \arrow[rr, "(\xi_T)^{-1}_M"] \arrow[ddd, "\delta^T_M"'] & & TR \otimes M \arrow[d, "\delta^T_R \otimes M"] \\
& & TTR \otimes M \arrow[d, "(\xi_T)^{-1}_{TR} \otimes M"] \\
& & (TR \otimes TR) \otimes M \arrow[d, "a_{TR,TR,M}"] \\
TTM \arrow[r, "(\xi_T)^{-1}_{TM}"'] & TR \otimes TM \arrow[r, "TR \otimes (\xi_T)^{-1}_M"'] & TR \otimes (TR \otimes M)
\end{tikzcd}$$

is commutative, where the lower composite is the M-component of $\xi_T^{-1} * \xi_T^{-1}$. By corollary 1.4.4 it suffices to check that the outer square of the diagram

$$\begin{tikzcd}
TR \arrow[r, "(\xi_T)_R^{-1} = r^{-1}"] \arrow[d, "\delta_R^T"'] & TR \otimes R \arrow[d, "\delta_R^T \otimes R"] \\
TTR \arrow[r, "r^{-1}"] \arrow[dd, "(\xi_T)_{TR}^{-1}"'] & TTR \otimes R \arrow[d, "(\xi_T)_{TR}^{-1} \otimes R"] \\
 & (TR \otimes TR) \otimes R \arrow[d, "a_{TR,TR,R}"] \\
TR \otimes TR \arrow[r, "TR \otimes (\xi_T)_R^{-1} = TR \otimes r^{-1}"'] & TR \otimes (TR \otimes R)
\end{tikzcd}$$

is commutative. But this is a direct consequence of the fact that r^{-1} is natural and the observation that $TR \otimes r_{TR}^{-1} = a_{TR,TR,R} \circ r_{TR \otimes TR}^{-1}$. The counit of $\widehat{H}\widehat{e}(T)$ has as M-component the composite

$$TR \otimes M \xrightarrow{\varepsilon_R^T \otimes M} R \otimes M \xrightarrow{l} M.$$

Thus it remains to check that the diagram

$$\begin{tikzcd}
TM \arrow[r, "(\xi_T)_M^{-1}"] \arrow[d, "\varepsilon_M^T"'] & TR \otimes M \arrow[d, "\varepsilon_R^T \otimes M"] \\
M & R \otimes M \arrow[l, "l"]
\end{tikzcd}$$

is commutative for every R-module M, which is equivalent to commutativity of

$$\begin{tikzcd}
TR \arrow[r, "(\xi_T)_R^{-1} = r^{-1}"] \arrow[d, "\varepsilon_R^T"'] & TR \otimes R \arrow[d, "\varepsilon_R^T \otimes R"] \\
R & R \otimes R \arrow[l, "l_R"]
\end{tikzcd}$$

by corollary 1.4.4. This follows by naturality of r because $l_R = r_R \colon R \otimes R \to R$. $\square$

Lemma 2.2.5. *Let (C, δ, ε) be a coalgebra. Using the notations of the propositions 2.2.1, 2.2.2 and 2.2.3, $r_C \colon C \otimes R \to C$ is a morphism of coalgebras $\widehat{e}\widehat{H}(C) \to C$.*

Proof. Unraveling the definitions we find that the comultiplication of $\widehat{e}\widehat{H}(C)$ is given by the composite

$$C \otimes R \xrightarrow{\delta \otimes R} (C \otimes C) \otimes R \xrightarrow{a_{C,C,R}} C \otimes (C \otimes R) \xrightarrow{(\xi_{C \otimes -}^{-1})_{C \otimes R}} (C \otimes R) \otimes (C \otimes R)$$

and its counit is given by

$$C \otimes R \xrightarrow{\varepsilon \otimes R} R \otimes R \xrightarrow{l} R.$$

The diagram

$$\begin{tikzcd}
C \otimes R \arrow[r, "r_C"] \arrow[d, "\varepsilon \otimes R"'] & C \arrow[d, "\varepsilon"] \\
R \otimes R \arrow[r, "l_R = r_R"'] & R
\end{tikzcd}$$

is commutative by naturality of r, which shows that r_C is compatible with the counits. Since the R-component of the natural transformation $r_C \otimes - \colon (C \otimes R) \otimes - \Rightarrow C \otimes -$ is $r_C \otimes R = r_{C \otimes R} \colon (C \otimes R) \otimes R \to C \otimes R$, it follows that $\xi_{C \otimes -} = r_C \otimes -$ (see proposition 2.2.2). Therefore it suffices to check that the diagram

$$\begin{tikzcd}
C \otimes R \arrow[r, "\delta \otimes R"] \arrow[d, "r_C"'] & (C \otimes C) \otimes R \arrow[r, "a_{C,C,R}"] \arrow[d, "r_{C \otimes C}"] & C \otimes (C \otimes R) \arrow[d, "r_C^{-1} \otimes R"] \\
C \arrow[r, "\delta"'] & (C \otimes C) & (C \otimes R) \otimes (C \otimes R) \arrow[l, "r_C \otimes r_C"]
\end{tikzcd}$$

is commutative. This follows immediately from the coherence theorem. □

Proposition 2.2.6. *With the notation of the propositions 2.2.1, 2.2.2 and 2.2.3, the functors*

$$\mathbf{T} = \widehat{H} \colon \mathbf{Coalg}_R \to \mathbf{CC}_R$$

and

$$\mathbf{C} = \widehat{e} \colon \mathbf{CC}_R \to \mathbf{Coalg}_R$$

are mutually inverse equivalences, with natural isomorphisms $\pi \colon \mathrm{id} \Rightarrow \mathbf{T} \circ \mathbf{C}$ *given by* $\pi_T = \xi_T^{-1} \colon T \Rightarrow \mathbf{T} \circ \mathbf{C}(T)$ *and* $\beta \colon \mathbf{C} \circ \mathbf{T} \Rightarrow \mathrm{id}$ *given by* $\beta_C = r_C \colon C \otimes R \to C$. *Moreover,* π *and* β *are the unit and counit of the adjunction* $\mathbf{C} \dashv \mathbf{T}$.

Proof. It follows from the lemmas 2.2.4 and 2.2.5 that π_T and β_C are well-defined, i.e., that they really are morphisms in the desired categories. Naturality of β follows from naturality of r, and naturality of π_T in T reduces by corollary 1.4.4 to commutativity of

$$\begin{tikzcd}
TR \arrow[r, "(\xi_T^{-1})_R = r^{-1}"] \arrow[d, "\varphi_R"'] & TR \otimes R \arrow[d, "\varphi_R \otimes R"] \\
T'R \arrow[r, "(\xi_{T'}^{-1})_R = r^{-1}"'] & T'R \otimes R
\end{tikzcd}$$

where $\varphi \colon T \to T'$ is a morphism of comonoids in $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$. This is again a consequence of the naturality of r. It remains to check that the triangular identities hold for π and β. For every comonoid $(T, \delta^T, \varepsilon^T)$ in $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ we have

$$\beta_{\mathbf{C}(T)} \circ \mathbf{C}(\pi_T) = r_{TR} \circ (\xi_T^{-1})_R = r_{TR} \circ r_{TR}^{-1} = \mathrm{id}_{TR}$$

by definition of ξ_T^{-1} (see proposition 2.2.2). On the other hand, we have for every comonoid (C, δ, ε) in $\mathbf{Mod}_R$ the equality $\xi_{\mathbf{T}(C)}^{-1} = r_C^{-1} \otimes -$ between natural transformations, because their R-components are equal. It follows that

$$\big(\mathbf{T}(\beta_C) \circ \pi_{\mathbf{T}(C)}\big)_M = r_C \otimes M \circ r_C^{-1} \otimes M = \mathrm{id}_{C \otimes M}$$

for every R-module M. These two equations are precisely the triangular identities for π and β. □

2.3. **Coalgebras and comodules.** In this chapter we prove some basic facts about comonoids in $\mathbf{Mod}_R$, define the category of comodules of such a comonoid and introduce the standard terminology. In the next section we will introduce the standard terminology for comonoids in $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ and give basic results and constructions.

Definition 2.3.1. A comonoid (C, δ, ε) in $\mathbf{Mod}_R$ (see definition 2.2.1) is called a *coalgebra*. The coalgebra (C, δ, ε) is called *flat* if the R-module C is flat. We denote the category of coalgebras by $\mathbf{Coalg}_R$.

Definition 2.3.2. Let (C, δ, ε) be a coalgebra. A *(left) comodule* of C is an R-module M together with a *coaction* $\rho\colon M \to C \otimes M$, that is, a homomorphism $\rho\colon M \to C \otimes M$ of R-modules such that the diagrams

$$\begin{tikzcd}
M \arrow[rr, "\rho"] \arrow[d, "\rho"'] & & C \otimes M \arrow[d, "\delta\otimes M"] \\
C \otimes M \arrow[r, "M\otimes\rho"'] & C \otimes (C \otimes M) \arrow[r, "a^{-1}"'] & (C \otimes C) \otimes M
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\mathrm{id}"'] & C \otimes M \arrow[d, "\varepsilon\otimes M"] \\
M & R \otimes M \arrow[l, "l"]
\end{tikzcd}$$

are commutative. A comodule M is called a *Cauchy comodule* if the R-module M is Cauchy, i.e., if it is finitely generated and projective. A morphism of comodules $\varphi\colon (M, \rho) \to (M', \rho')$ is a homomorphism $\varphi\colon M \to M'$ such that the diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\rho"'] & M' \arrow[d, "\rho'"] \\
C \otimes M \arrow[r, "C\otimes\varphi"'] & C \otimes M'
\end{tikzcd}$$

is commutative. The category of C-comodules and morphisms between them is denoted by $\mathbf{Comod}(C)$, and the full subcategory of Cauchy comodules is denoted by $\mathbf{Comod}^c(C)$.

Proposition 2.3.1. *For any coalgebra (C, δ, ε), the functor $W = C \otimes -\colon \mathbf{Mod}_R \to \mathbf{Comod}(C)$ which sends a module N to the C-comodule $(C \otimes N, \delta \otimes N)$ is right adjoint to the forgetful functor $V\colon \mathbf{Comod}(C) \to \mathbf{Mod}_R$ which sends a comodule (M, ρ) to the underlying module M. The unit of this adjunction is given by $\rho\colon (M, \rho) \to (C \otimes M, \delta \otimes M)$.*

Proof. It follows directly from the definition of a coalgebra that $(C \otimes N, \delta \otimes N)$ is a C-comodule. The natural bijection

$$f\colon \mathbf{Comod}(C)\big((M, \rho), (C \otimes N, \delta \otimes N)\big) \cong \mathbf{Mod}_R(M, N)$$

is given by $f(\varphi) = \varepsilon \otimes N \circ \varphi$ and $f^{-1}(\psi) = C \otimes \psi \circ \rho$. In particular, $\eta_M = f^{-1}(\mathrm{id}_M) = \rho$. ☐

Proposition 2.3.2. *Let (C, δ, ε) be a coalgebra. Then the forgetful functor $V : \mathbf{Comod}(C) \to \mathbf{Mod}_R$ creates colimits. This means that for any diagram $F : \mathscr{D} \to \mathbf{Comod}(C)$, if the colimit X of the diagram $VF : \mathscr{D} \to \mathbf{Mod}_R$ exists, then there is a unique coaction ρ on X such that the structure morphisms become morphisms of comodules and (X, ρ) with these structure morphisms is a colimit of F.*

Proof. If we demand that the structure maps $\eta_D : FD \to X$ be comodule morphisms, the coaction ρ on X must make the diagram

$$\begin{tikzcd}
FD \arrow[r, "\eta_D"] \arrow[d, "\rho_{FD}"'] & X \arrow[d, "\rho"] \\
C \otimes FD \arrow[r, "C\otimes\eta_D"'] & C \otimes X
\end{tikzcd}$$

commutative for every object D of $\mathscr{D}$. Since X is the colimit of the diagram VF there is a unique such ρ, and one can easily check that (X, ρ) is the colimit of F in $\mathbf{Comod}(C)$. ☐

Proposition 2.3.3. *Let (C, δ, ε) be a coalgebra. Then the forgetful functor $V : \mathbf{Comod}(C) \to \mathbf{Mod}_R$ creates those limits which are preserved by the functors $C \otimes -\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ and $(C \otimes C) \otimes -\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$. In other words, if*

$F\colon \mathscr{D} \to \mathbf{Comod}(C)$ is a diagram such that the limit $(L, (\kappa_D)_{D\in\mathscr{D}})$ of the diagram $VF\colon \mathscr{D} \to \mathbf{Mod}_R$ exists and is preserved by $C \otimes -$ and $(C \otimes C) \otimes -$, then there is a unique coaction $\rho\colon L \to C \otimes L$ such that the structure morphisms become morphisms of comodules and (L, ρ) with these structure morphisms is a limit of F.

Proof. The fact that $C \otimes -$ preserves the limit L implies that $(C \otimes L, (C \otimes \kappa_D)_{D\in\mathscr{D}})$ is a limit of the diagram $C \otimes VF(-)$. Thus there is a unique morphism $\rho\colon L \to C \otimes L$ such that the diagram

$$\begin{tikzcd}
L \arrow[r, "\kappa_D"] \arrow[d, "\rho"'] & FD \arrow[d, "\rho_{FD}"] \\
C \otimes L \arrow[r, "C\otimes\kappa_D"'] & C \otimes FD
\end{tikzcd}$$

is commutative for every $D \in \mathscr{D}$. Because $(C \otimes C) \otimes -$ preserves the limit L it follows that ρ is a coaction. Proving that (L, ρ) is the limit of F is straightforward. $\square$

Corollary 2.3.4. *For any coalgebra (C, δ, ε), the category $\mathbf{Comod}(C)$ is cocomplete. If C is flat, then $\mathbf{Comod}(C)$ is abelian.*

Proof. This is a direct consequence of proposition 2.3.2 and proposition 2.3.3. $\square$

Proposition 2.3.5. *Let C be a flat coalgebra, M a C-comodule and let $m \in M$. Then there is a subcomodule $M_0 \subseteq M$ containing m and a submodule $N \subseteq M$ such that N is a finitely generated R-module and $M_0 \subseteq N$. In particular, if R is Noetherian, then M_0 is finitely generated as R-module.*

Proof. There are elements $n_i \in M$ and $c_i \in C$ such that $\rho(m) = \sum_{i=1}^n c_i \otimes n_i$. Let N be the submodule of M generated by the n_i, $i = 1, \ldots, n$ and write $i : N \to M$ for the inclusion. By proposition 2.3.1 the morphism $C \otimes i\colon C \otimes N \to C \otimes M$ is a morphism of comodules. Since C is flat, $C \otimes i$ is injective. Now let E be the pullback (in $\mathbf{Comod}(C)$) of $\rho : M \to C \otimes M$ and $C \otimes i$. The diagram

$$\begin{tikzcd}
E \arrow[r, "\sigma"] \arrow[d, "j"'] & C \otimes N \arrow[r, "N\otimes\varepsilon"] \arrow[d, "C\otimes i"] & N \arrow[d, "i"] \\
M \arrow[r, "\rho"] \arrow[rr, bend right, "\mathrm{id}"'] & C \otimes M \arrow[r, "M\otimes\varepsilon"] & M,
\end{tikzcd}$$

where the left square is a pullback diagram, is commutative. By proposition 2.3.3 it follows that σ and j are injective. Commutativity of the diagram implies that $M_0 = j(E)$ is a submodule of N, and m lies in M_0 because $\rho(m)$ lies in the image of $C \otimes i$. $\square$

2.4. **Comonads and comonadicity.**

Definition 2.4.1. A *comonad* on a category $\mathscr{C}$ is an endofunctor $T : \mathscr{C} \to \mathscr{C}$ together with natural transformations $\delta : T \Rightarrow T \circ T$ (the *comultiplication* of T) and $\varepsilon : T \Rightarrow \mathrm{id}_{\mathscr{C}}$ (the *counit* of T) such that the equations

$$T\delta \circ \delta = \delta T \circ \delta, \quad T\varepsilon \circ \delta = 1_T \quad \text{and} \quad \varepsilon T \circ \delta = 1_T$$

hold. A *morphism* between comonads (T, δ, ε) and $(T', \delta', \varepsilon')$ is a natural transformation $\alpha : T \Rightarrow T'$ satisfying the equations

$$\varepsilon' \circ \alpha = \varepsilon \quad \text{and} \quad \alpha * \alpha \circ \delta = \delta' \circ \alpha.$$

Remark 2.4.1. A comonoid in $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ is a comonad on $\mathbf{Mod}_R$ whose underlying functor is cocontinuous and R-linear, and a morphism of comonoids in $\mathrm{Cocts}[\mathbf{Mod}_R, \mathbf{Mod}_R]$ is precisely a morphism of the corresponding comonads.

Definition 2.4.2. Let (T, δ, ε) be a comonad on $\mathscr{C}$. A T-*comodule*[3] consists of an object C of $\mathscr{C}$, together with a morphism $\xi : C \to TC$ such that $\varepsilon_C \circ \xi = \mathrm{id}$ and $\delta_C \circ \xi = T(\xi) \circ \xi$. A morphism of T-comodules $(C, \xi) \to (C', \xi')$ is a morphism $\varphi : C \to C'$ in $\mathscr{C}$ such that $\xi' \circ \varphi = T(\varphi) \circ T$. The category of T-comodules and morphisms between them is denoted by $\mathrm{Comod}(T)$.

Recall from definition 1.2.1 that the category $\mathbf{cat}_R \,/\, \mathbf{Mod}^c_R$ has objects the pairs $(\mathscr{A}, \omega)$, where $\mathscr{A}$ is an essentially small R-linear category and $\omega\colon \mathscr{A} \to \mathbf{Mod}^c_R$ is an R-linear functor, and the morphisms between two objects $(\mathscr{A}, \omega)$ and $(\mathscr{A}', \omega')$ in $\mathbf{cat}_R \,/\, \mathbf{Mod}^c_R$ are the R-linear functors $F\colon \mathscr{A} \to \mathscr{A}'$ making the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[rr, "F"] \arrow[dr, "\omega"'] & & \mathscr{A}' \arrow[dl, "\omega'"] \\
& \mathbf{Mod}^c_R &
\end{tikzcd}$$

commutative.

Definition 2.4.3. Recall that the category of R-linear cocontinuous comonads $\mathbf{Mod}_R \to \mathbf{Mod}_R$ and morphisms of comonads is denoted by $\mathbf{CC}_R$ (see definition 2.2.2). For any $T \in \mathbf{CC}_R$, a T-comodule (M, ξ) is called *Cauchy* if the underlying module lies in $\mathbf{Mod}^c_R$ (see definition 1.2.1). The category of Cauchy T-comodules is denoted by $\mathrm{Comod}^c(T)$, and the forgetful functor is denoted by

$$V_T\colon \mathrm{Comod}^c(T) \longrightarrow \mathbf{Mod}_R\,.$$

Any morphism of comonads $\varphi\colon T \to T'$ in $\mathbf{CC}_R$ induces a morphism

$$\mathrm{Comod}^c(\varphi)\colon \mathrm{Comod}^c(T) \to \mathrm{Comod}^c(T'),$$

given on objects by $(M, \xi) \mapsto (M, \varphi_M \circ \xi)$ and on morphisms by the identity. This construction extends to a functor

$$\mathrm{Comod}^c(-)\colon \mathbf{CC}_R \longrightarrow \mathbf{cat}_R\,/\mathbf{Mod}^c_R,$$

which sends T to $(\mathrm{Comod}^c(T), V_T)$ and $\varphi\colon T \to T'$ to $\mathrm{Comod}^c(\varphi)$. We call this functor the *comodule functor.*

Remark 2.4.2. The comodule functor is well-defined: the category $(\mathrm{Comod}^c(T), V_T)$ is small since $\mathbf{Mod}^c_R$ is small (see definition 1.2.1), and for any R-module M the collection of all possible coactions $M \to T(M)$ forms a subset of the set of all homomorphisms $M \to T(M)$.

Proposition 2.4.3. *Let (C, δ, ε) be a coalgebra. With $\mathbf{T}(C)\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ as in proposition 2.2.6, the categories* $\mathrm{Comod}(\mathbf{T}(C))$ *of $\mathbf{T}(C)$-comodules is equal to the category* $\mathbf{Comod}(C)$ *of definition 2.3.2.*

Proof. This follows directly from the definition of a T-comodule (see definition 2.4.2) for a comonad T, the definition of $\mathbf{T}(C)$ in proposition 2.2.6, and the definition of a comodule of a coalgebra C (see definition 2.3.2). □

This suggests the following generalization of proposition 2.3.1.

Proposition 2.4.4. *Let (T, δ, ε) be a comonad on $\mathscr{C}$. Then the functor $W = W_T\colon \mathscr{C} \to$* $\mathrm{Comod}(T)$ *which sends an object C of $\mathscr{C}$ to the T-comodule (TC, δ_C) and a morphism $\varphi : C \to C'$ to $T(\varphi)$ is right adjoint to the forgetful functor $V = V_T :$* $\mathrm{Comod}(T) \to \mathscr{C}$ *which sends (C, ξ) to C. The unit of this adjunction is given by $\eta_{(M,\xi)} = \xi\colon M \to TM = VW(M)$.*

---

[3] Note that a T-comodule is usually called a T-coalgebra. However, in this context this terminology would be rather misleading, and our choice is for example justified by proposition 2.4.3

Proof. First we have to check that W is a well-defined functor, i.e., that (TC, δ_C) is a T-comodule and that $T(\varphi)$ is a morphism of T-comodules. By definition 2.4.1 we have $T\delta \circ \delta = \delta T \circ \delta$. Looking at the C-component of this natural transformation we find that $T(\delta_C) \circ \delta_C = \delta_{TC} \circ \delta_C$, which is one of the two axioms which must hold for (TC, δ_C) to be a T-comodule. Again by definition of a comonad we must have $\varepsilon T \circ \delta = 1_T$, which shows that $\varepsilon_{TC} \circ \delta_C = \mathrm{id}_{TC}$, i.e., that the second axiom holds. So (TC, δ_C) is indeed a T-comodule, and $T(\varphi)$ is a morphism of T-comodules by naturality of $\delta \colon T \Rightarrow T \circ T$.
The natural bijection

$$f : \mathscr{C}(V(C,\xi), C') \to \mathrm{Comod}(T)((C,\xi),(TC', \delta_{C'}))$$

is given by $f(\alpha) = T(\alpha) \circ \xi$ with inverse given by $f^{-1}(\beta) = \varepsilon_{C'} \circ \beta$. These are indeed mutually inverse: for any $\alpha : C \to C'$ the equalities

$$\begin{aligned} f^{-1}(f(\alpha)) &= \varepsilon_{C'} \circ T(\alpha) \circ \xi \\ &= \alpha \circ \varepsilon_C \circ \xi \\ &= \alpha \end{aligned}$$

hold, by naturality of ε and because (C, ξ) is a T-comodule respectively. For any $\beta : (C,\xi) \to (TC', \delta_{C'})$ we have

$$\begin{aligned} f(f^{-1}(\beta)) &= \circ T(\varepsilon_{C'} \circ \beta) \circ \xi \\ &= T(\varepsilon_{C'}) \circ T(\beta) \circ \xi \\ &= T(\varepsilon_{C'}) \circ \delta_C \circ \beta \\ &= \beta \end{aligned}$$

since β is a morphism of T-comodules and because $T\varepsilon \circ \delta = 1_T$. One can easily check that this bijection is natural in (C, ξ) and C'. ☐

In the previous proposition we have seen that for any comonad (T, δ, ε) on $\mathscr{C}$ we have an associated adjoint pair $V \dashv W \colon \mathrm{Comod}(T) \to \mathscr{C}$. The next proposition concerns the converse situation. It gives a construction for a comonad on $\mathscr{C}$ associated to a given adjoint pair $F \dashv G \colon \mathscr{B} \to \mathscr{C}$.

Proposition 2.4.5. *Let $F \colon \mathscr{B} \to \mathscr{C}$ be left adjoint to $G : \mathscr{C} \to \mathscr{B}$, with unit $\eta \colon \mathrm{id} \Rightarrow GF$ and counit $\varepsilon \colon FG \Rightarrow \mathrm{id}$. Then $(FG, F\eta G, \varepsilon)$ is a comonad on $\mathscr{C}$.*

Proof. The pasted composite of the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "G"'] & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[r, "F"'] & \cdot \\
& & \cdot \arrow[ur, "G"'] & & \cdot \arrow[ur, "G"'] & &
\end{tikzcd}$$

can be computed in two ways, either as the vertical composite

$$\begin{tikzcd}
\cdot \arrow[r, "G"'] & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[rr, bend left, "\mathrm{id}"] & & \cdot \arrow[r, "F"'] & \cdot \\
& & \cdot \arrow[ur, "G"'] & \circ & & & \\
\cdot \arrow[r, "G"'] & \cdot \arrow[dr, "F"'] & & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[r, "F"'] & \cdot \\
& & \cdot \arrow[ur, "G"'] & & \cdot \arrow[ur, "G"'] & &
\end{tikzcd}$$

or as the vertical composite

$$\begin{tikzcd}
\cdot \arrow[r, "G"'] & \cdot \arrow[rr, bend left, "\mathrm{id}"] & & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[r, "F"'] & \cdot \\
& & & \circ & \cdot \arrow[ur, "G"'] & & \\
\cdot \arrow[r, "G"'] & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] \arrow[rr, phantom, "\Downarrow\eta"] & & \cdot \arrow[dr, "F"'] & & \cdot \arrow[r, "F"'] & \cdot. \\
& & \cdot \arrow[ur, "G"'] & & \cdot \arrow[ur, "G"'] & &
\end{tikzcd}$$

Since the pasting composite is well-defined it follows that these two vertical composites are equal, i.e., that $FGF\eta G \circ F\eta G = F\eta GFG \circ F\eta G$. The triangular identities for the adjunction yield the equalities

$$\begin{tikzcd}
\cdot \arrow[r, "G"] & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"'] & \Downarrow\eta & \cdot \arrow[r, "F"] & \cdot & = & \cdot \arrow[r, bend left, "FG"] \arrow[r, bend right, "FG"'] \arrow[r, phantom, "\Downarrow 1"] & \cdot \\
 & & \cdot \arrow[ur, "G"] \arrow[urr, bend right, "\mathrm{id}"'] & \Downarrow\varepsilon & & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdot \arrow[r, "G"] \arrow[drr, bend right, "\mathrm{id}"'] & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "F"] & \Downarrow\eta & \cdot \arrow[r, "F"] & \cdot & = & \cdot \arrow[r, bend left, "FG"] \arrow[r, bend right, "FG"'] \arrow[r, phantom, "\Downarrow 1"] & \cdot, \\
 & \Downarrow\varepsilon & \cdot \arrow[ur, "G"'] & & & & &
\end{tikzcd}$$

hence that $\varepsilon FG \circ F\eta G = 1_{FG}$ and $FG\varepsilon \circ F\eta G = 1_{FG}$. ☐

Proposition 2.4.6. *Let $F\colon \mathscr{B} \to \mathscr{C}$ be left adjoint to $G : \mathscr{C} \to \mathscr{B}$, with unit $\eta\colon \mathrm{id} \Rightarrow GF$ and counit $\varepsilon\colon FG \Rightarrow \mathrm{id}$. Then there is a comparison functor $J\colon \mathscr{B} \to \mathrm{Comod}(FG)$ (where FG is endowed with the structure of a comonad as in proposition 2.4.5) which sends an object B of $\mathscr{B}$ to the FG-comodule $(FB, F\eta_B)$ and a morphism $\varphi\colon B \to B'$ to $F(\varphi)$.*

Proof. We only have to show that $J(B) = (FB, F\eta_B)$ is a FG-comodule and that $F(\varphi)$ is a morphism of FG-comodules. We have

$$\begin{aligned}(F\eta G)_{FB} \circ F\eta_B &= F(\eta_{GFB} \circ \eta_B) \\ &= F(GF(\eta_B) \circ \eta_B)\end{aligned}$$

by naturality of η, and a triangular identity shows that

$$\varepsilon_{FB} \circ F\eta_B = \mathrm{id}_{FB}\,,$$

so $(FB, F\eta_B)$ really is a FG-comodule. For any $\varphi\colon B \to B'$ we have $FGF(\varphi) \circ F\eta_B = F\eta_B \circ F(\varphi)$ by naturality of $\eta\colon \mathrm{id} \Rightarrow GF$, which shows that $F(\varphi)$ is indeed a morphism of FG-comodules. Functoriality of J follows immediately from the fact that F is a functor. ☐

It turns out that there is a nice characterization of the adjunctions $F \dashv G\colon \mathscr{B} \to \mathscr{C}$ for which the functor $J\colon \mathscr{B} \to \mathrm{Comod}(FG)$ from proposition 2.4.6 is an equivalence of categories. This will be useful when we consider the recognition problem.

Definition 2.4.4. A functor $F\colon \mathscr{B} \to \mathscr{C}$ with a right adjoint $G\colon \mathscr{C} \to \mathscr{B}$ is called *comonadic* if the comparison functor $J\colon \mathscr{B} \to \mathrm{Comod}(FG)$ from proposition 2.4.6 is an equivalence of categories.

Definition 2.4.5. The diagram

$$\begin{tikzcd}
E \arrow[r, "s"] & A \arrow[l, bend right, "p"'] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[l, bend right, "q"']
\end{tikzcd}$$

in a category $\mathscr{C}$ is called a *split equalizer* or *contractible equalizer* if $us = vs$, $ps = \mathrm{id}_E$, $qu = \mathrm{id}_A$ and $qv = sp$.

Remark 2.4.7. If

$$\begin{tikzcd}
E \arrow[r, "s"] & A \arrow[l, bend right, "p"'] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[l, bend right, "q"']
\end{tikzcd}$$

is a split equalizer, s is the equalizer of u and v.

Proof. We have to check that for any morphism $a\colon X \to A$ with $ua = va$, there is a unique $b\colon X \to E$ with $a = sb$. If there is such a morphism, we must have$b = \mathrm{id}_E \circ B = psb = pa$, hence it remains to check that $s(pa) = a$. But we have

$$s(pa) = qva = qua = \mathrm{id}_A \circ a = a,$$

which concludes the proof. ☐

Theorem 2.4.8. *A functor $F : \mathscr{B} \to \mathscr{C}$ with left adjoint $G : \mathscr{C} \to \mathscr{B}$ is comonadic if and only if the following holds:*

- *i) The functor F reflects isomorphisms, that is, a morphism f in $\mathscr{B}$ is an isomorphism whenever Ff is an isomorphism in $\mathscr{C}$.*
- *ii) If $f, g\colon A \to B$ is a pair of morphisms in $\mathscr{B}$ for which there is a split equalizer*

$$\begin{tikzcd} E \arrow[r, "s"] & FA \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Fg"'] \arrow[l, bend right, "p"'] & FB \arrow[l, bend right, "q"'] \end{tikzcd}$$

in $\mathscr{C}$, then f and g have an equalizer in $\mathscr{B}$ which is preserved by F.

Proof. This is the dual result of Beck's monadicity theorem. There are many proofs of this theorem in the literature, for example in [BW05], theorem 3.14, or in [Bor94b], theorem 4.4.4. ☐

We will later use the following special case of this result.

Corollary 2.4.9. *Let $F : \mathscr{B} \to \mathscr{C}$ be a left adjoint. If*

(a) F reflects isomorphisms,
(b) the category $\mathscr{B}$ has equalizers and
(c) the functor F preserves equalizers,

then F is comonadic.

2.5. **The comodule functor has a left adjoint.**

Proposition 2.5.1. *If $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^{\mathrm{c}}$ is an R-linear functor, the associated functor $\widetilde{\omega} : \mathbf{Mod}_R \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ (see definition 1.5.1) is cocontinuous.*

Proof. Since colimits in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ are computed pointwise, it is sufficient to show that for any object $A \in \mathscr{A}$, the functor $[\omega(A), -] : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is cocontinuous. But $\omega(A)$ is by assumption a Cauchy module, i.e., finitely generated and projective, hence this functor is naturally isomorphic to $\omega(A)^\vee \otimes - = [\omega(A), R] \otimes -$, which is indeed cocontinuous. ☐

Corollary 2.5.2. *For any essentially small R-linear category $\mathscr{A}$ and any R-linear functor $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^{\mathrm{c}}$, the comonad $\mathrm{Lan}_Y \omega \circ \widetilde{\omega}$ on $\mathbf{Mod}_R$ associated to the adjunction*

$$\mathrm{Lan}_Y \omega \dashv \widetilde{\omega}\colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \longrightarrow \mathbf{Mod}_R$$

(see proposition 1.5.3 and proposition 2.4.5) is cocontinuous.

Proof. This follows immediately from proposition 2.5.1 because $\mathrm{Lan}_Y \omega$, as a left adjoint, is cocontinuous. ☐

From now on we fix an R-linear functor $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^{\mathrm{c}}$, where $\mathscr{A}$ is an essentially small R-linear category, and we abbreviate $\mathrm{Lan}_Y \omega$ as L_ω.

Proposition 2.5.3. *Let $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ be an R-linear functor. The map*

$$(\Phi_0)_F\colon \mathrm{Nat}(L_\omega \widetilde{\omega}, F) \to \mathrm{Nat}(L_\omega, F L_\omega)$$

given by

$$\begin{tikzcd}
& {} \arrow[dr, "L_\omega"] & & & & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, phantom, "\Downarrow\eta"] \arrow[dr, "L_\omega"'] & & {} \arrow[r, "L_\omega"] & {} \\
{} \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & {} & {} \arrow[r, mapsto] & {} & & {} \arrow[ur, "\widetilde{\omega}"] \arrow[urr, bend right, "F"'] \arrow[urr, phantom, "\Downarrow\varphi"] & &
\end{tikzcd}$$

is inverse to

$$\begin{tikzcd}
{} \arrow[rr, bend left, "L_\omega"] \arrow[rr, phantom, "\Downarrow\psi"] \arrow[dr, "L(\omega)"'] & & {} & & & {} \arrow[r, "\widetilde{\omega}"] \arrow[dr, bend right, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow\varepsilon"] & {} \arrow[rr, bend left, "L_\omega"] \arrow[rr, phantom, "\Downarrow\psi"] \arrow[d, "L_\omega"] & & {} \\
& {} \arrow[ur, "F"'] & & {} \arrow[r, mapsto] & {} & & {} \arrow[urr, "F"'] & &
\end{tikzcd}.$$

Moreover, the $(\Phi_0)_F$ *are natural in* F.

Proof. The above assignments are mutually inverse since the triangular identities hold for η and ε (see proposition 1.5.3), and for any $\alpha\colon F \Rightarrow F'$ we have

$$(\Phi_0)_{F'}(\alpha \circ \varphi) = (\alpha \circ \varphi)L_\omega \circ L_\omega\eta = \alpha L_\omega \circ \varphi L_\omega \circ L_\omega\eta = \alpha L_\omega \circ (\Phi_0)_F(\varphi),$$

which shows that the $(\Phi_0)_F$ are natural in F. □

Proposition 2.5.4. *Let* $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ *be a cocontinuous* R*-linear functor. Then the map*

$$\Phi_F\colon \operatorname{Nat}(L_\omega\widetilde{\omega}, F) \longrightarrow \operatorname{Nat}(\omega, F\omega)$$

given by

$$\begin{tikzcd}
& {} \arrow[dr, "L_\omega"] & & & & {} \arrow[r, "Y"] \arrow[rrrr, bend left=40, "\omega"] \arrow[rrrr, phantom, bend left=20, "\Downarrow\alpha_\omega"] \arrow[dr, bend right, "\omega"'] \arrow[dr, phantom, "\Downarrow\alpha_\omega^{-1}"] & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[rr, phantom, "\Downarrow\eta"] \arrow[d, "L_\omega"] & & {} \arrow[r, "L_\omega"] & {} \\
{} \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Downarrow\varphi"] & & {} & {} \arrow[r, mapsto] & {} & & {} \arrow[urr, "\widetilde{\omega}"] \arrow[urrr, bend right, "F"'] \arrow[urrr, phantom, "\Downarrow\varphi"] & & &
\end{tikzcd}$$

is a natural bijection.

Proof. The morphism Φ_F is equal to the composite

$$\operatorname{Nat}(L_\omega\widetilde{\omega}, F) \xrightarrow{(\Phi_0)_F} \operatorname{Nat}(L_\omega, FL_\omega) \xrightarrow{-Y} \operatorname{Nat}(L_\omega Y, FL_\omega Y)$$

$$\xrightarrow[\operatorname{Nat}(\alpha_\omega, F\alpha_\omega^{-1})]{} \operatorname{Nat}(\omega, FL_\omega Y)$$

where $-Y$ stands for whiskering with Y (see definition 1.3.1) and $\operatorname{Nat}(\alpha_\omega, F\alpha_\omega^{-1})$ sends β to $F\alpha_\omega^{-1} \circ \beta \circ \alpha_\omega$. The map $(\Phi_0)_F$ is an bijection by proposition 2.5.3, whiskering by Y gives a bijection since Y is dense and the involved functors are cocontinuous by assumption (see proposition 1.6.1 and proposition 1.6.4). Finally, $\operatorname{Nat}(\alpha_\omega^{-1}, F\alpha_\omega)$ is inverse to $\operatorname{Nat}(\alpha_\omega, F\alpha_\omega^{-1})$. Naturality in F follows immediately from the fact that pasting composites are well-defined. □

Lemma 2.5.5. *Let* $(T, \delta^T, \varepsilon^T)\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ *be a cocontinuous* R*-linear comonad, and let* $\varphi\colon L_\omega\widetilde{\omega} \Rightarrow T$ *be a natural transformation. Then* φ *is a morphism of comonads*

$$(L_\omega\widetilde{\omega}, L_\omega\eta\widetilde{\omega}, \varepsilon) \to (T, \delta^T, \varepsilon^T)$$

if and only if for every object $A \in \mathscr{A}$*, the pair* $(\omega(A), \Phi_T(\varphi)_A)$ *is a* T*-comodule.*

Proof. In order to increase readability we use single arrows when forming commutative diagrams of natural transformations. Since T is fixed we write Φ and Φ_0 for the morphisms Φ_T and $(\Phi_0)_T$ from proposition 2.5.4 and proposition 2.5.3 respectively. First note that $(L_\omega\widetilde{\omega}, L_\omega\eta\widetilde{\omega}, \varepsilon)$ really is a comonad by proposition 2.4.5, and by definition 2.4.2, the pair $(\omega(A), \Phi(\varphi)_A)$ is a T-comodule if and only if the diagrams

$$\begin{tikzcd}
\omega(A) \arrow[r, "\Phi(\varphi)_A"] \arrow[d, "\Phi(\varphi)_A"'] & T\omega(A) \arrow[d, "\delta^T_{\omega(A)}"] \\
T\omega(A) \arrow[r, "T\Phi(\varphi)_A"'] & TT\omega(A)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\omega(A) \arrow[r, "\Phi(\varphi)_A"] \arrow[dr, "\mathrm{id}"'] & T\omega(A) \arrow[d, "\varepsilon^T_{\omega(A)}"] \\
& \omega(A)
\end{tikzcd}$$

are commutative. We prove the lemma in two steps. First we show that the left diagram is commutative for every $A \in \mathscr{A}$ if and only if φ is compatible with the comultiplications (i.e., if and only if $\varphi * \varphi \circ L_\omega \eta \widetilde{\omega} = \delta^T \circ \varphi$), and in the second step we will see that the right diagram is commutative if and only if $\varepsilon^T \circ \varphi = \varepsilon$.
The left diagram above is commutative for every $A \in \mathscr{A}$ if and only if the inner square of the diagram of natural transformations

$$\begin{tikzcd}
L_\omega Y \arrow[rrr, "\Phi_0(\varphi)Y"] \arrow[dr, "\alpha^{-1}"] \arrow[ddd, "\Phi_0(\varphi)Y"'] & & & TL_\omega \arrow[ddd, "\delta^T L_\omega Y"] \\
& \omega \arrow[r, "\Phi(\varphi)"] \arrow[d, "\Phi(\varphi)"'] & T\omega \arrow[ur, "T\alpha"'] \arrow[d, "\delta^T \omega"] & \\
& T\omega \arrow[r, "T\Phi(\varphi)"'] & TT\omega \arrow[dr, "TT\alpha"] & \\
TL_\omega Y \arrow[ur, "T\alpha"] \arrow[rrr, "T\Phi_0(\varphi)Y"'] & & & TTL_\omega Y
\arrow[from=1-2, to=2-3, phantom, "(1)"]
\arrow[from=2-1, to=3-2, phantom, "(2)"]
\arrow[from=2-3, to=3-4, phantom, "(3)"]
\arrow[from=3-2, to=4-3, phantom, "(4)"]
\end{tikzcd}$$

is commutative. The parts (1), (2) and (4) are commutative by definition of Φ (see proposition 2.5.4), and part (3) is commutative since the two composites give precisely the two ways to compute the horizontal composite $\delta^T * \alpha$ (see proposition 1.3.1). Since α is an isomorphism it follows that commutativity of the inner square is equivalent to commutativity of the outer square. Since Y is dense and because L_ω, T are cocontinuous, it follows by proposition 1.6.4 that commutativity of the outer square is equivalent to commutativity of

$$\begin{tikzcd}
L_\omega \arrow[r, "\Phi_0(\varphi)"] \arrow[d, "\Phi_0(\varphi)"'] & TL_\omega \arrow[d, "\delta^T L_\omega"] \\
TL_\omega \arrow[r, "T\Phi_0(\varphi)"'] & TTL_\omega,
\end{tikzcd}$$

i.e., to the equality

$$\begin{tikzcd}
& & \cdot \arrow[dr, "L_\omega"] & \\
\cdot \arrow[r, "L_\omega"'] \arrow[urr, bend left, "\mathrm{id}"] & \cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, "T"] \arrow[dr, "T"'] & & \cdot \\
& & \cdot \arrow[ur, "T"'] &
\arrow[from=1-2, to=2-2, phantom, "\Downarrow\eta"]
\arrow[from=1-3, to=2-3, phantom, "\Downarrow\varphi"]
\arrow[from=2-3, to=3-3, phantom, "\Downarrow\delta^T"]
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "L_\omega"'] & & \cdot \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "L_\omega"] & & \cdot \arrow[dr, "L_\omega"] & \\
& \cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "T"'] & & \cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "T"'] & & \cdot
\arrow[from=1-2, to=2-2, phantom, "\Downarrow\eta"]
\arrow[from=1-3, to=2-3, phantom, "\Downarrow\varphi"]
\arrow[from=1-4, to=2-4, phantom, "\Downarrow\eta"]
\arrow[from=1-5, to=2-5, phantom, "\Downarrow\varphi"]
\end{tikzcd}$$

of pasted composites. Applying the bijection $(\Phi_0)^{-1}_{TT}$ (see proposition 2.5.3) we find that this equality holds if and only if the equality

$$\begin{tikzcd}
& \cdot \arrow[dr, "L_\omega"] & \\
\cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, "T"] \arrow[dr, "T"'] & & \cdot \\
& \cdot \arrow[ur, "T"'] &
\arrow[from=1-2, to=2-2, phantom, "\Downarrow\varphi"]
\arrow[from=2-2, to=3-2, phantom, "\Downarrow\delta^T"]
\end{tikzcd}
=
\begin{tikzcd}
& \cdot \arrow[rr, bend left, "\mathrm{id"] \arrow[dr, "L_\omega"] & & \cdot \arrow[dr, "L_\omega"] & \\
\cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "T"'] & & \cdot \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "T"'] & & \cdot
\arrow[from=1-2, to=2-2, phantom, "\Downarrow\varphi"]
\arrow[from=1-3, to=2-3, phantom, "\Downarrow\eta"]
\arrow[from=1-4, to=2-4, phantom, "\Downarrow\varphi"]
\end{tikzcd}$$

holds, that is, if and only if $\delta^T \circ \varphi = \varphi * \varphi \circ L_\omega \eta \widetilde{\omega}$, as claimed.
It remains to show that the diagram

$$\begin{tikzcd}
\omega(A) \arrow[r, "\Phi(\varphi)_A"] \arrow[dr, "\mathrm{id}"'] & T\omega(A) \arrow[d, "\varepsilon^T_{\omega(A)}"] \\
& \omega(A)
\end{tikzcd}$$

is commutative for every $A \in \mathscr{A}$ if and only if $\varepsilon^T \circ \varphi = \varepsilon$. The above diagram is commutative if and only if the inner triangle of the diagram

$$\begin{tikzcd}
& & TL_\omega Y \arrow[ddrr, "\varepsilon^T L_\omega Y"] & & \\
& & T\omega \arrow[u, "T\alpha"] \arrow[dr, "\varepsilon^T \omega"] & & \\
L_\omega Y \arrow[uurr, "\Phi_0(\varphi)Y"] \arrow[r, "\alpha^{-1}"'] \arrow[rrrr, bend right=20, "1_{L_\omega Y}"'] & \omega \arrow[ur, "\Phi(\varphi)"] \arrow[rr, "1_\omega"'] & & \omega \arrow[r, "\alpha"'] & L_\omega Y
\end{tikzcd}$$

of natural transformations is commutative. This is clearly equivalent to commutativity of the outer triangle, and density of Y implies that the outer triangle commutes if and only if

$$\begin{tikzcd}
& TL_\omega \arrow[dr, "\varepsilon^T L_\omega"] & \\
L_\omega \arrow[ur, "\Phi_0(\varphi)"] \arrow[rr, "1_{L_\omega}"'] & & L_\omega
\end{tikzcd}$$

is commutative. By definition of Φ_0 this diagram commutes if and only if the equality

$$\begin{tikzcd}
& & {} \arrow[dr, "L_\omega"] & \\
{} \arrow[urr, bend left, "\mathrm{id}"{name=I}] \arrow[r, "L_\omega"'] & {} \arrow[ur, ""{name=W}] \arrow[rr, "T"{name=T}] \arrow[rr, bend right=50, "\mathrm{id}"'{name=J}] & & {}
\arrow[Rightarrow, from=I, to=W, "\eta\,\widetilde{\omega}"]
\arrow[Rightarrow, from=1-3, to=T, "\varphi"]
\arrow[Rightarrow, from=T, to=J, "\varepsilon^T"]
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[r, bend left, "L_\omega"{name=U}] \arrow[r, bend right, "L_\omega"'{name=D}] & {}
\arrow[Rightarrow, from=U, to=D, "1"]
\end{tikzcd}$$

between pasted composites holds. Applying the bijection $(\Phi_0)^{-1}_{\mathrm{id}_{\mathbf{Mod}_R}}$ we find that this equality holds if and only if

$$\begin{tikzcd}
& {} \arrow[dr, "L_\omega"] & \\
{} \arrow[ur, "\widetilde{\omega}"] \arrow[rr, "T"{name=T}] \arrow[rr, bend right=50, "\mathrm{id}"'{name=J}] & & {}
\arrow[Rightarrow, from=1-2, to=T, "\varphi"]
\arrow[Rightarrow, from=T, to=J, "\varepsilon^T"]
\end{tikzcd}
=
\begin{tikzcd}
& {} \arrow[dr, "L_\omega"] & \\
{} \arrow[ur, "\widetilde{\omega}"] \arrow[rr, bend right, "\mathrm{id}"'{name=J}] & & {}
\arrow[Rightarrow, from=1-2, to=J, "\varepsilon"]
\end{tikzcd}$$

that is, if and only if $\varepsilon^T \circ \varphi = \varepsilon$. □

Proposition 2.5.6. *We use the notation of definition 2.4.3. The map*

$$\widehat{\Phi}_T \colon \mathbf{CC}_R(L_\omega \widetilde{\omega}, T) \to \mathbf{cat}_R \,/\, \mathbf{Mod}^c_R\big((\mathscr{A}, \omega), (\mathrm{Comod}^c(T), V_T)\big)$$

which sends a morphism of comonads $\varphi \colon L_\omega \widetilde{\omega} \Rightarrow T$ to the functor $F \colon (\mathscr{A}, \omega) \to (\mathrm{Comod}^c(T), V_T)$ given by $F(A) = (\omega(A), \Phi_T(\varphi)_A)$ and $Ff = \omega(f)$ is a bijection. Moreover, this bijection is natural in T. In other words, the assignment $(\mathscr{A}, \omega) \mapsto L_\omega \widetilde{\omega}$ uniquely extends to a left adjoint for the comodule functor (see definition 2.4.3). We denote this adjunction by

$$E_{(-)} \dashv \mathrm{Comod}^c(-) \colon \mathbf{cat}_R \,/\, \mathbf{Mod}^c_R \longrightarrow \mathbf{CC}_R \,.$$

We write $N = N_{(A,\omega)} \colon (\mathscr{A}, \omega) \to (\mathrm{Comod}^c(L_\omega \widetilde{\omega}), V_{L_\omega \widetilde{\omega}})$ *and* $\nu_T \colon E_{(\mathrm{Comod}^c(T), V_T)} \Rightarrow T$ *for the unit and the counit of this adjunction.*

Proof. This is follows from the observation that giving an R-linear functor $F \colon \mathscr{A} \to \mathrm{Comod}^c(T)$ with $V_T \circ F = \omega$ is equivalent to giving a natural transformation $\rho \colon \omega \Rightarrow T\omega$ such that $(\omega(A), \rho_A)$ is a T-comodule. Indeed, if ρ is such a natural transformation, we let $G(\rho)$ be the functor $\mathscr{A} \to \mathrm{Comod}^c(T)$ which sends A to $(\omega(A), \rho_A)$ and $f \colon A \to A'$ to $\omega(f)$. The fact that $G(\rho)$ is a well-defined functor follows by naturality of ρ, and we clearly have $V_T G(\rho) = \omega$. Conversely, given a functor $F \colon \mathscr{A} \to \mathrm{Comod}^c(T)$ with $V_T F = \omega$, for every $A \in \mathscr{A}$ there is a module $F_0 A$ and a homomorphism $\xi(F)_A \colon F_0 A \to T F_0 A$ such that $FA = (F_0 A, \xi(F)_A)$. We find that $F_0 A = V_T F A = \omega(A)$ and, for any morphism $f \colon A \to A'$, $Ff = V_T F f = \omega(f)$. By assumption, $Ff = \omega(f)$ is a morphism of comodules, which shows that the $\xi(F)_A$ are natural in A. Moreover, we have $G(\xi(F)) = F$ and $\xi(G(\rho)) = \rho$, hence the map

$$\widehat{\Phi}_T = G \circ \Phi_T \colon \ \mathbf{CC}_R(L_\omega \widetilde{\omega}, T) \to \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c\big((\mathscr{A}, \omega), (\mathrm{Comod}^c(T), V_T)\big)$$

is a bijection, and naturality in T follows directly from the definition of the comodule functor (see definition 2.4.3) and of Φ_T (see proposition 2.5.4). ☐

Definition 2.5.1. We denote the composite

$$\mathbf{C}\, E_{(-)} \dashv \mathrm{Comod}^c(\mathbf{T}(-)) \colon \ \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c \to \mathbf{Coalg}_R$$

of the adjunction $\mathbf{C} \dashv \mathbf{T}$ from proposition 2.2.6 with the adjunction $E_{(-)} \dashv \mathrm{Comod}^c(-)$ from proposition 2.5.6 by

$$\mathbf{E}_{(-)} \dashv \mathrm{Comod}^c(\mathbf{T}(-)) = \mathbf{Comod}^c(-) \colon \ \mathbf{cat}_R \,/\, \mathbf{Mod}_R^c \longrightarrow \mathbf{Coalg}_R \,.$$

Using notations from the propositions 2.2.6 and 2.5.6, the unit and counit of $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ are given by the composites

$$\begin{tikzcd}
{} \arrow[rrrr, bend left=20, "\mathrm{id}"] \arrow[dr, "E_{(-)}"'] & & \Downarrow N & & {} \\
& {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "\mathbf{C}"'] & \Downarrow \pi & {} \arrow[ur, "\mathrm{Comod}^c(-)"'] & \\
& & {} \arrow[ur, "\mathbf{T}"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & {} \arrow[dr, "E_{(-)}"] & & \\
& {} \arrow[ur, "\mathrm{Comod}^c(-)"] \arrow[rr, bend right, "\mathrm{id}"'] & \Downarrow \nu & {} \arrow[dr, "\mathbf{C}"] & \\
{} \arrow[ur, "\mathbf{T}"] \arrow[rrrr, bend right=20, "\mathrm{id}"'] & & \Downarrow \beta & & {}
\end{tikzcd}$$

respectively.

Proposition 2.5.7. *Let* $\mathscr{A}$ *be a small* R*-linear category and let* $\omega \colon \mathscr{A} \to \mathbf{Mod}_R^c$ *be an* R*-linear functor. Then the unit* $N \colon (\mathscr{A}, \omega) \to \mathrm{Comod}^c(E_{(\mathscr{A},\omega)})$ *of the adjunction* $E_{(-)} \dashv \mathrm{Comod}^c(-)$ *from proposition 2.5.6 is naturally isomorphic to the composite*

$$\mathscr{A} \xrightarrow{\ Y\ } [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \xrightarrow{\ J\ } \mathrm{Comod}(L_\omega \widetilde{\omega}),$$

where J *denotes the comparison functor associated to the adjunction* $L_\omega \dashv \widetilde{\omega}$ *(see proposition 2.4.6). Consequently, the unit of the adjunction* $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ *(see definition 2.5.1) is naturally isomorphic to the composite*

$$\mathscr{A} \xrightarrow{\quad Y \quad} [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \xrightarrow{\quad J \quad} \mathrm{Comod}(L_\omega \widetilde{\omega}) \xrightarrow{\mathrm{Comod}(\pi_{L_\omega \widetilde{\omega}})} \mathbf{Comod}(\mathbf{C}(L_\omega \widetilde{\omega})).$$

Proof. The second statement follows directly from the first, so it suffices to construct a natural isomorphism $N \cong JY$. Recall that JY sends an object $A \in \mathscr{A}$ to $(L_\omega Y(A), L_\omega \eta_{Y(A)})$ and N sends A to $(\omega(A), \Phi(1_{L_\omega \widetilde{\omega}})_A)$. We claim that $\alpha_A \colon \omega(A) \to L_\omega Y(A)$ gives the desired isomorphism. First we have to check that α_A is a morphism of $L_\omega\widetilde{\omega}$-comodules. This follows since $\Phi(1_{L_\omega \widetilde{\omega}})_A$ is given by the composite $L_\omega \widetilde{\omega} \alpha_A^{-1} \circ L_\omega \eta_{Y(A)} \circ \alpha_A$ (see proposition 2.5.4), which shows that the diagram

$$\begin{tikzcd}
\omega(A) \arrow[r, "\alpha_A"] \arrow[d, "\Phi(1_{L_\omega \widetilde{\omega}})_A"'] & L_\omega Y(A) \arrow[d, "L_\omega \eta_{Y(A)}"] \\
L_\omega \widetilde{\omega} \omega(A) \arrow[r, "L_\omega \widetilde{\omega} \alpha_A"'] & L_\omega \widetilde{\omega} L_\omega Y(A)
\end{tikzcd}$$

is commutative, i.e., that α_A really is a morphism $N(A) \to JY(A)$. This gives a natural isomorphism $N \Rightarrow JY$ because $\alpha \colon \omega \Rightarrow L_\omega Y$ is a natural isomorphism. □

2.6. **Reconstruction of coalgebras.** The goal of this section is to give a necessary and sufficient condition for the counit of the adjunction $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ from definition 2.5.1 to be an isomorphism. In order to do this we have to find a suitable description of this counit, hence we fix some notation first. Let $T \colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ be a cocontinuous R-linear comonad. We write $\mathscr{A}$ for the full subcategory $\mathrm{Comod}^c(T)$ of Cauchy T-comodules. We denote the unit and counit of the adjunction $V_T \vdash W_T$ by η^T and ε^T respectively. We denote the inclusion functor $\mathscr{A} \to \mathrm{Comod}^c(T)$ by K, and we let $\omega \colon \mathscr{A} \to \mathbf{Mod}_R$ be the composite $V_T \circ K$. By proposition 1.5.3, the functors K and ω induce adjunctions $L_K \dashv \widetilde{K} \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathscr{C}$ and $L_\omega \dashv \widetilde{\omega} \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathbf{Mod}_R$. We denote their units and counits by η^K, ε^K and η^ω, ε^ω respectively. The situation can be summarized in the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "K"] \arrow[dr, "\omega"'] & \mathrm{Comod}(T) \arrow[r, shift right, "\widetilde{K}"'] \arrow[d, shift left, "V"] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]} \arrow[l, shift right, "L_K"'] \arrow[dl, shift left, "L_\omega"] \\
& \mathbf{Mod}_R \arrow[u, shift left, "W"] \arrow[ur, shift left, "\widetilde{\omega}"] &
\end{tikzcd}.$$

Furthermore, we let $\alpha_K \colon K \Rightarrow L_K Y$ and $\alpha_\omega \colon \omega \Rightarrow L_\omega Y$ be the natural isomorphisms from proposition 1.5.4. By proposition 1.6.4 and proposition 1.6.1 there is a unique natural isomorphism $\sigma \colon L_\omega \Rightarrow V_T L_K$ such that

$$\sigma Y = \begin{tikzcd}
& & {} \arrow[dr, "L_\omega"] & \\
{} \arrow[urr, bend left, "Y"] \arrow[rr, "K"] \arrow[dr, "Y"'] & \Downarrow \alpha_\omega^{-1} & {} \arrow[r, "V_T"'] & {} \\
& {} \arrow[ur, "L_K"'] \arrow[u, phantom, "\Downarrow \alpha_K"] & &
\end{tikzcd}.$$

Recall that the natural transformations

$$\begin{tikzcd}
{} \arrow[rrrr, bend left=30, "\mathrm{id}"] \arrow[r, "L_K"'] & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] & \Downarrow \eta^T & {} \arrow[r, "\widetilde{K}"'] & {} \\
& & {} \arrow[ur, "W_T"'] & &
\end{tikzcd}
\qquad \Downarrow \eta^K$$

and

$$\begin{tikzcd}
& & {} \arrow[dr, "L_K"] & & \\
{} \arrow[rrrr, bend right=30, "\mathrm{id}"'] \arrow[r, "W_T"] & {} \arrow[ur, "\widetilde{K}"] \arrow[rr, bend right, "\mathrm{id}"'] & \Downarrow \varepsilon^K & {} \arrow[r, "V_T"] & {}
\end{tikzcd}
\qquad \Downarrow \varepsilon^T$$

give the unit and counit of the adjunction $V_T L_K \dashv \widetilde{K} W_T \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathbf{Mod}_R$. We let $\tau \colon \widetilde{\omega} \Rightarrow \widetilde{K} W_T$ be the mate of $\sigma^{-1} \colon V_T L_K \Rightarrow L_\omega$ under the adjunctions $L_\omega \dashv \widetilde{\omega}$ and $V_T L_K \dashv \widetilde{K} W_T$ (see proposition 1.3.2).

Proposition 2.6.1. *The T-component ν_T of the counit ν of the adjunction $E_{(-)} \dashv$ $\mathrm{Comod}^c(-)$ from proposition 2.5.6 is given by*

$$\chi := \begin{tikzcd}
{} \arrow[rr, bend left, "\widetilde{\omega}"] \arrow[dr, "W_T"'] \arrow[rr, phantom, "\Downarrow \tau"] & & {} \arrow[rr, bend left, "L_\omega"] \arrow[dr, "L_K"] \arrow[rr, phantom, "\Downarrow \sigma"] & & {} \\
& {} \arrow[ur, "\widetilde{K}"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow \varepsilon^K"] & & {} \arrow[ur, "V_T"'] &
\end{tikzcd} .$$

Proof. We use the notation introduced at the beginning of this section. We have to check that the bijection $\widehat{\Phi}$ defined in proposition 2.5.6 sends χ to the identity functor of $(\mathrm{Comod}^c(T), V_T)$, i.e., to $\mathrm{id}_{(\mathscr{A},\omega)}$ with the above notation. In order to do this we first have to compute $\Phi_T(\chi)$. Since τ is the mate of σ^{-1}, proposition 1.3.3 implies that the natural transformation

$$\zeta := \begin{tikzcd}
{} \arrow[r, "Y"] & {} \arrow[r, "L_\omega"'] \arrow[rrr, bend left=60, "\mathrm{id}"] \arrow[rrr, phantom, bend left=30, "\Downarrow \eta^\omega"] & {} \arrow[rr, bend left, "\widetilde{\omega}"] \arrow[dr, "W_T"'] \arrow[rr, phantom, "\Downarrow \tau"] & & {} \arrow[rr, bend left, "L_\omega"] \arrow[dr, "L_K"] \arrow[rr, phantom, "\Downarrow \sigma"] & & {} \\
& & & {} \arrow[ur, "\widetilde{K}"] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\Downarrow \varepsilon^K"] & & {} \arrow[ur, "V_T"'] &
\end{tikzcd}$$

is equal to the pasted composite of

$$\begin{tikzcd}
{} \arrow[r, "Y"] & {} \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[rrrr, phantom, "\Downarrow \eta^K"] \arrow[dr, "L_K"] \arrow[ddrr, bend right, "L_\omega"'] \arrow[ddrr, phantom, "\Downarrow \sigma^{-1}"] & & & & {} \arrow[rr, bend left, "L_\omega"] \arrow[dr, "L_K"] \arrow[rr, phantom, "\Downarrow \sigma"] & & {} \\
& & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] \arrow[rr, phantom, "\Downarrow \eta^T"] & & {} \arrow[ur] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\widetilde{K} \ \Downarrow \varepsilon^K"] & & {} \arrow[ur, "V_T"'] & \\
& & & {} \arrow[ur, "W_T"'] & & & &
\end{tikzcd}$$

which in turn is equal to

$$\begin{tikzcd}
{} \arrow[r, "Y"] & {} \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[dr, "L_K"] \arrow[ddrr, bend right, "L_\omega"'] \arrow[ddrr, phantom, "\Downarrow \sigma^{-1}"] & & & & {} \arrow[rr, bend left, "L_\omega"] \arrow[dr, "L_K"] \arrow[rr, phantom, "\Downarrow \sigma"] & & {} \\
& & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] \arrow[rr, phantom, "\Downarrow \eta^T"] & & {} \arrow[rr, bend right, "\mathrm{id}"'] & & {} \arrow[ur, "V_T"'] & \\
& & & {} \arrow[ur, "W_T"'] & & & &
\end{tikzcd}$$

because the triangular identities hold for η^K and ε^K. This last natural transformation is obviously equal to the pasted composite of

$$\begin{tikzcd}
{} \arrow[r, "Y"] & {} \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[rrrr, phantom, "\Downarrow \sigma"] \arrow[dr, "L_K"] \arrow[ddrr, bend right, "L_\omega"'] \arrow[ddrr, phantom, "\Downarrow \sigma^{-1}"] & & & & {} \\
& & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] \arrow[rr, phantom, "\Downarrow \eta^T"] & & {} \arrow[ur, "V_T"'] & \\
& & & {} \arrow[ur, "W_T"'] & &
\end{tikzcd} ,$$

and by definition of σ it follows that this, and therefore ζ, is equal to

$$\begin{tikzcd}
& & {} \arrow[drrr, bend left, "L_\omega"] \arrow[d, phantom, "\alpha_\omega^{-1}\Downarrow"] & & & \\
{} \arrow[urr, bend left, "Y"] \arrow[rr, bend left=60, "K"] \arrow[r, "Y"] \arrow[rr, bend right=60, "K"'] \arrow[ddrr, bend right, "Y"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha_K"] \arrow[r, phantom, shift right=4, "\Downarrow\alpha_K^{-1}"] & {} \arrow[r, "L_K"] & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] \arrow[rr, phantom, "\Downarrow\eta^T"] & & {} \arrow[r, "V_T"'] & {} . \\
& & & {} \arrow[ur, "W_T"'] & & \\
& & {} \arrow[ur, "L_\omega"'] \arrow[uu, phantom, shift right=4, "\Downarrow\alpha_\omega"] & & &
\end{tikzcd}$$

Since $\Phi_T(\chi) = V_T W_T \alpha_\omega^{-1} \circ \zeta \circ \alpha_\omega$ (see proposition 2.5.4 and the definition of ζ above), we find that $\Phi_T(\chi)$ is equal to the natural transformation

$$\begin{tikzcd}
{} \arrow[r, "K"] & {} \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, "V_T"'] & & {} \arrow[r, "V_T"] & {} . \\
& & {} \arrow[ur, "W_T"'] \arrow[u, phantom, "\Downarrow\eta^T"] & &
\end{tikzcd}$$

By proposition 2.5.6 it follows that $\widehat{\Phi}(\nu_T)$ is the functor $\mathscr{A} \to \mathscr{A} = \mathrm{Comod}^c(T)$ which sends an object $A = (M, \xi)$ of $\mathscr{A}$ to $(M, \eta^T_{(M,\xi)})$ and a morphism $f\colon (M,\xi) \to (M',\xi')$ to $\omega(f) = f$. But $\eta^T_{(M,\xi)} = \xi$ by proposition 2.4.4, which shows that $\widehat{\Phi}(\chi)$ is indeed the identity functor $\mathscr{A} \to \mathscr{A}$. □

Definition 2.6.1. Let (C, δ, ε) be a coalgebra. The *category of Cauchy comodules over C* is the category with objects the pairs (M, φ) of a Cauchy comodule M together with a morphism of C-comodules $\varphi\colon (M, \rho_M) \to (C, \delta)$, and with morphisms $(M, \varphi) \to (M', \varphi')$ the morphisms of Comodules ψ which make the diagram

$$\begin{tikzcd}
(M, \rho) \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & (M', \rho') \arrow[dl, "\varphi'"] \\
& (C, \delta) &
\end{tikzcd}$$

commutative. We write $\mathscr{D}$ for this category, and we let $D_C\colon \mathscr{D} \to \mathbf{Comod}(C)$ be the functor which sends (M, φ) to M and a morphism of $\mathscr{D}$ to itself. The functor D_C is called the *diagram of Cauchy comodules over C*. Finally, we let $\kappa_{(M,\varphi)} = \varphi\colon D_C(M,\varphi) = M \to C$. The $\kappa_{(M,\varphi)}$ constitute a cocone on D_C, called the *canonical cocone* on D_C.

Remark 2.6.2. With the notation introduced in definition 1.6.2 and at the beginning of this section we have $\mathscr{D} = (K \downarrow (C, \delta))$.

Proposition 2.6.3. *Let (C, δ, ε) be a coalgebra. Then the canonical cocone*

$$\big((C, \delta), (\kappa_{(M,\varphi)})_{(M,\varphi)\in\mathscr{D}}\big)$$

on the diagram D_C of Cauchy comodules over C exhibits C as colimit of D_C if and only if the cocone

$$\big(C, (\varphi)_{(M,\varphi)\in\mathscr{D}}\big)$$

exhibits the module C as colimit of $VD_C\colon \mathscr{D} \to \mathbf{Mod}_R$, where $V\colon \mathbf{Comod}(C) \to \mathbf{Mod}_R$ denotes the forgetful functor.

Proof. This follows directly from the fact that V reflects colimits (see proposition 2.3.2). □

Definition 2.6.2. Let (C, δ, ε) be a coalgebra. We say that C has *enough* Cauchy comodules if the following holds:

> *For every C-comodule M and every element $m \in M$ there is a Cauchy comodule $\widetilde{M}$ and a morphism of comodules $\varphi : \widetilde{M} \to M$ such that the image of φ contains the element m.*

Proposition 2.6.4. *Let (C, δ, ε) be a coalgebra. If*

- *i) C is a Cauchy module, or*
- *ii) if C is flat and has enough Cauchy comodules (see definition 2.6.2),*

then the canonical cocone on the diagram D_C of Cauchy comodules over C exhibits (C, δ) as the colimit of D_C.

Proof. If C is a Cauchy module, then (C, id) is a terminal object of $\mathscr{D}$, and the claim *i)* follows immediately from this fact. To see *ii)* we use proposition 2.6.3, i.e., we show that

$$\big(C, (\varphi)_{(M,\varphi)\in\mathscr{D}}\big)$$

is a colimit of $VD_C \colon \mathscr{D} \to \mathbf{Mod}_R$. To see this, we let

$$\big(N, (\lambda_{(M,\varphi)})_{(M,\varphi)\in\mathscr{D}}\big)$$

be a cocone on VD_C. We construct a homomorphism $\gamma \colon C \to N$ as follows: For any $c \in C$ we choose a Cauchy module M and a morphism $\varphi \colon M \to C$ together with an element $m \in M$ with $\varphi(m) = c$, and we let $\gamma(c) = \lambda_{(M,\varphi)}(m)$. We claim that this is a well-defined homomorphism of R-modules. To see this, we let $\varphi_i \colon M_i \to C$, $i = 0, 1$, be two morphisms of C-comodules, together with a elements $m_i \in M_i$ such that $\varphi_i(m_i) = c$ for $i = 0, 1$. Since C is flat, the pullback E of φ_0 and φ_1 in $\mathbf{Comod}(C)$ is computed as in $\mathbf{Mod}_R$ (see proposition 2.3.3). By assumption we have $(m_0, m_1) \in E$, and because C has enough Cauchy comodules it follows that there is a Cauchy comodule M and a morphism of comodules $\psi \colon M \to E$ together with an element $m \in M$ such that $\psi(m) = (m_0, m_1)$. Writing ψ_i for $\mathrm{pr}_i \circ \psi$ we get morphisms

$$\psi_i \colon (M, \psi) \to (M_i, \varphi_i),$$

$i = 0, 1$, in $\mathscr{D}$. Since λ is a cocone on VD_C it follows that

$$\lambda_{(M_0,\varphi_0)}(m_0) = \lambda_{(M_0,\varphi_0)}\psi_0(m) = \lambda_{(M,\psi)}(m) = \lambda_{(M_1,\varphi_1)}\psi_1(m) = \lambda_{(M_1,\varphi_1)}(m_1)$$

i.e., that γ is well-defined. The fact that γ is a morphism of R-modules is an immediate consequence of the fact that γ is well-defined. □

Theorem 2.6.5. *We use the notation introduced at the beginning of this section. Let (C, δ, ε) be a coalgebra. The counit morphism $\mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$ of the adjunction $\mathbf{E}_{(-)} \dashv \mathrm{Comod}^c(-)$ (see definition 2.5.1) is an isomorphism of coalgebras if and only if the canonical cocone on the diagram D_C of Cauchy comodules over C (see definition 2.6.1) exhibits (C, δ) as colimit of D_C.*

Proof. By definition 2.5.1, the counit morphism $\mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$ is given by the composite

$$\beta_C \circ \mathbf{C}(\nu_{\mathbf{T}(C)}).$$

Since β_C is an isomorphism and because $\mathbf{C}$ is an equivalence (see proposition 2.2.6) it follows that the counit morphism is an isomorphism if and only if $\nu_{\mathbf{T}(C)}$ is an isomorphism. By proposition 2.6.1, $\nu_{\mathbf{T}(C)}$ is given by the pasted composite of

$$\begin{tikzcd}
{} \arrow[rr, bend left, "\widetilde{\omega}"] \arrow[dr, "W_{\mathbf{T}(C)}"'] & & {} \arrow[rr, bend left, "L_\omega"] \arrow[dr, "L_K"'] & & {} \\
& {} \arrow[ur, "\widetilde{K}"'] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[u, phantom, "\Downarrow \tau"] & & {} \arrow[ur, "V_{\mathbf{T}(C)}"'] \arrow[u, phantom, "\Downarrow \sigma"] \arrow[ll, phantom, "\Downarrow \varepsilon^K"] &
\end{tikzcd},$$

where we use the notation introduced at the beginning of this section. By corollary 1.4.4, this is an isomorphism if and only if its R-component

$$(\nu_{\mathbf{T}(C)})_R = V_{\mathbf{T}(C)}(\varepsilon^K_{W_{\mathbf{T}(C)}(R)}) \circ V_{\mathbf{T}(C)} L_K \tau_R \circ \sigma_{\tilde{\omega} R}$$

is an isomorphism. Since τ and σ are isomorphisms and because $V_{\mathbf{T}(C)}$ reflects isomorphisms, this is equivalent to the fact that $\varepsilon^K_{W_{\mathbf{T}(C)}(R)}$ is an isomorphism. By definition we have equalities of $\mathbf{T}(C)$-comodules

$$W_{\mathbf{T}(C)}(R) = (\mathbf{T}(C)(R), \delta^{\mathbf{T}(C)}_R) = (C \otimes R, \delta \otimes R)$$

(see proposition 2.4.4 and proposition 2.2.6 respectively); and $r_C \colon (C \otimes R, \delta \otimes R) \to (C, \delta)$ clearly is an isomorphism of $\mathbf{T}(C)$-comodules. By naturality of ε^K it follows that $\varepsilon^K_{W_{\mathbf{T}(C)}(R)}$ is an isomorphism if and only if $\varepsilon^K_{(C,\delta)}$ is an isomorphism. By proposition 1.6.6, this is equivalent to the fact that the canonical cocone on $D_{(C,\delta)} \colon (K \downarrow (C,\delta))$ exhibits (C,δ) as colimit of $D_{(C,\delta)}$. With the notation of definition 2.6.1, this is equivalent to the fact that the canonical cocone on the diagram D_C of Cauchy comodules over C exhibits (C,δ) as colimit of $D_C \colon \mathscr{D} \to \mathbf{Comod}(C)$. ☐

Corollary 2.6.6. *Let* (C, δ, ε) *be a coalgebra. The counit morphism*

$$\mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$$

of the adjunction $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ *(see definition 2.5.1) is an isomorphism of coalgebras if and only if the cocone* $\varphi \colon M \to C$ *on* VD_C *(cf. proposition 2.6.3) exhibits* C *as colimit of* $VD_C \colon \mathscr{D} \to \mathbf{Mod}_R$.

Proof. This follows from theorem 2.6.5 and proposition 2.6.3. ☐

Corollary 2.6.7. *Let* (C, δ, ε) *be a coalgebra. If the inclusion functor*

$$K\colon\ \mathbf{Comod}^c(C) \to \mathbf{Comod}(C)$$

is dense, then the counit morphism $\mathbf{E}_{(\mathbf{Comod}^c(C),V)} \to C$ *of the adjunction* $\mathbf{E}_{(-)} \dashv \mathrm{Comod}^c(-)$ *(see definition 2.5.1) is an isomorphism of coalgebras.*

Proof. This follows from theorem 2.6.5 and proposition 1.6.6. ☐

Proposition 2.6.8. *If* R *is a Noetherian hereditary ring, then every flat coalgebra* (C, δ, ε) *has enough Cauchy comodules.*

Proof. Let (M, ρ_M) be a C-comodule and let $m \in M$. Then there are elements $c_1, \ldots, c_k \in C$ and elements $n_1, \ldots, n_k \in N$ such that $\rho_M(m) = \sum_{i=1}^k c_i \otimes n_i$. Let $\varphi : R^k \to M$ be the morphism which sends e_i to n_i. Then $C \otimes \varphi \colon C \otimes R^k \to C \otimes M$ sends the element $x = \sum_{i=1}^k c_i \otimes e_i$ to $\sum_{i=1}^k c_i \otimes n_i$, so $(m, x) \in E = \{(a, b) \in M \oplus (C \otimes R^k); \rho_M(a) = C \otimes \varphi(b)\}$, the pullback of M along $C \otimes \varphi$. Since C is flat, there is a unique coaction $\rho_E : E \to C \otimes E$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "\mathrm{pr}_{C \otimes R^k}"] \arrow[d, "\mathrm{pr}_M"'] & C \otimes R^k \arrow[d, "C \otimes \varphi"] \\
M \arrow[r, "\rho_M"] & C \otimes M
\end{tikzcd}$$

becomes a pullback diagram in the category of C-comodules (see proposition 2.3.3). Let E_0 be a finitely generated subcomodule of E which contains (m, x) (such a subcomodule exists by proposition 2.3.5). Then m lies in the image of pr_M restricted to E_0, and we are done if we can show that E_0 is in fact projective. But the morphism $\mathrm{pr}_{R^k \otimes C} : E \to R^k \otimes C$ is a monomorphism as pullback of the monomorphism ρ_M, hence E_0 is (up to isomorphism) a finitely presented submodule of the flat R-module $R^k \otimes C$. Since R is hereditary it follows that E_0 is projective. ☐

Corollary 2.6.9. *Let R be a noetherian hereditary ring, and let (C, δ, ε) be a flat R-coalgebra. Then the counit morphism $\mathbf{E}_{(\mathbf{Comod}^c(C), V)} \to C$ of the adjunction $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ (see definition 2.5.1) is an isomorphism of coalgebras.*

Proof. This follows from proposition 2.6.4 and theorem 2.6.5. ☐

In the entire section we rarely used properties of the category $\mathbf{Mod}_R$ besides the fact that it is a complete cocomplete symmetric monoidal closed category. For an arbitrary complete cocomplete symmetric monoidal closed category $\mathscr{V}$ we write $\mathscr{V}^c$ for the full subcategory of the *Cauchy objects*, i.e., the objects of $\mathscr{V}$ which have duals, and we use the notation from [Kel82]. In order to define a *comodule functor*

$$\mathrm{Comod}^c(-) \colon \mathrm{Comon}(\mathscr{V}) \to \mathscr{V}\text{-}\mathbf{cat}\,/\mathscr{V}^c$$

as in definition 2.4.3, we must assume that the category $\mathscr{V}^c$ is small.

Open question. *Under which conditions on $\mathscr{V}$ do the above results generalize? More precisely, is there a left adjoint*

$$E \colon \mathscr{V}\text{-}\mathbf{cat}\,/\mathscr{V}^c \to \mathrm{Comon}(\mathscr{V})$$

to the comodule functor? If such an adjoint exists, is the counit an isomorphism if and only if the canonical morphisms

$$\mathrm{Comod}(C)(KA, C) \otimes KA \to C$$

exhibit C as coend of the functor

$$\mathrm{Comod}(C)(K-, C) \otimes K- \colon \mathrm{Comod}^c(C)^{\mathrm{op}} \otimes \mathrm{Comod}^c(C) \to \mathscr{V}\,?$$

As far as I can see, there are no requirements besides smallness of $\mathscr{V}^c$.

3. The recognition problem

3.1. **Overview.** The goal of this chapter is to give a sufficient condition for the unit

$$(\mathscr{A}, \omega) \to \mathbf{Comod}(\mathbf{E}_{(\mathscr{A}, \omega)})$$

of the adjunction $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ (see definition 2.5.1) to be an equivalence of categories. Proposition 2.5.7 implies that this functor is an equivalence if and only if the composite

$$\mathscr{A} \xrightarrow{Y} [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \xrightarrow{J} \mathrm{Comod}(L_\omega \widetilde{\omega}),$$

gives an equivalence between $\mathscr{A}$ and $\mathrm{Comod}^c(L_\omega \widetilde{\omega})$, where J denotes the comparison functor of the adjunction $L_\omega \dashv \widetilde{\omega}$ (see proposition 2.4.6). Since the Yoneda embedding is fully faithful, comonadicity of $L_\omega \dashv \widetilde{\omega}$ would imply that the comparison functor is fully faithful. However, this is not to be expected in the general situation, because the category $\mathbf{Comod}(C)$ is usually not equivalent to a category of R-linear functors. Hence we first have to analyze the situation where $(\mathscr{A}, \omega)$ is equal to $(\mathrm{Comod}^c(T), V_T)$ more carefully. Writing $K \colon \mathscr{A} \to \mathrm{Comod}(T)$ for the inclusion functor, this situation is summarized by the diagram

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "K"] \arrow[dr, "\omega"'] & \mathrm{Comod}(T) \arrow[r, shift right, "\widetilde{K}"'] \arrow[d, shift left, "V"] & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R].} \arrow[l, shift right, "L_K"'] \arrow[dl, shift left, "L_\omega"] \\
& \mathbf{Mod}_R \arrow[u, shift left, "W"] \arrow[ur, shift left, "\widetilde{\omega}"] &
\end{tikzcd}$$

Under the additional assumption that K is dense we find that $\mathrm{Comod}(T)$ is a reflective subcategory of $[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$, and that the comonads $L_\omega \widetilde{\omega}$ and $V_T W_T$ are isomorphic (cf. proposition 2.6.1). Moreover, it follows directly from definition 1.5.1 that $\widetilde{K} \circ K = Y$. This suggests that we adopt the following strategy:

(a) Find conditions for $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^c$ which imply the existence of a reflective subcategory $\mathscr{C}$ of $[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$, together with a comonadic adjunction $V \dashv W\colon \mathscr{C} \to \mathbf{Mod}_R$ such that the comonad VW is isomorphic to $L_\omega\widetilde{\omega}$, and

(b) find conditions for ω which imply that the Yoneda embedding

$$Y\colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$$

factors through the embedding $\mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ from (a).

In order to do this we need some background material on locally presentable and accessible categories, which we summarize in section 3.2 and section 3.3. We then state the sufficient conditions for $\omega\colon \mathscr{A} \to \mathbf{Mod}_R^c$ to be of the form $V_T\colon \mathrm{Comod}^c(T) \to \mathbf{Mod}_R^c$ for some cocontinuous comonad T in section 3.4.

3.2. Locally presentable and accessible categories.

This is a summary of the terminology and of certain results found in [AR94].

Definition 3.2.1. An infinite cardinal λ is called *regular* if it can not be written as a union

$$\bigcup_{i\in I} \lambda_i$$

such that each λ_i has cardinality less than λ and $|I| < \lambda$

For example, $\aleph_0 = \omega$, the first infinite cardinal, is regular; for an infinite set can not be written as a finite union of finite sets. More importantly, *there are enough regular cardinals*, meaning that for any cardinal α there is a regular cardinal $\lambda \geq \alpha$ (see e.g. [Cam99]).

Definition 3.2.2. A category $\mathscr{D}$ is called λ-filtered (λ a regular cardinal) if

i) it is non-empty,
ii) for any family $(D_i)_{i\in I}$ of objects in $\mathscr{D}$, if $|I| < \lambda$, there exists an object D and morphisms $f_i\colon D_i \to D$, and
iii) for any family $(g_i\colon D_0 \to D_1)_{i\in I}$, if $|I| < \lambda$, there exists an object D and a morphism $g\colon D_1 \to D$ such that $g \circ g_i = g \circ g_j$ for all $i, j \in I$.

Definition 3.2.3. An object C of a category $\mathscr{C}$ is called λ-*presentable* (λ a regular cardinal) if the functor $\mathscr{C}(C, -)\colon \mathscr{C} \to \mathbf{Set}$ preserves λ-filtered colimits.

Definition 3.2.4. Let λ be a regular cardinal. A colimit $\mathrm{colim}_{i\in\mathscr{D}} D_i$ is called λ-*small* if $\mathscr{D}$ has less than λ morphisms.

Proposition 3.2.1. *A λ-small colimit of λ-presentable objects is λ-presentable.*

Proof. See [AR94], proposition 1.16. □

Definition 3.2.5. A category $\mathscr{C}$ is called λ-*accessible*, where λ is a regular cardinal, if

i) it has λ-filtered colimits, and
ii) there is a set $\mathscr{A}$ of λ-presentable objects of $\mathscr{C}$ such that each object of $\mathscr{C}$ can be written as a λ-filtered colimit of objects from $\mathscr{A}$.

The category $\mathscr{C}$ is called *locally λ-presentable* if it is λ-accessible and cocomplete. A category is called *accessible* (resp. *locally presentable*) if there exists a regular cardinal λ such that $\mathscr{C}$ is λ-accessible (resp. locally λ-presentable).

Proposition 3.2.2. *Let $\mathscr{S}$ be a set of objects of a locally presentable category $\mathscr{C}$. Then there exists a regular cardinal μ such that all elements of $\mathscr{S}$ are μ-presentable.*

Proof. By assumption there is a regular cardinal λ such that every object C in $\mathscr{S}$ is a λ-filtered colimit $C = \mathrm{colim}_{i \in \mathscr{D}_C} D_i$ such that the D_i are λ-presentable. Let μ be a regular cardinal which is bigger than the set of morphisms of the categories $\mathscr{D}_C$, $C \in \mathscr{S}$ and such that $\mu \geq \lambda$. The latter condition ensures that a μ-filtered diagram is automatically λ-filtered as well, and it follows that all $C \in \mathscr{S}$ are μ-small colimits of μ-presentable objects. Thus proposition 3.2.1 shows that all C are μ-presentable. □

Proposition 3.2.3. *For each locally λ-presentable category $\mathscr{C}$, each functor category $\mathscr{C}^{\mathscr{A}}$ ($\mathscr{A}$ small) is locally λ-presentable.*

Proof. See [AR94], corollary 1.54. □

In order to give examples of finitely presentable categories we will use a characterization involving strong generators.

Definition 3.2.6. An epimorphism $p\colon E \to B$ in a category $\mathscr{C}$ is called *extremal* if it does not factor through a proper subobject of B, i.e., if $p = iq$ for some monomorphism $i\colon B' \to B$, then i is an isomorphism.
It is called a *strong epimorphism* if for any monomorphism $i\colon A \to X$ in $\mathscr{C}$ and any commutative diagram of solid arrows

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "m"'] & A \arrow[d, "i"] \\ B \arrow[r, "g"'] \arrow[ur, dotted] & X \end{tikzcd}$$

there exists a dotted arrow such that the diagram is commutative.

Lemma 3.2.4. *Any strong epimorphism is extremal.*

Proof. If $p : E \to B$ is strong, and if $p = iq$ for a monomorphism $i : B' \to B$, there exists a morphism $j : B \to B'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[d, "m"'] & B' \arrow[d, "i"] \\ B \arrow[r, "\mathrm{id}"'] \arrow[ur, "j"] & B \end{tikzcd}$$

is commutative. Thus $ij = \mathrm{id}$, and $iji = i$. Since i is a monomorphism, the latter equation implies that $ji = \mathrm{id}$, which shows that i is an isomorphism. □

The following definition is from [Bor94a]; by the previous lemma it follows that a strong generator in the sense of [Bor94a] is also a strong generator in the sense of [AR94].

Definition 3.2.7. A set $\{G_i | i \in I\}$ of objects of a cocomplete category $\mathscr{C}$ is called a *strong generator* if for any $C \in \mathscr{C}$ the induced morphism

$$\coprod_{\varphi\colon G_i \to C} G_i \longrightarrow C$$

is a strong epimorphism, where the coproduct runs over all morphisms $\varphi\colon G_i \to C$ for all $i \in I$. In [AR94] it is only required that the induced morphism be an extremal epimorphism.

Proposition 3.2.5. *If the cocomplete category $\mathscr{C}$ has finite limits, then a set $\{G_i | i \in I\}$ is a strong generator if and only if the functors $\mathscr{C}(G_i, -)\colon \mathscr{C} \to$ **Set** collectively reflect isomorphisms.*

Proof. See [Bor94a], proposition 4.5.10. ☐

Proposition 3.2.6. *If a cocomplete category $\mathscr{C}$ has a strong generator consisting of λ-presentable objects, then $\mathscr{C}$ is λ-presentable.*

Proof. See [AR94], theorem 1.20. ☐

Corollary 3.2.7. *For any small R-linear category $\mathscr{A}$, the category $[\mathscr{A}, \mathbf{Mod}_R]$ of R-linear functors $\mathscr{A} \rightarrow \mathbf{Mod}_R$ is ω-presentable.*

Proof. The category is complete and cocomplete, and the representable functors form a strong generator consisting of ω-small objects. Indeed, by Yoneda we have a natural isomorphism $[\mathscr{A}, \mathbf{Mod}_R](\mathscr{A}(A,-), F) \cong FA$, and since colimits are computed pointwise it follows that the functors

$$F \mapsto [\mathscr{A}, \mathbf{Mod}_R](\mathscr{A}(A,-), F)$$

preserve ω-filtered colimits (in fact they preserve all colimits). Recall that a natural transformation α has an inverse if and only if all its components α_A are isomorphisms. Proposition 3.2.5 and the Yoneda lemma therefore imply that the representable functors do indeed form a strong generator. ☐

Definition 3.2.8. A functor $F \colon \mathscr{A} \rightarrow \mathscr{B}$ is called λ-*accessible* (where λ is a regular cardinal) if $\mathscr{A}$ and $\mathscr{B}$ are λ-accessible and F preserves λ-filtered colimits. The functor F is called *accessible* if there exists a regular cardinal λ such that F is λ-accessible.

Definition 3.2.9. A subcategory $\mathscr{A}$ of $\mathscr{C}$ is called *accessibly embedded* if it is full and if there is a regular cardinal λ such that $\mathscr{A}$ is closed under λ-filtered colimits in $\mathscr{C}$.

The following proposition (resp. its corollary) is crucial for our description result.

Proposition 3.2.8. *If $F \colon \mathscr{K} \rightarrow \mathscr{L}$ is an accessible functor, and if $\mathscr{L}_1$ is an accessible, accessibly embedded subcategory of $\mathscr{L}$, then $F^{-1}(\mathscr{L}_1)$ is an accessible, accessibly embedded subcategory of $\mathscr{K}$, where $F^{-1}(\mathscr{L}_1)$ denotes the full subcategory of $\mathscr{K}$ consisting of those objects $K \in \mathscr{K}$ with $F(K)$ lying in $\mathscr{L}_1$.*

Proof. See [AR94], remark 2.50. ☐

Corollary 3.2.9. *Let $\mathscr{A}$ be a small R-linear category and let $L \colon [\mathscr{A}, \mathbf{Mod}_R] \rightarrow \mathbf{Mod}_R$ a cocontinuous functor. Denote by Σ the class of morphisms of $[\mathscr{A}, \mathbf{Mod}_R]$ which are sent to isomorphisms by L. Then the full subcategory of* $\mathrm{Mor}([\mathscr{A}, \mathbf{Mod}_R])$ *generated by Σ is an accessible, accessibly embedded subcategory.*

Proof. The following argument is from [AR94], section 2.60. Recall that $\mathrm{Mor}(\mathscr{C})$ is the category of morphisms of $\mathscr{C}$, i.e., the category of functors $\mathbf{2} \rightarrow \mathscr{C}$, where $\mathbf{2}$ is the category with two objects 0,1 and one morphism $0 \leq 1$. By proposition 3.2.3 and corollary 3.2.7 it follows that $\mathscr{K} = \mathrm{Mor}([\mathscr{A}, \mathbf{Mod}_R])$ and $\mathscr{L} = \mathrm{Mor}(\mathbf{Mod}_R)$ are locally ω-presentable. We write $F \colon \mathscr{K} \rightarrow \mathscr{L}$ for the functor induced by L. Since colimits in $\mathrm{Mor}(\mathscr{C})$ are computed pointwise whenever $\mathscr{C}$ is cocomplete, it follows that F is cocontinuous. Thus $F \colon \mathscr{K} \rightarrow \mathscr{C}$ is an ω-accessible functor. Writing $\mathscr{L}_1 = \mathrm{Iso}(\mathbf{Mod}_R)$ for the full subcategory of $\mathscr{L}$ generated by the isomorphisms, we find that the full subcategory of $\mathscr{K}$ generated by Σ is precisely $F^{-1}(\mathscr{L}_1)$. The result follows if we can show that proposition 3.2.8 can be applied.

Since we already know that F is accessible it suffices to show that $\mathrm{Iso}(\mathbf{Mod}_R)$ is an accessible, accessibly embedded subcategory of $\mathrm{Mor}(\mathbf{Mod}_R)$. But the functor $\mathrm{Iso}(\mathbf{Mod}_R) \rightarrow \mathbf{Mod}_R$ which sends an isomorphism to its domain is an equivalence of categories, hence $\mathrm{Iso}(\mathbf{Mod}_R)$ is accessible; and $\mathrm{Iso}(\mathbf{Mod}_R)$ is closed under arbitrary colimits in $\mathrm{Mor}(\mathbf{Mod}_R)$, which shows that $\mathrm{Iso}(\mathbf{Mod}_R)$ is indeed accessibly embedded. ☐

3.3. **Orthogonality and the orthogonal reflection construction.**

Definition 3.3.1. Let Σ be a class of morphisms in a category $\mathscr{C}$. An object C of $\mathscr{C}$ is called *orthogonal* to Σ if for all solid arrow diagrams

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "s"'] & C \\
B \arrow[ur, dotted] &
\end{tikzcd}$$

where $s \in \Sigma$, there exists a unique dotted arrow making it commutative. The class of all objects which are orthogonal to Σ is denoted by $\Sigma^{\perp}$. We call this the *orthogonality class* of Σ.

We fix a locally presentable category $\mathscr{C}$, together with a *set* Σ of morphisms of $\mathscr{C}$. By proposition 3.2.2 there is a regular cardinal λ such that all domains and all codomains of the morphisms in Σ are λ-presentable. For any object X of $\mathscr{C}$ we construct a functor $X_{(-)} \colon \lambda \to \mathscr{C}$ by transfinite induction:

i) First step: We let $X_0 = X$.
ii) Successor step: If $X_{(-)}$ is defined on the subset $\{\beta \in \lambda | \beta \leq \alpha\}$ of λ, we let $X_{\alpha+1}$ be the colimit of the diagram

$$\begin{tikzcd}
 & C \arrow[dr, shift left, "q"] \arrow[dr, shift right, "p"'] & \cdots & C' \arrow[dl, shift left, "q'"] \arrow[dl, shift right, "p'"'] \\
A \arrow[rr, "f"] \arrow[dd, "s"'] \arrow[dr, phantom, "\ddots"] & & X_\alpha & \\
 & A' \arrow[ur, "f'"'] \arrow[dd, "s'"] & & \\
B & & & \\
 & B' & &
\end{tikzcd}$$

where $B \xleftarrow{s} A \xrightarrow{f} X_\alpha, \ldots, B' \xleftarrow{s'} A' \xrightarrow{f'} X_\alpha$ runs over all spans whose left leg lies in Σ, and the pairs $(p,q), \ldots, (p',q')$ run over all pairs of morphisms for which there exists a morphism $s \in \Sigma$ with $p \circ s = q \circ s$. We let $i_{\alpha,\alpha+1}$ be the structure morphism from the above diagram, which extends $X_{(-)}$ to a functor $\{\beta \in \lambda | \beta \leq \alpha+1\} \to \mathscr{C}$.
iii) Limit step: If $X_{(-)}$ is defined on $\{\beta \in \lambda | \beta < \mu\}$ for some limit ordinal $\mu \in \lambda$, we let X_μ be the colimit of the functor
$$X_{(-)} \colon \{\beta \in \lambda | \beta < \mu\} \longrightarrow \mathscr{C}.$$
We extend $X_{(-)}$ to a functor $\{\beta \in \lambda | \beta \leq \mu\} \to \mathscr{C}$ by letting $i_{\beta,\mu} : X_\beta \to X_\mu$ be the structure maps of this colimit.

Definition 3.3.2. For any X, the colimit rX of the functor $X_{(-)} \colon \lambda \to \mathscr{C}$ defined above is a called the *orthogonal reflection* of X. We denote the structure morphisms by $i_\alpha \colon X_\alpha \to rX$, and we let $\eta_X = i_0 \colon X \to rX$.

Since our definition differs slightly from the one in [AR94] and because the construction is so important for our description result, we provide a proof of the following proposition. All the arguments can be found in [AR94], section 1.37.

Proposition 3.3.1. *The orthogonal reflection construction has the following properties:*

(1) for any object X of $\mathscr{C}$, the object rX is orthogonal to Σ, and

(2) if Y is orthogonal to Σ and $f\colon X \to Y$ is any morphism, there is a unique morphism $\overline{f}\colon rX \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & rX \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

is commutative.

In other words, the orthogonal reflection construction provides a left adjoint to the inclusion $\Sigma^\perp \to \mathscr{C}$

Proof. First we show that rX is orthogonal to Σ. So let $s\colon A \to B$ be any morphism of Σ, together with a morphism $f\colon A \to rX$. Note that λ, considered as a category, is λ-filtered. It follows that for any λ-presentable object C of $\mathscr{C}$, $\mathscr{C}(C, rX)$ is the colimit of $\mathscr{C}(C, X_{(-)})\colon \lambda \to \mathbf{Set}$. Equivalently, this means that

a) any morphism $C \to rX$ factors via some $i_\alpha\colon X_\alpha \to rX$, and
b) if $i_\alpha \circ a = i_\alpha \circ b$ for $a, b\colon C \to X_\alpha$, there exists a $\beta \in \lambda$, $\beta \geq \alpha$ such that $i_{\alpha,\beta} \circ a = i_{\alpha,\beta} \circ b$.

In particular, there is a morphism $g\colon A \to X_\alpha$ such that $f = i_\alpha \circ g$. By definition of $X_{(-)}$ it follows that there is a morphism $B \to X_{\alpha+1}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "s"'] & X_\alpha \arrow[r] \arrow[d] & rX \\ B \arrow[r] & X_{\alpha+1} \arrow[ur] & \end{tikzcd}$$

is commutative. Indeed, the span $B \xleftarrow{s} A \xrightarrow{g} X_\alpha$ occurs in the defining diagram of $X_{\alpha+1}$, and the structure map $B \to X_{\alpha+1}$ gives the desired morphism. This shows existence of the dotted arrow in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & rX, \\ B \arrow[ur, dotted] & \end{tikzcd}$$

and it remains to show uniqueness. Thus let $a, b\colon B \to rX$ are two morphisms with $as = bs = f$. By a) it follows that both a and b factor as $a = i_\alpha p$ and $b = i_\alpha q$ for some morphisms $p, q\colon X_\alpha \to rX$. Since $i_\alpha ps = i_\alpha qs$, the property b) above implies that there is an element β of λ such that $i_{\alpha,\beta} ps = i_{\alpha,\beta} qs$. But this means that the pair $(i_{\alpha,\beta} p, i_{\alpha,\beta} q)$ occurs in the defining diagram of $X_{\beta+1}$, and therefore that

$$i_{\beta,\beta+1} \circ i_{\alpha,\beta} \circ p = i_{\beta,\beta+1} \circ i_{\alpha,\beta} \circ q,$$

which shows that $a = i_\alpha p = i_\alpha q = b$.

We now turn to the proof of (2). Since rX is the colimit of the functor $X_{(-)}\colon \lambda \to \mathscr{C}$, the statement in (2) is equivalent to the following: For each morphism $f\colon X \to Y$ there is a unique cocone $f_\alpha\colon X_\alpha \to Y$ with $f_0 = f$. We prove this by transfinite induction.

i) First step: demanding $f_0 = f$ uniquely determines a cocone on $X_{(-)}$ restricted to $\{0\} \subseteq \lambda$
ii) Successor step: Given a cocone $f_\beta\colon X_\beta \to Y$ on $X_{(-)}$ restricted to $\{\beta \in \lambda | \beta \leq \alpha\}$, we have to show that there is a unique extension $f_{\alpha+1}\colon X_{\alpha+1} \to X$ to a cocone of $X_{(-)}$ restricted to $\{\beta \in \lambda | \beta \leq \alpha + 1\}$. Equivalently, we have to show

that the morphism $f_\alpha \colon X_\alpha \to Y$ uniquely extends to a cocone on the solid arrow diagram

$$\begin{tikzcd}
 & & C \arrow[dr, shift left, "q"] \arrow[dr, shift right, "p"'] & \cdots & C' \arrow[dl, shift right, "p'"'] \arrow[dl, shift left, "q'"] & \\
A \arrow[rrr, "f"] \arrow[dd, "s"'] & & & X_\alpha \arrow[drr, "f_\alpha"] & & \\
 & \ddots & A' \arrow[ur, "f'"'] \arrow[dd, "s'"] & & & Y. \\
B \arrow[urrrrr, dotted, no head] & & & & & \\
 & & B' \arrow[uurrr, dotted, no head] & & &
\end{tikzcd}$$

Since Y is orthogonal to Σ there is a unique dotted arrows for all the spans in the above diagram. The dotted arrows together with f_α constitute a cocone if and only if $f_\alpha p = f_\alpha q$ for all the pairs (p,q) occurring in the above diagram. If (p,q) is such a pair, there is a morphism $s \in \Sigma$ such that $ps = qs$; hence $f_\alpha ps = f_\alpha qs$, and orthogonality of Y implies that $f_\alpha p = f_\alpha q$.

iii) Limit step: If $\mu \in \lambda$ is a limit ordinal and if the cocone $f_\beta \colon X_\beta$ is defined for the restriction of $X_{(-)}$ to $\{\beta \in \lambda | \beta < \mu\}$, we have to show that there is a unique extension to a cocone defined on $\{\beta \in \lambda | \beta \leq \mu\}$. This follows immediately from the definition of X_μ as a colimit.

The morphism $\overline{f} \colon rX \to Y$ is now given by the morphism which is induced by the unique cocone extending $f \colon X \to Y$. ☐

3.4. **Recognition of categories of Cauchy comodules.** We need one more concept in order to state our description theorem.

Definition 3.4.1. Let $\mathscr{A}$ be a small R-linear category. An R-linear functor $\omega \colon \mathscr{A} \to \mathbf{Mod}_R$ is called *flat* if the left Kan extension $\mathrm{Lan}_Y \omega \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathbf{Mod}_R$ of ω along Y (see definition 1.5.3) is left exact, i.e., if $\mathrm{Lan}_Y \omega$ preserves finite limits.

This definition is a generalization of flat R-modules: if $\mathscr{A}$ is the R-linear category $\mathscr{I}$ with one object $*$ and $\mathscr{I}(*,*) = R$, giving an R-linear functor $\omega \mathscr{I}^{\mathrm{op}} \to \mathbf{Mod}_R$ is the same as giving an R-module $M = \omega(*)$. Under this equivalence $[\mathscr{I}^{\mathrm{op}}, \mathbf{Mod}_R] \cong \mathbf{Mod}_R$, the left Kan extension $\mathrm{Lan}_Y \omega$ corresponds to $M \otimes - \colon \mathbf{Mod}_R \to \mathbf{Mod}_R$. Hence ω is flat if and only M is flat in the usual sense.

Definition 3.4.2. Let $\mathscr{A}$ be a small R-linear category, and let $F \colon \mathscr{A} \to \mathbf{Mod}_R$ be an R-linear functor. The category $\mathrm{el}(F)$ of *elements of* F has objects the pairs (A,a) with $A \in \mathscr{A}$ and $a \in FA$, and morphisms $(A,a) \to (A',a')$ the morphisms $f \colon A \to A'$ in $\mathscr{A}$ with $Ff(a) = a'$.

Proposition 3.4.1. *If $\mathscr{A}$ is an additive R-linear category, a functor $F \colon \mathscr{A} \to \mathbf{Mod}_R$ is flat if and only if the category $\mathrm{el}(F)$ of elements of F is cofiltered, i.e., if and only if $\mathrm{el}(F)^{\mathrm{op}}$ is filtered.*

Proof. This follows from [OR70], theorem 3.2. ☐

Proposition 3.4.2. *Let C be a flat coalgebra. If C has enough Cauchy comodules, then the forgetful functor $V \colon \mathbf{Comod}^c(C) \to \mathbf{Mod}_R$ is flat.*

Proof. By proposition 3.4.1 it suffices to show that $\mathrm{el}(V)$ is cofiltered. Since C has enough Cauchy comodules it follows immediately that $\mathrm{el}(V)$ is non-empty. Let

(M_i, m_i), $i = 0, 1$ be two objects of $\mathrm{el}(V)$. By proposition 2.3.3 it follows that $(M_0 \oplus M_1, (m_0, m_1))$ is an object of $\mathrm{el}(V)$ and that

$$\mathrm{pr}_i \colon (M_0 \oplus M_1, (m_0, m_1)) \to (M_i, m_i),$$

$i = 0, 1$, are morphisms in $\mathrm{el}(V)$. It remains to show that for two morphisms $\varphi_0, \varphi_1 \colon (M, m) \to (N, n)$ in $\mathrm{el}(V)$ there is a Cauchy comodule L, an element $l \in L$ and a morphism $\psi \colon L \to M$ of Cauchy comodules with $\psi(l) = m$, such that $\varphi_0 \circ \psi = \varphi_1 \circ \psi$. Let $\iota \colon K \to M$ be the equalizer of φ_0 and φ_1. There is a unique coaction ρ_K on K such that (K, ρ_K) is the equalizer of φ_0 and φ_1 in $\mathbf{Comod}(C)$ (see proposition 2.3.3), and by assumption we know that $m \in K$. Since C has enough Cauchy comodules there is a Cauchy comodule L, a morphism $\pi \colon L \to K$ and an element $l \in L$ such that $\pi(l) = m$. Now $\psi = \iota\pi$ gives the desired morphism $\psi \colon (L, l) \to (M, m)$ in $\mathrm{el}(V)$. □

Theorem 3.4.3. *Let $\mathscr{A}$ be a small R-linear additive category and let $\omega \colon \mathscr{A} \to \mathbf{Mod}_R^c$ be an R-linear functor. If*

i) ω (considered as a functor with domain $\mathbf{Mod}_R$) is flat and

ii) the functor $\mathscr{A} \xrightarrow{\omega} \mathbf{Mod}_R^c \longrightarrow \mathbf{Mod}_R$ reflects colimits,

then the unit

$$(\mathscr{A}, \omega) \longrightarrow \big(\mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)}), V\big)$$

of the adjunction $\mathbf{E}_{(-)} \dashv \mathbf{Comod}^c(-)$ (see definition 2.5.1) is fully faithful. If in addition

iii) the functor $\mathscr{A} \xrightarrow{\omega} \mathbf{Mod}_R^c \longrightarrow \mathbf{Mod}_R$ creates those colimits which happen to lie in $\mathbf{Mod}_R^c$,

then the unit

$$(\mathscr{A}, \omega) \longrightarrow \big(\mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)}), V\big)$$

is an equivalence of categories.

We split the proof into several lemmas, and we fix some notation. Let $\omega \colon \mathscr{A} \to \mathbf{Mod}_R^c$ be an R-linear functor. We consider the left Kan extension

$$\mathrm{Lan}_Y\, \omega \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathbf{Mod}_R$$

of $\omega \colon \mathscr{A} \to \mathbf{Mod}_R$ along Y (see definition 1.5.3), which we abbreviate as $L := \mathrm{Lan}_Y\, \omega$. Let $\Sigma \subseteq \mathrm{Mor}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R])$ be the class of morphisms in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ which are sent to isomorphisms by L. Since L is cocontinuous it follows that the full subcategory of $\mathrm{Mor}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R])$ generated by Σ is cocomplete, with colimits computed as in $\mathrm{Mor}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R])$. Let $\mathscr{C}$ be the full subcategory $\Sigma^{\perp}$ of $[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ consisting of the objects which are orthogonal to Σ (see definition 3.3.1). By corollary 3.2.9 it follows that Σ is an accessible, accessibly embedded subcategory of $\mathrm{Mor}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R])$. Accessibility of Σ implies that there is a regular cardinal λ' and a subset $\Sigma_0 \subseteq \Sigma$ such that every element of Σ can be written as a λ'-filtered colimit of elements in Σ_0 (see definition 3.2.5). It follows that the full subcategory $\mathscr{C} = \Sigma^{\perp}$ is equal to $\Sigma_0^{\perp}$. Moreover, there is a regular cardinal λ such that the domains and codomains of all morphisms in Σ_0 are λ-presentable (see proposition 3.2.2), hence the orthogonal reflection construction applied to Σ_0 (see definition 3.3.2) gives a left adjoint $r \colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \to \mathscr{C}$ to the inclusion $i \colon \mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$.

Lemma 3.4.4. *For any R-module M, $\widetilde{\omega}(M)$ is orthogonal to Σ. In other words, the functor $\widetilde{\omega} \colon \mathbf{Mod}_R \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ factors via the inclusion*

$$i \colon \mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R].$$

We write $W \colon \mathbf{Mod}_R \to \mathscr{C}$ for the unique functor with $iW = \widetilde{\omega}$.

Proof. Let $s \colon A \to B$ be any morphism in Σ. The existence of a unique dotted arrow in the left diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & \widetilde{\omega}(M) & \leftrightsquigarrow & LA \arrow[r, "f^{\#}"] \arrow[d, "L(s)"'] & M \\
B \arrow[ur, dotted] & & & LB \arrow[ur, dotted] &
\end{tikzcd}$$

is by adjunction equivalent to the existence of a unique dotted arrow in the right diagram. But the latter is evident since $L(s)$ is an isomorphism, so $\widetilde{\omega}(M)$ really does lie in $\mathscr{C}$. □

Lemma 3.4.5. *Let $V = Li \colon \mathscr{C} \to \mathbf{Mod}_R$. Using the notation from lemma 3.4.4, the functor V is left adjoint to W, with unit and counit given by the unit η and counit ε of the adjunction $L \dashv \widetilde{\omega}$ (see proposition 1.5.3). Moreover, the comonad associated to the adjunction $V \dashv W$ (see proposition 2.4.5) is equal to the comonad associated to the adjunction $L \dashv \widetilde{\omega}$.*

Proof. For every $X \in \mathscr{C}$, $\widetilde{\omega}(L(X))$ lies in $\mathscr{C}$ by lemma 3.4.4. Since the subcategory $\mathscr{C}$ is full it follows that $\eta \colon X \to \widetilde{\omega}(L(X)) = WV(X)$ is a morphism in $\mathscr{C}$. Furthermore, we have $VW(M) = LiW(M) = L\widetilde{\omega}(M)$ for every R-module M, which implies that ε_M is a morphism $VW(M) \to M$. The triangular identities obviously hold, because η and ε are the unit and counit of $L \dashv \widetilde{\omega}$. The statement about the associated comonads follows directly from the definition in proposition 2.4.5. □

Lemma 3.4.6. *The class Σ is equal to*

$$\{f \in \mathrm{Mor}([\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]) | r(f) \textit{ is an isomorphism}\}.$$

Proof. To see this, we will first show that for any $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$, the morphism $\eta_X \colon X \to rX$ (see definition 3.3.2) is in Σ, i.e., that $L(\eta_X)$ is an isomorphism. This follows from the fact that the morphisms $L(i_{\alpha,\beta}) \colon LX_\alpha \to LX_\beta$ are isomorphisms for all $\alpha, \beta \in \lambda$, $\alpha \leq \beta$, which can be seen by transfinite induction:

i) First step: there is nothing to show.
ii) Successor step: If $L(i_{\beta,\beta'}) \colon LX_\beta \to LX_{\beta'}$ is an isomorphism for all $\beta, \beta' \leq \alpha$, we have to show that $L(i_{\alpha,\alpha+1}) \colon LX_\alpha \to LX_{\alpha+1}$ is an isomorphism. Since L preserves colimits, $LX_{\alpha+1}$ is the colimit of the diagram

$$\begin{tikzcd}
 & LC \arrow[dr, shift left, "L(q)"] \arrow[dr, shift right, "L(p)"'] & \cdots & LC' \arrow[dl, shift right, "L(p')"'] \arrow[dl, shift left, "L(q')"] \\
LA \arrow[rr, "L(f)"] \arrow[ddd, "L(s)"'] & \ddots & LX_\alpha & \\
 & LA' \arrow[ur, "L(f')"'] \arrow[dd, "L(s')"] & & \\
 & & & \\
LB & LB' & &
\end{tikzcd}$$

with morphisms defined as in section 3.3. In particular, s, s' lie in Σ_0, hence $L(s), L(s')$ are isomorphisms; and for any pair (p, q) occurring in the above diagram there is a $s'' \in \Sigma_0$ such that $ps'' = qs''$, which implies that $L(p) =$

$L(q)$. It follows that

$$\begin{tikzcd}
& & LC \arrow[dr, shift left, "L(q)"] \arrow[dr, shift right, "L(p)"'] & \cdots & LC' \arrow[dl, shift right, "L(p')"'] \arrow[dl, shift left, "L(q')"] & \\
LA \arrow[rrr, "L(f)"] \arrow[dd, "L(s)"'] & & & LX_\alpha \arrow[drr, dotted, "\mathrm{id}"] & & \\
& \ddots & LA' \arrow[ur, "L(f')"'] \arrow[dd, "L(s')"'] & & & LX_\alpha . \\
LB \arrow[urrrrr, dotted, "L(f)\circ L(s)^{-1}"'] & & & & & \\
& & LB' \arrow[uurrr, dotted, "L(f')\circ L(s')^{-1}"'] & & &
\end{tikzcd}$$

is a colimit cocone, too. But $L(i_{\alpha,\alpha+1})$ obviously is the comparison morphism between the two colimit cocones, hence it is an isomorphism.

iii) Limit step: We have to show that for any limit ordinal $\mu \in \lambda$, if $L(i_{\beta,\beta'})$ is an isomorphism for all $\beta, \beta' < \mu$, then $L(i_{\beta,\mu})$ is an isomorphism for all $\beta \leq \mu$. The induction assumption implies that one way to compute the colimit of the μ-chain

$$LX_0 \longrightarrow LX_1 \longrightarrow \cdots \longrightarrow LX_\beta \longrightarrow \cdots$$

is given by LX_0, with structure morphisms $L(i_{0,\beta})^{-1}\colon LX_\beta \to LX_0$. Cocontinuity of L implies that LX_μ also is a colimit of this chain, hence its structure maps $L(i_{\beta,\mu})\colon LX_\beta \to LX_\mu$ must be isomorphisms, too.

Now, applying L to the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & rX \arrow[d, "r(f)"] \\
Y \arrow[r, "\eta_Y"'] & rY
\end{tikzcd}$$

we find that $L(f)$ is an isomorphism if and only if $L(r(f))$ is an isomorphism. Equivalently, this means that $f \in \Sigma$ if and only if $r(f) \in \Sigma$. But the latter is equivalent to the fact that $r(f)$ is an isomorphism, for if $r(f)$ is an isomorphism, then so is $L(r(f))$; and conversely, if $r(f) \in \Sigma$, there is a unique morphism $g\colon rY \to rX$ such that

$$\begin{tikzcd}
rX \arrow[r, "\mathrm{id}"] \arrow[d, "r(f)"'] & rX \\
rY \arrow[ur, dotted, "g"'] &
\end{tikzcd}$$

because rX is orthogonal to Σ. It follows that $L(g)$ is an isomorphism, and hence that the dotted arrow in

$$\begin{tikzcd}
rY \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & rY \\
rX \arrow[ur, dotted] &
\end{tikzcd}$$

exists, which shows that g is an isomorphism, and thus that $r(f) = g^{-1}$ is an isomorphism. This concludes the proof of our claim that $\Sigma = \{f | r(f) \text{ is an isomorphism}\}$.
□

Lemma 3.4.7. *If $\omega\colon \mathscr{A} \to \mathbf{Mod}_R$ is flat, then the adjunction $V \dashv W : \mathscr{C} \to \mathbf{Mod}_R$ (see lemma 3.4.5) is comonadic.*

Proof. The left adjoint V reflects isomorphisms, for if $f\colon X \to Y$ is a morphism in $\mathscr{C}$ such that $V(f) = L(f)$ is an isomorphism, it follows that $f \in \Sigma$; and in lemma 3.4.6 we have seen that this implies that $r(f)$ is an isomorphism. Since X and Y already lie in $\mathscr{C}$ and because $r\colon [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$ is a reflection, it follows that f is an isomorphism, too. Furthermore, our assumptions imply that the functor $V = Li$ is left exact: L is left exact because ω is flat, and i, as right adjoint to r, is automatically left exact. The claim now follows from corollary 2.4.9. $\square$

Definition 3.4.3. Let Σ_1 be the set of those morphisms $s_1\colon F \to \mathscr{A}(-,A)$ for which there exists a morphism $s_0\colon F_0 \to G_0$ in Σ_0, an object $A \in \mathscr{A}$ and a morphism $\overline{a}\colon \mathscr{A}(-,A) \to G_0$ which fit in a pullback diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "s_1"'] & F_0 \arrow[d, "s_0"] \\
\mathscr{A}(-,A) \arrow[r, "\overline{a}"'] & G_0.
\end{tikzcd}$$

In other words, Σ_1 consists of the pullbacks of elements of Σ_0 along morphisms with domain $\mathscr{A}(-,A)$ for some object $A \in \mathscr{A}$.

Lemma 3.4.8. *If ω is flat, the orthogonality classes $\Sigma_0^{\perp}$ and $\Sigma_1^{\perp}$ are equal.*

Proof. Flatness of ω implies that L preserves finite limits. It follows that $\Sigma_1 \subseteq \Sigma$, hence that $\Sigma^{\perp} \subseteq \Sigma_1^{\perp}$. Therefore it suffices to show that for any $X \in \Sigma_1$, any $s_0\colon F_0 \to G_0$ in Σ_0 and an arbitrary morphism $g\colon F_0 \to X$, there exists a unique dotted arrow such that the diagram

$$\begin{tikzcd}
F_0 \arrow[r, "G"] \arrow[d, "s_0"'] & X \\
G_0 \arrow[ur, dotted] &
\end{tikzcd}$$

is commutative. The assumption $X \in \Sigma_1^{\perp}$ implies that we have unique dotted arrows

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "s_1"'] & F_0 \arrow[r, "g"] \arrow[d, "s_0"] & X \\
\mathscr{A}(-,A) \arrow[r, "\overline{a}"'] \arrow[urr, dotted] & G_0 &
\end{tikzcd}$$

for any $\overline{a}\colon \mathscr{A}(-,A) \to G_0$, where the square on the left is a fixed pullback diagram. Uniqueness of the dotted arrows immediately implies that they constitute a cocone on the diagram D_F from definition 1.6.2. By corollary 1.6.7 it follows there is a unique morphism $h\colon G_0 \to X$ such that the diagrams

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "s_1"'] & F_0 \arrow[r, "g"] & X \\
\mathscr{A}(-,A) \arrow[r, "\overline{a}"'] & G_0 \arrow[ur, "h"'] &
\end{tikzcd}$$

are commutative for every $\overline{a}\colon \mathscr{A}(-,A) \to G_0$. This already shows that if the desired lift exists, it must be equal to h. It remains to show that $hs_0 = g$, i.e., that for every $x \in F_0A$, the equality

$$h_A \circ (s_0)_A(x) = g_A(x)$$

holds. If we fix some $x \in FA$ and let $a = (s_0)_A(x)$, the diagram

$$\begin{tikzcd}
\mathscr{A}(-,A) \arrow[r, "\overline{x}"] \arrow[d, equal] & F_0 \arrow[d, "s_0"] \\
\mathscr{A}(-,A) \arrow[r, "\overline{a}"'] & G_0
\end{tikzcd}$$

is commutative (recall that $\overline{x}$ stands for the unique natural transformation with $\overline{x}_A(\mathrm{id}_A) = x$). Since $s_1 \colon F \to \mathscr{A}(-,A)$ is the pullback of s_0 along $\overline{a}$, it follows that there is a unique *dashed* arrow making the diagram

$$\begin{tikzcd}
\mathscr{A}(-,A) \arrow[dr, dashed] \arrow[drr, bend left, "\overline{x}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& F \arrow[r] \arrow[d, "s_1"'] & F_0 \arrow[d, "s_0"] \\
& \mathscr{A}(-,A) \arrow[r, "\overline{a}"'] & G_0
\end{tikzcd}$$

commutative. Putting everything together we find that the diagram

$$\begin{tikzcd}
\mathscr{A}(-,A) \arrow[dr, dashed] \arrow[drr, bend left, "\overline{x}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & & \\
& F \arrow[r] \arrow[d, "s_1"'] & F_0 \arrow[r, "g"] & X \\
& \mathscr{A}(-,A) \arrow[r, "\overline{a}"'] & G_0 \arrow[ur, "h"'] &
\end{tikzcd}$$

is commutative, which shows that $g\overline{x} = h\overline{a}$ and thus, by definition of a, that

$$g_A(x) = g_A\big(\overline{x}_A(\mathrm{id}_A)\big) = h_A\big(\overline{a}_A(\mathrm{id}_A)\big) = h_A\big((s_0)_A(x)\big).$$

Since $x \in F_0A$ was arbitrary, this implies that $hs_0 = g$, i.e., that X is orthogonal to s_0. This concludes the proof of our claim that $\Sigma_1^{\perp} = \Sigma^{\perp}$. □

Lemma 3.4.9. *Assume that ω is flat and that $\omega \colon \mathscr{A} \to \mathbf{Mod}_R$ reflects colimits. Then for every object A of $\mathscr{A}$, the functor $\mathscr{A}(-,A)$ lies in $\mathscr{C}$. In other words, the Yoneda embedding factors through the inclusion $i \colon \mathscr{C} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]$.*

Proof. By lemma 3.4.8 it suffices to show that $\mathscr{A}(-,A')$ is orthogonal to Σ_1 for all $A' \in \mathscr{A}$. In other words, we have to show that for any morphism $s \colon F \to \mathscr{A}(-,A)$ such that $L(s)$ is an isomorphism, and for any morphism $f \colon F \to \mathscr{A}(-,A')$, there exists a unique dotted arrow

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[d, "s"'] & \mathscr{A}(-,A') \\
\mathscr{A}(-,A) \arrow[ur, dotted] &
\end{tikzcd}$$

making the diagram commutative. Let $(Y \downarrow F)$ be the category from definition 1.6.2. We consider the diagram $D \colon (Y \downarrow F) \to \mathscr{A}$ which sends $(B, \overline{b})$ to B and $f \colon (B, \overline{b}) \to (B', \overline{b'})$ to f. With the notation of definition 1.6.2 we find that $Y \circ D = D_F$. We let $\kappa_{(B,\overline{b})} \colon D(B, \overline{b}) \to A$ be the morphism $s_B(b) \colon B \to A$

(this makes sense because $b = \overline{b}_B(\mathrm{id}_B) \in FB$, and $s\colon F \to \mathscr{A}(-,A)$ is a natural transformation). We find that

$$\begin{aligned} Y(\kappa_{(B,\overline{b})})_B(\mathrm{id}_B) &= \mathscr{A}(B,\kappa_{(B,\overline{b})})(\mathrm{id}_B) \\ &= \kappa_{(B,\overline{b})} \\ &= s_B(b) \\ &= s_B\big(\overline{b}_B(\mathrm{id}_B)\big), \end{aligned}$$

hence the Yoneda lemma implies that $Y(\kappa_{(B,\overline{b})}) = s \circ \overline{b}$. The right-hand side is a cocone on $D_F = YD$ because the $\overline{b}$ constitute a cocone. Since Y is fully faithful, it follows that the $\kappa_{(B,\overline{b})}$ constitute a cocone on D. Similarly, we let $\xi_{(B,\overline{b})} : D(B,\overline{b}) \to A'$ be the morphism $f_B(b)$. An analogous argument shows that $Y(\xi_{(B,\overline{b})}) = f \circ \overline{b}$ and thus that the $\xi_{(B,\overline{b})}$ constitute a cocone on D. Since L preserves colimits and because the $\overline{b}\colon YD(B,b) \to F$ exhibit F as colimit of YD (see corollary 1.6.7) it follows that the $L(\overline{b})$ exhibit LF as colimit of the diagram LYD. But $L(s)$ is an isomorphism by assumption, which implies that the $L(s \circ \overline{b}) = LY(\kappa_{(B,\overline{b})})$ exhibit $LY(A)$ as (another) colimit of the same diagram LYD. Now the natural isomorphism $LY \cong \omega$ (see proposition 1.5.4) implies that the $\omega(\kappa_{(B,\overline{b})})$ exhibit $\omega(A)$ as the colimit of the diagram $\omega \circ D$. But ω reflects colimits by assumption, which shows that the $\kappa_{(B,\overline{b})} : D(B,\overline{b}) \to A$ constitute a colimit cocone. Hence there is a unique morphism $g : A \to A'$ in $\mathscr{A}$ such that $g \circ \kappa_{(B,b)} = \xi_{(B,b)}$ for all objects $(B,\overline{b})$ of $(Y \downarrow F)$. Applying the Yoneda embedding we find that, for every $b \in FB$,

$$\begin{aligned} Y(g) \circ s \circ \overline{b} &= Y(g) \circ Y(\kappa_{(B,\overline{b})}) \\ &= Y(\xi_{(B,b)}) \\ &= f \circ \overline{b}. \end{aligned}$$

Since the $\overline{b}$ exhibit F as colimit of the diagram $D_F = YD$, this implies that $Y(g) \circ s = f$. This shows the existence of a dotted arrow making

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "s"'] & \mathscr{A}(-,A') \\ \mathscr{A}(-,A) \arrow[ru, dotted] & \end{tikzcd}$$

commutative, and if h is any other arrow with $hs = f$, it must be of the form $Y(k)$ for a unique $k\colon A \to A'$ (since Y is fully faithful), and the fact that $hs\overline{b} = f\overline{b}$ implies that $Y(k \circ \kappa_{(B,b)}) = Y(\xi_{(B,b)})$, hence that $k \circ \kappa_{(B,\overline{b})} = \xi_{(B,\overline{b})}$. But the morphism g is unique with this property, which shows that $k = g$ and $h = Y(k) = Y(g)$. □

Proof of theorem 3.4.3. By proposition 2.5.7 it follows that the unit

$$(\mathscr{A},\omega) \longrightarrow \big(\mathbf{Comod}^c(\mathbf{E}_{(\mathscr{A},\omega)}), V_C\big)$$

is a fully faithful if and only if the composite

$$\mathscr{A} \xrightarrow{Y} [\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R] \xrightarrow{J} \mathrm{Comod}(L_\omega \widetilde{\omega})$$

is fully faithful, where J denotes the comparison functor of the adjunction $L_\omega \dashv \widetilde{\omega}$ (see proposition 2.4.6). With $V \dashv W\colon \mathscr{C} \to \mathbf{Mod}_R$ as in lemma 3.4.5 we get by proposition 2.4.6 a comparison functor $J'\colon \mathscr{C} \to \mathrm{Comod}(VW)$. Since the comonads VW and $L_\omega\widetilde{\omega}$ are equal (see lemma 3.4.5), this gives in fact a functor $J'\colon \mathscr{C} \to$

$\mathrm{Comod}(L_\omega\widetilde{\omega})$. Moreover, the diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "i"] \arrow[dr, "J'"'] & & {[\mathscr{A}^{\mathrm{op}}, \mathbf{Mod}_R]} \arrow[dl, "J"] \\
& \mathrm{Comod}(L_\omega\widetilde{\omega}) &
\end{tikzcd}$$

is commutative by definition of J and J'. If ω satisfies *i*), then J' is fully faithful (see lemma 3.4.7). By *ii*) it follows that the Yoneda embedding factors via $\mathscr{C}$ (see lemma 3.4.9). This shows that JY is equal to $J'Y$, and therefore that JY is fully faithful.

It remains to show that if ω creates those colimits which are finitely generated and projective, then the functor $JY = J'Y$ gives an equivalence between $\mathscr{A}$ and $\mathrm{Comod}^c(L_\omega\widetilde{\omega})$. Since J' is compatible with the forgetful functors, this is equivalent to showing that for every object X of $\mathscr{C}$ with $V(X)$ finitely generated and projective, there is an object A of $\mathscr{A}$ and an isomorphism $X \cong \mathscr{A}(-, A)$. By corollary 1.6.7 it follows that X is the colimit of some diagram $D\colon \mathscr{D} \to \mathscr{C}$ which is of the form $D = Y \circ D'$ for a unique $D'\colon \mathscr{D} \to \mathscr{A}$. Since V preserves colimits, we find that $V(X)$ is the colimit of $VD = VYD' = LYD' \cong \omega D'$ (see proposition 1.5.4). By *iii*) it follows that there is an object A of $\mathscr{A}$ and morphisms $\kappa_d\colon D'(d) \to A$ expressing A as colimit of D', such that ω preserves this colimit. The $Y(\kappa_d)$ constitute a cocone on $YD' = D$, hence there is a unique morphism $f\colon X \to \mathscr{A}(-, A)$ which is compatible with the respective cocones. Since both cocones are sent to colimit cocones by V it follows that $V(f)$ is an isomorphism, and by lemma 3.4.7 it follows that f is an isomorphism. $\square$

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