A theory of subsets of topological spaces

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Introduction

The following observation gave the motivation for this thesis:

Lemma: If X is any topological space, then starting from an arbitrary subset of X one can obtain at most 7 subsets of X, using only the operators $C : Y \mapsto \overline{Y}$ and $I : Y \mapsto Y^\circ$.

Proof: Note that if $U \subset X$ is open and $V \subset X$ is any subset containing U, then $U \subset V^\circ$. Similarly, if $A \subset X$ is closed and $B \subset X$ is contained in A, then $\overline{B} \subset A$. Therefore, for all $Y \subset X$ we have $(C \circ I \circ C \circ I)(Y) \subset (C \circ I)(Y)$ and $(I \circ C)(Y) \subset (I \circ C \circ I \circ C)(Y)$. On the other hand, for the same reason, $I(Y) \subset (I \circ C \circ I)(Y)$ and hence $(C \circ I)(Y) \subset (C \circ I \circ C \circ I)(Y)$, as well as $(C \circ I \circ C)(Y) \subset C(Y)$ and hence $(I \circ C \circ I \circ C)(Y) \subset (I \circ C)(Y)$. Thus, the sets $(C \circ I \circ C \circ I)(Y)$ and $(C \circ I)(Y)$ are equal and the sets $(I \circ C \circ I \circ C)(Y)$ and $(I \circ C)(Y)$ are equal, and the sets

$$Y, C(Y), I(Y), (C \circ I)(Y), (I \circ C)(Y), (C \circ I \circ C)(Y), (I \circ C \circ I)(Y)$$

are all the potentially different sets that can be obtained starting from Y. $\square$

In a suitable topological space X and a suitable starting set Y, these 7 sets really can be mutually distinct, as the following example shows: Let

$$Y := \{0\} \cup \big((1,2) \setminus \{\tfrac{3}{2}\}\big) \cup \big([3,4] \cap \mathbb{Q}\big) \subset \mathbb{R}.$$

Then

i. $C(Y) = \{0\} \cup [1,2] \cup [3,4]$,

ii. $I(Y) = (1,2) \setminus \{\frac{3}{2}\}$,

iii. $(C \circ I)(Y) = [1,2]$,

iv. $(I \circ C)(Y) = (1,2) \cup (3,4)$,

v. $(C \circ I \circ C)(Y) = [1,2] \cup [3,4]$ and

vi. $(I \circ C \circ I)(Y) = (1,2)$.

It is natural to ask how the situation changes if the choice of the operations is different, in particular if the operator ${}^c : Y \mapsto X \setminus Y$ and the binary operators $\cap$ and $\cup$ are added (i.e. if we use all the usual operators that appear in point-set topology). The structures in which to best examine this sort of problems are *closure algebras*; they represent an abstract algebraic description of exactly these operations. More precisely, a closure algebra is a Boolean algebra (that is, a set containing two distinguished minimal and maximal elements and two binary and one unary operations satisfying certain axioms, mimicking $\cap, \cup$ and ^{c}) with a closure operator (i.e. a map from the algebra to itself, satisfying some axioms mimicking C). The question of how many sets can be generated from a single subset of a topological space is closely related to the notion of *free* closure algebras, meaning closure algebras, in which equations involving a set of generators hold if and only if they hold in every closure algebra. Such closure algebras can be shown to exist for any non-empty set of generators, and to be infinite regardless of the number of generators. Furthermore, every closure algebra can be realized inside a topological space; this implies that there are subsets of some topological space X from which one obtains infinitely many distinct sets, using the operators C, ^{c} and $\cap$. Finally, one can show (through a more general statement), that X can be taken to be $\mathbb{R}^n$ for an arbitrary $n > 0$; so there is a subset of $\mathbb{R}^n$ that produces infinitely many subsets of $\mathbb{R}^n$ using these operators. All these results can be found in the article *The Algebra of Topology* by J.C.C. McKinsey and A. Tarski, Annals of Mathematics, Vol. 45, (1944), pp. 141 – 181. The contents of this thesis roughly follow this article.

In the first and second section, we recall the notions of Boolean algebras and closure algebras. We emphasise the results that every Boolean algebra can be embedded in the algebra of subsets of some set (following from Stone's representation theorem, which we prove) and that every closure algebra can be embedded in the closure algebra over some topological space. In the third section, we define closure formulas and treat equations of closure formulas. This constitutes the link to our questions about sets that can be obtained from some sets using some operations in a topological space. An important result in this section is that the theory of equations of closure formulas is decidable. In the fourth section we present the notion of free closure algebras and functionally free closure algebras and results about $\mathbb{R}^n$ in that matter. In the fifth section, we give examples to our questions, in particular a subset of $\mathbb{R}$ that generates infinitely many sets via the operators C, ^{c} and $\cap$. In the sixth section we show a rudimentary attempts to describe the free closure formula generated by 1 element, by listing equivalent closure formulas up to a limited size.

Contents

1 Boolean algebras

(1.1) **Definition:** A *Boolean algebra* is a tuple $(A, \wedge, \vee, \neg, 0, 1)$, where A is a set with $0, 1 \in A$, $\neg : A \to A$ is a unary operation and $\wedge, \vee : A \times A \to A$ are binary operations, such that

i. $\wedge, \vee$ are associative,

ii. $\wedge, \vee$ are commutative,

iii. $\wedge$ and $\vee$ are distributive, i.e. $\forall x, y, z \in A : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and

iv. $\forall x \in A : x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

(1.2) **Example:** If S is any set and $\mathcal{P}(S)$ is its power set, then

$$(\mathcal{P}(S), \cap, \cup, C, \varnothing, S),$$

where C is the map $\mathcal{P}(S) \to \mathcal{P}(S), X \mapsto S \setminus X$, is a Boolean algebra.

(1.3) **Remark:** Note that the axioms of Boolean algebras are symmetric in $\wedge$ and $\vee$, i.e. if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra, then so is $(A, \vee, \wedge, \neg, 1, 0)$, called its *dual* Boolean algebra.

Throughout the rest of this section, let $(A, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra.

(1.4) **Lemma:** The following statements hold:

i. $\forall x \in A : x \wedge x = x$ and $x \vee x = x$,

ii. $\forall x \in A : x \wedge 1 = x$ and $x \vee 0 = x$,

iii. $\forall x \in A : x \wedge 0 = 0$ and $x \vee 1 = 1$,

iv. $\forall x \in A : \neg(\neg x) = x$,

v. $\neg 0 = 1$ and $\neg 1 = 0$ and

vi. $\forall x, y \in A : \neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$.

(1.5) **Definition:** Let $B \subset A$ be a subset. Then B is called a *subalgebra* of A, if $0, 1 \in B$ and if B is closed with respect to the operations $\wedge, \vee, \neg$, i.e. if $(B, \wedge|_{B \times B}, \vee|_{B \times B}, \neg|_B, 0, 1)$ is itself a Boolean algebra. If $S \subset B$ is a subset, B is said to be *generated* by S, if B is the smallest subalgebra of A containing S.

(1.6) **Definition:** Let $(B, \wedge', \vee', \neg', 0', 1')$ be another Boolean algebra. A map $\varphi : A \to B$ is called a *homomorphism* of Boolean algebras, if

i. $\varphi(0) = 0'$ and $\varphi(1) = 1'$,

ii. $\forall x, y \in A : \varphi(x \wedge y) = \varphi(x) \wedge' \varphi(y)$ and $\varphi(x \vee y) = \varphi(x) \vee' \varphi(y)$ and

iii. $\forall x \in A : \varphi(\neg x) = \neg' \varphi(x)$.

(1.7) **Remark:** If $x, y \in A$, we write $x \leq y$, if $x \wedge y = x$ (or equivalently, if $x \vee y = y$). Note that $\leq$ is a partial ordering on A.

(1.8) **Definition:** The Boolean algebra A is called *completely additive*, if for any subset $X \subset A$ there exists a unique smallest element

$$y =: \bigvee_{x \in X} x \in A,$$

such that $x \leq y$ for all $x \in X$; or, equivalently if for any subset $X \subset A$ there exists a unique largest element

$$z =: \bigwedge_{x \in X} x \in A,$$

such that $z \leq x$ for all $x \in X$.

(1.9) **Definition:** An element $x \in A$ is called an *atom* of A, if $x \neq 0$ and if there is no $y \in A \setminus \{0, x\}$, such that $y \leq x$. The Boolean algebra A is called *atomic*, if every non-zero element contains an atom.

(1.10) **Example:** Let $\mathcal{X}$ be the set of subsets of $\mathbb{Q}$ that are both open and closed with respect to the topology induced from $\mathbb{R}$. Then $(\mathcal{X}, \cap, \cup, C, \varnothing, \mathbb{Q})$ is a non-atomic Boolean algebra.

(1.11) **Definition:** An *ultrafilter* on A is a set $U \subset A$, such that

i. $\forall x, y \in A : x \in U, x \leq y \Rightarrow y \in U$,

ii. $\forall x, y \in U : x \wedge y \in U$,

iii. $0 \notin U$ and

iv. $\forall x \in A : x \notin U \Rightarrow \neg x \in U$.

The set of all ultrafilters on A is denoted by $\mathcal{U}(A)$.

(1.12) **Remark:** A set which only satisfies the first three properties is called a *filter*. If U is any filter, then it is an ultrafilter if, and only if, there is no finer filter than U, i.e. if F is a filter, such that $U \subset F$, then $U = F$.
Also (using ii. and iii.), iv. actually implies

$$\forall x \in A : x \notin U \iff \neg x \in U.$$

(1.13) **Remark:** The set of ultrafilters on A can be thought of as something similar to the set S in the Boolean algebra of sets from (1.2), and and ultrafilter as something similar to an element of S. For instance, if A is already a Boolean algebra of subsets of some set S', then any $s \in S'$ generates an ultrafilter $U_s = \{x \in A \mid s \in x\}$. However, not every ultrafilter is really of that form; for example, the set of all co-finite subsets of $\mathbb{N}$ is a filter on $\mathcal{P}(\mathbb{N})$ and (by Zorn's Lemma) can be refined to an ultrafilter U. The intersection of all elements of U is empty.

(1.14) **Remark:** If U is an ultrafilter on A and $x_1, \ldots, x_n \in A$, such that $\bigvee_{i=1}^n x_i \in U$, then $x_i \in U$ for some i. Indeed, if $x, y \in A$, such that $x, y \notin U$, then $\neg x, \neg y \in U$ by (1.12). Hence $\neg x \wedge \neg y \in U$, i.e. $x \vee y = \neg(\neg x \wedge \neg y) \notin U$. From this, the claim follows by induction.

(1.15) **Definition:** The *Stone topology* on $\mathcal{U}(A)$ is the topology induced by the base consisting of all sets of the form

$$V_x := \{U \in \mathcal{U}(A) \mid x \in U\},$$

for some $x \in A$.

(1.16) **Remark:** Note that these sets really are a base of a topology on $\mathcal{U}(A)$, since they obviously cover $\mathcal{U}(A)$ and since $V_x \cap V_y = V_{x \wedge y}$.

(1.17) **Proposition:** With respect to the Stone topology, $\mathcal{U}(A)$ is a totally disconnected compact Hausdorff space.

Proof:

i. By (1.12), the complement of V_x is $V_{\neg x}$, so the sets V_x are also closed. Furthermore, if $X \subset \mathcal{U}(A)$ is any subset with at least two points, then $\exists x \in A$ such that $V_x \cap X$ and $V_{\neg x} \cap X$ are non-empty. Clearly, they are also disjoint and open in X (with the subspace topology). Hence $\mathcal{U}(A)$ is totally disconnected.

ii. Let $U, U' \in \mathcal{U}(A)$, with $U \neq U'$. By (1.12), the ultrafilters U and U' are not contained in each other, i.e. $\exists x, y \in A$, such that $x \in U \setminus U'$ and $y \in U' \setminus U$. Clearly, $x \not\leq y$ and $y \not\leq x$, so $x \wedge (\neg y)$ and $y \wedge (\neg x)$ are non-zero. Therefore, $V_{x \wedge (\neg y)}$ contains U, but not U', and $V_{y \wedge (\neg x)}$ contains U', but not U. Hence, $\mathcal{U}(A)$ is Hausdorff.

iii. Let $\mathcal{V} = (V_x)_{x \in X}$ be a basic open covering of $\mathcal{U}(A)$, where $X \subset A$. Suppose that $\nexists x_1, \ldots, x_n \in X$, such that $\bigvee_{i=1}^n x_i = 1$. Then, there exists an ultrafilter U, such that $U \cap X = \varnothing$. This implies in particular that $\forall x \in X : U \notin V_x$, i.e. $U \notin \bigcup_{x \in X} V_x$; contradiction. So let $x_1, \ldots, x_n \in X$, such that $\bigvee_{i=1}^n x_i = 1$. Then $\forall U \in \mathcal{U}(A) : \bigvee_{i=1}^n x_i \in U$, so $x_i \in U$ for some i by (1.14), i.e. $U \in V_{x_i}$; therefore $\{V_{x_1}, \ldots, V_{x_n}\}$ is a finite subcovering of $\mathcal{V}$. Hence $\mathcal{U}(A)$ is compact.

□

(1.18) **Stone's representation theorem:** There is an equivalence of the category of Boolean algebras and the category of totally disconnected compact Hausdorff spaces, given by the contra-variant functor

$$F : \begin{array}{rcl} A & \mapsto & \mathcal{U}(A) \\ (\varphi : A \to B) & \mapsto & \big(\mathcal{U}(\varphi) : \mathcal{U}(B) \to \mathcal{U}(A), U \mapsto \varphi^{-1}(U)\big) \end{array}$$

from Boolean algebras to disconnected compact Hausdorff spaces. The converse functor is given by

$$G : \begin{array}{rcl} X & \mapsto & \mathcal{C}(X) \\ (f : X \to Y) & \mapsto & \big(\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X), V \mapsto f^{-1}(V)\big), \end{array}$$

where $\mathcal{C}(X)$ denotes the Boolean algebra of all subsets of X, that are both open and closed.

Proof: As seen in the previous remark, $\mathcal{U}(A)$ is a totally disconnected compact Hausdorff space, and if $\varphi : A \to B$ is a homomorphism of Boolean algebras and $U \subset B$ is an ultrafilter, then so is $\varphi^{-1}(U) \subset A$. Furthermore, for any $x \in A$, the preimage of V_x with respect to $\mathcal{U}(\varphi)$ is $V_{\varphi(x)}$; hence $\mathcal{U}(\varphi)$ is continuous. Therefore, F is a functor.
On the other hand, if $V, V' \subset X$ are open and closed, then so are $V \cap V', V \cup V'$ and $X \setminus V$, so $\mathcal{C}(X)$ is a subalgebra of $\mathcal{P}(X)$. If $f : X \to Y$ is a continuous map, and $V \subset Y$ is open and closed, then so is $f^{-1}(V)$. Furthermore, clearly $f^{-1}(V \cap V') = f^{-1}(V) \cap f^{-1}(V')$, as well as $f^{-1}(V \cup V') = f^{-1}(V) \cup f^{-1}(V')$ and $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$; hence $\mathcal{C}(f)$ is a homomorphism of Boolean algebras. Therefore, G is a functor.
Let A be a Boolean algebra and let $\varphi : A \to \mathcal{C}\big(\mathcal{U}(A)\big)$ be the map $x \mapsto V_x$. Clearly, $\varphi(0) = \varnothing$ and $\varphi(1) = \mathcal{U}(A)$. It is also clear that $\varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$ and $\varphi(\neg x) = \mathcal{U}(A) \setminus \varphi(x)$, and it follows from (1.14) that $\varphi(x \vee y) = \varphi(x) \cup \varphi(y)$; so φ is a homomorphism of Boolean algebras. Furthermore, let $x, y \in A$, such

that $x \neq y$. Without loss of generality $x \not\leq y$ and hence $x \wedge \neg y \neq 0$. Therefore, $x \wedge \neg y$ is contained in an ultrafilter U, which also contains x, but not y; so $\varphi(x) \neq \varphi(y)$, i.e. φ is injective. Now, if $V \in \mathcal{C}\big(\mathcal{U}(A)\big)$, i.e. $V \subset \mathcal{U}(A)$ is open and closed, then it is a union of basic open subsets V_x, and since V is compact (being a closed subset of a compact space), there are $x_1, \dots, x_n$, such that $V = \bigcup_{i=1}^n V_{x_i} = V_x = \varphi(x)$, where $x = \bigvee_{i=1}^n x_i$; so φ is also surjective. Therefore, φ is an isomorphism.

Conversely, let X be a totally disconnected compact Hausdorff space, and let $f : X \to \mathcal{U}\big(\mathcal{C}(X)\big)$ be the map $x \mapsto \{V \in \mathcal{C}(X) \mid x \in V\} \in \mathcal{U}\big(\mathcal{C}(X)\big)$. Let $W \subset \mathcal{U}\big(\mathcal{C}(X)\big)$ be a basic open subset, i.e.

$$W = W_V = \{U \in \mathcal{U}\big(\mathcal{C}(X)\big) \mid V \in U\},$$

where $V \subset X$ is some open and closed subset. The preimage of W_V is V, so f is continuous. Conversely, if $V \subset X$ is a basic open subset, then $f(V) = W_V$, so f is open. Now let $x, y \in X$, such that $x \neq y$. Because X is Hausdorff and totally disconnected there exists an open and closed subset of X containing x, but not y, so $f(x) \neq f(y)$, i.e. f is injective. On the other hand, let $U \in \mathcal{U}\big(\mathcal{C}(X)\big)$ be an ultrafilter. Suppose that $\bigcap_{V \in U} V = \varnothing$. Then, of course, $\bigcup_{V \in U} X \setminus V = X$, and because X is compact, there are $V_1, \dots, V_n \in U$, such that $\bigcup_{i=1}^n X \setminus V_i = X$, or equivalently $\bigcap_{i=1}^n V_i = \varnothing$; contradiction. So let $x \in \bigcap_{V \in U} V$. By definition, $U \subset f(x)$, and hence $U = f(x)$ by (1.12); so f is also surjective. Therefore, f is a homeomorphism. □

(1.19) **Corollary:** Every Boolean algebra is isomorphic to a subalgebra of the Boolean algebra $\mathcal{P}(S)$ of subsets of some set S, namely $S = \mathcal{U}(A)$.

2 Closure algebras

(2.1) **Definition:** Let A be a Boolean algebra. A map

$$C : A \to A$$

is called *closure operator*, if

i. $C(0) = 0$,

ii. $\forall x \in A : x \leq Cx$,

iii. $\forall x \in A : Cx = CCx$ and

iv. $\forall x, y \in A : C(x \vee y) = Cx \vee Cy$.

(2.2) **Remark:** If $A = \mathcal{P}(X)$ for some set X, then these are exactly the Kuratowski closure axioms for a topological space on X. It is well known that they are equivalent to giving the collection of all closed sets, or equivalently a topology, on X.

(2.3) **Remark:** Note that for the definition of closure operators, the operations $\wedge$ and $\neg$ on A are irrelevant; so closure operators can be defined on any partially ordered set S containing a minimal element $0 \in S$, and with the property that for any $x, y \in S$ there is a unique smallest upper bound $x \vee y$ for x and y.

(2.4) **Definition:** A *closure algebra* is a pair (A, C), where A is a Boolean algebra and $C : A \to A$ is a closure operator.

(2.5) **Remark:** We will usually denote a closure algebra simply by A instead of (A, C), without specifying the closure operator, unless it is necessary.

(2.6) **Example:** Let X be a topological space. Then $\mathcal{P}(X)$ together with the closure operator $C : \mathcal{P}(X) \to \mathcal{P}(X), Y \mapsto \overline{Y}$ is a closure algebra, called the closure algebra over X.

(2.7) **Definition:** Let A be a closure algebra. Let I denote the map

$$I : A \to A, x \mapsto \neg(C(\neg x)).$$

For any element $x \in A$, the element Cx is called *closure* and Ix is called *interior* of x. The element x is called *closed*, if $x = Cx$ and it is called open, if $x = Ix$.

(2.8) **Remark:** It can easily be shown that usual properties for open and closed elements hold; for instance, any union and any finite intersection of open elements is open and any intersection and any finite union of closed elements is closed. Furthermore, if $\mathcal{P}(X)$ is a closure algebra over a topological space X, then an element of the closure algebra is open, closed etc. if, and only if, it is as subset of the topological space.

(2.9) **Definition:** Let A be a closure algebra. A subset $B \subset A$ is called a *subalgebra* of A, if it is a subalgebra as a Boolean algebra and $C|_B$ is well-defined as a map $B \to B$. If $S \subset B$ is a subset, B is said to be generated by S, if B is the smallest subalgebra of A containing S.

(2.10) **Definition:** Let A, A' be closure algebras. A map $\varphi : A \to A'$ is called a *homomorphism* of closure algebras, if it is a homomorphism of Boolean algebras and if $\varphi(Cx) = C'\varphi(x)$ for all $x \in A$.

(2.11) **Definition:** Let A be a closure algebra and $x \in A$. The closure algebra

$$A_x := \{y \in A \mid y \leq x\}$$

with the original operations $\wedge_x = \wedge, \vee_x = \vee$ and $\neg_x y := \neg y \wedge x$, as well as $0_x = 0$ and $1_x = x$ and with the closure operator

$$C_x : A_x \to A_x, y \mapsto Cy \wedge x$$

is called closure algebra relative to x.

(2.12) **Remark:** Note that a closure algebra relative to some element x is not a subalgebra, unless $x = 1$. Instead, if A is the closure algebra over a topological space X, a closure algebra relative to some element of A, i.e. a subset $Y \subset X$, is the closure algebra over the subspace Y.

(2.13) **Definition:** A closure algebra A is called

i. *connected*, if 0 and 1 are the only elements of A that are both open and closed, or

ii. *totally disconnected*, if there is no non-zero open element $x \in A$, such that the closure algebra A_x relative to x is connected.

(2.14) **Lemma:** Let A be a completely additive Boolean algebra and let $B \subset A$ be a subset containing the element 0 and with the property that the union $x \vee y$ of any two elements x, y of B is also contained in B. Let $\widetilde{C} : B \to B$ be a closure operator, in the sense of (2.3). Then there exists a closure operator $C : A \to A$, such that $C|_B = \widetilde{C}$.

Proof: Set $C0 := 0$ and for an arbitrary non-zero element $x \in A$ set

$$Cx := \bigwedge_{\substack{y \in B \\ x \leq \widetilde{C}y}} \widetilde{C}y.$$

By definition, $C0 = 0$, and obviously $\forall x \in A : x \leq Cx$.
In particular, this implies that $\forall x \in A : Cx \leq CCx$. Conversely, if $x \in A$ and $y \in B$, such that $x \leq \widetilde{C}y$, then by definition,

$$Cx = \widetilde{C}y \wedge \bigwedge_{\substack{z \in B \setminus \{y\} \\ x \leq \widetilde{C}z}} \widetilde{C}z,$$

hence $Cx \leq \widetilde{C}y$ and therefore $CCx \leq Cx$.
Furthermore, if $x, y \in A$, then

$$\begin{aligned}
Cx \vee Cy &= \bigwedge_{\substack{z \in B \\ x \leq \widetilde{C}z}} \widetilde{C}z \vee \bigwedge_{\substack{w \in B \\ y \leq \widetilde{C}w}} \widetilde{C}w \\
&= \bigwedge_{\substack{z \in B \\ x \leq \widetilde{C}z}} \bigwedge_{\substack{w \in B \\ y \leq \widetilde{C}w}} (\widetilde{C}z \vee \widetilde{C}w) \\
&= \bigwedge_{\substack{t \in B \\ x \vee y \leq \widetilde{C}t}} \widetilde{C}t \\
&= C(x \vee y).
\end{aligned}$$

(Note that for the second equality, the fact is used that infinite distributive laws hold in a Boolean algebra, if all involved intersections and unions exist; this follows for example from Stone's representation theorem.)
So $C : A \to A$ is a closure operator.
Finally, let $x \in B$. Then $x \leq \widetilde{C}x$, hence $Cx \leq \widetilde{C}x$. But on the other hand, if $y \in B$, such that $x \leq \widetilde{C}y$, then $\widetilde{C}x \leq \widetilde{C}y$, because $\widetilde{C}$ is a closure operator, so $\widetilde{C}x \leq Cx$. □

(2.15) **Remark:** Note that the closure operator C defined in the proof is the maximal extension of $\widetilde{C}$ in the sense of the lemma. Indeed, if C' is another such closure operator, then whenever an element x is contained in $\widetilde{C}y$ for some y, then so is $C'x$. Therefore,

$$C'x \leq \bigwedge_{\substack{y\in B \\ x\leq \widetilde{C}y}} \widetilde{C}y = Cx.$$

(2.16) **Proposition:** Every closure algebra is isomorphic to a subalgebra of the closure algebra over a topological space.

Proof: Let A be a closure algebra with closure operator C. By (1.18), there is an isomorphism φ of Boolean algebras from A to a subalgebra B of the Boolean algebra $\mathcal{P}(X)$ of subsets of the set $X = \mathcal{U}(A)$. For $x \in B$ define $\widetilde{C}x := \varphi\big(C(\varphi^{-1}(x))\big)$. Clearly, $\widetilde{C}$ is a closure operator on B, so by the previous lemma it can be extended to a closure operator $\overline{C} : \mathcal{P}(X) \to \mathcal{P}(X)$. By (2.2), $\overline{C}$ defines a topology on X and by construction, B is a subalgebra of the closure algebra over X and A is isomorphic to B. $\square$

(2.17) **Remark:** There is actually a stronger statement, that every closure algebra is isomorphic to a subalgebra of the closure algebra over a Hausdorff space.

3 Formulas and equations in closure algebras

(3.1) **Notation:** Let $\mathcal{X} = \{X_1, X_2, \ldots\}$ be a countably infinite set. The elements $X_i \in \mathcal{X}$ are called symbols.

(3.2) **Definition:** For any $n \geq 1$, the set $\mathcal{F}_n$ of *closure formulas* in n variables is defined as the smallest set of words over the alphabet

$$\mathcal{X} \cup \{\wedge, \vee, \neg, (,), C\},$$

satisfying

i. the symbols X_i are elements of $\mathcal{F}_n$ for $1 \leq i \leq n$, and

ii. if $\alpha, \beta \in \mathcal{F}_n$, then $\neg\alpha, C\alpha, (\alpha \wedge \beta), (\alpha \vee \beta) \in \mathcal{F}_n$.

The set $\mathcal{F}$ of all closure formulas is defined as

$$\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$

(3.3) **Remark:** Note that the operations $\wedge, \vee, \neg, C$ are well-defined on $\mathcal{F}_n$ and $\mathcal{F}$.

(3.4) **Remark:** It can be shown that if α is a closure formula, then exactly one of the following statements is true:

i. The formula α is a symbol.

ii. There is a formula β, such that $\alpha = \neg\beta$.

iii. There is a formula β, such that $\alpha = C\beta$.

iv. There are formulas β, β', such that $\alpha = (\beta \wedge \beta')$.

v. There are formulas β, β', such that $\alpha = (\beta \vee \beta')$.

In other words, the way of *building* the formula α is unique.

(3.5) **Remark:** If A is a closure algebra, then any closure formula in n variables $\alpha \in \mathcal{F}_n$ defines a map

$$f_\alpha^A : A^n \to A$$

as follows: Let $(x_1, \ldots, x_n) \in A^n$. For $1 \leq i \leq n$, let $f_{X_i}^A(x_1, \ldots, x_n) = x_i$ and for any $\alpha, \beta \in \mathcal{F}_n$, let $f_{\alpha\wedge\beta}^A(x_1, \ldots, x_n) = f_\alpha^A(x_1, \ldots, x_n) \wedge f_\beta^A(x_1, \ldots, x_n)$ and define $f_{\neg\alpha}^A, f_{C\alpha}^A$ and $f_{\alpha\vee\beta}^A$ analogously.
By the previous remark, for any closure formula α the function f_α^A is well-defined.
Such a function f_α^A is called a *closure function* on A in n variables.

(3.6) **Definition:** Let $\alpha \in \mathcal{F}_n$ be a closure formula in n variables. A sequence $\beta_1, \ldots, \beta_r$ of closure formulas in n variables is called a *chain* of length r for α, if $\beta_r = \alpha$ and if for $1 \leq i \leq r$, the formula β_i is either a symbol X_j with $1 \leq j \leq n$, or $(\beta_k \wedge \beta_l), (\beta_k \vee \beta_l), \neg\beta_k$ or $C\beta_k$, with $k, l < j$.
The length of the shortest chain for α is called the *order* of α.

(3.7) **Definition:** Let A be a closure algebra. Two closure formulas α, β in n variables are called A-*equivalent*, if $f_\alpha^A = f_\beta^A$. We then write $\alpha \equiv_A \beta$. The formula α is said to *vanish* in A, if f_α^A is the constant zero function.

(3.8) **Remark:** Note that the relation $\equiv_A$ is an equivalence relation on $\mathcal{F}_n$, and any two closure formulas are in the same equivalence class if and only if they define the same closure function on A. The operations $\wedge, \vee, \neg, C$ are well-defined on the equivalence classes, and with respect to these operations, $\mathcal{F}_n/_{\equiv_A}$ is a closure algebra, that can be considered the closure algebra of closure functions on A in n variables. Furthermore, since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ is a filtration of $\mathcal{F}$, the equivalence relation $\equiv_A$ extends to $\mathcal{F}$, and as before, $\mathcal{F}/_{\equiv_A}$ is a closure algebra, that can be considered the closure algebra of all closure functions on A.

(3.9) **Lemma:** Let (A, C) be a closure algebra and let $x_1, \ldots, x_n \in A$. Then there exists a subset $B \subset A$ and a map $\widetilde{C} : B \to B$, such that

i. $(B, \widetilde{C})$ is a closure algebra,

ii. $x_1, \ldots, x_n \in B$,

iii. $\forall x \in B : Cx \in B \Rightarrow \widetilde{C}x = Cx$ and

iv. $|B| \leq 2^{2^n}$.

Proof: Let $B \subset A$ be the the set of elements of A that can be obtained from $x_1, \ldots, x_n$ using the operations $\wedge, \vee, \neg$, i.e. the Boolean subalgebra of A generated by $\{x_1, \ldots, x_n\}$. Since B is finitely generated, it must be atomic, and any atom $b \in B$ must be of the form

$$b = y_1 \wedge y_2 \wedge \ldots \wedge y_n,$$

where for $1 \leq i \leq n$, the element y_i is either x_i or $\neg x_i$. Therefore, B contains at most 2^n atoms and hence at most 2^{2^n} elements.
Let $\widetilde{B} := \{x \in B \mid Cx \in B\} \subset B$. Note that if $x \in \widetilde{B}$, then $CCx = Cx \in B$, so $Cx \in \widetilde{B}$, as well. Therefore, $C|_{\widetilde{B}} : \widetilde{B} \to \widetilde{B}$ is a closure operator, and since B is finite and therefore completely additive, by (2.14), $C|_{\widetilde{B}}$ can be extended to a closure operator $\widetilde{C} : B \to B$. Obviously B is a closure algebra with respect to $\widetilde{C}$, and by construction $\forall x \in B : Cx \in B \Rightarrow \widetilde{C}x = Cx$. □

(3.10) **Proposition:** Let α be a closure formula in n variables of order r, such that $\alpha \equiv_B 0$ for any closure algebra B with at most $2^{2^{n+r}}$ elements. Then $\alpha \equiv_A 0$ for every closure algebra A.

Proof: Suppose that A is a closure algebra such that $\alpha \not\equiv_A 0$. Then there exist $x_1, \ldots, x_n \in A$ with

$$f_\alpha^A(x_1, \ldots, x_n) \neq 0.$$

Since α is of order r, there is a chain $\beta_1, \ldots, \beta_r$ for α, and for $1 \leq i \leq r$ let

$$y_i := f_{\beta_i}^A(x_1, \ldots, x_n) \in A.$$

By the previous lemma, there is a subset $B \subset A$ with at most $2^{2^{n+r}}$ elements, containing $x_1, \ldots, x_n, y_1, \ldots y_r$, and a closure operator $\widetilde{C} : B \to B$, such that $\forall x \in B : Cx \in B \Rightarrow \widetilde{C}x = Cx$.
If $\beta_i = C\beta_j$, then $Cy_j = y_i \in B$, therefore $\widetilde{C}y_j = Cy_j$ and hence for $1 \leq i \leq r$:

$$y_i = f_{\beta_i}^B(x_1, \ldots, x_n) \in B.$$

Finally,

$$f_\alpha^B(x_1, \ldots, x_n) = f_{\beta_r}^B(x_1, \ldots, x_n) = y_r = f_{\beta_r}^A(x_1, \ldots, x_n) = f_\alpha^A(x_1, \ldots, x_n) \neq 0,$$

contradiction. Therefore $\alpha \equiv_A 0$. □

(3.11) **Corollary:** Let α, β be closure formulas in n variables of order r and s, respectively. If $\alpha \equiv_B \beta$ for any closure algebra B with at most $2^{2^{n+r+s+5}}$ elements, then $\alpha \equiv_A \beta$ for every closure algebra A.

Proof: Clearly,

$$\alpha \equiv_A \beta \qquad \Longleftrightarrow \qquad (\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta) \equiv_A 0.$$

If $\gamma_1, \ldots \gamma_r$ and $\gamma'_1, \ldots, \gamma'_s$ are chains for α and β, respectively, then

$$\gamma_1, \ldots, \gamma_r, \gamma'_1, \ldots, \gamma'_s, \neg\gamma'_s, (\gamma_r \wedge \neg\gamma'_s), \neg\gamma_r, (\neg\gamma_r \wedge \gamma'_s), (\gamma_r \wedge \neg\gamma'_s) \vee (\neg\gamma_r \wedge \gamma'_s)$$

is a chain for $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$, so the order of $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$ is at most $r + s + 5$, and the claim follows from the previous proposition. $\square$

(3.12) **Definition:** Let α, β be closure formulas in n variables. Then α, β are called *equivalent*, if they are A-equivalent in every closure algebra A. We then write $\alpha \equiv \beta$. The formula α is said to *vanish identically*, if vanishes in every closure algebra.

(3.13) **Corollary:** The theory of equations of closure formulas (without quantors) is decidable.

Proof: By (3.11), to see whether two given closure formulas are equivalent, it is sufficient to test if they are A-equivalent in a finite number of finite closure algebras A. $\square$

4 Free closure algebras

(4.1) **Definition:** Let F be a closure algebra and $S \subset F$ be a subset. Then F is called *free* closure algebra generated by S, if F is generated by S and if for all positive integers n, for any $\alpha, \beta \in \mathcal{F}_n$ and for all pairwise distinct $x_1, \dots, x_n \in S$:

$$f_\alpha^F(x_1, \dots, x_n) = f_\beta^F(x_1, \dots, x_n) \iff \alpha \equiv \beta.$$

(4.2) **Example:** The free closure algebra generated by $\varnothing$ has just one element. It can be identified with the closure algebra over the empty topological space.

(4.3) **Example:** As for $\equiv_A$, the relation $\equiv$ is an equivalence relation on $\mathcal{F}_n$, and two closure formulas are in the same equivalence class if and only if they define the same closure function in every closure algebra. The operations $\wedge, \vee, \neg$ and C are well-defined on the equivalence classes of $\equiv$, and with respect to these operations, $\mathcal{F}_n/_\equiv$ is a free closure algebra generated by $X_1, \dots, X_n$. Similarly, the set $\mathcal{F}/_\equiv$ of equivalence classes of all closure formulas is a free closure algebra generated by $\mathcal{X}$.

(4.4) **Proposition:** Let F be a closure algebra and $S \subset F$. Then F is a free closure algebra generated by S if, and only if, it satisfies the following universal property: If A is any closure algebra and $\varphi : S \to A$ is any map, then there is a unique homomorphism $\widehat{\varphi} : F \to A$, such that $\widehat{\varphi} \circ i = \varphi$, where $i : S \hookrightarrow F$ is the inclusion map.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "i"'] & A \\ F \arrow[ur, dashed, "\widehat{\varphi}"'] & \end{tikzcd}$$

Furthermore, F is unique with respect to this universal property up to canonical isomorphism.

Proof: Suppose that F is a free closure algebra generated by S. Let $x \in F$ be any element. Since F is generated by S, there are a closure formula α in n variables, with $n \le |S|$, and mutually distinct elements $x_1, \dots, x_n \in S$, such that $x = f_\alpha^F(x_1, \dots, x_n)$. For any $s \in S$, let $\overline{s} := \varphi(s) \in A$, and let

$$\widehat{\varphi}(x) := f_\alpha^A(\overline{x_1}, \dots, \overline{x_n}) \in A.$$

Note that $\widehat{\varphi}(x)$ does not depend on the choice of α. Indeed, let β be another closure formula in m variables, with $m \le |S|$, and let $x_1', \dots, x_m'$ be mutually different elements of S, such that $x = f_\beta^F(x_1', \dots, x_m')$. After possible renumeration of x_i and x_i', we can assume that $x_i = x_i'$ for $1 \le i \le k$ for some k, and that $x_1, \dots, x_n, x_{k+1}', \dots, x_m'$ are mutually distinct. The formulas α and β can be augmented to formulas $\widetilde{\alpha}$ and $\widetilde{\beta}$ in $n + m - k \le |S|$ variables, simply ignoring the additional arguments. Because F is a free closure algebra, $f_{\widetilde{\alpha}}^F(x_1, \dots, x_n, x_{k+1}', \dots x_m') = f_{\widetilde{\beta}}^F(x_1', \dots, x_m', x_{k+1}, \dots x_n)$ implies $f_{\widetilde{\alpha}}^A(\overline{x_1}, \dots, \overline{x_n}, \overline{x_{k+1}'}, \dots \overline{x_m'}) = f_{\widetilde{\beta}}^A(\overline{x_1'}, \dots, \overline{x_m'}, \overline{x_{k+1}}, \dots \overline{x_n})$ and hence

$$f_\alpha^A(\overline{x_1}, \dots, \overline{x_n}) = f_\beta^A(\overline{x_1'}, \dots, \overline{x_m'}).$$

Therefore, $\widehat{\varphi} : F \to A$ is a well-defined map, and it is clear that it satisfies $\widehat{\varphi} \circ i = \varphi$. Also, if $f_\alpha^F(x_1, \dots, x_n) = x$, then $\neg x = f_{\neg\alpha}^F(x_1, \dots, x_n)$, hence $\widehat{\varphi}(\neg x) = f_{\neg\alpha}^A(\overline{x_1}, \dots, \overline{x_n}) = \neg\widehat{\varphi}(x)$, and similarly $\widehat{\varphi}(Cx) = C\widehat{\varphi}(x)$, as well as $\widehat{\varphi}(x \wedge y) = \widehat{\varphi}(x) \wedge \widehat{\varphi}(y)$ and $\widehat{\varphi}(x \vee y) = \widehat{\varphi}(x) \vee \widehat{\varphi}(y)$; so $\widehat{\varphi}$ is a homomorphism. Since F is generated by S, every homomorphism is fixed by the images of elements from S, so $\widehat{\varphi}$ is unique in satisfying $\widehat{\varphi} \circ i = \varphi$. Therefore, F satisfies the universal

property.
Conversely, let F be a closure algebra satisfying the universal property. Let A be a free closure algebra generated by S and let $j : S \hookrightarrow A$ be the inclusion map. Then there is a unique homomorphism $\widehat{j} : F \to A$, such that $\widehat{j} \circ i = j$. Assume that there are $\alpha, \beta \in \mathcal{F}_n$, with $\alpha \not\equiv \beta$, and $x_1, \ldots, x_n$ mutually different, such that $f_\alpha^F(x_1, \ldots, x_n) = f_\beta^F(x_1, \ldots x_n)$. Then

$$f_\alpha^A(x_1, \ldots, x_n) = \widehat{j}\big(f_\alpha^F(x_1, \ldots, x_n)\big) = \widehat{j}\big(f_\beta^F(x_1, \ldots, x_n)\big) = f_\beta^A(x_1, \ldots, x_n),$$

contradiction; so F is a free closure algebra generated by S.
Finally, if F and F' are two closure algebras satisfying the universal property, and if $i : S \to F$ and $j : S \to F'$ are the inclusion maps, then there are homomorphisms $\widehat{j} : F \to F'$ and $\widehat{i} : F' \to F$, such that $\widehat{j} \circ i = j$ and $\widehat{i} \circ j = i$. In particular, $(\widehat{i} \circ \widehat{j})|_S$ and $(\widehat{j} \circ \widehat{i})|_S$ are equal to the identity map on S, and since F and F' are generated by S, this means that $\widehat{j} \circ \widehat{i} = \mathrm{id}_{F'}$ and $\widehat{i} \circ \widehat{j} = \mathrm{id}_F$, so these maps are canonical isomorphisms between F and F'. □

(4.5) **Proposition:** Every free closure algebra is infinite.

Proof: It is sufficient to give an infinite set of pairwise inequivalent closure formulas $\alpha_1, \alpha_2, \ldots \in \mathcal{F}_1$. The following such example is due to Prof. Pink.
Let $\alpha_1 := X_1$ and for $n \geq 2$ let

$$\alpha_n := \big(\alpha_{n-1} \cap C(C\alpha_{n-1} \cap \neg\alpha_{n-1})\big).$$

Let X be the set of positive integers and let any non-empty $A \subset X$ be closed if, and only if, there is an $n \in X$, such that $A = A_n := \{k \in X \mid k \geq n\}$. Clearly, this defines a topology on X. For any $n \geq 1$ let

$$Y_n := \{2k - 1 \mid k \geq n\} \subset X.$$

Then $CY_n = A_{2n-1}$ and $\neg Y_n = X \setminus Y_n = \{1, 2, \ldots, 2n-2, 2n, 2n+2, \ldots\}$, so $CY_n \cap \neg Y_n = \{2k \mid k \geq n\}$. Therefore, $C(CY_n \cap \neg Y_n) = A_{2n}$ and hence

$$Y_{n+1} = Y_n \cap C(CY_n \cap \neg Y_n).$$

Clearly $Y_i \neq Y_j$ if $i \neq j$, so $f_{\alpha_i}^X(Y_1) \neq f_{\alpha_j}^X(Y_1)$, i.e. α_i and α_j are not equal in the closure algebra over X if $i \neq j$. □

(4.6) **Definition:** Let A be a closure algebra and let $n \geq 1$. Then A is called *functionally free of order* n, if

$$\forall \alpha, \beta \in \mathcal{F}_n : \alpha \equiv_A \beta \iff \alpha \equiv \beta$$

Furthermore, A is called *functionally free*, if it is functionally free of every order.

(4.7) **Remark:** If F is a free closure algebra generated by S, then F is obviously functionally free of order n, if $n \leq |S|$. In particular, F is functionally free if S is infinite. On the other hand, it can be shown that F is not functionally free if S is finite. Conversely, the closure algebra

$$A = \bigoplus_{n=1}^{\infty} \mathcal{F}_n/_\equiv = \{(x_n)_{n=1}^\infty \mid x_n \in \mathcal{F}_n/_\equiv \text{ and } x_n = 0 \text{ for almost all } n\},$$

on which all operations are defined component-wise, is an example for a functionally free closure algebra that is not free (generated by any subset).

(4.8) **Remark:** Recall that a topological space is called *perfect*, if it has no isolated points.

(4.9) **Proposition:** Let X be a perfect second-countable normal topological space (e.g. $X = \mathbb{R}^n$ for an arbitrary $n > 0$). The following statements hold:

i. For any finite closure algebra A there exists an open subset $U \subset X$, such that A is isomorphic to a subalgebra of the closure algebra over U.

ii. If in addition, X is totally disconnected (e.g. $X = \mathbb{Q}$), then U can be taken to be equal to X.

iii. The closure algebra over X is functionally free.

iv. The closure algebra over X contains a free closure algebra with countably many generators.

Proof:

i. See McKinsey & Tarski, Theorem 3.5 and Theorem 3.12.

ii. See McKinsey & Tarski, Theorem 3.5 and Theorem 3.8.

iii. If $\alpha \in \mathcal{F}_n$ does not vanish identically, then there is a finite closure algebra A and elements $x_1, \dots, x_n \in A$ such that $f^A_\alpha(x_1, \dots, x_n) \neq 0$. By i., there is an open subset $U \subset X$ and an embedding $\varphi : A \hookrightarrow B$ into the closure algebra B over U. Clearly, $f^B_\alpha\big(\varphi(x_1), \dots, \varphi(x_n)\big) \neq 0$, and if $y_1, \dots, y_n \in B$, such that $y_i \cap a = \varphi(x_i)$ for $1 \leq i \leq n$, then

$$f^B_\alpha(y_1, \dots, y_n) = a \cap f^B_\alpha\big(\varphi(x_1), \dots, \varphi(x_n)\big) \neq 0.$$

Therefore, by contraposition, if $\alpha \in \mathcal{F}_n$ vanishes in the closure algebra over X, then α vanishes identically. From this follows the claim, since $\forall \alpha, \beta \in \mathcal{F}_n : \alpha \equiv_A \beta \iff (\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta) \equiv_A 0$.

iv. See McKinsey & Tarski, Theorem 5.17.

□

(4.10) **Proposition:** There are subsets $Y_1, Y_2, \dots \subset \mathbb{R}$ (and equally of $\mathbb{R}^m$ for any $m \geq 1$) with the property that any equation of closure formulas involving n variables is satisfied for all subsets of all topological spaces if and only if it is satisfied for $Y_1, \dots, Y_n$.

Proof: Note that the sets $\mathcal{F}_n$ are countably infinite and that there are therefore only countably many equations $(\mathcal{E}_i)_{i \geq 1}$ of closure formulas. If some equation $\mathcal{E}_i$ is not satisfied for all subsets of all topological spaces, then by (3.11), there are elements of a finite closure algebra A_i for which $\mathcal{E}_i$ is not satisfied. Then, by i. from the previous proposition, there are subsets of $\mathbb{R}$ (and equally of $\mathbb{R}^m$ for any $m \geq 1$) for which the equation $\mathcal{E}_i$ does not hold. Because there are only countably many equations $\mathcal{E}_i$, these sets can be mapped homeomorphically into disjoint intervals (a_i, b_i). This way, we get subsets $Y_1, Y_2, \dots \subset \mathbb{R}$ with the desired property. □

5 Examples

(5.1) **Remark:** Recall that in the proof of (4.5), the following example for a subset of a topological space X, which generates an infinite subalgebra of the closure algebra over X, was given: Let $X = \mathbb{Z}^{\geq 1}$, and let $A \subset X$ be closed if and only if there is an $n \in X$, such that $A = A_n := \{x \in X \mid x \geq n\}$. Then the subsets $Y_n := \{2k-1 \mid k \geq n\} \subset X$ are obviously pairwise distinct, and furthermore, $Y_n = Y_{n-1} \cap C(C(Y_{n-1}) \setminus Y_{n-1})$, for any $n \geq 2$. We will use this example to find an example of subsets $Y_1, Y_2 \ldots \subset \mathbb{R}$ that satisfy the same recursive formula; hence, Y_1 will be an example of a subset of $\mathbb{R}$ that generates an infinite subalgebra of the closure algebra over $\mathbb{R}$.

(5.2) **Example:** Note that it is possible to replace the formula $X \wedge C(CX \wedge \neg X)$ in (4.5) by the simpler formula $CX \wedge \neg X$. Instead of the sets

$$Y_n = \{2k-1 \mid k \geq n\}$$

we will then obtain the sets

$$Y_n = \begin{cases} \{2k-1 \mid k \geq \frac{n+1}{2}\}, & \text{if } n \text{ is odd,} \\ \{2k \mid k \geq \frac{n}{2}\}, & \text{if } n \text{ is even.} \end{cases}$$

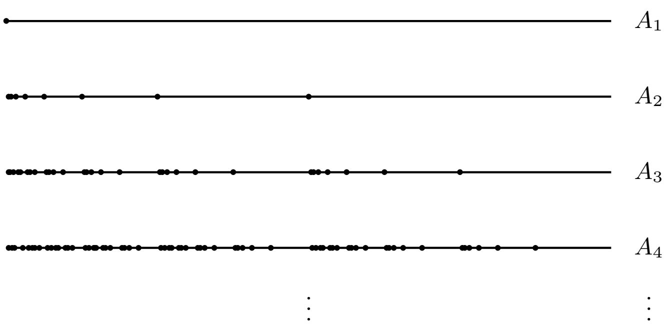
We have chosen to use the somewhat more complicated formula in (4.5) and in the following example for the sake of a slightly easier description.

(5.3) **Example:** Define the subsets $A_1, A_2, \ldots \subset [0,1)$ as follows: Let $A_1 := \{0\}$ and for $n \geq 2$ let A_n be the union of the monotone convergent sequences (not including their limit points)

$$\{x_0 + (x_1 - x_0)2^{-k} \mid k \in \mathbb{Z}^{\geq 1}\},$$

for $x_0 \in A_{n-1}$ and $x_1 = \sup\{x \in [0,1) \mid (x_0, x) \cap A_{n-1} = \varnothing\}$. Explicitly,

$$A_n = \{2^{-k_2}(1 + 2^{-k_3}(1 + 2^{-k_4}(1 + \ldots (1 + 2^{-k_n}) \ldots))) \mid k_2, \ldots, k_n \in \mathbb{Z}^{\geq 1}\}.$$



Note that the sets A_j are mutually exclusive and that the set of limit points of A_n is $A_1 \cup \ldots \cup A_{n-1}$. Therefore, the closure of A_n in $[0,1)$ is $A_1 \cup \ldots \cup A_n$. Let

$$X_n := \bigcup_{i=1}^{n} A_{2i-1}.$$

By what was just said, the closure $C(X_n)$ of X_n is $A_1 \cup \ldots \cup A_n$, so

$$C(X_n) \setminus X_n = \bigcup_{i=1}^{n-1} A_{2i},$$

and hence

$$X_n \cap C\big(C(X_n) \setminus X_n\big) = X_{n-1}.$$

Let

$$Y := \bigcup_{n=1}^{\infty} (n + X_n) \subset \mathbb{R}.$$

Furthermore, let $Y_1 := Y$ and

$$Y_{n+1} := Y_n \cap C\big(C(Y_n) \setminus Y_n\big).$$

Then for all $i, j \geq 1$, the sets Y_i and Y_j are unequal, whenever $i \neq j$. Indeed, if $j > i$, then $Y_j \cap [j, j+1) = j + X_j \neq j + X_i = Y_i \cap [j, j+1)$. This means that Y is a subset of R that generates an infinite subalgebra of the closure algebra over $\mathbb{R}$.

(5.4) **Lemma:** Let Z_n be the closure algebra over $\{1, \ldots, n\}$ with the induced topology from $\mathbb{Z}^{\geq 1}$ from (4.5), i.e. an element of Z_n is closed if and only if it is of the form $\{k, k+1, \ldots, n\}$ for some k. Let B_n be the subalgebra of the closure algebra over $\mathbb{R}$, that is generated by $A_1, \ldots, A_n$. Then the following statements are true:

i. The atoms of Z_n are precisely the elements $\{k\} \in Z_n$, with $1 \leq k \leq n$.

ii. The atoms of B_n are precisely the element $A_k \in B_n$, with $1 \leq k \leq n$.

iii. The map $h : B_n \to Z_n$ that is defined on atoms as $A_i \mapsto \{n - i\}$ is an isomorphism.

Proof:

i. Obvious.

ii. This follows directly from the definition of B_n, since the elements A_j are mutually exclusive.

iii. Because h is defined on the atoms of B_n, it is clear that it is a homomorphism of Boolean algebras. Let $x \in B_n$, i.e. $x = \bigcup_{m=1}^{k} A_{i_m}$ for $1 \leq k \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Then $Cx = \bigcup_{i=1}^{i_k} A_i$. Therefore,

$$h(Cx) = h\Big(\bigcup_{i=1}^{i_k} A_i\Big) = \bigcup_{i=1}^{i_k} \underbrace{h(A_i)}_{\{n-i\}} = \{n - i_k, n - i_k + 1, \ldots, n\}.$$

On the other hand, $h(x) = \{n - i_k, n - i_{k-1}, \ldots, n - i_1\}$, so

$$Ch(x) = \{n - i_k, n - i_k + 1, \ldots n\},$$

i.e. $h(Cx) = Ch(x)$. Hence, h is a homomorphism. It is obvious that h is bijective; thus it is an isomorphism.

□

(5.5) **Example:** The closure algebra Z over $\mathbb{Z}^{\geq 1}$ and the subalgebra B of the closure algebra over $\mathbb{R}$ that is generated by the subset $Y_1 \subset \mathbb{R}$ are not free. In both cases, there is an open atom: In Z it is the element $\{1\}$, and in B the element can be obtained accordingly, using the map h. Furthermore, this atom is dense; therefore, in both cases every element x satisfies

$$C\big(Ix \cup I(X \setminus x)\big) = X,$$

where $X = \mathbb{Z}^{\geq 1}$ in the first case and $X = \mathbb{R}$ in the second case. In other words, the closure formulas

$$C\big(\neg C \neg X_1 \vee \neg C X_1\big) \quad \text{and} \quad X_1 \vee \neg X_1$$

are Z-equivalent and B-equivalent. But they clearly cannot be equivalent in every closure algebra; over $\mathbb{R}$, for instance, $C\big(I\mathbb{Q} \cup I(\mathbb{R} \setminus \mathbb{Q})\big) = \varnothing$.

6 Equivalence classes of closure formulas

(6.1) **Definition:** A closure formula in n variables is said to have *depth* 1 if it is a symbol, and for any other closure formula α we define its *depth* recursively as the unique integer k such that there is a closure formula β of depth $k-1$ with $\alpha = \neg\beta$ or $\alpha = C\beta$, or such that there are closure formulas β, β' with $\max\{\operatorname{depth}\beta, \operatorname{depth}\beta'\} = k-1$, with $\alpha = (\beta \wedge \beta')$ or $\alpha = (\beta \vee \beta')$.

(6.2) **Remark:** In (4.5) we have seen that there are infinitely many equivalence classes of closure formulas of any number of variables, and it is obvious that they all contain infinitely many elements. In this section we will list closure formulas in just one variable in ascending depth and decide if they are equivalent to a closure formula already given or not.

(6.3) **Example:** There is only one closure formula of depth 1 in $\mathcal{F}_1$, namely the symbol $X = X_1$. We call its equivalence class $\mathcal{K}_1$. There are two closure formulas of order 2, namely $\neg X$ and CX. They are clearly not equivalent to X or to each other. We call their equivalence classes $\mathcal{K}_2$ and $\mathcal{K}_3$, respectively.

(6.4) **Remark:** There are actually two more closure formulas of depth 2, namely $X \wedge X$ and $X \vee X$; they are clearly equivalent to X. We are going to omit such superfluous closure formulas, when their equivalence to some easier formula is more than obvious and they are really nothing but a silly way of writing something in a more complicated way. Hence, we will not list (nor give any commentary about) formulas that include a union or intersection of a formula with itself or its closure, nor formulas that include one of the operators $\neg$ and C applied twice in a row. Also, we will choose a closure formula (with minimal order) from every equivalence class, and allow no other closure formulas of this equivalence class to be used in chains for other closure formulas. Furthermore, we use commutativity where it is possible and we will not write brackets when they are not needed.

(6.5) **Example:** There are two closure formulas of depth 3 obtained by application of a unary operator, namely $\neg CX$ and $C\neg X$. They are clearly not equivalent to each other or to a previously listed closure formula (take e.g. $x = [0,1) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_4$ and $\mathcal{K}_5$, respectively. There are four closure formulas of depth 3 obtained by application of either $\wedge$ or $\vee$, namely $0 = X \wedge \neg X$, $1 = X \vee \neg X$, $\neg X \wedge CX$ and $\neg X \vee CX$. The last one is equivalent to $X \vee \neg X$, the first three are not equivalent to each other or to a previously listed closure formula (again, take e.g. $x = [0,1) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_6, \mathcal{K}_7$ and $\mathcal{K}_8$, respectively.

(6.6) **Remark:** We will not list any closure formula that has $X \wedge \neg X$ or $X \vee \neg X$ in a minimal chain, because these will clearly be equivalent to a formula of smaller depth, since all operations are trivial on those two formulas.

(6.7) **Example:** There are four closure formulas of depth 4 obtained by application of a unary operator, namely $IX = \neg C \neg X$, $\neg(\neg X \wedge CX)$, $C\neg CX$ and $C(\neg X \wedge CX)$. The first two are not equivalent to each other or to any previously listed formula (take e.g $x = [0,1) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_9$ and $\mathcal{K}_{10}$, respectively. The third one is not equivalent to a previously listed formula either (take e.g. $x = [0,1) \setminus \{\frac{1}{2}\} \subset \mathbb{R}$); we call its equivalence class $\mathcal{K}_{11}$. Finally, the fourth one is not equivalent to any previously listed formula (take e.g. $x = [0,1) \cap \mathbb{Q} \subset \mathbb{R}$); we call its equivalence class $\mathcal{K}_{12}$.

(6.8) **Example:** There are 11 closure formulas of depth 4 obtained by application of the operator $\wedge$, namely $X \wedge \neg CX \in \mathcal{K}_6$, $X \wedge C\neg X$, $X \wedge (\neg X \wedge CX) \in \mathcal{K}_6$, $\neg X \wedge \neg CX \in \mathcal{K}_4$, $\neg X \wedge (\neg X \wedge CX) \in \mathcal{K}_8$, $CX \wedge \neg CX \in \mathcal{K}_6$, $CX \wedge C\neg X$,

$CX \wedge (\neg X \wedge CX) \in \mathcal{K}_8$, $\neg CX \wedge C\neg X \in \mathcal{K}_4$, $\neg CX \wedge (\neg X \wedge CX) \in \mathcal{K}_6$ and $C\neg X \wedge (\neg X \wedge CX) \in \mathcal{K}_8$. The formulas $X \wedge C\neg X$ and $\partial X = CX \wedge C\neg X$ are not equivalent to each other or to any previously listed formula (take e.g. $x = [0,1) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_{13}$ and $\mathcal{K}_{14}$, respectively.

(6.9) **Example:** There are 11 closure formulas of depth 4 obtained by application of the operator $\vee$, namely $X \vee \neg CX$, $X \vee C\neg X \in \mathcal{K}_7$, $X \vee (\neg X \wedge CX)$, $\neg X \vee \neg CX \in \mathcal{K}_2$, $\neg X \vee (\neg X \wedge CX) \in \mathcal{K}_2$, $CX \vee \neg CX \in \mathcal{K}_7$, $CX \vee C\neg X \in \mathcal{K}_7$, $CX \vee (\neg X \wedge CX) \in \mathcal{K}_3$, $\neg CX \vee C\neg X \in \mathcal{K}_5$, $\neg CX \vee (\neg X \wedge CX)$ and $C\neg X \vee (\neg X \wedge CX) \in \mathcal{K}_5$. By de Morgan's law, $\neg(\neg X \wedge CX) \equiv X \vee \neg CX$, so $X \vee \neg CX \in \mathcal{K}_{10}$. Furthermore, $X \vee (\neg X \wedge CX) \in \mathcal{K}_3$, because of the equivalence $X \vee (\neg X \wedge CX) \equiv (X \wedge CX) \vee (\neg X \wedge CX) \equiv (X \vee \neg X) \wedge CX \equiv CX$. Finally, $\neg CX \vee (\neg X \wedge CX) \equiv (\neg CX \vee \neg X) \wedge (\neg CX \vee CX) \equiv \neg X \in \mathcal{K}_2$.

(6.10) **Example:** There are 4 closure formulas of depth 5 obtained by application of the operator $\neg$, namely $\neg C\neg CX = (I \circ C)X$, $\neg C(\neg X \wedge CX)$, $\neg(X \wedge C\neg X)$ and $\neg(CX \wedge C\neg X) = \neg\partial X$. None of them are equivalent to each other or any previously listed formula (take e.g. $x = [0,1) \setminus \{\frac{1}{2}\} \subset \mathbb{R}$ for the first, third and fourth and $x = [0,1) \cap \mathbb{Q} \subset \mathbb{R}$ for the second). We call their equivalence classes $\mathcal{K}_{15}$ through $\mathcal{K}_{18}$.

(6.11) **Example:** There are 4 closure formulas of depth 5 obtained by application of the operator C, namely $C\neg C\neg X = (C \circ I)X$, $C(X \vee \neg CX) \equiv CX \vee C\neg CX \in \mathcal{K}_7$, $C(X \wedge C\neg X)$ and $C(CX \wedge C\neg X) = C\partial X \equiv \partial X \in \mathcal{K}_{14}$. The first one and the third one are not equivalent to each other or to any previously listed formula (take e.g. $x = [0,1) \cup ([2,3) \cap \mathbb{Q}) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_{19}$ and $\mathcal{K}_{20}$, respectively.

(6.12) **Example:** There are 50 closure formulas of depth 5 obtained by application of the operator $\wedge$, namely $X \wedge \neg C\neg X \in \mathcal{K}_9$, $X \wedge (X \vee \neg CX) \in \mathcal{K}_1$, $X \wedge C\neg CX$, $X \wedge C(\neg X \wedge CX)$, $X \wedge (X \wedge C\neg X) \in \mathcal{K}_{13}$, $X \wedge (CX \wedge C\neg X) \equiv X \wedge C\neg X \in \mathcal{K}_{13}$, $\neg X \wedge \neg C\neg X \in \mathcal{K}_6$, $\neg X \wedge (X \vee \neg CX) \equiv \neg X \wedge \neg CX \in \mathcal{K}_4$, $\neg X \wedge C\neg CX$, $\neg X \wedge C(\neg X \wedge CX)$, $\neg X \wedge (X \wedge C\neg X) \in \mathcal{K}_6$, $\neg X \wedge (CX \wedge C\neg X) \equiv CX \wedge \neg X \in \mathcal{K}_8$, $CX \wedge \neg C\neg X \in \mathcal{K}_9$, $CX \wedge (X \vee \neg CX) \equiv X \in \mathcal{K}_1$, $CX \wedge C\neg CX$, $CX \wedge C(\neg X \wedge CX) \in \mathcal{K}_{12}$, $CX \wedge (X \wedge C\neg X) \in \mathcal{K}_{13}$, $CX \wedge (CX \wedge C\neg X) \in \mathcal{K}_{14}$, $\neg CX \wedge \neg C\neg X \in \mathcal{K}_6$, $\neg CX \wedge (X \vee \neg CX) \in \mathcal{K}_4$, $\neg CX \wedge C\neg CX \in \mathcal{K}_4$, $\neg CX \wedge C(\neg X \wedge CX) \in \mathcal{K}_6$, $\neg CX \wedge (X \wedge C\neg X) \in \mathcal{K}_6$, $\neg CX \wedge (CX \wedge C\neg X) \in \mathcal{K}_6$, $C\neg X \wedge \neg C\neg X \in \mathcal{K}_6$, $C\neg X \wedge (X \vee \neg CX)$, $C\neg X \wedge C\neg CX \in \mathcal{K}_{11}$, $C\neg X \wedge C(\neg X \wedge CX) \in \mathcal{K}_{12}$, $C\neg X \wedge (X \wedge C\neg X) \in \mathcal{K}_{13}$, $C\neg X \wedge \partial X \in \mathcal{K}_{14}$, $(\neg X \wedge CX) \wedge \neg C\neg X \in \mathcal{K}_6$, $(\neg X \wedge CX) \wedge (X \vee \neg CX) \in \mathcal{K}_6$, $(\neg X \wedge CX) \wedge C\neg CX$, $(\neg X \wedge CX) \wedge (X \wedge C\neg X) \in \mathcal{K}_6$, $(\neg X \wedge CX) \wedge (CX \wedge C\neg X) \equiv \neg X \wedge CX \in \mathcal{K}_8$, $IX \wedge (X \vee \neg CX) \equiv IX \in \mathcal{K}_9$, $\neg C\neg X \wedge C\neg CX \in \mathcal{K}_6$, $IX \wedge C(\neg X \wedge CX)$, $IX \wedge (X \wedge C\neg X) \in \mathcal{K}_6$, $IX \wedge (CX \wedge C\neg X) \in \mathcal{K}_6$, $(X \vee \neg CX) \wedge C\neg CX$, $(X \vee \neg CX) \wedge C(\neg X \wedge CX)$, $(X \vee \neg CX) \wedge (X \wedge C\neg X) \in \mathcal{K}_{13}$, $(X \vee \neg CX) \wedge \partial X$, $C\neg CX \wedge C(\neg X \wedge CX)$, $C\neg CX \wedge C\neg X$, $C\neg CX \wedge \partial X$, $C(\neg X \wedge CX) \wedge (X \wedge C\neg X)$, $C(\neg X \wedge CX) \wedge (CX \wedge C\neg X) \in \mathcal{K}_{14}$ and $(X \wedge C\neg X) \wedge (CX \wedge C\neg X) \in \mathcal{K}_{13}$.
The formulas $X \wedge C\neg CX$, $X \wedge C(\neg X \wedge CX)$, $\neg X \wedge C\neg CX$, $CX \wedge C\neg CX$, $C\neg X \wedge (X \vee \neg CX)$, $(\neg X \wedge CX) \wedge C\neg CX$, $(X \vee \neg CX) \wedge C\neg CX$ and $C\neg CX \wedge C(\neg X \wedge CX)$ are not equivalent to each other or to any previously listed formula (take e.g. $x = [0,1) \setminus \{\frac{1}{2}\} \cup ([2,3) \cap \mathbb{Q}) \subset \mathbb{R}$); we call their equivalence classes $\mathcal{K}_{21}$ through $\mathcal{K}_{28}$.
On the other hand, $\neg x \wedge C(\neg x \wedge Cx) \leq \neg x \wedge Cx$ for any element x of any closure algebra, since $C(\neg x \wedge Cx) \leq Cx$, and conversely, $\neg x \wedge Cx \leq \neg x \wedge C(\neg x \wedge Cx)$, because $\neg x \wedge Cx \leq \neg x$ and $\neg x \wedge Cx \leq C(\neg x \wedge Cx)$; therefore we have $\neg X \wedge C(\neg X \wedge CX) \equiv \neg X \wedge CX \in \mathcal{K}_8$. Furthermore, for any x as before, $Ix \wedge C(\neg x \wedge Cx) \leq Ix \wedge C\neg x = 0$, so $IX \wedge C(\neg X \wedge CX) \in \mathcal{K}_6$. Also,

$$\begin{aligned}(X \vee \neg CX) \wedge C(\neg X \wedge CX) &\equiv (X \wedge C(\neg X \wedge CX)) \vee (\neg CX \wedge C(\neg X \wedge CX)) \\ &\equiv X \wedge C(\neg X \wedge CX) \in \mathcal{K}_{22}.\end{aligned}$$

Furthermore,

$$\begin{aligned}(X \vee \neg CX) \wedge (CX \wedge C\neg X) &\equiv (X \wedge C\neg X) \vee (\neg CX \wedge CX \wedge C\neg X)\\ &\equiv X \wedge C\neg X \in \mathcal{K}_{13},\end{aligned}$$

and $C\neg CX \wedge (X \wedge C\neg X) \equiv C\neg CX \wedge X \in \mathcal{K}_{21}$. Similarly, we have $C\neg CX \wedge (CX \wedge C\neg X) \equiv C\neg CX \wedge CX \in \mathcal{K}_{24}$. Finally $C(\neg x \wedge CX) \leq C\neg x$ for any x as before, and hence $C(\neg X \wedge CX) \wedge (X \wedge C\neg X) \equiv C(\neg X \wedge CX) \wedge X \in \mathcal{K}_{22}$.

(6.13) **Example:** There are 50 closure formulas of depth 5 obtained by application of the operator $\vee$, namely $X \vee \neg C\neg X \in \mathcal{K}_1$, $X \vee (X \vee \neg CX) \in \mathcal{K}_{10}$, $X \vee C\neg CX$, $X \vee C(\neg X \wedge CX)$, $X \vee (X \wedge C\neg X) \in \mathcal{K}_1$, $X \vee \partial X \equiv CX \in \mathcal{K}_3$, $\neg X \vee IX$, $\neg X \vee (X \vee \neg CX) \in \mathcal{K}_7$, $\neg X \vee C\neg CX$, $\neg X \vee C(\neg X \wedge CX)$, $\neg X \vee (X \wedge C\neg X) \in \mathcal{K}_5$, $\neg X \vee \partial X \equiv C\neg X \in \mathcal{K}_5$, $CX \vee IX \in \mathcal{K}_3$, $CX \vee (X \vee \neg CX) \in \mathcal{K}_7$, $CX \vee C\neg CX \in \mathcal{K}_7$, $CX \vee C(\neg X \wedge CX) \in \mathcal{K}_3$, $CX \vee (X \wedge C\neg X) \in \mathcal{K}_3$, $CX \vee \partial X \in \mathcal{K}_3$, $\neg CX \vee IX \equiv \neg(CX \wedge C\neg X) \in \mathcal{K}_{18}$, $\neg CX \vee (X \vee \neg CX) \in \mathcal{K}_{10}$, $\neg CX \vee C(\neg X \wedge CX)$, $\neg CX \vee (X \wedge C\neg X) \in \mathcal{K}_{25}$, $\neg CX \vee \partial X \in \mathcal{K}_5$, $C\neg X \vee \neg C\neg X \in \mathcal{K}_7$, $C\neg X \vee (X \vee \neg CX) \in \mathcal{K}_7$, $C\neg X \vee C\neg CX \in \mathcal{K}_5$ $C\neg X \vee C(\neg X \wedge CX) \in \mathcal{K}_5$, $C\neg X \vee (X \wedge C\neg X) \in \mathcal{K}_5$, $C\neg X \vee \partial X \in \mathcal{K}_5$, $(\neg X \wedge CX) \vee IX$, $(\neg X \wedge CX) \vee (X \vee \neg CX) \in \mathcal{K}_7$, $(\neg X \wedge CX) \vee C\neg CX$, $(\neg X \wedge CX) \vee C(\neg X \wedge CX) \in \mathcal{K}_{12}$, $(\neg X \wedge CX) \vee (X \wedge C\neg X) \equiv \partial X \in \mathcal{K}_{14}$, $(\neg X \wedge CX) \vee (CX \wedge C\neg X) \in \mathcal{K}_{14}$, $IX \vee (X \vee \neg CX) \in \mathcal{K}_{10}$, $IX \vee C\neg CX$, $IX \vee C(\neg X \wedge CX)$, $IX \vee (X \wedge C\neg X) \equiv X \wedge (IX \vee C\neg X) \in \mathcal{K}_1$, $IX \vee \partial X \in \mathcal{K}_3$, $(X \vee \neg CX) \vee C\neg CX \equiv X \vee C\neg CX \in \mathcal{K}_{21}$, $(X \vee \neg CX) \vee C(\neg X \wedge CX) \in \mathcal{K}_7$, $(X \vee \neg CX) \vee (X \wedge C\neg X) \in \mathcal{K}_{10}$, $(X \vee \neg CX) \vee \partial X \in \mathcal{K}_7$, $C\neg CX \vee C(\neg X \wedge CX)$, $C\neg CX \vee (X \wedge C\neg X)$, $C\neg CX \vee \partial X$, $C(\neg X \wedge CX) \vee (X \wedge C\neg X)$, $C(\neg X \wedge CX) \vee \partial X \in \mathcal{K}_{14}$ and $(X \wedge C\neg X) \vee \partial X \in \mathcal{K}_{14}$.
The formulas $X \vee C\neg CX$, $\neg X \vee C\neg CX$, $\neg X \vee C(\neg X \wedge CX)$, $(\neg X \wedge CX) \vee IX$, $IX \vee C\neg CX$, $IX \vee C(\neg X \wedge CX)$ and $C\neg CX \vee (X \wedge C\neg X)$ are not equivalent to each other or to any previously listed formula (again, take e.g. $x = [0,1) \setminus \{\frac{1}{2}\} \cup ([2,3) \cap \mathbb{Q}) \subset \mathbb{R}$). We call their equivalence classes $\mathcal{K}_{29}$ through $\mathcal{K}_{35}$.
On the other hand, $Cx = x \vee (\neg x \wedge Cx) \leq x \vee C(\neg x \wedge Cx) \leq Cx$ for any element x of any closure algebra, hence $X \vee C(\neg X \wedge CX) \equiv CX \in \mathcal{K}_3$. Furthermore, $\neg X \vee IX \equiv \neg(X \wedge C\neg X) \in \mathcal{K}_{17}$ and $\neg CX \vee C(\neg X \wedge CX) \equiv \neg X \vee C(\neg X \wedge CX)$, because for any x as before,

$$\big(\neg x \vee C(\neg x \wedge Cx)\big) \wedge \neg\big(\neg Cx \vee (\neg x \wedge Cx)\big) = (\neg x \wedge Cx) \wedge \neg C(\neg x \wedge Cx) = 0,$$

i.e. $\neg x \vee C(\neg x \wedge Cx) \leq \neg Cx \vee C(\neg x \wedge Cx)$; the converse is clear. Hence $\neg CX \vee C(\neg X \wedge CX) \in \mathcal{K}_{31}$. Also,

$$(\neg X \wedge CX) \vee C\neg CX \equiv (\neg X \vee C\neg CX) \wedge (CX \vee C\neg CX) \equiv (\neg X \vee C\neg CX) \in \mathcal{K}_{30},$$

and

$$C\neg CX \vee C(\neg X \wedge CX) \equiv C(\neg CX \vee (\neg X \wedge CX)) \equiv C\neg X \in \mathcal{K}_5.$$

Finally, $C\neg CX \vee (CX \wedge C\neg X) \equiv (C\neg CX \vee CX) \wedge C\neg X \in \mathcal{K}_5$ and for any x as before, $\partial x = (\neg x \wedge Cx) \vee (x \wedge C\neg x) \leq C(\neg x \wedge Cx) \vee (x \wedge C\neg x)$ and $C(\neg x \wedge Cx) \vee (x \wedge C\neg x) \leq C((\neg x \wedge Cx) \vee (x \wedge C\neg x)) = \partial x$, so we have $C(\neg X \wedge CX) \vee (X \wedge C\neg X) \in \mathcal{K}_{14}$.

(6.14) **Remark:** Let α be a closure formula in one variable. If A is a closure algebra and $x_1 \in A$ is any element, define the sequence $(x_n)_{n \geq 1}$ of elements of A recursively as $x_n := f_\alpha^A(x_{n-1})$, for $n \geq 2$. Note that this sequence actually depends only on the equivalence class of α.
Clearly, either the elements x_i are pairwise different, or the sequence becomes periodic after finitely many iterations. The answer to the question of which is the case may differ, depending on the closure algebra A and the element $x_1 \in A$. However, we make the following observations: If the sequence becomes periodic

in every closure algebra and for every starting element, then there is a maximal period length. Indeed, if there were closure algebras A_i and elements $x_1^{(i)}$, such that the period length of the sequence $(x_n^{(i)})_{n\geq 1}$ is at least i, then the sequence in $\prod_{i=1}^{\infty} A_i$ starting with $(x_1^{(1)}, x_1^{(2)}, \ldots)$ would not be periodic. On the other hand, if the sequence becomes periodic in every closure algebra and for every starting element, then the period length of every such sequence is a divisor of the maximal period length. Similarly, if the sequence becomes periodic in every closure algebra and for every starting element, then there is a maximal preperiod.
Therefore, for any equivalence class $\mathcal{K}$ of closure formulas, we can define $d(\mathcal{K})$ as the maximal period length and $r(\mathcal{K})$ as the maximal preperiod of a sequence $(x_n)_{n\geq 1}$ as above, starting from any element of any closure algebra. We set $d(\mathcal{K}) = \infty$ and $r(\mathcal{K}) = \infty$, if there is an element $x_1 \in A$ in a closure algebra A, such that the sequence $(x_n)_{n\geq 1}$ is not periodic.

(6.15) **Example:** Clearly, $d(\mathcal{K}_1) = 1$ and $r(\mathcal{K}_1) = 0$. Furthermore, $d(\mathcal{K}_2) = 2$ and $r(\mathcal{K}_2) = 0$, as well as $d(\mathcal{K}_3) = 1$ and $r(\mathcal{K}_3) = 1$.

(6.16) **Example:** Note that for any element x of any closure algebra, $\neg C\neg x$ is equal to Ix; hence $\neg C\neg Cx = (I \circ C)x$. We have seen already in the introduction that $(I \circ C \circ I \circ C)x = (I \circ Cx)$, therefore, $d(\mathcal{K}_4) \leq 2$ and $r(\mathcal{K}_4) \leq 2$, and the example $x = \mathbb{R} \setminus [0,1] \cup \{\frac{1}{2}\} \subset \mathbb{R}$ shows that we have equality in both cases.
Similarly, $C\neg C\neg x = (C \circ I)x$ and $(C \circ I \circ C \circ I)x = (C \circ I)x$, therefore $d(\mathcal{K}_5) \leq 2$ and $r(\mathcal{K}_5) \leq 2$, and the example $x = [0,1] \setminus \{\frac{1}{2}\} \subset \mathbb{R}$ shows that we have equality in both cases.
Obviously, $d(\mathcal{K}_6) = 1$ and $r(\mathcal{K}_6) = 1$, as well as $d(\mathcal{K}_7) = 1$ and $r(\mathcal{K}_7) = 1$.
We have seen in the example (5.2), that $d(\mathcal{K}_8) = r(\mathcal{K}_8) = \infty$.

(6.17) **Example:** The formula $\neg C\neg X \in \mathcal{K}_9$ is equivalent to IX and I is idempotent (and non-trivial), so $d(\mathcal{K}_9) = 1$ and $r(\mathcal{K}_9) = 1$. Clearly, $r(\mathcal{K}_{10}) \geq 1$, and $(X \vee \neg CX) \vee \neg(X \vee \neg CX) \equiv X \vee \neg CX \vee (\neg CX \wedge (I \circ C)X) \equiv X \vee \neg CX$, so $r(\mathcal{K}_{10}) = 1$ and $d(\mathcal{K}_{10}) = 1$. We have $C\neg C(C\neg CX) \equiv (C \circ I \circ C)X$ and as seen in the introduction $(C \circ I \circ C)\big((C \circ I \circ C)X\big) \equiv (C \circ I \circ C)X$, so $d(\mathcal{K}_{11}) \leq 2$ and $r(\mathcal{K}_{11}) \leq 1$, and the example $x = [0,1) \subset \mathbb{R}$ shows that we have equality in both cases. Furthermore, $r(\mathcal{K}_{12}) \geq 1$ and

$$\begin{aligned} C\big(\neg C(\neg X \wedge CX) \wedge CC(\neg X \wedge CX)\big) &\equiv C\big(\neg C(\neg X \wedge CX) \wedge C(\neg X \wedge CX)\big) \\ &\equiv C(0) \equiv 0 \end{aligned}$$

and $C(\neg 0 \wedge C(0)) = 0$, hence $d(\mathcal{K}_{12}) = 1$ and $r(\mathcal{K}_{12}) = 1$. Clearly, $r(\mathcal{K}_{13}) \geq 1$ and

$$\begin{aligned} (X \wedge C\neg X) \wedge C\neg(X \wedge C\neg X) &\equiv (X \wedge C\neg X) \wedge (C\neg X \vee (C \circ I)X) \\ &\equiv X \wedge C\neg X, \end{aligned}$$

since $(X \wedge C\neg X) \leq C\neg X \leq (C\neg X \vee (C \circ I)X)$, so $d(\mathcal{K}_{13}) = 1$ and $r(\mathcal{K}_{13}) = 1$.
Finally, we claim that $\partial\partial\partial X \equiv \partial\partial X$. Indeed,

$$\begin{aligned} \partial\partial\partial X &\equiv CX \wedge C\neg X \wedge C\neg(CX \wedge C\neg X) \\ &\qquad \wedge C\neg\big(CX \wedge C\neg X \wedge C(\neg(CX \wedge C\neg X))\big) \\ &\equiv \partial\partial X \wedge \neg I(CX \wedge C\neg X \wedge C\neg(CX \wedge C\neg X)) \\ &\equiv \partial\partial X \wedge \neg\big((I \circ C)X \wedge (I \circ C)\neg X \wedge (I \circ C)(\neg(CX \wedge C\neg X))\big) \end{aligned}$$

and hence $\partial\partial\partial X \equiv \partial\partial X$ as claimed, because for any element x of any closure algebra

$$\begin{aligned} \partial\partial x \;\wedge\; &\big((I \circ C)x \wedge (I \circ C)\neg x \wedge (I \circ C)\neg(Cx \wedge C\neg x)\big) \\ &= Cx \wedge C\neg x \wedge (I \circ C)x \wedge (I \circ C)\neg x \wedge C\neg(Cx \wedge C\neg x) \wedge (I \circ C)\neg(Cx \wedge C\neg x) \\ &= (I \circ C)x \wedge (I \circ C)\neg x \wedge (I \circ C)\neg(Cx \wedge C\neg x) \\ &= I\partial x \wedge (I \circ C)\neg\partial x = 0. \end{aligned}$$

Therefore, $d(\mathcal{K}_{14}) = 1$ and $r(\mathcal{K}_{14}) \leq 2$, and the example $x = [0,1] \cap \mathbb{Q} \subset \mathbb{R}$ shows that we have equality.

(6.18) **Example:** We have seen in the introduction that $d(\mathcal{K}_{15}) = d(\mathcal{K}_{19}) = 1$ and $r(\mathcal{K}_{15}) = r(\mathcal{K}_{19}) = 1$. Because for any element x_1 of any closure algebra, the element $x_2 = \neg C(\neg x_1 \wedge Cx_1)$ is open and hence $(\neg x_1 \wedge Cx_1)$ is closed, starting from x_2 the sequence generated by $\mathcal{K}_{16}$ is the same as the sequence generated by $\mathcal{K}_{10}$. Therefore, $d(\mathcal{K}_{16}) = 1$ and $r(\mathcal{K}_{16}) = 2$. With the same argument with $\mathcal{K}_{20}$ and $\mathcal{K}_{13}$, using that $C(x_1 \wedge C\neg x_1)$ is always closed, we see that $d(\mathcal{K}_{20}) = 1$ and $r(\mathcal{K}_{20}) = 2$. Furthermore, $\neg\partial\neg\partial\neg\partial X \equiv \neg\partial\partial\partial X \equiv \neg\partial\partial X \equiv \neg\partial\neg\partial X$, so $d(\mathcal{K}_{18}) = 1$ and $r(\mathcal{K}_{18}) \leq 2$, and the example $x = [0,1] \cap \mathbb{Q} \subset \mathbb{R}$ shows that we have equality. It is easy to check that $d(\mathcal{K}_{24}) = r(\mathcal{K}_{24}) = 1$, $d(\mathcal{K}_{28}) = r(\mathcal{K}_{28}) = 1$ and $d(\mathcal{K}_{29}) = r(\mathcal{K}_{29}) = 1$. On the other hand, we have already seen that $d(\mathcal{K}_{22}) = r(\mathcal{K}_{22}) = \infty$, and the same example $x = \{2k-1 \mid k \in \mathbb{Z}^{\geq 1}\} \subset \mathbb{Z}^{\geq 1}$ with the topology from (4.5) also shows that $d(\mathcal{K}_{17}) = r(\mathcal{K}_{17}) = \infty$, as well as $d(\mathcal{K}_{25}) = r(\mathcal{K}_{25}) = \infty$ and $d(\mathcal{K}_{32}) = r(\mathcal{K}_{32}) = \infty$. Moreover, the example $x = \{2k-1 \mid k \in \mathbb{Z}^{\geq 2}\} \subset \mathbb{Z}^{\geq 1}$ shows that $d(\mathcal{K}_{26}) = r(\mathcal{K}_{26}) = \infty$.
For the remaining equivalence classes, we do not calculate d and r, but only give a lower bound using the examples that have been used throughout this section, as well as combinations of them; see the following table.

(6.19) **Corollary:** The following is a complete list of equivalence classes of closure formula that contain a closure formula of depth at most 5:

Depth	Number	Representant	d	r
1	$\mathcal{K}_1$	X	1	0
2	$\mathcal{K}_2$	$\neg X$	2	0
2	$\mathcal{K}_3$	CX	1	1
3	$\mathcal{K}_4$	$\neg CX$	2	2
3	$\mathcal{K}_5$	$C\neg X$	2	2
3	$\mathcal{K}_6$	$X \wedge \neg X$	1	1
3	$\mathcal{K}_7$	$X \vee \neg X$	1	1
3	$\mathcal{K}_8$	$\neg X \wedge CX$	∞	∞
4	$\mathcal{K}_9$	$\neg C\neg X$	1	1
4	$\mathcal{K}_{10}$	$(X \vee \neg CX)$	1	1
4	$\mathcal{K}_{11}$	$C\neg CX$	1	2
4	$\mathcal{K}_{12}$	$C(\neg X \wedge CX)$	1	1
4	$\mathcal{K}_{13}$	$X \wedge C\neg X$	1	1
4	$\mathcal{K}_{14}$	$CX \wedge C\neg X$	1	2
5	$\mathcal{K}_{15}$	$\neg C\neg CX$	1	1
5	$\mathcal{K}_{16}$	$\neg C(\neg X \wedge CX)$	1	2
5	$\mathcal{K}_{17}$	$\neg(X \wedge C\neg X)$	∞	∞
5	$\mathcal{K}_{18}$	$\neg(CX \wedge C\neg X)$	1	2
5	$\mathcal{K}_{19}$	$C\neg C\neg X$	1	1
5	$\mathcal{K}_{20}$	$C(X \wedge C\neg X)$	1	2
5	$\mathcal{K}_{21}$	$X \wedge C\neg CX$	≥ 1	≥ 1
5	$\mathcal{K}_{22}$	$X \wedge C(\neg X \wedge CX)$	∞	∞
5	$\mathcal{K}_{23}$	$\neg X \wedge C\neg CX$	≥ 2	≥ 4
5	$\mathcal{K}_{24}$	$CX \wedge C\neg CX$	1	1
5	$\mathcal{K}_{25}$	$C\neg X \wedge (X \vee \neg CX)$	∞	∞
5	$\mathcal{K}_{26}$	$(\neg X \wedge CX) \wedge C\neg CX$	∞	∞
5	$\mathcal{K}_{27}$	$(X \vee \neg CX) \wedge C\neg CX$	≥ 2	≥ 2
5	$\mathcal{K}_{28}$	$C\neg CX \wedge C(\neg X \wedge CX)$	1	1
5	$\mathcal{K}_{29}$	$X \vee C\neg CX$	1	1
5	$\mathcal{K}_{30}$	$\neg X \vee C\neg CX$	≥ 2	≥ 3
5	$\mathcal{K}_{31}$	$\neg X \vee C(\neg X \wedge CX)$	≥ 2	≥ 2
5	$\mathcal{K}_{32}$	$(\neg X \wedge CX) \vee \neg C\neg X$	∞	∞
5	$\mathcal{K}_{33}$	$\neg C\neg X \vee C\neg CX$	≥ 2	≥ 1
5	$\mathcal{K}_{34}$	$\neg C\neg X \vee C(\neg X \wedge CX)$	≥ 2	≥ 2
5	$\mathcal{K}_{35}$	$C\neg CX \vee (X \wedge C\neg X)$	≥ 2	≥ 2.